Research Article

A Modified Proximal Point Algorithm and Some Convergence Results

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In this paper, the convergence to minimizers of a convex function of a modified proximal point algorithm involving a single-valued nonexpansive mapping and a multivalued nonexpansive mapping in CAT(0) spaces is studied and a numerical example is given to support our main results.

1. Introduction

In daily life, no matter what we do, there are always many options available and many possible outcomes. When we do these things, we always consciously or unconsciously choose an optimal solution in order to achieve the optimal result. The discipline of seeking the best solution to achieve the best result is optimization. The way to find the optimal solution is the optimization method.

Given a real number \( \kappa \) (curvature), let \( M_{\kappa}^2 \) denote the following space: if \( \kappa < 0 \), then \( M_{\kappa}^2 \) is a real hyperbolic space \( H^2 \) with the distance function scaled by a factor of \((1/\sqrt{-\kappa})\); if \( \kappa = 0 \), then \( M_{\kappa}^2 \) is the Euclidean plane; if \( \kappa > 0 \), then \( M_{\kappa}^2 \) is the 2-sphere \( S^2 \) with the metric scaled by a factor \((1/\sqrt{\kappa})\). Let \( D_{\kappa} \) denote the diameter of \( M_{\kappa}^2 \). Let \( \Delta \) be a geodesic triangle in \( X \) with a perimeter less than \( 2D_{\kappa} \). Let \( \overline{\Delta} \in M_{\kappa}^2 \) be a comparison triangle for \( \Delta \). Then, \( \Delta \) is said to satisfy the CAT(\( \kappa \)) inequality if for \( x, y \in \Delta \) and all comparison points \( \overline{x}, \overline{y} \in \overline{\Delta} \),

\[
d(x, y) \leq d(\overline{x}, \overline{y}).
\]

(1)

If \( \kappa \leq 0 \), then \( X \) is called a CAT(\( \kappa \)) space if \( X \) is a geodesic space all of whose geodesic triangles satisfy the CAT(\( \kappa \)) inequality. If \( \kappa > 0 \), then \( X \) is called a CAT(\( \kappa \)) space if \( X \) is \( D_{\kappa} \)-geodesic and all geodesic triangles in \( X \) of perimeter less than \( 2D_{\kappa} \) satisfy the CAT(\( \kappa \)) inequality. Thus, it can be seen that a CAT(0) space is a special case of CAT(\( \kappa \)) spaces when the curvature \( \kappa = 0 \). Furthermore, it is possible that the metric on CAT(\( \kappa \)) spaces \( (\kappa > 0) \) may take infinite values.

A metric space is said to be a geodesic metric space if every two points of \( X \) are joined by a geodesic in this metric space; a geodesic metric space \( (X, d) \) is said to be a CAT(0) space if each geodesic triangle of geodesic metric space \( (X, d) \) is at least as “thin” as its comparison triangle in \( R^2 \). In addition, a CAT(0) space is said to be a Hadamard space if it is complete; see for more details in [1–8]. Let \( X \) be a geodesic metric space and \( D \) be a nonempty subset of \( X \). One of the major problems for optimization is to find a point \( x \in X \) such that

\[
f(x) = \min_{y \in X} f(y),
\]

(2)

where \( f \) is a proper convex lower semicontinuous function and the set of all minimizers of \( f \) on \( X \) is denoted by \( \text{argmin}_{y \in X} f(y) \). There are many ways to study this problem. For any \( \lambda > 0, \forall x \in X \), the Moreau–Yosida resolvent of \( f \) is defined in CAT(0) spaces \( X \) as

\[
J_\lambda(x) = \text{argmin}_{y \in X} \left[ f(y) + \frac{1}{2\lambda}d^2(y, x) \right].
\]

(3)
Motivated and inspired by the above research work, Ishikawa iteration process, multivalued mapping, nonexpansive mapping, and convex function are considered as some elements for a new idea; then, the plan is to make use of these elements to reconstruct an algorithm; thus in this article, a modified proximal point algorithm involving a convex function and two nonexpansive mappings will be proposed. Under some suitable conditions, the convergence of the proposed algorithm is studied and its convergence analysis in the end is given.

