Research Article

A Hybrid Power Mean Involving the Dedekind Sums and Cubic Gauss Sums

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The main purpose of this paper is using analytic methods and the properties of the Dedekind sums to study one kind hybrid power mean calculating problem involving the Dedekind sums and cubic Gauss sum and give some interesting calculating formulae for it.

1. Introduction

For any integer $q \geq 3$ and integer m , the classical cubic Gauss sum $A(m, q)$ is defined as follows:

$$A(m, q) = \sum_{a=0}^{q-1} e\left(\frac{ma^3}{q}\right), \quad (1)$$

where as usual, $e(x) = e^{2\pi i x}$ and $i^2 = -1$.

This sum plays a very important role in the study of the elementary number theory and analytic number theory, so there are many people who had studied the arithmetical properties of $A(m, q)$ and related contents (see [1–6]). For example, if $q = p$ is an odd prime, Cao and Wang [5] proved the following interesting conclusion: let p be an odd prime with $p \equiv 1 \pmod{3}$. If 3 is not a cubic residue modulo p , then we have the identity

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right|^2 \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^4 = p^2(4p^2 - 19p - d^2 - 1), \quad (2)$$

where $4p = d^2 + 27 \cdot b^2$ and d is uniquely determined as follows: $d \equiv 1 \pmod{3}$

On the other hand, we define the Dedekind sums $S(h, q)$ as follows.

Let q be a natural number and h be an integer prime to q . The classical Dedekind sum $S(h, q)$ is defined as

$$S(h, q) = \sum_{a=1}^q \left(\left(\frac{a}{q} \right) \right) \left(\left(\frac{ah}{q} \right) \right), \quad (3)$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer,} \\ 0, & \text{if } x \text{ is an integer.} \end{cases} \quad (4)$$

About the various properties of $S(h, q)$, many authors had studied them and obtained a series of interesting results, and related works can be found in [7–14].

The main purpose of this paper is using the analytic method and the properties of Dedekind sums to study the computational problem of the hybrid power mean:

$$\frac{\sum_{m=1}^{q'} S^h(m, q)}{A^k(m, q)}, \quad (5)$$

and give some exact computational formulae for (5) with $q = p$, an odd prime, where h and k are any two fixed positive integers.

About this problem, so far no one seems to consider; at least, we have not seen any related papers before. Of course, this problem is meaningful, and it can describe the mean value distribution properties of the two different sums. It is clear that if $(p-1, 3) = 1$, then from the properties of the complete residue system modulo p , we have $A(m) = 0$. This time (5) is meaningless. So, in the following, we only

consider the case $p \equiv 1 \pmod{3}$. The main purpose of this paper is to prove the following several results.

Theorem 1. *Let p be a prime with $p = 12h + 1$. Then, for any positive integers k and h with $(2, h) = 1$, we have the identity*

$$\sum_{m=1}^{p-1} \frac{S(m^2, p)}{A^k(m, p)} = \sum_{m=1}^{p-1} \frac{S^h(m, p)}{A^k(m, p)} = 0. \quad (6)$$

Theorem 2. *Let p be a prime with $p = 12h + 7$. Then, for any positive integer k , we have the identity*

$$\sum_{m=1}^{p-1} \frac{S(m^2, p)}{A^k(m, p)} = \frac{1}{\pi^2} \cdot \frac{p}{p-1} \cdot \left(\frac{p-1}{A^k(1, p)} - H(k, p) \right) \cdot |L(1, \chi_2 \lambda)|^2 + \frac{h_p^2}{p-1} \cdot H(k, p), \quad (7)$$

where χ_2 is Legendre's symbol modulo p , λ denotes any third-order character modulo p , $h_p = \sqrt{p}/\pi \cdot |L(1, \chi_2)|$ denotes the class number of the quadratic field $F_p(\sqrt{-p})$, and

$$H(k, p) = \sum_{m=1}^{p-1} \frac{1}{A^k(m, p)}, \quad (8)$$

satisfies the third-order linear recursive formula as follows:

$$H(k, p) = -\frac{3}{d} H(k-1, p) + \frac{1}{dp} H(k-3, p), \quad (9)$$

with the initial values $H(0, p) = p-1$, $H(1, p) = -p-1/d$, and $H(2, p) = 3(p-1)/d^2$, and $4p = d^2 + 27 \cdot b^2$, where d is uniquely determined by $d \equiv 1 \pmod{3}$.

