Hindawi
Journal of Mathematics
Volume 2021, Article ID 7069556, 8 pages
https://doi.org/10.1155/2021/7069556

Research Article

Constructions and Properties for a Finite-Dimensional Modular Lie Superalgebra $K(n,m)$

Dan Mao and Keli Zheng

Department of Mathematics, School of Science, Northeast Forestry University, Harbin 150040, China

Correspondence should be addressed to Keli Zheng; zhengkl561@nenu.edu.cn

Received 18 August 2021; Accepted 15 September 2021; Published 29 September 2021

Academic Editor: Wenpeng Zhang

In this paper, a finite-dimensional Lie superalgebra $K(n,m)$ over a field of prime characteristic is constructed. Then, we study some properties of $K(n,m)$. Moreover, we prove that $K(n,m)$ is an extension of a simple Lie superalgebra, and if $m=n-1$, then it is isomorphic to a subalgebra of a restricted Lie superalgebra.

1. Introduction

In the 1970s, physicists introduced the concept of Lie superalgebra in order to describe supersymmetry (see [1]). Since Lie superalgebra is an important mathematical model of supersymmetry, the research on it has been very active and rich results have been obtained (see [2]). In mathematics, Lie superalgebra is also a natural generalization of Lie algebra. In 1977, Kac completed the classification of finite-dimensional simple Lie superalgebras over a field of characteristic zero (see [3]). The research on Lie superalgebras over a field of characteristic zero has been quite systematic (see [4–6]), but the research on modular Lie superalgebras remains to be perfected (see [7]). Although some mathematicians try to study the classification of modular Lie superalgebras (see [7–13]), the classification problem has still been open. Therefore, it is very important to construct new finite-dimensional modular Lie superalgebras.

The finite-dimensional modular Lie superalgebras $W(n,m)$, $H(n,m)$ were constructed in [14, 15], respectively. Their natural filtrations are investigated in [16]. The finite-dimensional modular Lie superalgebra $S(n,m)$ was given in [17]. Modular Lie superalgebra $K(n)$, which takes the Grassmann algebra as base algebra, was constructed in [18]. Inspired by the above mentioned literatures, this paper constructs a finite-dimensional modular Lie superalgebra of contact type, which is denoted by $K(n,m)$.

The remainder of this paper is arranged as follows. A brief summary of the relevant concepts and notations is presented in Section 2. In Section 3, we construct the finite-dimensional modular Lie superalgebra $K(n,m)$. In Section 4, we obtain some properties of $K(n,m)$. Moreover, we prove that $K(n,m)$ is an extension of $\mathscr{K}(n,0)$, and if $m=n-1$, then it is isomorphic to a subalgebra of $KO(n-1,n,\underline{1})$.

2. Preliminaries

Throughout this paper, $\mathbb{F}$ denotes an algebraic closed field of characteristic $p\geq 3$; n is an integer greater than 3. Apart from the standard notation $\mathbb{Z}$, the sets of positive integers and nonnegative integers are denoted by $\mathbb{N}$ and $\mathbb{N}_0$, respectively. $\mathbb{Z}_2=\{\overline{0},\overline{1}\}$ denotes the ring of integers modulo 2.

Let $\Lambda(n)$ be the Grassmann algebra over $\mathbb{F}$ in n variables $x_1,x_2,\ldots,x_n$. Set $\mathbb{B}_k=\{\langle i_1,i_2,\ldots,i_k\rangle \mid 1\leq i_1<i_2<\cdots<i_k\leq n\}$ and $\mathbb{B}(n)=\cup_{k=0}^{n}\mathbb{B}_k$, where $\mathbb{B}_0=\varnothing$. For $u=\langle i_1, i_2,\ldots,i_k\rangle\in\mathbb{B}_k$, set $|u|=k$, $\{u\}=\{i_1,i_2,\ldots,i_k\}$ and $x^u=x_{i_1}x_{i_2},\ldots,x_{i_k}$ ($|\varnothing|=0, x^{\varnothing}=1$). Then, $\{x^u \mid u\in\mathbb{B}(n)\}$ is an $\mathbb{F}$-basis of $\Lambda(n)$.

