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Research Article

A Modified Proximal Point Algorithm and Some Convergence Results

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In this paper, the convergence to minimizers of a convex function of a modified proximal point algorithm involving a single-valued nonexpansive mapping and a multivalued nonexpansive mapping in CAT(0) spaces is studied and a numerical example is given to support our main results.

1. Introduction

In daily life, no matter what we do, there are always many options available and many possible outcomes. When we do these things, we always consciously or unconsciously choose an optimal solution in order to achieve the optimal result. The discipline of seeking the best solution to achieve the best result is optimization. The way to find the optimal solution is the optimization method.

Given a real number κ (curvature), let M_κ^2 denote the following space: if $\kappa < 0$, then M_κ^2 is a real hyperbolic space H^2 with the distance function scaled by a factor of $(1/\sqrt{-\kappa})$; if $\kappa = 0$, then M_κ^2 is the Euclidean plane; if $\kappa > 0$, then M_κ^2 is the 2-sphere S^2 with the metric scaled by a factor $(1/\sqrt{\kappa})$. Let D_κ denote the diameter of M_κ^2. Let Δ be a geodesic triangle in X with a perimeter less than $2D_\kappa$. Let $\overline{\Delta} \in M_\kappa^2$ be a comparison triangle for $\overline{\Delta}$. Then, Δ is said to satisfy the CAT(κ) inequality if for $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x, y) \leq d(\overline{x}, \overline{y}). \tag{1}$$

If $\kappa \leq 0$, then X is called a CAT(κ) space if X is a geodesic space all of whose geodesic triangles satisfy the CAT(κ) inequality. If $\kappa > 0$, then X is called a CAT(κ) space if X is D_κ-geodesic and all geodesic triangles in X of perimeter less than $2D_\kappa$ satisfy the CAT(κ) inequality. Thus, it can be seen that a CAT(0) space is a special case of CAT(κ) spaces when the curvature $\kappa = 0$. Furthermore, it is possible that the metric on CAT(κ) spaces ($\kappa > 0$) may take infinite values.

A metric space is said to be a geodesic metric space if every two points of X are joined by a geodesic in this metric space; a geodesic metric space (X, d) is said to be a CAT(0) space if each geodesic triangle of geodesic metric space (X, d) is at least as "thin" as its comparison triangle in R^2. In addition, a CAT(0) space is said to be a Hadamard space if it is complete; see for more details in [1–8]. Let X be a geodesic metric space and D be a nonempty subset of X. One of the major problems for optimization is to find a point $x \in X$ such that

$$f(x) = \min_{y \in X} f(y), \tag{2}$$

where f is a proper convex lower semicontinuous function and the set of all minimizers of f on X is denoted by $\operatorname{argmin}_{y\in X} f(y)$. There are many ways to study this problem.

For any $\lambda > 0, \forall x \in X$, the Moreau–Yosida resolvent of f is defined in CAT(0) spaces X as

$$J_\lambda(x) = \arg\min_{y \in X}\left[f(y) + \frac{1}{2\lambda} d^2(y, x)\right]. \tag{3}$$

In fact, here f is a proper convex and lower semicontinuous function. The concept of Moreau – Yosida resolvent of f appeared in [9]. Furthermore, the fact that the set $F(J_\lambda)$ of fixed points of the resolvent associated with f coincides with the set $\operatorname{argmin}_{y\in X} f(y)$ of minimizers of f is shown in [10, 11]. Also, for any $\lambda > 0$, the resolvent J_λ of f is nonexpansive [12].

In 1970, Martinet [13] published an article and a new algorithm was proposed to solve this optimization problem; the new algorithm is called as the proximal point algorithm. In 1976, in a Hilbert space, Rockafellar [14] studied the convergence to a solution of the convex minimization problem by the proximal point algorithm and proved and obtained a main conclusion that the sequence $\{x_n\}$ converges weakly to a minimizer of a convex function f such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. In 2013, the proximal point algorithm was introduced by Bačák [15] into CAT(0) spaces (X, d) as follows: $x_1 \in X$, for each $n \in N$,

$$x_{n+1} = \arg\min_{y\in X}\left[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)\right]. \tag{4}$$

Here, $\lambda_n > 0$, $\forall n \in N$, and this shows that, if f has a minimizer, $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}\Delta$-converges to its minimizer. In fact, the proximal point algorithm has been combined with many iterative methods, and a new construction algorithm is further proposed to find approximating fixed points of nonlinear mappings and a proper convex lower semicontinuous function f. In 1953, a known iteration method was proposed by Mann, and it was named as Mann iteration [16]; the Mann iteration process is defined as follows: $x_1 \in D$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n. \tag{5}$$

