Research Article

Some Dimensional Results of a Class of Homogeneous Moran Sets

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In this paper, we construct a class of special homogeneous Moran sets: $\{m_k\}$ -quasi-homogeneous perfect sets, and obtain the Hausdorff dimension of the sets under some conditions. We also prove that the upper box dimension and the packing dimension of the sets can get the maximum value of the homogeneous Moran sets under the condition $\sup_{k \geq 1} m_k < \infty$ and estimate the upper box dimension of the sets in two cases.

1. Introduction

The fractal dimensions of the Moran sets have been studied by many authors and close connected with many subjects, such as the multifractals (see [1–3]), the quasi-conformal mappings (see [4, 5]), the ergodic theory (see [6]), and the number theory (see [7]). In these applications, the homogeneous Moran sets play an important role. There are many classical results about the fractal dimensions of the homogeneous Moran sets (see [8, 9]), but Wen et al. [8, 9] only showed the maximal values and the minimal values of the Hausdorff dimension, the upper box dimension, and the packing dimension (see [10]) of the family of the homogeneous Moran sets, and we need to get some more exact expressions of the fractal dimensions for some special homogeneous Moran sets. Wen and Wu [11] constructed a class of homogeneous Moran sets by the gaps, which are called homogeneous perfect sets, and obtained the Hausdorff dimension of the homogeneous perfect sets under some conditions. Later, Wang and Wu [12] obtained the upper box dimension and the packing dimension of the homogeneous perfect sets under some conditions. Hu [13, 14] constructed a special kind of homogeneous Moran sets which is called $\{m_k\}$ -quasi-homogeneous Cantor sets by the

connected components and the gaps and obtained its Hausdorff dimension, upper box dimension, and packing dimension.

In this paper, a class of special homogeneous Moran sets, which is called $\{m_k\}$ -quasi-homogeneous perfect sets, is constructed by the connected components and the gaps, and several results about the fractal dimensions of the sets are obtained, which extend the results in [11, 12, 14] under some conditions.

This paper is organized as follows. In Section 2, we give the definitions of the homogeneous Moran sets, the homogeneous perfect sets, and the $\{m_k\}$ -quasi-homogeneous perfect sets. Our main results are stated in Section 3. Section 4 gives the proofs of our theorems.

2. Preliminaries

First, we recall the definition of homogeneous Moran sets.

Let $\{n_k\}_{k \geq 1}$ be a sequence of positive integers and $\{c_k\}_{k \geq 1}$ be a sequence of positive real numbers, such that $n_k \geq 2$, $0 < c_k < 1$ and $n_k c_k < 1$, for any $k \geq 1$. Let $D_k = \{\sigma_1 \sigma_2 \dots \sigma_k: 1 \leq \sigma_j \leq n_j, 1 \leq j \leq k\}$, for any $k \geq 1$, and let $D_0 = \{\emptyset\}$ and $D = \bigcup_{k=0}^{\infty} D_k$. If $\sigma = \sigma_1 \sigma_2 \dots \sigma_k \in D_k$ and $1 \leq j \leq n_{k+1}$, let $\sigma * j = \sigma_1 \sigma_2 \dots \sigma_k j \in D_{k+1}$.

Definition 1 (homogeneous Moran sets, (see [8])). Let $I \subset \mathbb{R}$ be a nonempty closed interval and $\mathcal{F} = \{I_\sigma: \sigma \in D\}$ be a collection of closed subintervals of I . We say that $\mathcal{F}$ has homogeneous Moran structure if it satisfies

- (i) $I_\emptyset = I$
- (ii) For any $k \geq 0$, $\sigma \in D_k$, $I_{\sigma*1}, I_{\sigma*2}, \dots, I_{\sigma*n_{k+1}}$ are subintervals of I_σ , and the interiors of $I_{\sigma*i}$ and $I_{\sigma*j}$ are disjoint for any $i \neq j$
- (iii) For any $k \geq 1$, $\sigma \in D_{k-1}$, and $1 \leq j \leq n_k$, we have $(|I_{\sigma*j}|/|I_\sigma|) = c_k$, where $|A|$ denotes the diameter of the set $A \subset \mathbb{R}$

Let

$$E = E(\mathcal{F}) := \bigcap_{k \geq 1} \bigcup_{\sigma \in D_k} I_\sigma. \quad (1)$$

Then, we call E a homogeneous Moran set satisfying $(I, \{n_k\}, \{c_k\})$ and denote by $\mathcal{M}(I, \{n_k\}, \{c_k\})$ the class of all homogeneous Moran sets associated with $I, \{n_k\}$, and $\{c_k\}$.

Let $\mathcal{F}_k = \{I_\sigma: \sigma \in D_k\}$; then, $\mathcal{F} = \bigcup_{k \geq 0} \mathcal{F}_k$ and any I_σ in $\mathcal{F}_k$ is called a k -order basic interval of E . Without loss of generality, we set $I = [0, 1]$ and assume that $I_{\sigma*1}, \dots, I_{\sigma*n_k}$ are located from left to right for any $k \geq 1$ and any $\sigma \in D_{k-1}$.

Based on the basic intervals and the gaps, Wen and Wu [11] constructed a special kind of homogeneous Moran sets: homogeneous perfect sets.

Definition 2 (homogeneous perfect sets, (see [11])). Let $E \in \mathcal{M}(I, \{n_k\}, \{c_k\})$, and we say that $\mathcal{F} = \{I_\sigma: \sigma \in D\}$ satisfies homogeneous perfect structure if it satisfies the following: there exists a sequence of real numbers $\{\eta_{k,j}: k \geq 1, 0 \leq j \leq n_k\}$ such that, for any $k \geq 1, 0 \leq j \leq n_k$, we have $\eta_{k,j} \geq 0$, and for any $k \geq 0, \sigma \in D_k$, and $1 \leq i \leq n_{k+1} - 1$, we have $\min(I_{\sigma*1}) - \min(I_\sigma) = \eta_{k+1,0}$, $\min(I_{\sigma*(i+1)}) - \max(I_{\sigma*i}) = \eta_{k+1,i}$, and $\max(I_\sigma) - \max(I_{\sigma*n_{k+1}}) = \eta_{k+1,n_{k+1}}$. Then, we call $E = E(\mathcal{F}) = E(I, \{n_k\}, \{c_k\}, \{\eta_{k,j}\}) := \bigcap_{k \geq 1} \bigcup_{\sigma \in D_k} I_\sigma$ a homogeneous perfect set satisfying $(I, \{n_k\}, \{c_k\}, \{\eta_{k,j}\})$ and denote by $\mathcal{J}(I, \{n_k\}, \{c_k\}, \{\eta_{k,j}\})$ the class of all homogeneous perfect sets associated with $I, \{n_k\}, \{c_k\}, \{\eta_{k,j}\}$.

