Hindawi
Journal of Mathematics
Volume 2021, Article ID 6756908, 7 pages
https://doi.org/10.1155/2021/6756908

Research Article

On the Recurrent C_0-Semigroups, Their Existence, and Some Criteria

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Received 4 July 2021; Accepted 3 September 2021; Published 16 September 2021

Academic Editor: Tuncer Acar

In this paper, recurrent C_0-semigroups are introduced and investigated. It is proved that, despite hypercyclic C_0-semigroups, recurrent C_0-semigroups can be found on finite-dimensional Banach spaces. Some criteria are stated for recurrence, which is based on open sets, neighborhoods of zero, and special eigenvectors. It is established that having a dense set of recurrent vectors is a sufficient and necessary condition for a C_0-semigroup to be recurrent. Moreover, the direct sum of recurrent C_0-semigroups is investigated.

1. Introduction

The study of the dynamical system is a notable branch in mathematics. Hypercyclicity, recurrency, chaoticity, and mixing are investigated in this branch. For a Banach space X, an operator is named hypercyclic if $\overline{\text{orb}(T, x)} = X$ for some $x \in X$ or equivalently $T^{-n}(U) \cap V \neq \phi$ for some integer $n \geq 1$, where U and V are arbitrary open sets in X. If for any open set U in X, a positive integer n can be chosen such that $T^{-n}(U) \cap U \neq \phi$; then, T is called a recurrent operator. Recurrence is a remarkable case of hypercyclicity. In fact, in this case, the inverse image of any open set under the operator intersects with itself. More information are accessible in [1, 2].

A hypercyclic operator T on X, with a dense set of periodic points is named a chaotic operator. Also, T is called mixing if for any open set U and V of X, $T^n(U) \cap V \neq \phi$ for any n greater or equal a natural number N. References [3, 4] contain valuable information about the above notions and results.

One of the significant structures that are considered by mathematicians is C_0-semigroups. Suppose that $(T_t)_{t\geq 0}$ be a family of operators. $(T_t)_{t\geq 0}$ is called a C_0-semigroup if for any s and $t \geq 0$, $T_0 = I$, $T_{s+t} = T_s T_t$, and $\lim_{s\longrightarrow t} T_s x = T_t x$, for any $x \in X$.

Similar to operators, hypercyclic C_0-semigroups, chaotic C_0-semigroups, and mixing C_0-semigroups are defined.

A C_0-semigroup $(T_t)_{t\geq 0}$ on X is named hypercyclic if it has a dense orbit or equivalently for any open sets U and V of X, there is $t > 0$ such that $T_t^{-1}(U) \cap V \neq \phi$. One can see [5] either. Moreover, there are remarkable criteria in this matter that can be found in [6]. Also, new points about constructive approximation of semigroups can be observed in [7]. A hypercyclic C_0-semigroup $(T_t)_{t\geq 0}$ is called chaotic if it has a dense set of periodic points. It means there are points like $x \in X$ with this property that $T_t x = x$ for some $t > 0$. Also, a C_0-semigroup $(T_t)_{t\geq 0}$ on X is called mixing if for any open sets U and V of X, $s > 0$ can be chosen such that, for any $t \geq s$, $T_t(U) \cap V \neq \phi$.

In the following, we mention hypercyclicity criterion and a recurrent hypercyclicity criterion for C_0-semigroups.

Definition 1 (see [8])(HCC). A semigroup $(T_t)_{t\geq 0}$ on X fulfills the hypercyclicity criterion if and only if $t > 0$ can be found such that

$$\begin{aligned} &T_t(U) \cap W \neq \phi, \\ &T_t(W) \cap V \neq \phi, \end{aligned} \tag{1}$$

where U and V are arbitrary nonempty and open sets and W is an arbitrary neighborhood of zero in X.

Definition 2 (see [8])(RHCC). A semigroup $(T_t)_{t\geq 0}$ on X fulfills the recurrent hypercyclicity criterion if and only if for any nonempty and open sets U and V and any neighborhood W_1 of zero in X, $L_1 > 0$, and $L_2 > 0$ can be found such that, for any $t\geq 0$, $s_1 \in J \cap [t, t + L_1)$ and $s_2 \in J \cap [t, t + L_2)$ can be found such that

$$\begin{aligned} &T_{s_1}(U) \cap W \neq \phi, \\ &T_{s_2}(W) \cap V \neq \phi. \end{aligned} \tag{2}$$

Desch and Schappacher [8] proved that if a semigroup is RHCC, then it is HCC. In [9, 10], more matters can be found about properties of C_0-semigroups.