2. Preliminaries

If $z, x, y$ are three points in CAT(0) spaces and if $(x \oplus y)/2$ is the midpoint of a geodesic segment $[x, y]$, then the CAT(0) inequality implies

$$d^2(z, x) \leq \frac{1}{2}d^2(z, x) + \frac{1}{2}d^2(z, y) - \frac{1}{4}d^2(x, y),$$

which is the $(CN)$ inequality (see [19]).

Let $X$ be a CAT(0) space; a subset $D \in X$ is convex, if for any $x, y \in D$, $[x, y] \in D$, where $[x, y] = \{\lambda x \oplus (1 - \lambda)y : 0 \leq \lambda \leq 1\}$ and $[x, y]$ is the unique geodesic joining $x$ and $y$. Indeed, a geodesic space $(X, d)$ is said to be a CAT(0) space, if and only if the inequality $((CN^*)$ inequality [20]),

$$d^2((1 - \lambda)x \oplus \lambda y, z) \leq (1 - \lambda)d^2(x, z) + \lambda d^2(y, z) - \lambda(1 - \lambda)d^2(x, y),$$

is satisfied for all $x, y, z \in X$ and $\lambda \in [0, 1]$. Moreover, if $x, y, z$ are points in a CAT(0) space $(X, d)$ and $\lambda \in [0, 1]$, then

$$d((1 - \lambda)x \oplus \lambda y, z) \leq (1 - \lambda)d(x, z) + \lambda d(y, z).$$

Let $CB(D)$ and $KC(D)$ denote the families of nonempty closed bounded subsets and compact convex subsets of $D$, respectively. The Pompeiu – Hausdorff distance [21] on $CB(D)$ is defined by

$$H(A, B) = \max_{x \in A} \sup_{y \in B} \text{dist}(x, B), \quad \text{sup dist}(y, A),$$

for $A, B \in CB(D)$, where $\text{dist}(x, D) = \inf \{d(x, y) : y \in D\}$ is the distance from a point $x$ to a subset $D$. Let $S : D \longrightarrow CB(D)$ be a multivalued mapping. An element $x \in D$ is said to be a fixed point of mapping $S$, if an element $x \in Sx$. The set of fixed points of mapping $S$ is denoted by $F(S)$; in brief, $F(S) = \{x \in D : x \in Sx\}$. More references for a multivalued mapping can be seen in [18, 22, 23].

**Definition 1** (see [18]). A single-valued mapping $T : D \longrightarrow D$ is nonexpansive if $\forall x, y \in D, d(Tx, Ty) \leq d(x, y)$.

**Definition 2** (see [18]). A multivalued mapping $S : D \longrightarrow CB(D)$ is nonexpansive if $\forall x, y \in D, H(Sx, Sy) \leq d(x, y)$.

**Definition 3** (see [18]). Let $\{x_n\}$ be a bounded sequence in a CAT(0) space $X$. For any $x \in X$, we define a mapping

$$z_n = \arg \min_{y \in X} \left\{ f(y) + \frac{1}{2\lambda_n}d^2(y, x_n) \right\},$$

$$y_n = \beta_n z_n \oplus (1 - \beta_n)w_n, \quad w_n \in S y_n,$$

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T z_n, \quad \forall n \in N,$$

where $\alpha_n, \beta_n \in [0, 1]$, $T$ is a single-valued nonexpansive mapping, and $S$ is a multi-valued nonexpansive mapping.
Lemma 1 (see [25]). Every bounded sequence in a CAT(0) space has a ∆-convergent subsequence.

Lemma 2 (see [26]). Let D be a nonempty closed convex subset of a CAT(0) space X. If \([x_n]\) is a bounded sequence in D, then the asymptotic center of \([x_n]\) is in D.

Lemma 3 (see [20]). If \([x_n]\) is a bounded sequence in a complete CAT(0) space with \(A([x_n]) = \{x\}\), \([u_n]\) is a subsequence of \([x_n]\) with \(A([u_n]) = \{u\}\), and the sequence \(d(x_n, u)\) converges, then \(x = u\).