Based on the connected components of the basic intervals and their gaps, we can construct a special class of homogeneous Moran sets: $\{m_k\}$ -quasi-homogeneous perfect sets.

Definition 3 ($\{m_k\}$ -quasi-homogeneous perfect sets). Let $E \in \mathcal{M}(I, \{n_k\}, \{c_k\})$, if for all $k \geq 1$ and $\sigma \in D_{k-1}$, the k -order basic intervals $I_{\sigma*1}, I_{\sigma*2}, \dots, I_{\sigma*n_k}$ arbitrarily connect forming m_k ($1 \leq m_k \leq n_k$) connected components (denote by $J_{\sigma*1}, J_{\sigma*2}, \dots, J_{\sigma*m_k}$, which are called k -order connected components). We say that $\mathcal{F} = \{I_\sigma: \sigma \in D\}$ satisfies homogeneous perfect structure if it satisfies

- (i) $I_\emptyset = J = I$
- (ii) There exists a sequence of real numbers $\{\eta_{k,l}^*: k \geq 1, 0 \leq l \leq m_k\}$ such that, for any $k \geq 1, 0 \leq l \leq m_k$, we have $\eta_{k,l}^* \geq 0$ and $\min_{1 \leq l \leq m_{k-1}} \eta_{k,l}^* > 0$, and for any $k \geq 0, \sigma \in D_k$, and $1 \leq i \leq m_{k+1} - 1$, we

$$\text{have } \min(J_{\sigma*1}) - \min(I_\sigma) = \eta_{k+1,0}^*, \min(J_{\sigma*(i+1)}) - \max(J_{\sigma*i}) = \eta_{k+1,i}^*, \text{ and } \max(I_\sigma) - \max(J_{\sigma*m_{k+1}}) = \eta_{k+1,m_{k+1}}^*$$

Let $E = E(\mathcal{F}) = E(I, J, \{n_k\}, \{m_k\}, \{c_k\}, \{\eta_{k,l}^*\}) := \bigcap_{k \geq 1} \bigcup_{\sigma \in D_k} I_\sigma$; then, we call E a $\{m_k\}$ -quasi-homogeneous perfect set satisfying $(I, J, \{n_k\}, \{m_k\}, \{c_k\}, \{\eta_{k,l}^*\})$ and denote by $\mathcal{J}^*(I, J, \{n_k\}, \{m_k\}, \{c_k\}, \{\eta_{k,l}^*\})$ the class of all $\{m_k\}$ -quasi-homogeneous perfect sets associated with $I, J, \{n_k\}, \{c_k\}, \{\eta_{k,l}^*\}$. For any $\sigma \in D_{k-1}$, $1 \leq j \leq m_k$, denote by $\# \{i: I_{\sigma*i} \subset J_{\sigma*j}\}$ the number of the k -order basic intervals contained in the k -order connected components $J_{\sigma*j}$.

Remark 1. Let $E \in \mathcal{J}^*(I, J, \{n_k\}, \{m_k\}, \{c_k\}, \{\eta_{k,l}^*\})$; then, we have

- (i) If $m_k = n_k$, for any $k \geq 1$, then E is a homogeneous perfect set and $E \in \mathcal{J}(I, \{n_k\}, \{c_k\}, \{\eta_{k,l}^*\})$, where $\eta_{k,l} = \eta_{k,l}^*$, for any $k \geq 1, 0 \leq l \leq m_k$
- (ii) If $m_k = n_k, \eta_{k,0}^* = \eta_{k,m_k}^* = 0$, and $\eta_{k,1}^* = \dots = \eta_{k,m_k-1}^*$, for any $k \geq 1$, then E is a homogeneous Cantor set [8]

3. Main Results

Let

$$\begin{aligned} s_* &= \liminf_{k \rightarrow \infty} \frac{\log n_1 n_2 \dots n_k}{-\log c_1 c_2 \dots c_k}, \\ s^* &= \limsup_{k \rightarrow \infty} \frac{\log n_1 n_2 \dots n_k}{-\log c_1 c_2 \dots c_k}, \\ t^* &= \limsup_{k \rightarrow \infty} \frac{\log n_1 n_2 \dots n_{k+1}}{-\log c_1 c_2 \dots c_k + \log n_{k+1}}. \end{aligned} \quad (2)$$

Our main results are stated as follows.

Theorem 1. Let $E \in \mathcal{J}^*(I, J, \{n_k\}, \{m_k\}, \{c_k\}, \{\eta_{k,l}^*\})$ with $\eta_{k,0}^* = \eta_{k,m_k}^* = 0$. For any $k \geq 1, \sigma \in D_{k-1}$, and $1 \leq j \leq m_k$, there exists a positive integer α such that $\# \{i: I_{\sigma*i} \subset J_{\sigma*j}\} \leq \alpha$ for the k -order connected component $J_{\sigma*j}$. If there exist positive constants $\tilde{c}_1, \tilde{c}_2$ ($\tilde{c}_2 \geq 1$) such that at least one of the following two conditions is satisfied for any $k \geq 1$:

- (A) $\max_{1 \leq l \leq m_{k-1}} \eta_{k,l}^* \leq \tilde{c}_1 \cdot c_1 c_2 \dots c_k$.
- (B) $\max_{1 \leq l \leq m_{k-1}} \eta_{k,l}^* \leq \tilde{c}_2 \cdot \min_{1 \leq l \leq m_{k-1}} \eta_{k,l}^*$; then,

$$\dim_H E = \liminf_{k \rightarrow \infty} \frac{\log n_1 n_2 \dots n_{k-1} m_k}{-\log c_1 c_2 \dots c_k (n_k/m_k)} = s_*. \quad (3)$$

Example 1. Let E be a $\{m_k\}$ -quasi-homogeneous perfect set satisfying the conditions of Theorem 1, then we have $\dim_H E = s_* = \liminf_{k \rightarrow \infty} (\log n_1 n_2 \dots n_{k-1} n_k / (-\log c_1 c_2 \dots c_k))$, which is equivalent to Theorem 1.2 (1) in [11]. Notice that if $E \in \mathcal{J}(I, \{n_k\}, \{c_k\}, \{\eta_{k,l}^*\})$ ($0 \leq l \leq n_k$) is a homogeneous perfect set, then $E \in \mathcal{J}^*(I, J, \{n_k\}, \{m_k\}, \{c_k\}, \{\eta_{k,l}^*\})$ ($1 \leq m_k \leq n_k, 0 \leq l \leq m_k$) and satisfies $\eta_{k,0}^* = \eta_{k,m_k}^* = 0$ (for any $k \geq 1$) if $\eta_{k,0} = \eta_{k,n_k} = 0$. Thus, Theorem 1

generalizes Theorem 1.2 (A) and (B) in [11] under the condition $\eta_{k,0} = \eta_{k,n_k} = 0$ (for any $k \geq 1$).