Let $\Lambda(n-1)$ be the Grassmann algebra over $\mathbb{F}$ in $n-1$ variables $x_1, x_2, \ldots, x_{n-1}$. Obviously, $\Lambda(n) = \Lambda(n-1) \wedge \langle x_n \rangle$.

Let $\overline{D}_k\colon \Lambda(n) \longrightarrow W(n)$ be the linear map such that for any $f \in \Lambda(n)_\theta$, $\theta \in \mathbb{Z}_2$,

$$\overline{D}_k(f) = \sum_{i=1}^{n-1} f_i D_i + f_n x_n D_n, \tag{1}$$

where $f_i = (-1)^\theta (x_i x_n D_n(f) + D_i(f))$ and $f_n = 2f - \sum_{i=1}^{n-1} x_i D_i(f)$.

Let $\overline{K}(n) = \{\overline{D}_k(f) | f \in \Lambda(n)\}$. Then, $\overline{K}(n)$ is a finite-dimensional Lie superalgebra according to the operations in $W(n)$. Let $K(n) = [\overline{K}(n), \overline{K}(n)]$. Then, $K(n) = \{f | f \in \Lambda(n), f \neq x^{\widehat{u}}\}$, where $\widehat{u} = \langle 1, \ldots, n-1 \rangle$. In [18], Xin proves that $K(n)$ is not a simple Lie superalgebra.

Let $\mathscr{U} = \Lambda(n) \otimes \mathbb{T}(m)$ be the tensor product, where $\mathbb{T}(m)$ is the truncated polynomial algebra satisfying $y_i^p = 1$ for all $i = 1, 2, \ldots, m$ (see [17]). Then, $\mathscr{U}$ is an associative superalgebra with $\mathbb{Z}_2$-gradation induced by the trivial $\mathbb{Z}_2$-gradation of $\mathbb{T}(m)$ and the natural $\mathbb{Z}_2$-gradation of $\Lambda(n)$. Namely, $\mathscr{U} = \mathscr{U}_{\overline{0}} \oplus \mathscr{U}_{\overline{1}}$, where $\mathscr{U}_{\overline{0}} = \Lambda(n)_{\overline{0}} \otimes \mathbb{T}(m)$ and $\mathscr{U}_{\overline{1}} = \Lambda(n)_{\overline{1}} \otimes \mathbb{T}(m)$.

For $f \in \Lambda(n)$ and $\alpha \in \mathbb{T}(m)$, we abbreviate $f \otimes \alpha$ as $f\alpha$. Then, the elements $x^u y^\lambda$ with $u \in \mathbb{B}(n)$ and $\lambda \in G$ form an $\mathbb{F}$-basis of $\mathscr{U}$. It is easy to see that $\mathscr{U} = \oplus_n^{i=0} \mathscr{U}_i$ is a $\mathbb{Z}$-graded superalgebra, where $\mathscr{U}_i = \operatorname{span}_{\mathbb{F}}\{x^u y^\lambda | u \in \mathbb{B}(n), |u| = i, \lambda \in G\}$. In particular, $\mathscr{U}_0 = \mathbb{T}(m)$ and $\mathscr{U}_n = \operatorname{span}_{\mathbb{F}}\{x^\pi y^\lambda | \lambda \in G\}$, where $\pi := \langle 1, 2, \ldots, n \rangle \in \mathbb{B}(n)$.

In this paper, let $\operatorname{hg}(A) = A_{\overline{0}} \cup A_{\overline{1}}$, where $A = A_{\overline{0}} \oplus A_{\overline{1}}$ is a superalgebra. If x is a $\mathbb{Z}_2$-homogeneous element of A, then $\deg x$ denotes the $\mathbb{Z}_2$-degree of x.