Theorem 3. *Let p be a prime with $p \equiv 1 \pmod{3}$. Then, for any positive integer k , we have the asymptotic formula as follows:*

$$\sum_{m=1}^{p-1} \frac{S^2(m, p)}{A^k(m, p)} = \left(\frac{5p}{144} + \frac{p}{\pi^4} \cdot \sum_{n=1}^{\infty} \frac{|\sum_{d|n} \lambda(d)|^2}{n^2} \right) \cdot H(k, p) + O\left(\frac{p}{|d|^k} \cdot \exp\left(\frac{4 \ln p}{\ln \ln p} \right) \right), \quad (10)$$

where $\exp(y) = e^y$ and λ is a three-order character modulo p .

From Theorem 3, we may immediately deduce the following two corollaries.

Corollary 1. *Let p be a prime with $p \equiv 1 \pmod{3}$, then we have*

$$\sum_{m=1}^{p-1} \frac{S^2(m, p)}{A(m, p)} = \frac{p^2}{d} \cdot \left(\frac{5}{144} + \frac{1}{\pi^4} \cdot \sum_{n=1}^{\infty} \frac{|\sum_{d|n} \lambda(d)|^2}{n^2} \right) + O\left(\frac{p}{|d|} \cdot \exp\left(\frac{4 \ln p}{\ln \ln p} \right) \right). \quad (11)$$

Corollary 2. *Let p be a prime with $p \equiv 1 \pmod{3}$, then we have*

$$\sum_{m=1}^{p-1} \left| \frac{S(m, p)}{A(m, p)} \right|^2 = \frac{3p^2}{d^2} \cdot \left(\frac{5}{144} + \frac{1}{\pi^4} \cdot \sum_{n=1}^{\infty} \frac{|\sum_{d|n} \lambda(d)|^2}{n^2} \right) + O\left(\frac{p}{d^2} \cdot \exp\left(\frac{4 \ln p}{\ln \ln p} \right) \right). \quad (12)$$

2. Several Lemmas

To complete the proofs of our all theorems, we need to prove several simple lemmas. Hereinafter, we shall use some properties of the character sums and Gauss sums, and all of these contents can be found in [15], so they will not be repeated here.

Lemma 1. *Let p be an odd prime with $p \equiv 1 \pmod{3}$ and λ be any third-order character modulo p , then we have the identity*

$$\tau^3(\lambda) + \tau^3(\bar{\lambda}) = dp, \quad (13)$$

where $\tau(\lambda) = \sum_{a=1}^{p-1} \lambda(a)e(a/p)$ denotes the classical Gauss sums, and $4p = d^2 + 27 \cdot b^2$, where d is uniquely determined by $d \equiv 1 \pmod{3}$.

Proof. The proof of this lemma is shown in the study by Zhang and Hu [2] or Berndt and Evans [16]. $\square$

Lemma 2. *Let p be an odd prime with $p \equiv 1 \pmod{3}$. Then, for any integer $k \geq 3$ and*

$$H(k, p) = \sum_{m=1}^{p-1} \frac{1}{A^k(m, p)}, \quad (14)$$

we have the third-order linear recursive formula as follows:

$$H(k, p) = -\frac{3}{d}H(k-1, p) + \frac{1}{dp}H(k-3, p), \quad (15)$$

with the initial values $H(0, p) = p-1$, $H(1, p) = -p-1/d$, and $H(2, p) = 3(p-1)/d^2$, where d is defined as in Lemma 1.

Proof. Let λ be any third-order character modulo p ; then, for any integer $1 \leq m \leq p-1$, from the properties of the third-order characters and the classical Gauss sums modulo p , we have the identity

$$\begin{aligned} A(m, p) &= \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) = 1 + \sum_{a=1}^{p-1} \left(1 + \lambda(a) + \bar{\lambda}(a)\right) e\left(\frac{ma}{p}\right), \\ &= \sum_{a=0}^{p-1} e\left(\frac{ma}{p}\right) + \bar{\lambda}(m)\tau(\lambda) + \lambda(m)\tau(\bar{\lambda}) = \bar{\lambda}(m)\tau(\lambda) + \lambda(m)\tau(\bar{\lambda}). \end{aligned} \quad (16)$$