Theorem 2. Suppose $E \in \mathcal{F}^*(I, J, \{n_k\}, \{m_k\}, \{c_k\}, \{\eta_{k,l}^*\})$. If $\sup_{k \geq 1} m_k < \infty$, then

$$\overline{\dim}_B E = \dim_p E = s^*. \quad (4)$$

Example 2. Let E be a $\{m_k\}$ -quasi-homogeneous Cantor set (refer to [14]); then, $E \in \mathcal{F}^*(I, J, \{n_k\}, \{m_k\}, \{c_k\}, \{\eta_{k,l}^*\})$ and satisfies $\eta_{k,0}^* = \eta_{k,m_k}^* = 0$ and $\eta_{k,1}^* = \dots = \eta_{k,m_k-1}^*$, for any $k \geq 1$. Since the conclusion of Theorem 2 holds under the condition $\sup_{k \geq 1} m_k < \infty$, Theorem 2 generalizes Theorem 2.3 in [14].

Theorem 3. Let $E \in \mathcal{F}^*(I, J, \{n_k\}, \{m_k\}, \{c_k\}, \{\eta_{k,l}^*\})$ with $\eta_{k,0}^* = \eta_{k,m_k}^* = 0$ and $a_k = [n_k/m_k]$, for any $k \geq 1$. If there exists a positive constant $\tilde{c}_3$ such that, for any $k \geq 1$, there exists a sequence of nonnegative integers $\{N_k\}_{k \geq 1}$ with $N_k < a_k$ and $\sup_{k \geq 1} N_k < \infty$, satisfying that, for any k -order connected component J_{σ^*j} ($\sigma \in D_{k-1}$, $1 \leq j \leq m_k$), $\#\{i: I_{\sigma^*i} \subset J_{\sigma^*j}\}$ is one of $a_k - N_k, \dots, a_k, \dots, a_k + N_k$, and the following condition is satisfied:

$$\max_{1 \leq l \leq m_k-1} \eta_{k,l}^* \leq \tilde{c}_3 \cdot c_1 c_2, \dots, c_k. \quad (5)$$

Then,

$$\overline{\dim}_B E = \limsup_{k \rightarrow \infty} \frac{\log n_1 n_2, \dots, n_{k-1} m_k}{-\log(c_1 c_2, \dots, c_{k-1} / m_k)}. \quad (6)$$

Remark 2. The conditions of N_k and $\#\{i: I_{\sigma^*i} \subset J_{\sigma^*j}\}$ in Theorem 3 imply that there exist $0 < \varphi_1, \varphi_2 < \infty$, such that, for any $k \geq 1, \sigma \in D_{k-1}, 1 \leq i, j \leq m_k$, we have $\varphi_1 \leq (|J_{\sigma^*i}|/|J_{\sigma^*j}|) \leq \varphi_2$.

Theorem 4. Let $E \in \mathcal{F}^*(I, J, \{n_k\}, \{m_k\}, \{c_k\}, \{\eta_{k,l}^*\})$ with $\eta_{k,0}^* = \eta_{k,m_k}^* = 0$, for any $k \geq 1$. Suppose that there exist positive constants $\tilde{c}_4, \tilde{c}_5, \tilde{c}_6, \gamma$ ($\tilde{c}_6 \geq 1, \gamma \geq 1$) such that, for any $k \geq 1$, at least one of the following three conditions is satisfied:

$$(C) \max_{1 \leq l \leq m_k-1} \eta_{k,l}^* \leq \tilde{c}_4 \cdot c_1 c_2, \dots, c_k, \text{ and } (n_k/m_k) \leq \gamma.$$

$$(D) m_k \cdot \min_{1 \leq l \leq m_k-1} \eta_{k,l}^* \geq \tilde{c}_5 \cdot c_1 c_2, \dots, c_{k-1}.$$

$$(E) \max_{1 \leq l \leq m_k-1} \eta_{k,l}^* \leq \tilde{c}_6 \cdot \min_{1 \leq l \leq m_k-1} \eta_{k,l}^* \text{ and } (n_k/m_k) \leq \gamma; \text{ then,}$$

$$\overline{\dim}_B E = \limsup_{k \rightarrow \infty} \frac{\log n_1 n_2, \dots, n_{k-1} m_k}{-\log(c_1 c_2, \dots, c_{k-1} / m_k)}. \quad (7)$$

Example 3. We obtain the following conclusions about the relationships between theorems in this paper and Theorem 1.4 in [12]:

- (i) If Theorem 1.4 (A) in [12] holds for E which satisfies $\eta_{k,0} = \eta_{k,n_k} = 0$ (for any $k \geq 1$) and there exists $\gamma \geq 1$ such that $(n_k/m_k) \leq \gamma$ (for any $k \geq 1$) or Theorem 1.4 (B) or (C) holds for E (it means $(n_k/m_k) = 1$) which satisfies $\eta_{k,0} = \eta_{k,n_k} = 0$, then we have $E \in \mathcal{F}^*(I, J, \{n_k\}, \{m_k\}, \{c_k\}, \{\eta_{k,l}^*\})$ ($1 \leq m_k \leq n_k, 0 \leq l \leq m_k$) and satisfies the conditions of Theorem 4, then (7) holds for E . Since $\eta_{k,0} = \eta_{k,n_k} = 0$ and $m_k \leq n_k \leq \gamma m_k$, (7) is equivalent to (1.4) in [12] for the upper box dimension.
- (ii) If Theorem 1.4 (D) in [12] holds for E which satisfies $\eta_{k,0} = \eta_{k,n_k} = 0$ (for any $k \geq 1$), then $\sup_{k \geq 1} m_k < \infty$ and $\sup_{k \geq 1} n_k < \infty$. Thus, $E \in \mathcal{F}^*(I, J, \{n_k\}, \{m_k\}, \{c_k\}, \{\eta_{k,l}^*\})$ ($1 \leq m_k \leq n_k, 0 \leq l \leq m_k$) and satisfies the conditions of Theorem 2; then, (4) holds for E . Since $\sup_{k \geq 1} n_k < \infty$, we have $s^* = t^*$ and (4) is equivalent to (1.4) in [12].