Here, $\{\alpha_n\}, n \in N$, is a real sequence in (0,1). In 1974, another well-known iteration method was proposed by Ishikawa, and this iteration method was named as Ishikawa iteration [17]; the Ishikawa iteration process concretely is expressed as follows: $x_1 \in D$ and

$$\begin{cases} y_n = (1-\alpha_n)Tx_n + \alpha_n x_n, \\ x_{n+1} = (1-\beta_n)Ty_n + \beta_n x_n, \end{cases} \tag{6}$$

for each $n \in N$, where $\{\alpha_n\}, \{\beta_n\}$ is a real sequence in (0,1).

In 2017, Suthep Suantai and Withum Phuengtattana [18] proposed a proximal point algorithm for a hybrid pair of nonexpansive single-valued and multivalued mappings in geodesic metric spaces as follows:

$$\begin{cases} z_n = \arg\min_{y\in X}\left[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)\right], \\ y_n = \beta_n z_n \oplus (1-\beta_n) w_n, \quad w_n \in S y_n, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) T z_n, \quad \forall n \in N, \end{cases} \tag{7}$$

where $\alpha_n, \beta_n \in [0,1]$, T is a single-valued nonexpansive mapping, and S is a multi-valued nonexpansive mapping.

Motivated and inspired by the above research work, Ishikawa iteration process, multivalued mapping, nonexpansive mapping, and convex function are considered as some elements for a new idea; then, the plan is to make use of these elements to reconstruct an algorithm; thus in this article, a modified proximal point algorithm involving a convex function and two nonexpansive mappings will be proposed. Under some suitable conditions, the convergence of the proposed algorithm is studied and its convergence analysis in the end is given.

2. Preliminaries

If z, x, and y are three points in CAT(0) spaces and if $((x\oplus y)/2)$ is the midpoint of a geodesic segment $[x, y]$, then the CAT(0) inequality implies

$$d^2\left(z, \frac{x\oplus y}{2}\right) \le \frac{1}{2} d^2(z, x) + \frac{1}{2} d^2(z, y) - \frac{1}{4} d^2(x, y), \tag{8}$$

which is the (CN) inequality (see [19]).

Let X be a CAT(0) space; a subset $D \in X$ is convex, if for any $x, y \in D$, $[x, y] \in D$, where $[x, y] = \{\lambda x \oplus (1-\lambda) y\colon 0 \le \lambda \le 1\}$ and $[x, y]$ is the unique geodesic joining x and y. Indeed, a geodesic space (X, d) is said to be a CAT(0) space, if and only if the inequality ((CN^*) inequality [20]),

$$\begin{aligned} d^2((1-\lambda)x \oplus \lambda y, z) \le{} & (1-\lambda) d^2(x, z) \\ & + \lambda d^2(y, z) - \lambda(1-\lambda) d^2(x, y), \end{aligned} \tag{9}$$

is satisfied for all $x, y, z \in X$ and $\lambda \in [0,1]$. Moreover, if x, y, and z are points in a CAT(0) space (X, d) and $\lambda \in [0,1]$, then

$$d((1-\lambda)x \oplus \lambda y, z) \le (1-\lambda) d(x, z) + \lambda d(y, z). \tag{10}$$

Let CB(D) and KC(D) denote the families of nonempty closed bounded subsets and compact convex subsets of D, respectively. The Pompeiu – Hausdorff distance [21] on CB(D) is defined by

$$H(A, B) = \max\left\{ \sup_{x\in A} \operatorname{dist}(x, B), \sup_{y\in B} \operatorname{dist}(y, A)\right\}, \tag{11}$$

for $A, B \in \mathrm{CB}(D)$, where $\operatorname{dist}(x, D) = \inf\{d(x, y)\colon y \in D\}$ is the distance from a point x to a subset D. Let $S\colon D \longrightarrow \mathrm{CB}(D)$ be a multivalued mapping. An element $x \in D$ is said to be a fixed point of mapping S, if an element $x \in Sx$. The set of fixed points of mapping S is denoted by $F(S)$; in brief, $F(S) = \{x \in D\colon x \in Sx\}$. More references for a multivalued mapping can be seen in [18, 22, 23].

Definition 1 (see [18]). A single-valued mapping $T\colon D \longrightarrow D$ is nonexpansive if $\forall x, y \in D$, $d(Tx, Ty) \le d(x, y)$.

Definition 2 (see [18]). A multivalued mapping $S\colon D \longrightarrow$ CB (D) is nonexpansive if $\forall x, y \in D$, $H(Sx, Sy) \le d(x, y)$.