Lemma 4 (see [20]). Let D be a nonempty closed convex subset of a complete CAT(0) space X and \(T : D \to D\) be a nonexpansive mapping. If \([x_n]\) is a bounded sequence in D such that \(\lim_{n \to \infty} d(x_n, Tx_n) = 0\) and \(\Delta - \lim_{n \to \infty} x_n = x\), then \(x = Tx\).

Lemma 5 (see [12]). Let \((X, d)\) be a complete CAT(0) space and \(f : X \to (-\infty, \infty)\) be a proper convex and lower semicontinuous function. Then, the following identity holds:
\[
J_{\lambda} x = f_{\lambda} \left( \frac{\lambda - y}{\lambda - x} x + \frac{y}{\lambda} \right), \quad \forall x \in X, \lambda > \mu > 0,
\]
where \(J_{\lambda}\) is the Moreau – Yosida resolvent of \(f\).

Lemma 6 (see [27]). Let \((X, d)\) be a complete CAT(0) space and \(f : X \to (-\infty, \infty)\) be a proper convex and lower semicontinuous function. Then, for all \(x, y \in X\) and \(\lambda > 0\), the following inequality holds:
\[
\frac{1}{\lambda^2} J_{\lambda}^2 (x, y) - \frac{1}{\lambda} J_{\lambda} (x, y) + \frac{1}{\lambda} J_{\lambda}^2 (x, J_{\lambda} x) + f(J_{\lambda} x) \leq f(y),
\]
where \(J_{\lambda}\) is the Moreau – Yosida resolvent of \(f\).

Lemma 7 (see [28, 29]). Let \(X\) be a CAT(0) space and \(D\) be a nonempty closed and convex subset of \(X\). Let \([x_n]_{n=1}^\infty\) be any finite subset of \(D\) and \(\alpha_i \in (0, 1), i = 1, 2, \ldots, n\) such that \(\sum_{i=1}^n \alpha_i = 1\). Then, the following inequalities hold:
\[
(i) \frac{d(\oplus_{i=1}^n \alpha_i x_i, z)}{\sum_{i=1}^n \alpha_i} \leq \frac{\sum_{i=1}^n \alpha_i d(x_i, z)}{\sum_{i=1}^n \alpha_i}, \forall z \in D
\]
\[
(ii) \frac{d(\bigoplus_{j=1}^m \beta_j \oplus \bigoplus_{j=1}^m \gamma_j P_n, p)}{\sum_{j=1}^m \beta_j + \sum_{j=1}^m \gamma_j} \leq \frac{\sum_{j=1}^m \beta_j d(w_j, p) + \sum_{j=1}^m \gamma_j d(T_p, p)}{\sum_{j=1}^m \beta_j + \sum_{j=1}^m \gamma_j},
\]
where \(\oplus\) and \(\bigoplus\) denote convex and weakly closed convex hulls, respectively, and \(P_n\) is the projection onto \(D\).

3. Main Results

Theorem 1. Let \((X, d)\) denote a CAT(0) space and be complete, assuming that the subset \(D \subseteq X\) is nonempty, closed, and convex. Suppose that \(S : D \to CB(D)\) is a multivalued nonexpansive mapping, \(T\) is a single-valued nonexpansive mapping, and \(f : D \to (-\infty, \infty)\) is a proper convex and lower semicontinuous function. Suppose that the set
\[
\Omega = F(T) \cap F(S) \cap \arg \min_{y \in C} f(y) \neq \emptyset
\]
and \(Sp = \{p\} \in \Omega\). For \(x_1 \in D\), let the sequence \([x_n]\) be defined by
\[
\begin{align*}
\omega_n &= \arg \min_{y \in C} \left[ f(y) + \frac{1}{2\mu} d^2(y, x_n) \right], \\
x_{n+1} &= \alpha_n x_n + \beta_n p_n + \gamma_n T_p, \quad p_n \in Sp,
\end{align*}
\]
the sequences \(\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0, 1), 0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1, \alpha_n + \beta_n + \gamma_n = 1 \in N\), and \(\lambda_n\) is a sequence such that \(\lambda_n \geq \lambda > 0\) for all \(n \in N\) and some \(\lambda\). Then, the following statements hold:
\[
(i) \lim_{n \to \infty} d(x_n, p) \exists \forall p \in \Omega
\]
\[
(ii) \lim_{n \to \infty} d(x_n, \omega_n) = 0
\]
\[
(iii) \lim_{n \to \infty} d(x_n, Tx_n) = 0
\]
\[
(iv) \lim_{n \to \infty} \text{dist}(x_n, Sx_n) = 0
\]
\[
(v) \lim_{n \to \infty} d(x_n, J_{\lambda} x_n) = 0
\]
Proof. Let \(p \in \Omega\); then, we get that \(p = Tp \in Sp\) and \(f(p) \leq f(y)\) for all \(y \in D\). Thus, it shows that
\[
f(p) + \frac{1}{2\lambda} d^2(p, p) \leq f(y) + \frac{1}{2\lambda} d^2(y, p), \quad \forall y \in D
\]
and hence, \(p = J_{\lambda} p\) for each \(n \in N\).