By various types of C_0-semigroups and their notable properties, it sounds interesting to investigate C_0-semigroups with this property that returns back an open set to itself. If we name this new concept as recurrent C_0-semigroup, the question arises as to whether we can find interesting properties for them or whether we can get some relations between them and concepts such as hypercyclicity, chaoticity, and mixing for C_0-semigroups?

In this paper, we introduce the concept of recurrent C_0-semigroups and look into their properties. In Section 2, we state some preliminaries about this concept. We show that the recurrence of any operator in a C_0-semigroup implies the C_0-semigroup recurrence. Also, if $(T_t)_{t\geq 0}$ satisfies HCC, then the recurrence of T_t, can be concluded for any $t > 0$.

In Section 3, recurrent vectors for C_0-semigroups are defined. It is proved that a C_0-semigroup $(T_t)_{t\geq 0}$ on X is recurrent if and only if $\overline{\mathrm{Rec}(T_t)_{t\geq 0}} = X$. It is proved that recurrence preserves under conjugacy. We establish that, despite hypercyclic C_0-semigroups, recurrent C_0-semigroups can be constructed on finite-dimensional spaces. Also, we make various examples of recurrent C_0-semigroups.

In Section 4, various criteria for recurrence of C_0-semigroups are presented. The conditions of these criteria are based on open sets, neighborhoods of zero, dense sets, and special eigenvectors.

In Section 5, we investigate the recurrence of the direct sum of two C_0-semigroups. We establish that recurrence of the direct sum of two semigroups implies recurrence of each of them. Moreover, if at least one of them be mixing, the converse is also true. Additionally, it is proved that if one of the C_0-semigroups is RHCC and the other C_0-semigroup is HCC, then their direct sum is recurrent.

2. Preliminaries

We begin this section by defining the concept of recurrent C_0-semigroup.

Definition 3. We say a C_0-semigroup $(T_t)_{t\geq 0}$ is recurrent if for any open and nonempty set U, some $t > 0$ can be found such that $T_t^{-1}(U) \cap U \neq \phi$.

By Definition 3, it is not complicated to see that if $(T_t)_{t\geq 0}$ is a hypercyclic C_0-semigroup or a chaotic C_0-semigroup, then $(T_t)_{t\geq 0}$ is recurrent. Moreover, we establish that hypercyclicity (chaoticity) of an operator in a C_0-semigroup implies its hypercyclicity (chaoticity) as follows.

Corollary 1. *Suppose that $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. If T_s be hypercyclic or chaotic for some $s > 0$, then $(T_t)_{t\geq 0}$ is recurrent.*

Proof. Hypercyclicity (chaoticity) of T_s for some $s > 0$ indicates that $(T_t)_{t\geq 0}$ is hypercyclic (chaotic). Accordingly, $(T_t)_{t\geq 0}$ is recurrent.

The next theorem shows that the recurrence of any operator in a C_0-semigroup implies its recurrence. □

Theorem 1. *Consume $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. Then, recurrence of $(T_t)_{t\geq 0}$ is derived from recurrence of any of T_t's.*

Proof. Let T_{t_0} be recurrent for some $t_0 > 0$. Consider U be an open and nonempty set. Hence, there is $n \in \mathbb{N}$ such that $T_{t_0}^{-n}(U) \cap U \neq \phi$. Suppose $x \in T_{t_0}^{-n}(U) \cap U$. Then, $T_{t_0}^{n}x \in U$ and $x \in U$. But $(T_t)_{t\geq 0}$ is a C_0-semigroup. Hence, $T_{t_0}^{n} = T_{nt_0}$. So, $T_{nt_0}x \in U$. Therefore, $x \in T_{nt_0}^{-1}(U) \cap U$. Thus, $(T_t)_{t\geq 0}$ is recurrent.

Now, this question arises that if the converse of Theorem 1 is true?

The next theorem manifests that the answer is affirmative if $(T_t)_{t\geq 0}$ fulfills the hypercyclicity criterion. □

Theorem 2. *If $(T_t)_{t\geq 0}$ fulfills the hypercyclicity criterion, then T_t is a recurrent operator for any $t\geq 0$.*

Proof. Let $(T_t)_{t\geq 0}$ fulfills the hypercyclicity criterion. Thus, $(T_t)_{t\geq 0}$ is hypercyclic (Theorem 7.27 in [4]). Hence, T_t is hypercyclic, and so, it is recurrent for any $t\geq 0$ (Theorem 2.3 in [11]).