Note that $\tau(\lambda)\tau(\bar{\lambda}) = p$ and $\lambda^2 = \bar{\lambda}$; from (16), we have

$$A^2(m, p) = \lambda(m)\tau^2(\lambda) + 2p + \bar{\lambda}(m)\tau^2(\bar{\lambda}). \quad (17)$$

From (16) and Lemma 1, we also have

$$\begin{aligned} A^3(m, p) &= (\bar{\lambda}(m)\tau(\lambda) + \lambda(m)\tau(\bar{\lambda}))^3, \\ &= \tau^3(\lambda) + \tau^3(\bar{\lambda}) + 3pA(m, p) = dp + 3pA(m, p). \end{aligned} \quad (18)$$

From (17) and the properties of the character sums modulo p , we have

$$\sum_{m=1}^{p-1} A^2(m, p) = \sum_{m=1}^{p-1} (\lambda(m)\tau^2(\lambda) + 2p + \bar{\lambda}(m)\tau^2(\bar{\lambda})) = 2p(p-1). \quad (19)$$

Combining (18) and (19), we can deduce that

$$\sum_{m=1}^{p-1} A^2(m, p) = dp \cdot \sum_{m=1}^{p-1} \frac{1}{A(m, p)} + 3p \sum_{m=1}^{p-1} 1, \quad (20)$$

or

$$\sum_{m=1}^{p-1} \frac{1}{A(m, p)} = \frac{1}{dp} (2p(p-1) - 3p(p-1)) = -\frac{p-1}{d}. \quad (21)$$

Similarly, from (16), (18), and (21), we have

$$0 = \sum_{m=1}^{p-1} A(m, p) = \sum_{m=1}^{p-1} \frac{dp}{A^2(m, p)} + \sum_{m=1}^{p-1} \frac{3p}{A(m, p)} = \sum_{m=1}^{p-1} \frac{dp}{A^2(m, p)} - \frac{3p(p-1)}{d}, \quad (22)$$

or

$$\sum_{m=1}^{p-1} \frac{1}{A^2(m, p)} = \frac{3(p-1)}{d^2}. \quad (23)$$

If $k \geq 3$, then from (18), we have

$$\frac{1}{A^{k-3}(m, p)} = dp \cdot \frac{1}{A^k(m, p)} + 3p \cdot \frac{1}{A^{k-1}(m, p)}, \quad (24)$$

or

$$H(k, p) = -\frac{3}{d}H(k-1, p) + \frac{1}{dp}H(k-3, p). \quad (25)$$

Now, Lemma 2 follows from (21)–(25) and $H(0, p) = p-1$. $\square$

Lemma 3. Let p be a prime with $p \equiv 1 \pmod{3}$. For any positive integer k and any character $\chi \pmod{p}$, if $\chi^3 \neq \chi_0$, then we have the identity

$$\sum_{m=1}^{p-1} \frac{\chi(m)}{A^k(m, p)} = 0, \quad (26)$$

where χ_0 denotes the principal character modulo p .

Proof. Let g be a primitive root modulo p . Then, from the definition of $A(m, p)$ and the properties of the complete residue system modulo p , we have

$$A(mg^3, p) = \sum_{a=0}^{p-1} e\left(\frac{mg^3 a^3}{p}\right) = \sum_{a=0}^{p-1} e\left(\frac{m(ga)^3}{p}\right) = \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) = A(m, p). \quad (27)$$

From (27), we have

$$\sum_{m=1}^{p-1} \frac{\chi(m)}{A^k(m, p)} = \sum_{m=1}^{p-1} \frac{\chi(mg^3)}{A^k(mg^3, p)} = \chi^3(g) \cdot \sum_{m=1}^{p-1} \frac{\chi(m)}{A^k(m, p)}. \quad (28)$$

or

$$(1 - \chi^3(g)) \cdot \sum_{m=1}^{p-1} \frac{\chi(m)}{A^k(m, p)} = 0. \quad (29)$$