Definition 3 (see [18]). Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For any $x \in X$, we define a mapping

$r(\cdot,\{x_n\})\colon X \longrightarrow [0,\infty)$ by $r(x,\{x_n\}) = \lim_{n\longrightarrow\infty}\sup d(x, x_n)$: (i) the asymptotic radius of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{r(x,\{x_n\})\colon x \in X\}$; (ii) the asymptotic center of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X\colon r(x,\{x_n\}) = r(\{x_n\})\}$.

The asymptotic center $A(\{x_n\})$ in a complete CAT(0) space consists of exactly one point [24].

Definition 4 (see [18]). A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\longrightarrow\infty} x_n = x$ and call x as the Δ-limit of $\{x_n\}$.

In fact, given $\{x_n\}$ in X such that $\{x_n\}\Delta$-converges to x and given $y \in X$ with $x \neq y$, $\lim_{n\longrightarrow\infty}\inf d(x_n, x) < \lim_{n\longrightarrow\infty}\inf d(x_n, y)$. So, the Opial condition in Banach space is also satisfied in every CAT(0) space X.

Lemma 1 (see [25]). *Every bounded sequence in a CAT(0) space has a Δ-convergent subsequence.*

Lemma 2 (see [26]). *Let D be a nonempty closed convex subset of a CAT(0) space X. If $\{x_n\}$ is a bounded sequence in D, then the asymptotic center of $\{x_n\}$ is in D.*

Lemma 3 (see [20]). *If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$, and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 4 (see [20]). *Let D be a nonempty closed convex subset of a complete CAT(0) space X and $T\colon D \longrightarrow D$ be a nonexpansive mapping. If $\{x_n\}$ is a bounded sequence in D such that $\lim_{n\longrightarrow\infty} d(x_n, Tx_n) = 0$ and $\Delta - \lim_{n\longrightarrow\infty} x_n = x$, then $x = Tx$.*

Lemma 5 (see [12]). *Let (X,d) be a complete CAT(0) space and $f\colon X \longrightarrow (-\infty,\infty]$ be a proper convex and lower semicontinuous function. Then, the following identity holds:*

$$J_\lambda x = J_\mu\left(\frac{\lambda-\mu}{\lambda}J_\lambda x \oplus \frac{\mu}{\lambda}x\right), \quad \forall x \in X, \lambda > \mu > 0, \tag{12}$$

where J_λ is the Moreau – Yosida resolvent of f.

Lemma 6 (see [27]). *Let (X,d) be a complete CAT(0) space and $f\colon X \longrightarrow (-\infty,\infty]$ be a proper convex and lower semicontinuous function. Then, for all $x, y \in X$ and $\lambda > 0$, the following inequality holds:*

$$\frac{1}{2\lambda}d^2(J_\lambda x, y) - \frac{1}{2\lambda}d^2(x, y) + \frac{1}{2\lambda}d^2(x, J_\lambda x) + f(J_\lambda x) \leq f(y), \tag{13}$$

where J_λ is the Moreau – Yosida resolvent of f.

Lemma 7 (see [28, 29]). *Let X be a CAT(0) space and D be a nonempty closed and convex subset of X. Let $\{x_i\}_{i=1}^n$ be any finite subset of D and $\alpha_i \in (0,1), i = 1, 2, \ldots, n$ such that $\sum_{i=1}^n \alpha_i = 1$. Then, the following inequalities hold:*

(i) $d(\oplus_{i=1}^n \alpha_i x_i, z) \leq \sum_{i=1}^n \alpha_i d(x_i, z), \forall z \in D$

(ii) $d^2(\oplus_{i=1}^n \alpha_i x_i, z) \leq \sum_{i=1}^n \alpha_i d^2(x_i, z) - \sum_{i,j=1, i\neq j}^n d^2(x_i, x_j)$, $\forall z \in D$

3. Main Results

Theorem 1. *Let (X, d) denote a CAT(0) space and be complete, assuming that the subset $D \subseteq X$ is nonempty, closed, and convex. Suppose that $S\colon D \longrightarrow CB(D)$ is a multivalued nonexpansive mapping, T is a single-valued nonexpansive mapping, and $f\colon D \longrightarrow (-\infty,\infty]$ is a proper convex and lower semicontinuous function. Suppose that the set*

$$\Omega = F(T) \cap F(S) \cap \arg\min_{y\in C} f(y) \neq \varnothing \tag{14}$$

and $Sp = \{p\}$ for $p \in \Omega$. For $x_1 \in D$, let the sequence $\{x_n\}$ be defined by

$$\begin{cases} w_n = \arg\min_{y\in C}\left[f(y) + \dfrac{1}{2\lambda_n}d^2(y, x_n)\right], \\ x_{n+1} = \alpha_n x_n \oplus \beta_n p_n \oplus \gamma_n T p_n, \quad p_n \in S w_n, \end{cases} \tag{15}$$

the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0,1)$, and $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1, \alpha_n + \beta_n + \gamma_n = 1$ $n \in N$, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in N$ and some λ. Then, the following statements hold:

(i) $\lim_{n\longrightarrow\infty} d(x_n, p)$ *exists for all* $p \in \Omega$

(ii) $\lim_{n\longrightarrow\infty} d(x_n, w_n) = 0$

(iii) $\lim_{n\longrightarrow\infty} d(x_n, Tx_n) = 0$

(iv) $\lim_{n\longrightarrow\infty} dist(x_n, Sx_n) = 0$

(v) $\lim_{n\longrightarrow\infty} d(x_n, J_\lambda x_n) = 0$

Proof. Let $p \in \Omega$; then, we get that $p = Tp \in Sp$ and $f(p) \leq f(y)$ for all $y \in D$. Thus, it shows that

$$f(p) + \frac{1}{2\lambda_n}d^2(p, p) \leq f(y) + \frac{1}{2\lambda_n}d^2(y, p), \quad \forall y \in D, \tag{16}$$

and hence, $p = J_\lambda p$ for each $n \in N$.

(i) First of all, the first step is to prove the fact that for all $p \in \Omega$, $\lim_{n\longrightarrow\infty} d(x_n, p)$ exists. Since $w_n = J_{\lambda_n} x_n$, with the nonexpansiveness of J_{λ_n}, then

$$d(w_n, p) = d\left(J_{\lambda_n} x_n, J_{\lambda_n} p\right) \leq d(x_n, p). \tag{17}$$

For $p \in \Omega$, by $p \in Sp$, (17), and Lemma 7, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n x_n \oplus \beta_n p_n \oplus \gamma_n T p_n, p) \\ &\leq \alpha_n d(x_n, p) + \beta_n d(p_n, p) + \gamma_n d(T p_n, p) \\ &\leq \alpha_n d(x_n, p) + \beta_n H(S w_n, Sp) + \gamma_n d(T p_n, p) \\ &\leq \alpha_n d(x_n, p) + \beta_n d(w_n, p) + \gamma_n d(p_n, p) \\ &\leq \alpha_n d(x_n, p) + \beta_n d(w_n, p) + \gamma_n d(w_n, p) \\ &\leq \alpha_n d(x_n, p) + (1-\alpha_n) d(w_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{18}$$

This shows that the sequence $\{d(x_n, p)\}$ is decreasing and bounded. So, the limit $\lim_{n\longrightarrow\infty} d(x_n, p)$ exists for all $p \in \Omega$.

(ii) Next, prove that $\lim_{n\longrightarrow\infty} d(x_n, w_n) = 0$. Now, we let

$$\lim_{n\longrightarrow\infty} d(x_n, p) = c, \tag{19}$$

where c is a constant in $[0, \infty)$ and $p \in \Omega$. In fact, by the inequality of Lemma 6, it implies that

$$\begin{aligned}&\frac{1}{2\lambda_n} d^2(w_n, p) - \frac{1}{2\lambda_n} d^2(x_n, p) + \frac{1}{2\lambda_n} d^2(w_n, x_n)\\ &\quad\leq f(p) - f(w_n).\end{aligned} \tag{20}$$

Because of $f(p) \leq f(w_n)$ for all $n \geq 1$, it follows that

$$d^2(w_n, x_n) \leq d^2(x_n, p) - d^2(w_n, p). \tag{21}$$

Since $\lim_{n\longrightarrow\infty} d(x_n, p) = c$, in order to show the fact that $\lim_{n\longrightarrow\infty} d(x_n, w_n) = 0$, it is sufficient to show that

$$\lim_{n\longrightarrow\infty} d(w_n, p) = c. \tag{22}$$

From (18), we have

$$d(w_n, p) \geq \frac{1}{1-\alpha_n}\left[d(x_{n+1}, p) - \alpha_n d(x_n, p)\right]. \tag{23}$$

Then, this shows that

$$\begin{aligned}&\lim_{n\longrightarrow\infty} \inf d(w_n, p)\\ &\geq \lim_{n\longrightarrow\infty} \inf\left\{\frac{1}{1-\alpha_n}\left[d(x_{n+1}, p) - \alpha_n d(x_n, p)\right]\right\} = c.\end{aligned} \tag{24}$$