Set $Y = \{1, 2, \ldots, n\}$. Given $i \in Y$, let $\partial/\partial x_i$ be the partial derivative on $\Lambda(n)$ with respect to x_i. For $i \in Y$, let D_i be the linear transformation on $\mathscr{U}$ such that $D_i(x^u y^\lambda) = (\partial x^u / \partial x_i) y^\lambda$ for all $u \in \mathbb{B}(n)$ and $\lambda \in G$. Let $\operatorname{Der}\mathscr{U}$ denote the derivation superalgebra of $\mathscr{U}$ (see [11]). Then, $D_i \in \operatorname{Der}_{\overline{1}}\mathscr{U}$ for all $i \in Y$ since $\partial/\partial x_i \in \operatorname{Der}_{\overline{1}}(\Lambda(n))$ (see [7]).

Suppose that $u \in \mathbb{B}_k \subseteq \mathbb{B}(n)$ and $i \in Y$. When $i \in \{u\}$, $u - \langle i \rangle$ denotes the uniquely determined element of $\mathbb{B}_{k-1}$ satisfying $\{u - \langle i \rangle\} = \{u\} \setminus \{i\}$. Then, the number of integers less than i in $\{u\}$ is denoted by $\tau(u, i)$. When $i \notin \{u\}$, we set $\tau(u, i) = 0$ and $x^{u - \langle i \rangle} = 0$. Therefore, $D_i(x^u) = (-1)^{\tau(u,i)} x^{u - \langle i \rangle}$ for all $i \in Y$ and $u \in \mathbb{B}(n)$.

We define $(fD)(g) = fD(g)$ for $f, g \in \operatorname{hg}(\mathscr{U})$ and $D \in \operatorname{hg}(\operatorname{Der}\mathscr{U})$. Since the multiplication of $\mathscr{U}$ is supercommutative, fD is a derivation of $\mathscr{U}$. Let

$$W(n, m) = \operatorname{span}_{\mathbb{F}}\left\{x^u y^\lambda D_i | u \in \mathbb{B}(n), \quad \lambda \in G, i \in Y\right\}. \tag{2}$$

Then, $W(n, m)$ is a finite-dimensional Lie superalgebra contained in $\operatorname{Der}\mathscr{U}$. A direct computation shows that

$$\left[fD_i, gD_j\right] = fD_i(g)D_j - (-1)^{\deg(fD_i)\deg\left(gD_j\right)} gD_j(f)D_i, \tag{3}$$

where $f, g \in \operatorname{hg}(\mathscr{U})$ and $i, j \in Y$.

Definition 1 (see [4]). A Lie superalgebra L is called simple if it does not have any graded ideals which are different from $\{0\}$ and L and if, moreover, $[L, L] \neq \{0\}$.

3. Construction of $K(n, m)$

Set $J = \{1, \ldots, n-1\}$. Let $\widetilde{D}_k\colon \mathscr{U} \longrightarrow W(n, m)$ be the linear map such that

$$\widetilde{D}_k(f) = \sum_{i \in J} f_i D_i + f_n x_n D_n, \tag{4}$$

where $f \in \operatorname{hg}(\mathscr{U})$, $f_i = (-1)^{\deg f}(x_i x_n D_n(f) + D_i(f))$, $i \in J$, and $f_n = 2f - \sum_{i \in J} x_i D_i(f)$.

Let $\overline{K}(n, m) = \{\widetilde{D}_k(f) | f \in \mathscr{U}\}$. Then, $\overline{K}(n, m)$ is a subspace of $W(n, m)$.

Let

$$G_i = D_i + x_i x_n D_n, \quad \forall i \in J, \tag{5}$$

$$G_n = 2x_n D_n. \tag{6}$$

By direct calculation, we have

$$\begin{aligned} \left[G_i, G_j\right] &= \delta_{ij} G_n, \\ \left[G_n, G_j\right] &= 0, \end{aligned} \tag{7}$$

where $i, j \in J$ and δ_{ij} is the Kronecker delta.

Proposition 1. $\widetilde{D}_k(f) = \sum_{i \in J} (-1)^{\deg f} G_i(f) G_i + f G_n$.