The following lemma states that, for a C_0-semigroup $(T_t)_{t\geq 0}$ and any open and nonempty set U, $T_t^{-1}(U) \cap U \neq \phi$ for infinitely many $t > 0$. □

Lemma 1. *Let $(T_t)_{t\geq 0}$ be a recurrent C_0-semigroup. Then,*

$$\left\{t > 0\colon T_t^{-1}(U) \cap U \neq \phi\right\}, \tag{3}$$

which is an infinite subset of $\mathbb{R}^+$.

Proof. Assume, on the contrary, there is an open and nonempty set U such that

$$\left\{t > 0\colon T_t^{-1}(U) \cap U \neq \phi\right\} = \{t_1, t_2, \ldots, t_n\}. \tag{4}$$

Suppose that t_n be the greatest number in the right set of (4). Now, $T_{t_n}^{-1}(U) \cap U$ is nonempty and open. So, there is $t_s > 0$ such that

$$T_{t_s}^{-1}\left(T_{t_n}^{-1}(U) \cap U\right) \cap \left(T_{t_n}^{-1}(U) \cap U\right) \neq \phi. \tag{5}$$

Hence, $T^{-1}_{t_n+t_s}(U)\cap U\neq\phi$. That means t_n+t_s belongs to the left set of (4). But this is a contradiction since $t_n+t_s>t_n$. □

3. Recurrent Vectors, Frequently Recurrent Vectors, and Finite Dimensions

A vector $x\in X$ is named a recurrent vector for an operator T if $T^{n_k}x\longrightarrow x$, where (n_k) is an increasing sequence. By this notion, we can state the next lemma.

Lemma 2. *Suppose that $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. Then, $(T_t)_{t\geq 0}$ is recurrent if for some $t_0>0$, Rec(T_{t_0}) be dense in X.*

Proof. Suppose that $\overline{\text{Rec}(T_{t_0})}=X$ for some $t_0>0$. Then, T_{t_0} is recurrent (Proposition 2.1 in [1]). Now, by Theorem 1, $(T_t)_{t\geq 0}$ is recurrent.

The concept of the recurrent vector can be defined for C_0-semigroups as follows. □

Definition 4. A vector x is named a recurrent vector for $(T_t)_{t\geq 0}$ if $T_{t_n}x\longrightarrow x$ for some increasing sequence (t_n). We denote the set of recurrent vectors of $(T_t)_{t\geq 0}$ by Rec$(T_t)_{t\geq 0}$.

It is affirmed in the next theorem that having a dense set of recurrent vectors is an equivalent condition for a C_0-semigroup to be recurrent.

Theorem 3. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. Then, $(T_t)_{t\geq 0}$ is recurrent if and only if $\overline{\text{Rec}(T_t)_{t\geq 0}}=X$.*

Proof. Let $\overline{\text{Rec}(T_t)_{t\geq 0}}=X$. Suppose that U be an open and nonempty subset of X. So, $x\in\text{Rec}((T_t)_{t\geq 0})\cap U$ can be found. Hence, there is an increasing sequence (t_n) such that $T_{t_n}x\longrightarrow x$. Therefore, $n_0\in\mathbb{N}$ can be chosen such that $T_{t_{n_0}}x\in U$. Hence, $x\in T_{t_{n_0}}{}^{-1}(U)\cap U$.

Now, suppose that $(T_t)_{t\geq 0}$ $x\in X$. Consider $U=B(x,\epsilon)$, where $\epsilon<1$. By recurrence of $(T_t)_{t\geq 0}$, $t_0>0$ can be found such that $T^{-1}_{t_0}(U)\cap U\neq\phi$. So, there is $x_1\in T^{-1}_{t_0}(U)\cap U$.

Let $\epsilon_1<(1/2)$ be such that

$$U_1=B(x_1,\epsilon_1)\subseteq T^{-1}_{t_0}(U)\cap U. \tag{6}$$

Another by recurrence, $t_1>t_0$ can be found such that $T^{-1}_{t_1}(U_1)\cap U_1\neq\phi$. Therefore, $x_2\in X$ and $\epsilon_2<(1/2)^2$ can be chosen such that $x_2\in T^{-1}_{t_1}(U_1)\cap U_1$ and

$$U_2=B(x_2,\epsilon_2)\subseteq T^{-1}_{t_1}(U_1)\cap U_1. \tag{7}$$

Inductively, an increasing sequence (t_n) can be found such that, for any $n\geq 1$,

$$U_n=B(x_n,\epsilon_n)\subseteq T^{-1}_{t_{n-1}}(U_{n-1})\cap U_{n-1}. \tag{8}$$