By the argument above, Theorems 2 and 4 generalize Theorem 1.4 in [12] for the upper box dimension under the conditions $\eta_{k,0} = \eta_{k,n_k} = 0$ and $(n_k/m_k) \leq \gamma$ (for any $k \geq 1$) (notice that if E satisfies (B) or (C) or (D) of Theorem 1.4 in [12], we only need the condition $\eta_{k,0} = \eta_{k,n_k} = 0$).

Remark 3. The conditions of Theorem 3 and the conditions of Theorem 4 (C) do not contain each other.

If E satisfies the conditions of Theorem 3, such that $m_k \equiv 2$, $N_k \equiv 1$, and $\sup_{k \geq 1} n_k = \infty$, then $a_k = [n_k/m_k] \rightarrow \infty$ as $n_k \rightarrow \infty$, and E does not satisfy the conditions of Theorem 4 (C).

If E satisfies the conditions of Theorem 4 (C), such that $m_k = 2k$ and $n_k = 3k$ for any $k \geq 1$ (it means $(n_k/m_k) \equiv (3/2)$) and there exists a positive integer k_0 such that, for any $k \geq k_0$ and any $\sigma \in D_{k-1}$, the numbers of the k -order basic intervals contained in $J_{\sigma^*1}, J_{\sigma^*2}, \dots, J_{\sigma^*m_k}$ are $k+1, 1, 1, \dots, 1$, then E does not satisfy the conditions of Theorem 3.

Remark 4. The upper box dimension in Theorems 3 or 4 is different from s^* and t^* .

If E satisfies the conditions of Theorem 3, such that $n_1 = m_1 = 2$, $n_k = 2^{n_1 n_2, \dots, n_{k-1}}$ (for any $k \geq 2$), $m_k = 2^{n_{k-1}}$ (for any $k \geq 2$), $a_1 = [n_1/m_1] = 1$, $a_k = [n_k/m_k] = 2^{n_{k-1} (n_1 n_2, \dots, n_{k-2} - 1)}$ (for any $k \geq 2$), $N_k \equiv 0$, and $c_k = 2^{-n_k}$ (for any $k \geq 1$) or if E satisfies the conditions of Theorem 4 (D) such that $n_1 = m_1 = 2$, $n_k = 2^{n_1 n_2, \dots, n_{k-1}}$ (for any $k \geq 2$), $m_k = 2^{n_{k-1}}$ (for any $k \geq 2$), and $c_k = 2^{-n_k}$ (for any $k \geq 1$), then

$$\begin{aligned}
\overline{\dim}_B E &= \limsup_{k \rightarrow \infty} \frac{1 + n_1 + n_1 n_2 + \cdots + n_1, \dots, n_{k-2} + n_{k-1}}{n_1 + n_2 + \cdots + n_{k-1} + n_k} = \frac{1}{2}, \\
s^* &= \limsup_{k \rightarrow \infty} \frac{1 + n_1 + n_1 n_2 + \cdots + n_1, \dots, n_{k-2} + n_1, \dots, n_{k-2} n_{k-1}}{n_1 + n_2 + \cdots + n_{k-1} + n_k} = 0, \\
t^* &= \limsup_{k \rightarrow \infty} \frac{1 + n_1 + n_1 n_2 + \cdots + n_1, \dots, n_{k-2} + n_1, \dots, n_{k-1} + n_1, \dots, n_k}{n_1 + n_2 + \cdots + n_{k-1} + n_k + n_1, \dots, n_k} = 1.
\end{aligned} \tag{8}$$

It can be concluded that $\overline{\dim}_B E$ is different from s^* and t^* .

4. Proofs of the Theorems

We shall use the following lemmas to prove the theorems.

Lemma 1 (mass distribution principle, (see [10])). *Let E be a Borel set and μ be a Borel measure supported on E such that $0 < \mu(E) < \infty$. Suppose that, for some s , there are numbers $c > 0$ and $\delta > 0$ such that $\mu(U) \leq c|U|^s$, for all sets U with $|U| \leq \delta$. Then, $\dim_H E \geq s$.*

Lemma 2 (see [13]). *Let $E \in \mathcal{F}^*(I, J, \{n_k\}, \{m_k\}, \{c_k\}, \{\eta_{k,l}^*\})$; then,*

$$\dim_H E \leq \liminf_{k \rightarrow \infty} \frac{\log n_1 n_2, \dots, n_{k-1} m_k}{-\log c_1 c_2, \dots, c_k (n_k/m_k)}. \tag{9}$$

Lemma 3 (see [8]). *Let $E \in \mathcal{M}(I, \{n_k\}, \{c_k\})$, then*

$$s^* \leq \dim_P E \leq \overline{\dim}_B E \leq t^*. \tag{10}$$

Lemma 4 (see [15]) *Let $E \in \mathcal{F}^*(I, J, \{n_k\}, \{m_k\}, \{c_k\}, \{\eta_{k,l}^*\})$; then,*

$$\overline{\dim}_B E = \inf \left\{ s > 0: \sum_{k=1}^{\infty} n_1 n_2 \cdots n_k \sum_{i=0}^{m_{k+1}} (\eta_{k+1,i}^*)^s < \infty \right\}. \tag{11}$$

4.1. Proof of Theorem 1. Since the conditions of Theorem 1 imply that $(n_k/m_k) \leq \alpha$, for all $k \geq 1$, we have $\liminf_{k \rightarrow \infty} (\log n_1 n_2, \dots, n_{k-1} m_k / -\log c_1 c_2, \dots, c_k (n_k/m_k)) = \liminf_{k \rightarrow \infty} (\log n_1 n_2, \dots, n_{k-1} n_k / -\log c_1 c_2, \dots, c_k) = s^*$. We only need to prove $\dim_H E = \liminf_{k \rightarrow \infty} (\log n_1 n_2, \dots, n_{k-1} m_k / -\log c_1 c_2, \dots, c_k (n_k/m_k))$.

By Lemma 2, it suffices to prove that if E satisfies the conditions of Theorem 1, then $\dim_H E = \liminf_{k \rightarrow \infty} (\log n_1 n_2, \dots, n_{k-1} m_k / -\log c_1 c_2, \dots, c_k (n_k/m_k))$.