At the same time, from (17), we get that

$$\lim_{n\longrightarrow\infty} \sup d(w_n, p) \leq \lim_{n\longrightarrow\infty} \sup d(x_n, p) = c. \tag{25}$$

Thus, from (24) and (25), it is implied that

$$\lim_{n\longrightarrow\infty} d(w_n, p) = c. \tag{26}$$

Then, from (19), (21), and (26), it can be shown that

$$\lim_{n\longrightarrow\infty} d(x_n, w_n) = 0. \tag{27}$$

(iii) Next, prove that $\lim_{n\longrightarrow\infty} d(x_n, Tx_n) = 0$. By Lemma 7, we have

$$\begin{aligned}d^2(x_{n+1}, p) &= d^2(\alpha_n x_n \oplus \beta_n p_n \oplus \gamma_n T p_n, p)\\ &\leq \alpha_n d^2(x_n, p) + \beta_n d^2(p_n, p) + \gamma_n d^2(Tp_n, p)\\ &\quad - \alpha_n\beta_n d^2(x_n, p_n) - \alpha_n\gamma_n d^2(x_n, Tp_n) - \beta_n\gamma_n d^2(p_n, Tp_n)\\ &\leq \alpha_n d^2(x_n, p) + \beta_n H^2(Sw_n, Sp) + \gamma_n d^2(Tp_n, Tp)\\ &\quad - \alpha_n\beta_n d^2(x_n, p) - \alpha_n\gamma_n d(x_n, Tp_n) - \beta_n\gamma_n d^2(p_n, Tp_n)\\ &\leq \alpha_n d^2(x_n, p) + \beta_n d^2(w_n, p) + \gamma_n d^2(p_n, p)\\ &\quad - \alpha_n\beta_n d^2(x_n, p) - \alpha_n\gamma_n d(x_n, Tp_n) - \beta_n\gamma_n d^2(p_n, Tp_n)\\ &\leq d^2(x_n, p) - \alpha_n\beta_n d^2(x_n, p_n) - \alpha_n\gamma_n d(x_n, Tp_n) - \beta_n\gamma_n d^2(p_n, Tp_n),\end{aligned} \tag{28}$$

that is,

$$\begin{aligned}&\alpha_n\beta_n d^2(x_n, p_n) + \alpha_n\gamma_n d(x_n, Tp_n)\\ &\quad + \beta_n\gamma_n d^2(p_n, Tp_n) \leq d^2(x_n, p) - d^2(x_{n+1}, p).\end{aligned} \tag{29}$$

Then from (19) and (29), we get

$$\begin{aligned}&\lim_{n\longrightarrow\infty} d(x_n, p_n) = 0,\\ &\lim_{n\longrightarrow\infty} d(x_n, Tp_n) = 0,\\ &\lim_{n\longrightarrow\infty} d(p_n, Tp_n) = 0.\end{aligned} \tag{30}$$

Therefore, from (30), we can obtain

$$\begin{aligned}d(x_n, Tx_n) &\leq d(x_n, Tp_n) + d(Tp_n, Tx_n)\\ &\leq d(x_n, Tp_n) + d(p_n, x_n).\end{aligned} \tag{31}$$

From (30) and (31), it can be shown that

$$\lim_{n\longrightarrow\infty} d(x_n, Tx_n) = 0. \tag{32}$$

(iv) Thus,

$$\begin{aligned}\mathrm{dist}(x_n, Sx_n) &\leq d(x_n, p_n) + \mathrm{dist}(p_n, Sx_n)\\ &\leq d(x_n, p_n) + H(Sw_n, Sx_n)\\ &\leq d(x_n, p_n) + d(w_n, x_n).\end{aligned} \tag{33}$$

From (27), (30), and (33), it is implied that

$$\operatorname{dist}(x_n, Sx_n) = 0. \tag{34}$$

(v) Since $\lambda_n \geq \lambda > 0$, making use of $w_n = J_{\lambda_n} x_n$, the nonexpansiveness of J_λ, and Lemma 5, it can be seen that

$$\begin{aligned} d(x_n, J_\lambda x_n) &\leq d(x_n, w_n) + d(w_n, J_\lambda x_n) \\ &\leq d(x_n, w_n) + d\left(J_{\lambda_n} x_n, J_\lambda x_n\right) \\ &= d(x_n, w_n) + d\left(J_\lambda\left(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n \oplus \frac{\lambda}{\lambda_n} x_n\right), J_\lambda x_n\right) \\ &\leq d(x_n, w_n) + \frac{\lambda_n - \lambda}{\lambda_n} d\left(J_{\lambda_n} x_n, x_n\right) + \frac{\lambda}{\lambda_n} d(x_n, x_n) \\ &= \left(2 - \frac{\lambda}{\lambda_n}\right) d(x_n, w_n). \end{aligned} \tag{35}$$