So, for any $n\geq 1$,

$$T_{t_{n-1}}(B(x_n,\epsilon_n))\subseteq U_{n-1}. \tag{9}$$

Now, the Cantor theorem implies that $\cap_{n=1}^{\infty}U_n=\{z\}$ for some $z\in X$. Also, $T_{t_n}z\longrightarrow z$ by (9). Hence, z is a recurrent vector. Moreover, $\|x-z\|<\epsilon$.

Assume that $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ be two C_0-semigroups on spaces X and Y, respectively. If a continuous map φ: $Y\longrightarrow X$ can be found with this property that $T_t\circ\varphi=\varphi\circ S_t$ for any $t\geq 0$, then $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are named quasi-conjugate. We state in the following theorem that quasi-conjugacy preserves recurrence. □

Theorem 4. *Recurrence of C_0-semigroups preserves under quasi-conjugacy.*

Proof. Let $(S_t)_{t\geq 0}$ be a recurrent C_0-semigroup, and let $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ be quasi-conjugate. Let U be an open and nonempty set. Now, $\varphi^{-1}(U)$ is open by continuity of φ. Since $(S_t)_{t\geq 0}$ has a dense set of recurrent vectors, there exists $x\in\text{Rec}(S_t)_{t\geq 0}$ such that $x\in\varphi^{-1}(U)$. Hence, $n_0\in\mathbb{N}$ can be found such that $S_{t_{n_0}}x\in\varphi^{-1}(U)$. So, there is $n_0>0$ such that $\varphi\circ S_{t_{n_0}}x\in U$. Now, by conjugacy, $\varphi\circ S_{t_{n_0}}=T_{t_{n_0}}\circ\varphi$, and thus, we can conclude that $T_{t_{n_0}}\circ\varphi x\in U$. Hence, $\varphi(x)\in T_{t_{n_0}}{}^{-1}(U)\cap U$. Therefore, $(T_t)_{t\geq 0}$ is recurrent.

Pay attention to the fact that periodic points of a C_0-semigroup are recurrent. In fact, if x be a periodic point for C_0-semigroup $(T_t)_{t\geq 0}$, then $T_{t_0}x=x$ for some $t_0>0$. Hence, for any $n\in\mathbb{N}$, we have $T_{nt_0}x=x$. Therefore, $T_{nt_0}x\longrightarrow x$, and hence, x is a recurrent vector for $(T_t)_{t\geq 0}$.

By this fact, interesting examples can be constructed as follows. □

Example 1. Suppose that $(T_t)_{t\geq 0}$ be a C_0-semigroup on $\mathbb{T}$ so that, for any $t\geq 0$, T_t: $\mathbb{T}\longrightarrow\mathbb{T}$ is defined by $T_t(x)=e^{it}x$. Let $t_0=(2/N)\pi$. So, $T^N_{t_0}=I$, and hence, every point of $\mathbb{T}$ is a periodic point for T_{t_0}. Hence, the set of periodic points of T_{t_0} is dense, and hence, T_{t_0} is recurrent. So, by Theorem 3, $(T_t)_{t\geq 0}$ is recurrent.

Bermúdez et al. in [12] proved that there are hypercyclic C_0-semigroups that are not chaotic. Hence, there exist recurrent C_0-semigroups that do not have a dense set of periodic points.

By Theorem 2.4 in [12], hypercyclic C_0-semigroups can be found on any infinite-dimensional and separable Banach spaces. So, we can deduce that recurrent C_0-semigroups exist on these spaces since hypercyclic C_0-semigroups are recurrent. By Example 1, we can deduce the following theorem.

Theorem 5. *There are finite-dimensional Banach spaces that support recurrent C_0-semigroups.*

Proof. In Example 1, we make a recurrent C_0-semigroup on a finite-dimensional Banach space. So, recurrent C_0-semigroups can be found on finite-dimensional spaces.

As it is established in Theorem 7.15 in [4], hypercyclic C_0-semigroups cannot be built on finite-dimensional spaces. So, we can state the following corollary. □

Corollary 2. *The set of hypercyclic C_0-semigroups is a proper subset of the set of recurrent C_0-semigroups.*

Proof. The proof is evident by Theorem 5 and this fact that hypercyclic C_0-semigroups do not exist on finite-dimensional spaces.