For convenience, let

$$\begin{aligned}
\delta_k &= c_1 c_2, \dots, c_k, \\
\bar{\alpha}_k &= \max_{1 \leq l \leq m_k-1} \eta_{k,l}^*, \\
\underline{\alpha}_k &= \min_{1 \leq l \leq m_k-1} \eta_{k,l}^*, \quad (\text{for any } k \geq 1).
\end{aligned} \tag{12}$$

For any $0 < l < \liminf_{k \rightarrow \infty} (\log n_1 n_2, \dots, n_{k-1} m_k / -\log c_1 c_2, \dots, c_k (n_k/m_k))$, there exists a positive integer K such that, for any $k \geq K$, we have

$$n_1 n_2, \dots, n_{k-1} m_k (\delta_k \alpha)^l \geq n_1 n_2, \dots, n_{k-1} m_k \left(\delta_k \frac{n_k}{m_k} \right)^l \geq 1. \tag{13}$$

Let μ be a Borel probability measure supported on E , such that, for each k -order basic intervals $I_{\sigma^* i} (k \geq 1, \sigma \in D_{k-1}, 1 \leq i \leq n_k)$, $\mu(I_{\sigma^* i}) = (n_1 n_2, \dots, n_{k-1} n_k)^{-1}$. Then, for each k -order connected component $J_{\sigma^* j} (1 \leq j \leq m_k)$, $\mu(J_{\sigma^* j}) \leq (\alpha/n_1 n_2, \dots, n_{k-1} n_k) \leq (\alpha/n_1 n_2, \dots, n_{k-1} m_k)$.

Let U be an interval with $0 < |U| = \delta < \delta_K$; then, there exists a positive integer $k > K$ such that $\alpha \delta_k \leq \delta < \alpha \delta_{k-1}$. Thus, the number of the $(k-1)$ -order basic intervals that intersect U is at most $2\alpha \leq 2\alpha^2$. Denote by g the number of the k -order connected components that intersect U . In order to estimate $\mu(U)$, we divide our proof into the following two cases:

Case (i): $\alpha \delta_k \leq \delta < \underline{\alpha}_k$.

It is obvious that $g \leq 2$. Then,

$$\mu(U) \leq \frac{2\alpha}{n_1 n_2, \dots, n_{k-1} m_k} \leq 2\alpha (\delta_k \alpha)^l \leq 2\alpha \delta^l = 2\alpha |U|^l. \tag{14}$$

Case (ii): $\max\{\alpha \delta_k, \underline{\alpha}_k\} \leq \delta < \alpha \delta_{k-1}$

We discuss separately according to the two conditions (A) and (B) of Theorem 1.

Condition (A): $\bar{\alpha}_k \leq \tilde{c}_1 c_1 c_2, \dots, c_k$. Then, we have

$$\begin{aligned}
\delta_{k-1} &= \sum_{i=1}^{m_{k-1}} \eta_{k,i}^* + n_k \delta_k \leq (m_k - 1) \bar{\alpha}_k + n_k \delta_k \\
&\leq \tilde{c}_1 (m_k - 1) \delta_k + n_k \delta_k \leq \tilde{c}_1 m_k \delta_k + n_k \delta_k \leq (\tilde{c}_1 + 1) n_k \delta_k.
\end{aligned} \tag{15}$$

On the one hand, since the number of the $(k-1)$ -order basic intervals that intersect U is at most $2\alpha^2$, we have $g \leq 2\alpha^2 m_k$. On the other hand, the number of the k -order connected components that intersect U is at

most $(\delta/\delta_k) + 2 \leq 2((\delta/\delta_k) + 1) \leq (4\delta/\delta_k)$. Then, by (13) and (15), we have

$$\begin{aligned}
 \mu(U) &\leq \frac{\alpha}{n_1 n_2, \dots, n_{k-1} m_k} \cdot \min \left\{ 2\alpha^2 m_k, \frac{4\delta}{\delta_k} \right\} \\
 &\leq \frac{\alpha}{n_1 n_2, \dots, n_{k-1} m_k} (2\alpha^2 m_k)^{1-l} \left(\frac{4\delta}{\delta_k} \right)^l \\
 &\leq 8\alpha \delta^l \frac{1}{n_1 n_2, \dots, n_{k-1} m_k} m_k^{1-l} \left(\frac{1}{\delta_k} \right)^l \alpha^{2-2l} \\
 &\leq 8\alpha \delta^l \frac{1}{n_1 n_2, \dots, n_{k-1} m_k \alpha^l} n_k \left(\frac{1}{n_k \delta_k} \right)^l \alpha^{2-l} \\
 &\leq 8\alpha^3 \delta^l \frac{1}{n_1 n_2, \dots, n_{k-1} \alpha^l} \frac{n_k}{m_k} (\tilde{c}_1 + 1)^l \left(\frac{1}{\delta_{k-1}} \right)^l \\
 &\leq 8\alpha^4 \delta^l \frac{1}{n_1 n_2, \dots, n_{k-2} (\delta_{k-1} \alpha)^l} \frac{1}{n_{k-1}} (\tilde{c}_1 + 1)^l \\
 &\leq 8\alpha^4 (\tilde{c}_1 + 1)^l \delta^l \frac{m_{k-1}}{n_{k-1}} \\
 &\leq 8\alpha^4 (\tilde{c}_1 + 1)^l \delta^l \\
 &\leq 8\alpha^4 (\tilde{c}_1 + 1) \delta^l \\
 &= 8\alpha^4 (\tilde{c}_1 + 1) |U|^l.
 \end{aligned} \tag{16}$$

Condition (B): $\bar{\alpha}_k \leq \tilde{c}_2 \underline{\alpha}_k$. In this case, we have

$$\begin{aligned}
 \delta_{k-1} &\leq (m_k - 1) \bar{\alpha}_k + n_k \delta_k \leq \tilde{c}_2 m_k \underline{\alpha}_k + \alpha m_k \delta_k \\
 &= m_k (\tilde{c}_2 \underline{\alpha}_k + \alpha \delta_k) \leq m_k \tilde{c}_2 (\underline{\alpha}_k + \alpha \delta_k) \\
 &\leq 2\tilde{c}_2 m_k \max\{\alpha \delta_k, \underline{\alpha}_k\}.
 \end{aligned} \tag{17}$$

On the one hand, $g \leq 2\alpha^2 m_k$. On the other hand, by the definition of g , we have

$$\begin{aligned}
 (g - 2)\delta_k &\leq \delta, \\
 (g - 1)\underline{\alpha}_k &\leq \delta.
 \end{aligned} \tag{18}$$