(i) First of all, the first step is to prove the fact that for all \(p \in \Omega\), \(\lim_{n \to \infty} d(x_n, p) \exists\). Since \(\omega_n = J_{\lambda} x_n\), with the nonexpansiveness of \(J_{\lambda}\), then
\[
d(\omega_n, p) = d(J_{\lambda} x_n, J_{\lambda} p) \leq d(x_n, p).
\]
For \(p \in \Omega\), by \(p \in Sp\), (17), and Lemma 7, we have
\[
d(x_n, p) = d(\alpha_n x_n + \beta_n p_n + \gamma_n T_p, p_n)
\]
\[
\leq \alpha_n d(x_n, p) + \beta_n d(p_n, p) + \gamma_n d(T_p, p)
\]
\[
\leq \alpha_n d(x_n, p) + \beta_n H(\omega_n, Sp) + \gamma_n d(T_p, p)
\]
\[
\leq \alpha_n d(x_n, p) + \beta_n d(\omega_n, p) + \gamma_n d(p_n, p)
\]
\[
\leq \alpha_n d(x_n, p) + \beta_n d(p_n, p) + \gamma_n d(w_n, p)
\]
\[
\leq \alpha_n d(x_n, p) + (1 - \alpha_n) d(w_n, p)
\]
\[
\leq d(x_n, p).
\]
This shows that the sequence \( \{d(x_n, p)\} \) is decreasing and bounded. So, the limit \( \lim_{n \to \infty} d(x_n, p) \) exists for all \( p \in \Omega \).

(ii) Next, prove that \( \lim_{n \to \infty} d(x_n, w_n) = 0 \). Now, we let

\[
\lim_{n \to \infty} d(x_n, p) = c, \tag{19}
\]

where \( c \) is a constant in \([0, \infty)\) and \( p \in \Omega \). In fact, by the inequality of Lemma 6, it implies that

\[
\frac{1}{2\lambda_n}d^2(w_n, p) - \frac{1}{2\lambda_n}d^2(x_n, p) + \frac{1}{2\lambda_n}d^2(w_n, x_n) \leq f(p) - f(w_n).
\]

Because of \( f(p) \leq f(w_n) \) for all \( n \geq 1 \), it follows that

\[
d^2(w_n, x_n) \leq d^2(x_n, p) - d^2(w_n, p). \tag{21}
\]

Since \( \lim_{n \to \infty} d(x_n, p) = c \), in order to show the fact that \( \lim_{n \to \infty} d(x_n, w_n) = 0 \), it is sufficient to show that

\[
\lim_{n \to \infty} d(w_n, p) = c. \tag{22}
\]