Frequently recurrent vectors for operators are defined in [13]. Similarly, frequently recurrent vectors for C_0-semigroups can be defined as follows. □

Definition 5. A vector $x \in X$ is named a frequently recurrent vector for a C_0-semigroup $(T_t)_{t\geq 0}$ on X if

$$\bigcup_{t\geq 0}\{n \geq 0\colon T_{nt}x \in U\}, \tag{10}$$

has positive lower density for any open set U that contains x. We denote the set of frequently recurrent vectors of $(T_t)_{t\geq 0}$ by $F\mathrm{Rec}(T_t)_{t\geq 0}$.

By Definition 5, it is not complicated to see that

$$\mathrm{Per}(T_t)_{t\geq 0} \subseteq F\mathrm{Rec}(T_t)_{t\geq 0} \subseteq \mathrm{Rec}(T_t)_{t\geq 0}. \tag{11}$$

By Theorem 3 and (11), we can conclude the following corollary.

Corollary 3. *If a C_0-semigroup $(T_t)_{t\geq 0}$ has a dense set of frequently recurrent vectors, then it is recurrent.*

In the next example, a C_0-semigroup is constructed with a dense set of frequently recurrent vectors.

Example 2. Let $X = C_0(\mathbb{R}^+)$, where

$$C_0(\mathbb{R}^+) = \left\{f\colon f\colon \mathbb{R}^+ \longrightarrow \mathbb{C}, \quad \lim_{x\longrightarrow\infty} f(x) = 0\right\}. \tag{12}$$

Consider $\|f\| = \sup_{x\in\mathbb{R}^+}|f(x)|$. Let $\beta > 0$. If for any $f \in C_0(\mathbb{R}^+)$ and $x \in \mathbb{R}^+$ we define

$$T_t f(x) = e^{\beta t} f(x+t), \tag{13}$$

then $(T_t)_{t\geq 0}$ is a chaotic C_0-semigroup (Example 7.10 in [4]). Hence, it has a dense set of periodic points in X. Therefore, $(T_t)_{t\geq 0}$ has a dense set of frequently recurrent vectors.

4. Some Criteria for Recurrence of C_0-Semigroups

Some criteria for recurrency of C_0-semigroups are presented in this section. The conditions of the first theorem are based on open sets and neighborhoods of zero. Moreover, it has weaker conditions than conditions in HCC.

Theorem 6. *Assume $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. If for any open and nonempty set U and any neighborhood W of zero, there is some $t > 0$ such that*

$$\begin{aligned} T_t(U) \cap W &\neq \phi,\\ T_t(W) \cap U &\neq \phi, \end{aligned} \tag{14}$$

and then, $(T_t)_{t\geq 0}$ is recurrent.

Proof. Suppose that U be an open and nonempty set. So, there is an open set U_1 and a neighborhood W_1 of zero so that $U_1 + W_1 \subseteq U$ (Lemma 2.36 in [4]). By hypothesis, there is $t > 0$ such that

$$\begin{aligned} T_t(U_1) \cap W_1 &\neq \phi,\\ T_t(W_1) \cap U_1 &\neq \phi. \end{aligned} \tag{15}$$

Hence, there are $u \in U_1$ and $w \in W_1$ such that

$$\begin{aligned} T_t(u) &\in W_1,\\ T_t(w) &\in U_1. \end{aligned} \tag{16}$$

Now, by (16),

$$\begin{aligned} T_t(u+w) = T_t u + T_t w &\in U,\\ u + w &\in U. \end{aligned} \tag{17}$$

Therefore, $u + w \in U$ and $T_t(u+w) \in U$. Hence, $t > 0$ can be selected such that $T_t^{-1}(U) \cap U \neq \phi$. So, $(T_t)_{t\geq 0}$ is recurrent.

We can rewrite Theorem 6 as follows. □

Theorem 7. *Consider $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. If there is $\alpha > 0$ such that, for any open and nonempty set U and any neighborhood W of zero, $t > 0$ can be found so that*

$$\begin{aligned} T_t(U) \cap W &\neq \phi,\\ T_{t+\alpha}(W) \cap U &\neq \phi, \end{aligned} \tag{18}$$

and then, $(T_t)_{t\geq 0}$ is recurrent.