If $\chi^3 \neq \chi_0$, then $\chi^3(g) \neq 1$. So, from (29), we have the identity

$$\sum_{m=1}^{p-1} \frac{\chi(m)}{A^k(m, p)} = 0. \quad (30)$$

This proves Lemma 3. $\square$

Lemma 4. Let p be a prime with $p \equiv 1 \pmod{3}$ and λ be any third-order character modulo p . Then, for any positive integer k , we have the identity

$$\sum_{m=1}^{p-1} \frac{\lambda(m) + \bar{\lambda}(m)}{A^k(m, p)} = \frac{p-1}{A^k(1, p)} - H(k, p). \quad (31)$$

Proof. From the properties of the third-order character modulo p , we have

$$1 + \lambda(m) + \bar{\lambda}(m) = \begin{cases} 3, & \text{if } m \text{ is a 3 - the residue modulo } p, \\ 0, & \text{otherwise.} \end{cases} \quad (32)$$

From (32), the properties of the 3-th residue modulo p , and the identity $A^k(m^3, p) = A^k(1, p)$ where $(m, p) = 1$, we have

$$\begin{aligned} \sum_{m=1}^{p-1} \frac{\lambda(m) + \bar{\lambda}(m)}{A^k(m, p)} &= \sum_{m=1}^{p-1} \frac{1 + \lambda(m) + \bar{\lambda}(m)}{A^k(m, p)} - \sum_{m=1}^{p-1} \frac{1}{A^k(m, p)} \\ &= \sum_{m=1}^{p-1} \frac{1}{A^k(m^3, p)} - \sum_{m=1}^{p-1} \frac{1}{A^k(m, p)} = \frac{p-1}{A^k(1, p)} - H(k, p). \end{aligned} \quad (33)$$

This proves Lemma 4. $\square$

Lemma 5. Let $q > 2$ be an integer; then, for any integer a with $(a, q) = 1$, we have the identity

$$S(a, q) = \frac{1}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a) |L(1, \chi)|^2, \quad (34)$$

where $L(s, \chi)$ denotes the Dirichlet L -function corresponding to the character χ and $\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}$ denotes the summation over all odd characters modulo d .

Proof. See Lemma 2 of [14]. $\square$

3. Proofs of the Theorems

In this section, we shall complete the proofs of our all theorems. If p be a prime, then from Lemma 5, we have

$$S(a, p) = \frac{1}{\pi^2} \cdot \frac{p}{p-1} \cdot \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(a) |L(1, \chi)|^2. \quad (35)$$

It is clear that if h is an integer with $(2, h) = 1$, then note that the identities $S^h(-m, p) = -S^h(m, p)$ and $A(-m, p) = A(m, p)$, and we have

$$\sum_{m=1}^{p-1} \frac{S^h(m, p)}{A^k(m, p)} = \sum_{m=1}^{p-1} \frac{S^h(-m, p)}{A^k(-m, p)} = - \sum_{m=1}^{p-1} \frac{S^h(m, p)}{A^k(m, p)} = 0. \quad (36)$$

Now, for any positive integer k , from (35), we have the identity

$$\sum_{m=1}^{p-1} \frac{S(m^2, p)}{A^k(m, p)} = \frac{1}{\pi^2} \cdot \frac{p}{p-1} \cdot \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \sum_{m=1}^{p-1} \frac{\chi^2(m)}{A^k(m, p)} |L(1, \chi)|^2. \quad (37)$$

If $p = 12h + 1$, then for any odd character $\chi \bmod p$, $\chi^6 = \chi_0$ if and only if $\chi = \chi_0, \chi_2, \chi_2\lambda$ or $\chi_2\lambda^2$, where χ_2 is Legendre's symbol modulo p . In this time, χ is not an odd character modulo p . So, from (37) and Lemma 3, we have

$$\sum_{m=1}^{p-1} \frac{S(m^2, p)}{A^k(m, p)} = 0. \quad (38)$$

If $p = 12h + 7$, then for any odd character $\chi \bmod p$, $\chi^6 = \chi_0$ if and only if $\chi = \chi_2, \chi_2\lambda$ or $\chi = \chi_2\bar{\lambda}$. So, in this time, note that $|L(1, \chi_2\lambda)|^2 = |L(1, \chi_2\bar{\lambda})|^2$; applying (37) and Lemma 4, we have