So, this implies the fact that the limit

$$\lim_{n \longrightarrow \infty} d(x_n, J_\lambda x_n) = 0. \tag{36}$$

This completes the proof. □

Theorem 2. *Let D be a nonempty closed convex subset of a complete CAT(0) space (X, d). Let function $f\colon D \longrightarrow (-\infty, \infty]$ be a proper convex and lower semicontinuous function, T be a nonexpansive single-valued mapping, and $S\colon D \longrightarrow KC(D)$ be a multivalued nonexpansive mapping. Suppose that*

$$\Omega = F(T) \cap F(S) \cap \underset{y \in D}{\arg\min} f(y) \neq \varnothing \tag{37}$$

and $Sp = \{p\}$ for $p \in \Omega$. For $x_1 \in D$, let the sequence $\{x_n\}$ be defined by (15), the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0, 1)$, and $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1, \alpha_n + \beta_n + \gamma_n = 1\, n \in N$. In addition, $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in N$ and some λ. Then, the sequence $\{x_n\}\Delta$-converges to a point in Ω.

Proof. Let $\omega_\Delta = \bigcup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Let $p \in \omega_\Delta(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{p\}$. $D \in X$ is a nonempty closed convex subset, and it is easy to know from Theorem 1 that the sequence $\{x_n\}$ is bounded. Then, by Lemmas 1 and 2, it can be shown that there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that

$$\Delta - \lim_{n \longrightarrow \infty} v_n = v \in D. \tag{38}$$

From Theorem 1 (iii) and (v), we have $\lim_{n \longrightarrow \infty} d(v_n, Tv_n) = 0$ and $\lim_{n \longrightarrow \infty} d(v_n, J_\lambda v_n) = 0$. Then, by the nonexpansiveness of T and J_λ, through Lemma 4, it can be shown that the fact that $v = Tv = J_\lambda v$. Thus, we get

$$v \in F(T) \cap F(J_\lambda) = F(T) \cap \underset{u \in D}{\arg\min} f(u). \tag{39}$$

Since the mapping S is compact valued, for each $n \in N$, then there exist $r_n \in Sv_n$ and $\gamma_n \in Sv$ such that $d(v_n, r_n) = \operatorname{dist}(v_n, Sv_n)$ and $d(r_n, \gamma_n) = \operatorname{dist}(r_n, Sv)$. By Theorem 1 (iv), this implies the fact that

$$\lim_{n \longrightarrow \infty} d(v_n, r_n) = 0. \tag{40}$$

Owing to the fact Sv is compact, there exists a subsequence $\{\gamma_{n_i}\}$ of $\{\gamma_n\}$ such that $\lim_{i \longrightarrow \infty} \gamma_{n_i} = \gamma \in Sv$. Thus, this shows that

$$\begin{aligned} \lim_{i \longrightarrow \infty} \sup d\left(v_{n_i}, \gamma\right) &\leq \lim_{i \longrightarrow \infty} \sup\left(d\left(v_{n_i}, r_{n_i}\right) + d\left(r_{n_i}, \gamma_{n_i}\right) + d\left(\gamma_{n_i}, \gamma\right)\right) \\ &\leq \lim_{i \longrightarrow \infty} \sup\left(d\left(v_{n_i}, r_{n_i}\right) + \operatorname{dist}\left(r_{n_i}, Sv\right) + d\left(\gamma_{n_i}, \gamma\right)\right) \\ &\leq \lim_{i \longrightarrow \infty} \sup\left(d\left(v_{n_i}, r_{n_i}\right) + H\left(Sr_{n_i}, Sv\right) + d\left(\gamma_{n_i}, \gamma\right)\right) \\ &\leq \lim_{i \longrightarrow \infty} \sup\left(d\left(v_{n_i}, r_{n_i}\right) + d\left(r_{n_i}, v\right) + d\left(\gamma_{n_i}, \gamma\right)\right) \\ &= \lim_{i \longrightarrow \infty} \sup d\left(v_{n_i}, v\right). \end{aligned} \tag{41}$$

Through (38) and the uniqueness of asymptotic centers, we can obtain that $v = \gamma \in Sv$. Therefore, by (39), we can show that

$$v \in F(T) \cap F(S) \cap \underset{u \in D}{\arg\min} f(u) = \Omega. \tag{42}$$

It follows by Lemma 3 and Theorem 1 (i) that $p = v$, and hence, $\omega_\Delta(x_n) \subseteq \Omega$.