Proof. Assume that W is a neighborhood of zero. Consider $W' := W \cap T_\alpha^{-1}(W)$. Then, W' is a nonempty neighborhood of zero. By hypothesis, $t > 0$ can be selected such that

$$\begin{aligned} T_t(U) \cap W' &\neq \phi,\\ T_{t+\alpha}(W') \cap U &\neq \phi. \end{aligned} \tag{19}$$

Hence,

$$\begin{aligned} T_t(U) \cap \left(W \cap T_\alpha^{-1}(W)\right) &\neq \phi,\\ T_{t+\alpha}\left(W \cap T_\alpha^{-1}(W)\right) \cap U &\neq \phi. \end{aligned} \tag{20}$$

Therefore,

$$\begin{aligned} T_t(U) \cap T_\alpha^{-1}(W) &\neq \phi,\\ T_{t+\alpha}(W) \cap U &\neq \phi. \end{aligned} \tag{21}$$

So,

$$\begin{aligned} T_{t+\alpha}(U) \cap W &\neq \phi,\\ T_{t+\alpha}(W) \cap U &\neq \phi. \end{aligned} \tag{22}$$

Now, by Theorem 6, $(T_t)_{t\geq 0}$ is recurrent.

In Theorem 7, α is a positive and arbitrary scalar. So, by considering $\alpha = 1$, we get the following corollary. □

Corollary 4. *Suppose that $(T_t)_{t\geq 0}$ is a C_0-semigroup on X. If for any open and nonempty set U and any neighborhood W of zero, $t > 0$ can be found so that*

$$T_t(U) \cap W \neq \phi,$$
$$T_{t+1}(W) \cap U \neq \phi, \tag{23}$$

then $(T_t)_{t\geq 0}$ *is recurrent.*

Proof. It is enough to consider $\alpha = 1$ in Theorem 7

Like Theorem 4.1 in [14], we can define another recurrent criterion that is based on dense subsets. □

Theorem 8. *Assume that* $(T_t)_{t\geq 0}$ *be a* C_0*-semigroup on* X*. Suppose that there is a dense subset* Y *of* X *and there is a sequence* (t_n) *of positive real numbers such that*

(i) For any $y \in Y$, $T_{t_n} y \longrightarrow 0$

(ii) For any $y \in Y$*, there is* (y_n) *in* X *such that* $y_n \longrightarrow 0$ *and* $T_{t_n} y_n \longrightarrow y$

Then, $(T_t)_{t\geq 0}$ *is recurrent.*

Proof. Assume that U be an open and nonempty set. By density of Y, there is $y \in U \cap Y$. Hence, $\epsilon > 0$ can be chosen such that $B(y, \epsilon) \subseteq U$.

By (i), $T_{t_n} y \longrightarrow 0$, and by (ii), (y_n) can be found such that $y_n \longrightarrow 0$ and $T_{t_n} y_n \longrightarrow y$. So, for sufficiently large N, we have

$$\left\| T_{t_n} y \right\| < \frac{\epsilon}{3},$$
$$\left\| y_n \right\| < \frac{\epsilon}{3}, \tag{24}$$
$$\left\| T_{t_n} y_n - y \right\| < \frac{\epsilon}{3}.$$

Therefore, if we consider $x: = y + y_n$, then by (24),

$$\|x - y\| < \epsilon,$$
$$\left\| T_{t_n} x - y \right\| < \epsilon. \tag{25}$$

Hence, $x \in U$ and $T_{t_n} x \in U$. It means that $T_{t_n}^{-1}(U) \cap U \neq \phi$. Thus, $(T_t)_{t\geq 0}$ is recurrent.

By Theorem 2.3 in [15], the idea of the next theorem comes to mind. □

Theorem 9. *Assume* $(T_t)_{t\geq 0}$ *be a* C_0*-semigroup on* X*. Suppose that a dense subset* Y *of* X *exists such that*

(i) For any $y \in Y$, $T_t y \longrightarrow 0$ *when* $t \longrightarrow 0$

(ii) For any $y \in Y$ *and for any* $\epsilon > 0$, $x \in X$ *and* $t > 0$ *exist such that*

$$\|x\| < \epsilon,$$
$$\left\| T_t x - y \right\| < \epsilon. \tag{26}$$

Then, $(T_t)_{t\geq 0}$ *is recurrent.*

Proof. Suppose that U be an open and nonempty set. Let $y \in U \cap Y$, and let $\epsilon > 0$ be such that $B(y, \epsilon) \subseteq U$. By (i), there is $t_0 > 0$ such that

$$\left\| T_t y \right\| < \frac{\epsilon}{3}, \quad \text{for any } t \geq t_0. \tag{27}$$

By (ii), there is $x \in X$ such that

$$\|x\| < \epsilon,$$
$$\left\| T_t x - y \right\| < \frac{\epsilon}{3}. \tag{28}$$

So, if we consider $z := x + y$, by (27) and (28), similar to the proof of Theorem 8, $t > 0$ can be found such that $z \in T_t^{-1}(U) \cap U$.