From (18), we have

\[
d^2(x_{n+1}, p) = d^2(\alpha_n, x_n, \beta_n, p_n, \gamma_n, Tp_n, p) \leq \alpha_n d^2(x_n, p) + \beta_n d^2(p_n, p) + \gamma_n d^2(Tp_n, p) - \alpha_n \beta_n d^2(x_n, p_n) - \alpha_n \gamma_n d^2(x_n, Tp_n) - \beta_n \gamma_n d^2(p_n, Tp_n) \leq \alpha_n d^2(x_n, p) + \beta_n d^2(x_n, p) - \alpha_n \gamma_n d^2(x_n, Tp_n) - \beta_n \gamma_n d^2(p_n, Tp_n) \leq \alpha_n d^2(x_n, p) + \beta_n d^2(x_n, p) - \alpha_n \gamma_n d^2(x_n, Tp_n) - \beta_n \gamma_n d^2(p_n, Tp_n) \leq d^2(x_n, p) - \alpha_n \beta_n d^2(x_n, p_n) - \alpha_n \gamma_n d^2(x_n, Tp_n) - \beta_n \gamma_n d^2(p_n, Tp_n),
\]

that is,

\[
\alpha_n \beta_n d^2(x_n, p_n) + \alpha_n \gamma_n d(x_n, Tp_n) + \beta_n \gamma_n d^2(p_n, Tp_n) \leq d^2(x_n, p) - d^2(x_{n+1}, p). \tag{29}
\]

Then from (19) and (29), we get

\[
\lim_{n \to \infty} d(x_n, p_n) = 0,
\]

\[
\lim_{n \to \infty} d(x_n, Tp_n) = 0,
\]

\[
\lim_{n \to \infty} d(p_n, Tp_n) = 0. \tag{30}
\]

Therefore, from (30), we can obtain

\[
d(w_n, p) \geq \frac{1}{1 - \alpha_n} [d(x_{n+1}, p) - \alpha_n d(x_n, p)]. \tag{23}
\]

Then, this shows that

\[
\lim_{n \to \infty} \inf d(w_n, p) \geq \frac{1}{1 - \alpha_n} \left[ d(x_{n+1}, p) - \alpha_n d(x_n, p) \right] = c. \tag{24}
\]

At the same time, from (17), we get that

\[
\lim_{n \to \infty} \sup d(w_n, p) \leq \lim_{n \to \infty} \sup d(x_n, p) = c. \tag{25}
\]

Thus, from (24) and (25), it is implied that

\[
\lim_{n \to \infty} d(w_n, p) = c. \tag{26}
\]

Then, from (19), (21), and (26), it can be shown that

\[
\lim_{n \to \infty} d(x_n, w_n) = 0. \tag{27}
\]

(iii) Next, prove that \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \). By Lemma 7, we have

\[
d(x_n, Tx_n) \leq d(x_n, Tp_n) + d(Tp_n, Tx_n) \leq d(x_n, Tp_n) + d(p_n, x_n). \tag{31}
\]

From (30) and (31), it can be shown that

\[
\lim_{n \to \infty} d(x_n, Tx_n) = 0. \tag{32}
\]

(iv) Thus,

\[
\text{dist}(x_n, Sx_n) \leq d(x_n, p_n) + d(p_n, Sx_n) \leq d(x_n, p_n) + H(Sx_n, Sx_n) \leq d(x_n, p_n) + d(w_n, x_n). \tag{33}
\]

From (27), (30), and (33), it is implied that
be a multivalued nonexpansive mapping. Suppose that 
\( (v) \) Since \( \lambda_n \geq \lambda > 0 \), making use of \( w_n = J_{\lambda_n} x_n \), the nonexpansiveness of \( J_{\lambda_n} \), and Lemma 5, it can be shown that
\[
d(x_n, J_{\lambda_n} x_n) \leq d(x_n, w_n) + d(w_n, J_{\lambda_n} x_n)
\]
\[
\leq d(x_n, w_n) + d(J_{\lambda_n} x_n, J_{\lambda_n} x_n)
\]
\[
= d(x_n, w_n) + d\left( J_{\lambda_n} \left( \frac{\lambda_n - \lambda}{\lambda_n} x_n, x_n \right), J_{\lambda_n} x_n \right)
\]
\[
\leq d(x_n, w_n) + \frac{\lambda_n - \lambda}{\lambda_n} d(x_n, x_n) + \frac{\lambda}{\lambda_n} d(x_n, x_n)
\]
\[
= (2 - \frac{\lambda}{\lambda_n})d(x_n, w_n).
\]
So, this implies the fact that the limit
\[
\lim_{n \to \infty} d(x_n, J_{\lambda_n} x_n) = 0.
\]
This completes the proof.