Then,

$$g \leq \frac{\delta}{\max\{\delta_k, \underline{\alpha}_k\}} + 2 \leq \frac{4\delta}{\max\{\delta_k, \underline{\alpha}_k\}} \leq \frac{4\delta\alpha}{\max\{\alpha\delta_k, \underline{\alpha}_k\}}. \tag{19}$$

Combining (13), (17), and (19), we have

$$\begin{aligned}
 \mu(U) &\leq \frac{\alpha}{n_1 n_2, \dots, n_{k-1} m_k} \cdot \min \left\{ 2\alpha^2 m_k, \frac{4\delta\alpha}{\max\{\alpha\delta_k, \underline{\alpha}_k\}} \right\} \\
 &\leq \frac{\alpha}{n_1 n_2, \dots, n_{k-1} m_k} (2\alpha^2 m_k)^{1-l} \left(\frac{4\delta\alpha}{\max\{\alpha\delta_k, \underline{\alpha}_k\}} \right)^l \\
 &\leq 8\delta^l \frac{\alpha}{n_1 n_2, \dots, n_{k-1} m_k} \alpha^{2-2l} m_k^{1-l} \left(\frac{2\tilde{c}_2 m_k}{\delta_{k-1}} \right)^l \alpha^l \\
 &\leq 16\tilde{c}_2 \alpha^3 \delta^l \frac{1}{n_1 n_2, \dots, n_{k-2} n_{k-1} (\delta_{k-1} \alpha)^l} \\
 &\leq 16\tilde{c}_2 \alpha^3 \delta^l \frac{1}{n_1 n_2, \dots, n_{k-2} m_{k-1} (\delta_{k-1} \alpha)^l} \\
 &\leq 16\tilde{c}_2 \alpha^3 \delta^l \\
 &= 16\tilde{c}_2 \alpha^3 |U|^l.
 \end{aligned} \tag{20}$$

By (14), (16), (20), and Lemma 1, we have $\dim_H E \geq l$. By the arbitrariness of l , we obtain $\dim_H E \geq \liminf_{k \rightarrow \infty} (\log n_1 n_2, \dots, n_{k-1} m_k / -\log c_1 c_2, \dots, c_k (n_k / m_k))$ and complete the proof of Theorem 1.

4.2. Proof of Theorem 2. By Lemma 3, it suffices to prove $\dim_B E \leq s^*$. The conclusion is obvious if $s^* = 1$. Now, we assume $s^* < 1$. For any $1 > t > s^* = \limsup_{k \rightarrow \infty} (\log n_1 n_2, \dots, n_k / -\log c_1 c_2, \dots, c_k)$, there exist $\varepsilon > 0, k_1 > 0$ such that, for any $k \geq k_1$,

$$t > \frac{\log n_1 n_2, \dots, n_k}{-\log c_1 c_2, \dots, c_k} + \varepsilon. \tag{21}$$

Let $b_k = t - (\log n_1 n_2, \dots, n_k / -\log c_1 c_2, \dots, c_k) > \varepsilon$; then, $n_1 n_2, \dots, n_k (\delta_k)^t = (\delta_k)^{b_k}$. By the definitions of n_k and c_k , we have $\delta_k = c_1 c_2, \dots, c_k \leq 2^{-k}$. Since $m_k \geq 1$, for any $k \geq 1$, we have $m_k + 1 \geq 2$ and $m_k + 1 \leq 2m_k$. By the convexity of x^s ($0 < s < 1$), we obtain that

$$\begin{aligned}
 \sum_{i=0}^{m_{k+1}} (\eta_{k+1,i}^*)^t &\leq (m_{k+1} + 1) \left(\frac{\sum_{i=0}^{m_{k+1}} \eta_{k+1,i}^*}{m_{k+1} + 1} \right)^t \\
 &= (m_{k+1} + 1) \left(\frac{(1 - c_{k+1} n_{k+1}) \delta_k}{m_{k+1} + 1} \right)^t.
 \end{aligned} \tag{22}$$

Let $m = \sup_{k \geq 1} m_k < \infty$; by (21) and (22), we have

$$\begin{aligned}
\sum_{k=k_1}^{\infty} n_1 n_2, \dots, n_k \sum_{i=0}^{m_{k+1}} (\eta_{k+1,i}^*)^t &\leq \sum_{k=k_1}^{\infty} n_1 n_2, \dots, n_k (m_{k+1} + 1) \left(\frac{(1 - c_{k+1} n_{k+1}) \delta_k}{m_{k+1} + 1} \right)^t \\
&\leq 2m \sum_{k=k_1}^{\infty} n_1 n_2, \dots, n_k \left(\frac{\delta_k}{m_{k+1} + 1} \right)^t \\
&= 2m \sum_{k=k_1}^{\infty} n_1 n_2, \dots, n_k (\delta_k^t) (m_{k+1} + 1)^{-t} \\
&= 2m \sum_{k=k_1}^{\infty} (\delta_k^{b_k}) (m_{k+1} + 1)^{-t} \\
&\leq 2m \sum_{k=k_1}^{\infty} (\delta_k^{b_k}) 2^{-t} \\
&\leq 2m \sum_{k=k_1}^{\infty} 2^{-k\varepsilon} \cdot 1 < \infty.
\end{aligned} \tag{23}$$

By Lemma 4, we have $\overline{\dim}_B E \leq t$. By the arbitrariness of t , we have $\overline{\dim}_B E \leq s^*$ and complete the proof of Theorem 1.

4.3. Proof of Theorem 3. First, we use Lemma 4 to estimate the upper bound for $\overline{\dim}_B E$.

Proposition 1. Suppose $E \in \mathcal{F}^*(I, J, \{n_k\}, \{m_k\}, \{c_k\}, \{\eta_{k,l}^*\})$; then,

$$\overline{\dim}_B E \leq \limsup_{k \rightarrow \infty} \frac{\log n_1 n_2, \dots, n_{k-1} m_k}{-\log(c_1 c_2, \dots, c_{k-1} / m_k)}. \tag{24}$$

Proof. For any $1 > p > \limsup_{k \rightarrow \infty} (\log n_1 n_2, \dots, n_k m_{k+1} / -\log(c_1 c_2, \dots, c_k / m_{k+1}))$, there exist $\xi > 0$ and $k_2 > 0$ such that, for any $k \geq k_2$,

$$p > \frac{\log n_1 n_2, \dots, n_k m_{k+1}}{-\log(c_1 c_2, \dots, c_k / m_{k+1})} + \xi, \tag{25}$$

and it follows that

$$(\xi - p) \log \frac{c_1 c_2, \dots, c_k}{m_{k+1}} > \log n_1 n_2, \dots, n_k m_{k+1}, \tag{26}$$

which implies

$$n_1 n_2, \dots, n_k m_{k+1} < \left(\frac{c_1 c_2, \dots, c_k}{m_{k+1}} \right)^{\xi - p}. \tag{27}$$