It is considerable that if a C_0-semigroup fulfills the conditions of Theorem 9, then it fulfills the conditions of Theorem 8.

The next sufficient condition for recurrence is based on special eigenvectors of operators of a C_0-semigroup. □

Theorem 10. *Consume* $(T_t)_{t\geq 0}$ *is a* C_0*-semigroup on* X*. Assume that* $|\lambda| < 1$ *and* $(t_n) \subseteq \mathbb{R}^+$ *exist such that*

$$Y = \operatorname{span}\left\{ z \in X \colon T_{t_n} z = \lambda^n z, n \in \mathbb{N} \right\}, \tag{29}$$

be dense in X*. Then,* $(T_t)_{t\geq 0}$ *is recurrent.*

Proof. Assume that U is an open and nonempty subset of X. By density of Y, we can find $x \in U \cap Y$. So, we can write $x = \sum_{i=1}^{p} \alpha_i z_i$, where $T_{t_n} z_i = \lambda^n z_i$ and $\alpha_i \in \mathbb{C}$ for $1 \leq i \leq p$. Now,

$$T_{t_n} x = \sum_{i=1}^{p} \alpha_i T_{t_n}(z_i) = \sum_{i=1}^{p} \alpha_i \lambda^n z_i \longrightarrow 0. \tag{30}$$

If we consider $y_n = \sum_{i=1}^{p} (\alpha_i / \lambda^n) z_i$, then $y_n \longrightarrow 0$ and $T_{t_n} y_n \longrightarrow x$ since

$$T_{t_n} y_n = \sum_{i=1}^{p} \frac{\alpha_i}{\lambda^n} T_{t_n} z_i = \sum_{i=1}^{p} \alpha_i z_i = x. \tag{31}$$

Hence, $x + y_n \longrightarrow x$ and

$$T_{t_n}(x + y_n) = T_{t_n}(x) + T_{t_n}(y_n) \longrightarrow x. \tag{32}$$

So, for a large enough N, $x + y_N \in U$ and $T_{t_N}(x + y_n) \in U$. Accordingly, $(T_t)_{t\geq 0}$ is recurrent.

By Theorem 10, we can state the following interesting corollary. □

Corollary 5. *Consume* $(T_t)_{t\geq 0}$ *is a* C_0*-semigroup on* X*. Assume that* $|\lambda| < 1$ *is a scalar. If there is* $s > 0$ *such that*

$$Y = \operatorname{span}\{ z \in X \colon T_s z = \lambda z \}, \tag{33}$$

be dense in X*, then* $(T_t)_{t\geq 0}$ *is recurrent.*

Proof. Suppose that $y \in Y$. Hence, $y = \sum_{i=1}^{m} \alpha_i z_i$, where $T_s^n z_i = \lambda^n z_i$. But $T_s^n = T_{ns}$. So, if we consider $t_n := ns$, by Theorem 10, $(T_t)_{t\geq 0}$ is recurrent. □

5. Direct Sum of C_0-Semigroups

We investigate the direct sum of recurrent C_0-semigroups and their properties in this section. We begin this section by showing that recurrence of the direct sum of two C_0-semigroups implies recurrence of each of them.

Theorem 11. *If $(S_t \oplus T_t)_{t\geq 0}$ be a recurrent C_0-semigroup on $X\oplus Y$, then $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are recurrent on X and Y, respectively.*

Proof. Assume that U be an open and nonempty set in X. Then, $U\oplus\{0\}$ is an open set of $X\oplus Y$. By hypothesis, $(S_t\oplus T_t)_{t\geq 0}$ is recurrent. So, there exists $t>0$ so that

$$\left(S_t\oplus T_t\right)^{-1}(U\oplus\{0\})\cap(U\oplus\{0\})\neq\phi. \tag{34}$$

Hence, $S_t^{-1}(U)\cap U\neq\phi$. This indicates that $(S_t)_{t\geq 0}$ is recurrent. Similarly, $(T_t)_{t\geq 0}$ is recurrent.

It is proved that the recurrence of $(S_t\oplus T_t)_{t\geq 0}$ implies the recurrence of $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$. But if the converse is true?