**Theorem 2.** Let \( D \) be a nonempty closed convex subset of a complete CAT(0) space \( (X, d) \). Let function \( f : D \to (-\infty, \infty) \) be a proper convex and lower semicontinuous function, \( T \) be a nonexpansive single-valued mapping, and \( S : D \to KC(D) \) be a multivalued nonexpansive mapping. Suppose that
\[
\Omega = F(T) \cap F(S) \cap \arg \min_{y \in D} f(y) \neq \emptyset
\]
and \( Sp = \{ p \} \) for \( p \in \Omega \). For \( x_1 \in D \), let the sequence \( \{ x_n \} \) be defined by (15), the sequences \( \{ \alpha_n \}, \{ \beta_n \}, \{ y_n \} \in (0,1) \), and \( 0 < a \leq \alpha_n, p_n \leq b < 1, \alpha_n + p_n + y_n = 1, n \in N \). In addition, \( \{ \lambda_n \} \) is a sequence such that \( \lambda_n \geq \lambda > 0 \) for all \( n \in N \) and some \( \lambda \). Then, the sequence \( \{ x_n \} \Delta\)-converges to a point in \( \Omega \).

**Proof.** Let \( \omega_n = \bigcup A(\{ u_n \}) \), where the union is taken over all subsequences \( \{ u_n \} \) of \( \{ x_n \} \). Let \( p \in \omega_n \) such that
\[
\frac{\lambda_n - \lambda}{\lambda_n} d(x_n, w_n) = \frac{\lambda_n - \lambda}{\lambda_n} d(x_n, x_n) + \frac{\lambda}{\lambda_n} d(x_n, x_n) = (2 - \frac{\lambda}{\lambda_n})d(x_n, w_n).
\]
So, this implies the fact that the limit
\[
\lim_{n \to \infty} d(x_n, J_{\lambda_n} x_n) = 0.
\]
This completes the proof.

Owing to the fact \( Sv \) is compact, there exists a subsequence \( \{ y_n \} \) of \( \{ y_n \} \) such that \( \lim_{n \to \infty} y_n = y \in Sv \). Thus, this shows that

\[
\lim \sup_{n \to \infty} d(y_n, y) \leq \lim_{n \to \infty} \sup \left( d(y_n, r_n) + d(r_n, y_n) + d(y_n, y) \right)
\]
\[
\leq \lim_{n \to \infty} \sup \left( d(y_n, r_n) + \text{dist}(r_n, Sv) + d(y_n, y) \right)
\]
\[
\leq \lim_{n \to \infty} \sup \left( d(y_n, r_n) + \text{dist}(r_n, Sv) + d(y_n, y) \right)
\]
\[
= \lim_{n \to \infty} \sup d(y_n, v).
\]

Through (38) and the uniqueness of asymptotic centers, we can obtain that \( v = y \in Sv \). Therefore, by (39), we can show that
\[
v \in F(T) \cap F(S) \cap \arg \min_{u \in D} f(u) = \Omega.
\]

It follows by Lemma 3 and Theorem 1 (i) that \( p = v \), and hence, \( \omega_n = (x_n) \subseteq \Omega \).

In order to show that \( \{ x_n \} \Delta\)-converges to a point in \( \Omega \), it suffices to show that \( \omega_n = (x_n) \) consists of exactly one point. Suppose that \( \{ u_n \} \) is a subsequence of \( \{ x_n \} \) with \( A(\{ u_n \}) = \{ u^* \} \) and \( A(\{ x_n \}) = \{ x \} \). Since \( u^* \in \omega_n = (x_n) \subseteq \Omega \) and \( \{ d(x_n, u^*) \} \) converge, it implies by Lemma 3 that \( x = u^* \).

This completes the proof. \( \square \)

**Remark 1.**

(i) The results of Shuntau and Phuaengrattana [18] and Cholamjiak [30] are extended and improved by Theorem 2. In fact, a new proximal point algorithm can be used for solving the constrained convex
minimization problem as well as the fixed-point problem of a single-valued nonexpansive mapping and a multivalued nonexpansive mapping in a CAT(0) space.