In order to show that $\{x_n\}\Delta$-converges to a point in Ω, it suffices to show that $\omega_\Delta(x_n)$ consists of exactly one point. Suppose that $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u^*\}$ and $A(\{x_n\}) = \{x\}$. Since $u^* \in \omega_\Delta(x_n) \subseteq \Omega$ and $\{d(x_n, u^*)\}$ converge, it implies by Lemma 3 that $x = u^*$.

This completes the proof. □ □

Remark 1.

(i) The results of Shuntai and Phuaengrattana [18] and Cholamjiak [30] are extended and improved by Theorem 2. In fact, a new proximal point algorithm can be used for solving the constrained convex minimization problem as well as the fixed-point problem of a single-valued nonexpansive mapping and a multivalued nonexpansive mapping in a CAT(0) space.

(ii) Since every real Hilbert space H is a complete CAT(0) space, the above result can also be obtained in Hilbert spaces, so a convergence weakly theorem can be obtained in a real Hilbert space as follows.

Corollary 1. *Let D be a nonempty closed convex subset of a real Hilbert space H. Let $f\colon D \longrightarrow (-\infty, \infty]$ be a proper convex and lower semicontinuous function, T be a single-valued nonexpansive mapping, and $S\colon D \longrightarrow CB(D)$ be a multivalued nonexpansive mapping. Suppose that*

$$\Omega = F(T) \cap F(S) \cap \underset{y \in C}{\arg\min} f(y) \neq \varnothing \tag{43}$$

and $Sp = \{p\}$, $p \in \Omega$. For $x_1 \in D$, let the sequence $\{x_n\}$ be defined by

$$\begin{cases} w_n = \underset{y \in C}{\arg\min}\left[f(y) + \dfrac{1}{2\lambda_n} d^2(y, x_n)\right], \\ \\ x_{n+1} = \alpha_n x_n + \beta_n p_n + \gamma_n T p_n, \quad p_n \in S w_n, \end{cases} \tag{44}$$

the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0,1)$, and $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1, \alpha_n + \beta_n + \gamma_n = 1\, n \in N$, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0, \forall n \in N$ and some λ. Then, the sequence $\{x_n\}$ converges weakly to an element in Ω.

Remark 2. In fact, the construction of our proposed algorithm is rather peculiar and it is different from references [31–33]. Convex combination of the sequences $\{x_n\}, \{p_n\}$, and $\{Tp_n\}$ are given here, especially $p_n \in Sw_n$, and Tp_n is the nonexpansive mapping T that operates on the sequences $\{p_n\}$.

4. Numerical Experiments

In this section, a numerical example is given to illustrate reckoning the convergence of modified proximal point algorithm with iteration (15) by numerical experiment for supporting Theorems 1 and 2.

Let $X = \mathbb{R}^2$ with Euclidean norm and $D = \{x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^2\colon 0 \leq x^{(1)}, x^{(2)} \leq 1\}$. For each $x = (x^{(1)}, x^{(2)}) \in D$, the concrete definition of nonexpansive mappings T and S is shown as

$$\begin{aligned} Tx &= \left(\frac{2x^{(1)} + 1}{4}, \frac{x^{(2)} + 7}{8}\right), \\ Sx &= \left\{x^{(1)}\right\} \times \left[\frac{3x^{(2)} + 1}{4}, 1\right]. \end{aligned} \tag{45}$$

For each $x \in D$, assume that $\|x\|_1 = \sum_{i=1}^n |x_i|$, and $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$. The function $f\colon D \longrightarrow (-\infty, \infty)$ is defined in the following manner:

$$f(x) = \|x\|_1 + \frac{1}{2}\|x\|_2^2 + (-1.5, -2)x + 7. \tag{46}$$

From the fact that T and S are nonexpansive and f is a proper convex lower semicontinuous function easy to prove, we skip their proofs here. Furthermore, by making use of the soft thresholding operator [34] and the proximity operator [35], let $\lambda_n = 1$; we have

$$\begin{aligned} J_0 x &= \underset{v \in D}{\arg\min}\left[f(v) + \frac{1}{2}\|v - x\|^2\right] = \operatorname{prox}_f x \\ &= \operatorname{prox}_{(\|\cdot\|_1/2)}\left(\frac{x - (-1.5, -2)}{2}\right) \\ &= \left(\max\left\{\frac{\left|x^{(1)} + 1.5\right| - 1}{2}, 0\right\}\operatorname{sgn}\left(x^{(1)} + 1.5\right), \max\left\{\frac{\left|x^{(2)} + 2\right| - 1}{2}, 0\right\}\operatorname{sgn}\left(x^{(2)} + 2\right)\right), \end{aligned} \tag{47}$$

where $\operatorname{sgn}(\xi)$ is a signum function, that is,

$$\begin{cases} \operatorname{sgn}(\xi) = 1, & \xi > 0, \\ \operatorname{sgn}(\xi) = 0, & \xi = 0, \\ \operatorname{sgn}(\xi) = -1, & \xi < 0. \end{cases} \tag{48}$$