By $\delta_k = c_1 c_2, \dots, c_k \leq 2^{-k}$, for any $k \geq 0$, and the convexity of x^s ($0 < s < 1$), we have

$$\begin{aligned}
\sum_{i=0}^{m_{k+1}} (\eta_{k+1,i}^*)^p &\leq (m_{k+1} + 1) \left(\frac{\sum_{i=0}^{m_{k+1}} \eta_{k+1,i}^*}{m_{k+1} + 1} \right)^p \\
&= (m_{k+1} + 1) \left(\frac{(1 - c_{k+1} n_{k+1}) \delta_k}{m_{k+1} + 1} \right)^p.
\end{aligned} \tag{28}$$

Since $m_k \geq 1$ (for any $k \geq 1$), we have $m_k + 1 \leq 2m_k$ for any $k \geq 1$. Then, by (27) and (28), we have

$$\begin{aligned}
\sum_{k=k_2}^{\infty} n_1 n_2, \dots, n_k \sum_{i=0}^{m_{k+1}} (\eta_{k+1,i}^*)^p &\leq \sum_{k=k_2}^{\infty} n_1 n_2, \dots, n_k (m_{k+1} + 1) \left(\frac{(1 - c_{k+1} n_{k+1}) \delta_k}{m_{k+1} + 1} \right)^p \\
&\leq 2 \sum_{k=k_2}^{\infty} n_1 n_2, \dots, n_k m_{k+1} \left(\frac{\delta_k}{m_{k+1} + 1} \right)^p \\
&\leq 2 \sum_{k=k_2}^{\infty} \left(\frac{\delta_k}{m_{k+1}} \right)^{\xi - p} \left(\frac{\delta_k}{m_{k+1}} \right)^p \\
&= 2 \sum_{k=k_2}^{\infty} \left(\frac{\delta_k}{m_{k+1}} \right)^{\xi} \\
&\leq 2 \sum_{k=k_2}^{\infty} 2^{-k\xi} < \infty.
\end{aligned} \tag{29}$$

By Lemma 4, we have $\overline{\dim}_B E \leq p$. By the arbitrariness of p , we have $\overline{\dim}_B E \leq \limsup_{k \rightarrow \infty} (\log n_1 n_2, \dots, n_{k-1} m_k / -\log(c_1 c_2, \dots, c_{k-1} / m_k))$.

Now, we estimate the lower bound for $\overline{\dim}_B E$. $\square$

Proposition 2. Suppose $E \in \mathcal{F}^*(I, J, \{n_k\}, \{m_k\}, \{c_k\}, \{\eta_{k,l}^*\})$ satisfies the conditions of Theorem 3; then,

$$\overline{\dim}_B E \geq \limsup_{k \rightarrow \infty} \frac{\log n_1 n_2, \dots, n_{k-1} m_k}{-\log(c_1 c_2, \dots, c_{k-1} / m_k)}. \quad (30)$$

Proof. Let $r_1 = (1/m_1)$ and $r_k = (c_1 c_2, \dots, c_{k-1} / m_k)$ (for any $k \geq 2$). We estimate $(\log N_{r_k}(E) / -\log r_k)$ for a fixed k large enough.

By the condition of Theorem 3, we have $\bar{\alpha}_k \leq c_3 \cdot c_1 c_2, \dots, c_k$. Let M_k be the maximum number of the k -order connected components intersecting a ball of diameter r_k ; then, we have

$$\begin{aligned} (M_k - 2)(a_k - N_k)c_1 c_2, \dots, c_k &\leq r_k = \frac{c_1 c_2, \dots, c_{k-1}}{m_k} \\ &\leq \frac{n_k c_1 c_2, \dots, c_k + (m_k - 1)\bar{\alpha}_k}{m_k} \leq \frac{n_k c_1 c_2, \dots, c_k + (m_k - 1)\tilde{c}_3 \cdot c_1 c_2, \dots, c_k}{m_k} \\ &\leq \frac{n_k c_1 c_2, \dots, c_k + (n_k - 1)\tilde{c}_3 \cdot c_1 c_2, \dots, c_k}{m_k} \leq \frac{c_1 c_2, \dots, c_k (n_k + n_k \tilde{c}_3)}{m_k}, \end{aligned} \quad (31)$$

which implies

$$M_k - 2 \leq \frac{(n_k / m_k)(1 + \tilde{c}_3)}{a_k - N_k}. \quad (32)$$

Let $N = \sup_{k \geq 1} N_k$, then we have $((n_k / m_k) / (a_k - N_k)) \leq ((a_k + N_k) / (a_k - N_k)) = 1 + (2N_k / (a_k - N_k)) \leq 1 + 2N_k \leq 2N + 1$. Then, $M_k \leq 2 + (1 + \tilde{c}_3)(2N + 1)$. Thus,

$$\begin{aligned} \frac{\log N_{r_k}(E)}{-\log r_k} &\geq \frac{\log(1/M_k)n_1 n_2, \dots, n_{k-1} m_k}{-\log r_k} \\ &\geq \frac{\log n_1 n_2, \dots, n_{k-1} m_k - \log(2 + (1 + \tilde{c}_3)(2N + 1))}{-\log(c_1 c_2, \dots, c_{k-1} / m_k)}. \end{aligned} \quad (33)$$

Since $r_k \rightarrow 0$ as $k \rightarrow \infty$, then, by (33), we have

$$\begin{aligned} \overline{\dim}_B E &= \limsup_{k \rightarrow \infty} \frac{\log N_{r_k}(E)}{-\log r_k} \geq \limsup_{k \rightarrow \infty} \frac{\log N_{r_k}(E)}{-\log r_k} \\ &\geq \limsup_{k \rightarrow \infty} \frac{\log n_1 n_2, \dots, n_{k-1} m_k}{-\log(c_1 c_2, \dots, c_{k-1} / m_k)}. \end{aligned} \quad (34)$$

From Propositions 1 and 2, we have Theorem 3. $\square$

4.4. Proof of Theorem 4. It is obvious that the conclusion of Proposition 1 also holds if E satisfies the conditions of Theorem 4, then it suffices to estimate the lower bound of $\overline{\dim}_B E$ under the conditions of Theorem 4.