(ii) Since every real Hilbert space H is a complete CAT(0) space, the above result can also be obtained in Hilbert spaces, so a convergence weakly theorem can be obtained in a real Hilbert space as follows.

**Corollary 1.** Let D be a nonempty closed convex subset of a real Hilbert space H. Let $f : D \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function, $T$ be a single-valued nonexpansive mapping, and $S : D \to \text{CB}(D)$ be a multivalued nonexpansive mapping. Suppose that

$$\Omega = F(T) \cap F(S) \cap \text{arg min}_{y \in C} f(y) \neq \emptyset$$

and $Sp = \{p\}, p \in \Omega$. For $x_1 \in D$, let the sequence $\{x_n\}$ be defined by

$$\left\{ \begin{array}{l}
  w_n = \text{arg min}_{y \in C} \left[ f(y) + \frac{1}{2\lambda_n} d^2(y, x_n) \right], \\
  x_{n+1} = \alpha_n x_n + \beta_np_n + \gamma_n Tp_n, \\
  p_n \in S_{w_n},
\end{array} \right.\quad (43)$$

the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0, 1)$, and $0 < \alpha \leq \alpha_n, \beta_n, \gamma_n \leq 1$. Then, the sequence $\{x_n\}$ converges weakly to an element in $\Omega$.

**Remark 2.** In fact, the construction of our proposed algorithm is rather peculiar and it is different from references [31–33]. Convex combination of the sequences $\{x_n\}, \{p_n\}$, and $\{T_{p_n}\}$ are given here, especially $p_n \in S_{w_n}$, and $T_{p_n}$ is the nonexpansive mapping $T$ that operates on the sequences $\{p_n\}$.

### 4. Numerical Experiments

In this section, a numerical example is given to illustrate reckoning the convergence of modified proximal point algorithm with iteration (15) by numerical experiment for supporting Theorems 1 and 2.

Let $X = \mathbb{R}^2$ with Euclidean norm and $D = \{x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^2 : 0 \leq x^{(1)}, x^{(2)} \leq 1\}$. For each $x = (x^{(1)}, x^{(2)}) \in D$, the concrete definition of nonexpansive mappings $T$ and $S$ is shown as

$$Tx = \left( \frac{2x^{(1)} + 1}{4}, \frac{x^{(2)} + 7}{8} \right),$$

$$Sx = \left[ x^{(1)} \times \left( \frac{3x^{(2)} + 1}{4}, 1 \right) \right].$$

For each $x \in D$, assume that $\|x\|_1 = \sum_{i=1}^{n} |x_i|$, and $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$. The function $f : D \to (-\infty, \infty)$ is defined in the following manner:

$$f(x) = \|x\|_1 + \frac{1}{2}\|x\|_2^2 + (-1.5, -2)x + 7.$$  

(46)

From the fact that $T$ and $S$ are nonexpansive and $f$ is a proper convex lower semicontinuous function easy to prove, we skip their proofs here. Furthermore, by making use of the soft thresholding operator [34] and the proximity operator [35], let $\lambda_n = 1$; we have

$$J_{\lambda}x = \text{arg min}_{v \in D} \left[ f(v) + \frac{1}{2}\|v - x\|^2 \right] = \text{prox}_{\lambda f}x$$

$$= \text{prox}_{(1/2\lambda)} \left( \frac{x - (-1.5, -2)}{2} \right)$$

$$= \left( \max \left\{ \frac{x^{(1)} + 1.5 - 1}{2}, 0 \right\}, \text{sgn}(x^{(1)} + 1.5), \max \left\{ \frac{x^{(2)} + 2 - 1}{2}, 0 \right\}, \text{sgn}(x^{(2)} + 2) \right),$$

(47)

where $\text{sgn}(\xi)$ is a signum function, that is,

$$\left\{ \begin{array}{l}
  \text{sgn}(\xi) = 1, \\
  \xi > 0, \\
  \text{sgn}(\xi) = 0, \\
  \xi = 0, \\
  \text{sgn}(\xi) = -1, \\
  \xi < 0.
\end{array} \right. \quad (48)$$

Further simplification of the proposed iterative algorithm is in the following expression:

$$\left\{ \begin{array}{l}
  w_n = J_{\lambda}x_n, \\
  x_{n+1} = \alpha_n x_n + \beta_n p_n + \gamma_n T_{p_n}, \\
  p_n \in S_{w_n},
\end{array} \right. \quad (49)$$

where $x_n = (x_n^{(1)}, x_n^{(2)})$ and $w_n = (w_n^{(1)}, w_n^{(2)})$ are points in $\mathbb{R}^2$. In addition, we choose some points and the sequences of parameters as follows:
Next, we use Algorithm (49) with an initial point $x_1 = (0.1, 0.2)$ and obtain numerical results in Table 1.