$$\begin{aligned} \sum_{m=1}^{p-1} \frac{S(m^2, p)}{A^k(m, p)} &= \frac{1}{\pi^2} \cdot \frac{p}{p-1} \cdot \left(\sum_{m=1}^{p-1} \frac{\lambda(m)}{A^k(m, p)} + \sum_{m=1}^{p-1} \frac{\bar{\lambda}(m)}{A^k(m, p)} \right) \cdot |L(1, \chi_2\lambda)|^2 + \frac{h_p^2}{p-1} \sum_{m=1}^{p-1} \frac{1}{A^k(m, p)}, \\ &= \frac{1}{\pi^2} \cdot \frac{p}{p-1} \cdot \left(\frac{p-1}{A^k(1, p)} - H(k, p) \right) \cdot |L(1, \chi_2\lambda)|^2 + \frac{h_p^2}{p-1} \cdot H(k, p), \end{aligned} \quad (39)$$

where λ is any third-order character modulo p .

It is clear that Theorem 1 follows from (36) and (38).

Theorem 2 follows from (39) and Lemma 2.

Now, we prove Theorem 3. Note that $p \equiv 1 \pmod{3}$ and (see [17] or [18]):

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^4 = \frac{5\pi^4}{144} \cdot p + O\left(\exp\left(\frac{4 \ln p}{\ln \ln p}\right)\right), \quad (40)$$

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^2 \cdot |L(1, \chi\lambda)|^2 = \frac{p}{2} \cdot \sum_{n=1}^{\infty} \frac{|\sum_{d|n} \lambda(d)|^2}{n^2} + O\left(\exp\left(\frac{4 \ln p}{\ln \ln p}\right)\right). \quad (41)$$

From (35), (40), and (41) and Lemma 3, we have the asymptotic formula as follows:

$$\begin{aligned}
\sum_{m=1}^{p-1} \frac{S^2(m, p)}{A^k(m, p)} &= \frac{1}{\pi^4} \frac{p^2}{(p-1)^2} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \sum_{m=1}^{p-1} \frac{\chi(m)\eta(m)}{A^k(m, p)} |L(1, \chi)|^2 |L(1, \eta)|^2, \\
&= \frac{1}{\pi^4} \cdot \frac{p^2}{(p-1)^2} \cdot \left(\sum_{m=1}^{p-1} \frac{1}{A^k(m, p)} \right) \cdot \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} (|L(1, \chi)|^4 + 2|L(1, \chi)|^2 |L(1, \chi\lambda)|^2), \\
&= \left(\frac{5p}{144} + \frac{p}{\pi^4} \cdot \sum_{n=1}^{\infty} \frac{|\sum_{d|n} \lambda(d)|^2}{n^2} \right) \cdot H(k, p) + O\left(|H(k, p)| \cdot \exp\left(\frac{4 \ln p}{\ln \ln p}\right)\right).
\end{aligned} \tag{42}$$

Applying Lemma 2 and mathematical induction, we can easily deduce the estimate

$$|H(k, p)| \ll \frac{p}{|d|^k}. \tag{43}$$

In fact, from Lemma 2, we have $H(0, p) = p - 1$, $H(1, p) = -p - 1/d$, and $H(2, p) = 3(p - 1)/d^2$. So, (43) is true for $k = 0, 1$, and 2 . Note that the third-order linear recursive formula is as follows:

$$H(k, p) = -\frac{3}{d} H(k - 1, p) + \frac{1}{dp} H(k - 3, p). \tag{44}$$

So, for $k \geq 3$, from the mathematical induction, we can easily deduce the estimate

$$|H(k, p)| \leq \frac{3}{|d|} \cdot |H(k - 1, p)| + \frac{1}{|d| \cdot p} \cdot |H(k - 3, p)| \ll \frac{p}{|d|^k}. \tag{45}$$

Combining (42) and (43), we complete the proof of Theorem 3.

4. Conclusion

The main results in this paper are three theorems, which are closely related to Dedekind sums and cubic Gauss sums. They describe that when p is a prime, the hybrid power mean of the Dedekind sums and cubic Gauss sums has good mean distribution properties. In fact, we can give some exact calculating formulae (see Theorems 1 and 2) or asymptotic formula (see Theorem 3) for them.

Data Availability

The data generated or analysed during this study are included in this published article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors have equally contributed to this work. All authors read and approved the final manuscript.

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