Further simplification of the proposed iterative algorithm is in the following expression:

$$\begin{cases} w_n = J_0 x_n, \\ x_{n+1} = \alpha_n x_n + \beta_n p_n + \gamma_n T p_n, \quad p_n \in S w_n, \end{cases} \tag{49}$$

where $x_n = (x_n^{(1)}, x_n^{(2)})$ and $w_n = (w_n^{(1)}, w_n^{(2)})$ are points in $\mathbb{R}^2$. In addition, we choose some points and the sequences of parameters as follows:

Table 1: Numerical results of Algorithm (49).

n	$x_n = (x_n^{(1)}, x_n^{(2)})$	$\|x_n - x_{n-1}\|_2$	$f(x_n)$
1	(0.100000, 0.200000)	0.347205	6.525875
3	(0.310620, 0.705335)	0.124269	6.400753
5	(0.406401, 0.884487)	0.050838	6.379830
7	(0.453275, 0.954060)	0.021717	6.375968
9	(0.476569, 0.981610)	0.009587	6.375205
11	(0.488219, 0.992611)	0.004359	6.375045
13	(0.494066, 0.997023)	0.002034	6.375010
15	(0.497008, 0.998799)	0.000970	6.375002
17	(0.498490, 0.999514)	0.000471	6.375000
19	(0.499237, 0.999803)	0.000231	6.375000
21	(0.499614, 0.999920)	0.000115	6.375000
23	(0.499805, 0.999967)	0.000057	6.375000
25	(0.499901, 0.999986)	0.000028	6.375000
27	(0.499950, 0.999994)	0.000014	6.375000
29	(0.499974, 0.999997)	0.000007	6.375000
31	(0.499987, 0.999999)	0.000003	6.375000
33	(0.499993, 0.999999)	0.000001	6.375000
35	(0.499996, 0.999999)	0.000000	6.375000

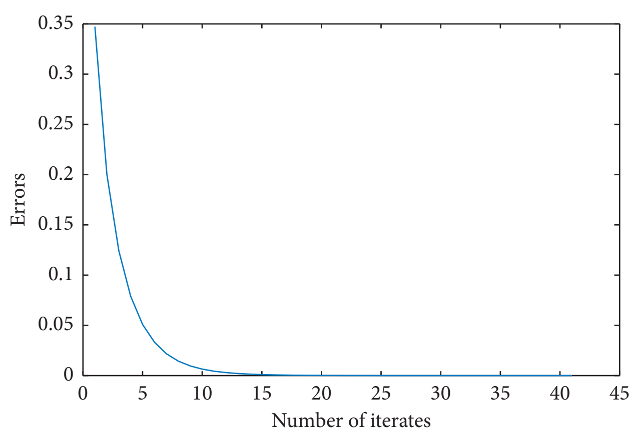

Figure 1: Number of iterations and errors.

$$
\begin{aligned}
p_n &= \left(w_n^{(1)}, \frac{3w_n^{(2)} + 1}{4} \right) \in S w_n, \\
\alpha_n &= \frac{50n - 3}{100n}, \\
\beta_n &= \frac{35n - 4}{100n}, \\
\gamma_n &= \frac{15n + 7}{100n}.
\end{aligned} \tag{50}
$$

Next, we use Algorithm (49) with an initial point $x_1 = (0.1, 0.2)$ and obtain numerical results in Table 1.

Remark 3

(i) From Figure 1, with the increase in the number of iterations, the errors between x_n and x_{n+1} decrease. When the iteration is 17 times, the minimum of f is obtained.

(ii) From Table 1 and Figure 1, it is observed that the sequence $\{x_n\}$ converges to a point (0.5, 1).

(iii) The point (0.5, 1) is a solution of the constrained convex minimization problems (46) and also a solution of the fixed-point problems of a pair of a nonexpansive single-valued mapping T and a nonexpansive multivalued mapping S.

Data Availability

Some data obtained by the authors themselves were used to support this study.

Conflicts of Interest

All authors declare no conflicts of interest.

Authors' Contributions

All authors contributed equally to this study.

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