In the following theorem, we state that the answer to the question is positive if at least one of them be mixing. □

Theorem 12. *Suppose that $(S_t)_{t\geq 0}$ is a recurrent C_0-semigroup on X, and suppose that $(T_t)_{t\geq 0}$ is a C_0-semigroup on Y.*

(i) If $(T_t)_{t\geq 0}$ be a mixing semigroup, then $(S_t\oplus T_t)_{t\geq 0}$ is recurrent

(ii) If there is $t_0>0$ such that $(T_{nt_0})_{n\in\mathbb{N}}$ be mixing, then $(S_t\oplus T_t)_{t\geq 0}$ is recurrent

Proof. To prove part (i), consume $U\oplus V$ is an open subset of $X\oplus Y$. By hypothesis, $(T_t)_{t\geq 0}$ is mixing. So, there is $t_1>0$ such that, for any $t\geq t_1$,

$$T_t^{-1}(V)\cap V\neq\phi. \tag{35}$$

Also, $(S_t)_{t\geq 0}$ is recurrent. So, there is $t_2>0$ such that

$$S_{t_2}^{-1}(V)\cap V\neq\phi. \tag{36}$$

By Lemma 1, we can assume that $t_2>t_1$. Hence, by (35) and (36),

$$\begin{aligned}&\left(T_{t_2}\oplus S_{t_2}\right)^{-1}(U\oplus V)\cap(U\oplus V)\\&=\left(T_{t_2}^{-1}(U)\oplus S_{t_2}^{-1}(V)\right)\cap(U\oplus V)\\&=\left(T_{t_2}^{-1}(U)\cap U\right)\oplus\left(S_{t_2}^{-1}(V)\cap V\right)\neq\phi.\end{aligned} \tag{37}$$

For proving part (ii), note that, by Proposition 7.21 in [4], mixing of $(T_{nt_0})_{n\in\mathbb{N}}$ implies that $(T_t)_{t\geq 0}$ is mixing. Hence, $(S_t\oplus T_t)_{t\geq 0}$ is recurrent by part (i).

The following corollary can be deduced from Theorem 12. □

Corollary 6. *Consume $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are mixing C_0-semigroups on X and Y, respectively. Then, $(S_t\oplus T_t)_{t\geq 0}$ is recurrent.*

Especially, $(S_t\oplus S_t)_{t\geq 0}$ and $(T_t\oplus T_t)_{t\geq 0}$ are recurrent.

So, notable examples can be made by using mixing C_0-semigroups as follows.

Example 3. Consider $(e_n)_{n\in\mathbb{N}}$ to be a sequence in X with this property that $\overline{\mathrm{span}\{e_n\colon n\in\mathbb{N}\}}=X$. Consider B is an operator on X so that $Be_1=0$ and $Be_n=e_{n-1}$. It is proved that the generated C_0-semigroup by B is mixing (Theorem 1.6 in [16]). So, if we denote this C_0-semigroup with $(T_t)_{t\geq 0}$, then by Corollary 6, $(T_t)_{t\geq 0}$ is recurrent.

Also, we can state the following lemmas by using RHCC and HCC.

Lemma 3. *If $(S_t)_{t\geq 0}$ is RHCC and $(T_t)_{t\geq 0}$ is HCC, then $(S_t\oplus T_t)_{t\geq 0}$ is recurrent.*

Proof. It is deduced from Lemma 5.3 in [8] that $(S_t\oplus T_t)_{t\geq 0}$ is HCC. Hence, it is recurrent. □

Lemma 4. *If $(T_t)_{t\geq 0}$ is HCC, then $(T_t\oplus T_t)_{t\geq 0}$ is recurrent. Especially, $(T_t)_{t\geq 0}$ is recurrent.*

Proof. If $(T_t)_{t\geq 0}$ is HCC, then by Theorem 2.5 in [16] and Theorem 7.28 in [4], $(T_t\oplus T_t)_{t\geq 0}$ is hypercyclic, and so, it is recurrent.

Also, we make the following example by Lemma 4. □

Example 4. Let $I=[0,\infty]$, and let ρ denote an admissible weight function on I. Assume that $X=L_p^{\rho}(I,\mathbb{R})$. If $\inf_{t\longrightarrow\infty}\rho(t)=0$, then $(T_t)_{t\geq 0}$ fulfills the hypercyclicity criterion (Proposition 4.4 in [8]). Then, $(T_t\oplus T_t)_{t\geq 0}$ is recurrent.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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