Proposition 3. Suppose $E \in \mathcal{F}^*(I, J, \{n_k\}, \{m_k\}, \{c_k\}, \{\eta_{k,l}^*\})$ satisfies the conditions of Theorem 4; then,

$$\overline{\dim}_B E \geq \limsup_{k \rightarrow \infty} \frac{\log n_1 n_2, \dots, n_{k-1} m_k}{-\log(c_1 c_2, \dots, c_{k-1} / m_k)}. \quad (35)$$

Proof. We discuss separately according to the three conditions (C), (D), and (E) of Theorem 4 to estimate $(\log N_{r_k}(E) / -\log r_k)$ for a fixed k large enough, and we use the definitions of r_k and M_k in the proof of Theorem 3.

Condition (C): in this case, $\bar{\alpha}_k \leq \tilde{c}_4 \cdot c_1 c_2, \dots, c_k$ and $(n_k / m_k) \leq \gamma$. Then, we have

$$\begin{aligned} (M_k - 2)c_1 c_2, \dots, c_k &\leq r_k = \frac{c_1 c_2, \dots, c_{k-1}}{m_k} \\ &\leq \frac{n_k c_1 c_2, \dots, c_k + (m_k - 1)\bar{\alpha}_k}{m_k} \leq \frac{n_k c_1 c_2, \dots, c_k + (m_k - 1)\tilde{c}_4 \cdot c_1 c_2, \dots, c_k}{m_k} \\ &\leq \frac{n_k c_1 c_2, \dots, c_k + (n_k - 1)\tilde{c}_4 \cdot c_1 c_2, \dots, c_k}{m_k} \leq \frac{c_1 c_2, \dots, c_k (n_k + n_k \tilde{c}_4)}{m_k}, \end{aligned} \quad (36)$$

it follows

$$M_k - 2 \leq \frac{n_k + n_k \tilde{c}_4}{m_k} \leq \gamma(1 + \tilde{c}_4), \quad (37)$$

which implies $M_k \leq 2 + \gamma(1 + \tilde{c}_4)$. Thus,

$$\begin{aligned} \frac{\log N_{r_k}(E)}{-\log r_k} &\geq \frac{\log(1/M_k)n_1n_2, \dots, n_{k-1}m_k}{-\log r_k} \\ &\geq \frac{\log n_1n_2, \dots, n_{k-1}m_k - \log(2 + \gamma(1 + \tilde{c}_4))}{-\log(c_1c_2, \dots, c_{k-1}/m_k)}. \end{aligned} \quad (38)$$

Condition (D): in this case, $m_k \cdot \underline{\alpha}_k \geq \tilde{c}_5 \cdot c_1c_2, \dots, c_{k-1}$. Then,

$$\underline{\alpha}_k \geq \frac{\tilde{c}_5 c_1c_2, \dots, c_{k-1}}{m_k} = \tilde{c}_5 r_k, \quad (39)$$

which implies that the number of the k -order connected components intersecting a ball of diameter $\tilde{c}_5 r_k$ is at most 2; then, the number of the k -order connected components intersecting a ball of diameter r_k is at most $2((1/\tilde{c}_5) + 1) := T_k$. Thus,

$$\begin{aligned} \frac{\log N_{r_k}(E)}{-\log r_k} &\geq \frac{\log(1/T_k)n_1n_2, \dots, n_{k-1}m_k}{-\log r_k} \\ &\geq \frac{\log n_1n_2, \dots, n_{k-1}m_k - \log(2((1/\tilde{c}_5) + 1))}{\log(c_1c_2, \dots, c_{k-1}/m_k)}. \end{aligned} \quad (40)$$

Condition (E): in this case, $\bar{\alpha}_k \leq \tilde{c}_6 \cdot \underline{\alpha}_k$. We distinguish two cases.

Case 1: $\underline{\alpha}_k < c_1c_2, \dots, c_k$; then, $\bar{\alpha}_k \leq \tilde{c}_6 \cdot c_1c_2, \dots, c_k$. Similar to the discussions in condition (C), we have

$$\begin{aligned} \frac{\log N_{r_k}(E)}{-\log r_k} &\geq \frac{\log(1/M_k)n_1n_2, \dots, n_{k-1}m_k}{-\log r_k} \\ &\geq \frac{\log n_1n_2, \dots, n_{k-1}m_k - \log(2 + \gamma(1 + \tilde{c}_6))}{-\log(c_1c_2, \dots, c_{k-1}/m_k)}. \end{aligned} \quad (41)$$

Case 2: $\underline{\alpha}_k \geq c_1c_2, \dots, c_k$. In this case, we have

$$\begin{aligned} c_1c_2, \dots, c_{k-1} &\leq n_k c_1c_2, \dots, c_k + (m_k - 1)\bar{\alpha}_k \\ &\leq n_k c_1c_2, \dots, c_k + (m_k - 1)\tilde{c}_6 \underline{\alpha}_k \\ &\leq n_k c_1c_2, \dots, c_k + n_k \tilde{c}_6 \underline{\alpha}_k \\ &\leq n_k \underline{\alpha}_k + n_k \tilde{c}_6 \underline{\alpha}_k, \end{aligned} \quad (42)$$

and it follows that

$$\underline{\alpha}_k \geq \frac{c_1c_2, \dots, c_{k-1}}{n_k(1 + \tilde{c}_6)} \geq \frac{c_1c_2, \dots, c_{k-1}}{\gamma(m_k(1 + \tilde{c}_6))} = \frac{1}{\gamma(1 + \tilde{c}_6)} \cdot r_k. \quad (43)$$

Similarly to the discussions in Condition (D), we have

$$\frac{\log N_{r_k}(E)}{-\log r_k} \geq \frac{\log n_1n_2, \dots, n_{k-1}m_k - \log 2(\gamma(\tilde{c}_6 + 1) + 1)}{\log(c_1c_2, \dots, c_{k-1}/m_k)}. \quad (44)$$

Notice that $r_k \rightarrow 0$ as $k \rightarrow \infty$. Then, by (38), (40), (41), and (44), we have

$$\begin{aligned} \overline{\dim}_B E &= \limsup_{\varepsilon \rightarrow 0} \frac{\log N_\varepsilon(E)}{-\log \varepsilon} \geq \limsup_{k \rightarrow \infty} \frac{\log N_{r_k}(E)}{-\log r_k} \\ &\geq \limsup_{k \rightarrow \infty} \frac{\log n_1n_2, \dots, n_{k-1}m_k}{-\log(c_1c_2, \dots, c_{k-1}/m_k)}. \end{aligned} \quad (45)$$

From Propositions 1 and 3, we have Theorem 4. $\square$

Data Availability

The data used to support the study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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