Table 1: Numerical results of Algorithm (49).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_n = (x_n^{(1)}, x_n^{(2)})$</th>
<th>$|x_n - x_{n-1}|_2$</th>
<th>$f(x_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0.100000, 0.200000)</td>
<td>0.347205</td>
<td>6.525875</td>
</tr>
<tr>
<td>2</td>
<td>(0.310620, 0.705335)</td>
<td>0.124269</td>
<td>6.400753</td>
</tr>
<tr>
<td>5</td>
<td>(0.406401, 0.884487)</td>
<td>0.050838</td>
<td>6.379830</td>
</tr>
<tr>
<td>7</td>
<td>(0.453275, 0.954060)</td>
<td>0.021717</td>
<td>6.375968</td>
</tr>
<tr>
<td>9</td>
<td>(0.476569, 0.981610)</td>
<td>0.009587</td>
<td>6.375205</td>
</tr>
<tr>
<td>11</td>
<td>(0.488219, 0.992611)</td>
<td>0.004359</td>
<td>6.375045</td>
</tr>
<tr>
<td>13</td>
<td>(0.494066, 0.997023)</td>
<td>0.002034</td>
<td>6.375010</td>
</tr>
<tr>
<td>15</td>
<td>(0.497008, 0.998799)</td>
<td>0.000970</td>
<td>6.375002</td>
</tr>
<tr>
<td>17</td>
<td>(0.498490, 0.999514)</td>
<td>0.000471</td>
<td>6.375000</td>
</tr>
<tr>
<td>19</td>
<td>(0.499237, 0.999803)</td>
<td>0.000231</td>
<td>6.375000</td>
</tr>
<tr>
<td>21</td>
<td>(0.499614, 0.999920)</td>
<td>0.000015</td>
<td>6.375000</td>
</tr>
<tr>
<td>23</td>
<td>(0.499805, 0.999967)</td>
<td>0.000005</td>
<td>6.375000</td>
</tr>
<tr>
<td>25</td>
<td>(0.499901, 0.999986)</td>
<td>0.000002</td>
<td>6.375000</td>
</tr>
<tr>
<td>27</td>
<td>(0.499950, 0.999999)</td>
<td>0.000001</td>
<td>6.375000</td>
</tr>
<tr>
<td>29</td>
<td>(0.499974, 0.999999)</td>
<td>0.000000</td>
<td>6.375000</td>
</tr>
<tr>
<td>31</td>
<td>(0.499987, 0.999999)</td>
<td>0.000000</td>
<td>6.375000</td>
</tr>
<tr>
<td>33</td>
<td>(0.499993, 0.999999)</td>
<td>0.000000</td>
<td>6.375000</td>
</tr>
<tr>
<td>35</td>
<td>(0.499996, 0.999999)</td>
<td>0.000000</td>
<td>6.375000</td>
</tr>
</tbody>
</table>

Remark 3

(i) From Table 1 and Figure 1, it is observed that the sequence $\{x_n\}$ converges to a point $(0.5, 1)$.

(ii) From Table 1 and Figure 1, it is observed that the sequence $\{x_n\}$ converges to a point $(0.5, 1)$.

(iii) The point $(0.5, 1)$ is a solution of the constrained convex minimization problems (46) and also a solution of the fixed-point problems of a pair of a nonexpansive single-valued mapping $T$ and a nonexpansive multivalued mapping $S$.

Data Availability

Some data obtained by the authors themselves were used to support this study.

Conflicts of Interest

All authors declare no conflicts of interest.

Authors’ Contributions

All authors contributed equally to this study.

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