Proof. For any $j \in J$ and $\lambda \in G$, we have

$$\begin{aligned} &\left(\sum_{i \in J} (-1)^{\deg f} G_i(f) G_i + f G_n\right)\left(x_j y^\lambda\right) \\ &= \left(\sum_{i \in J} (-1)^{\deg f} \left(D_i(f) + x_i x_n D_n(f)\right)\left(D_i + x_i x_n D_n\right)\right)\left(x_j y^\lambda\right) \\ &= (-1)^{\deg f}\left(D_j(f) + x_j x_n D_n(f)\right)\left(y^\lambda\right) \\ &= \widetilde{D}_k(f)\left(x_j y^\lambda\right), \end{aligned}$$

$$\begin{aligned}
&\left(\sum_{i\in J}(-1)^{\deg f}G_i(f)G_i + fG_n\right)\left(x_n y^\lambda\right)\\
&= \left(\sum_{i\in J}(-1)^{\deg f}\left(D_i(f) + x_i x_n D_n(f)\right)\left(D_i + x_i x_n D_n\right) + 2f x_n D_n\right)\left(x_n y^\lambda\right)\\
&= \sum_{i\in J}(-1)^{\deg f}\left(D_i(f) + x_i x_n D_n(f)\right)x_i x_n y^\lambda + 2f x_n y^\lambda\\
&= \sum_{i\in J}(-1)^{\deg f}D_i(f)x_i x_n y^\lambda + 2f x_n y^\lambda\\
&= \left(-\sum_{i\in J}x_i D_i(f) + 2f\right)\left(x_n y^\lambda\right)\\
&= \widetilde{D}_k(f)\left(x_n y^\lambda\right).
\end{aligned} \tag{8}$$

Therefore,

$$\widetilde{D}_k(f) = \sum_{i\in J}(-1)^{\deg f}G_i(f)G_i + fG_n. \tag{9}$$

□

Proposition 2. *Let* $f \in \mathscr{U}_\theta$ *and* $g \in \mathscr{U}_\mu$, *where* $\theta, \mu \in \mathbb{Z}_2$. *Then,*

$$\left[\widetilde{D}_k(f), \widetilde{D}_k(g)\right] = \widetilde{D}_k(\langle f, g\rangle), \tag{10}$$

where $\langle f, g\rangle = \widetilde{D}_k(f)(g) - G_n(f)(g)$.

Proof. Let $\widetilde{D}_k(f) = \sum_{i\in Y} f_i G_i, \widetilde{D}_k(g) = \sum_{j\in Y} g_j G_j$.

Since $\widetilde{D}_k(f) = \sum_{i\in J}(-1)^{\deg f}G_i(f)G_i + fG_n$, we get

$$\begin{aligned}
f_i &= (-1)^\theta G_i(f), \quad \forall i \in J, f_n = f,\\
g_i &= (-1)^\mu G_i(g), \quad \forall i \in J, g_n = g.
\end{aligned} \tag{11}$$

Therefore, $\deg(f_i) = \theta + 1, \deg(g_i) = \mu + 1$, for $i \in Y$. Hence,

$$G_i\left(f_j\right) = G_i\left((-1)^\theta G_j(f)\right) = (-1)^\theta G_i G_j(f). \tag{12}$$

Since $[G_i, G_j] = \delta_{ij}G_n$, where $i, j \in J$, we obtain

$$\begin{aligned}
G_i\left(f_j\right) &= -G_j\left(f_i\right) + (-1)^\theta \delta_{ij}G_n(f),\\
G_i\left(g_j\right) &= -G_j\left(g_i\right) + (-1)^\mu \delta_{ij}G_n(g).
\end{aligned} \tag{13}$$

Set $[\widetilde{D}_k(f), \widetilde{D}_k(g)] = \sum_{j\in Y} h_j G_j$. For any $h \in \mathscr{U}$, we have

$$\begin{aligned}
&\left[\widetilde{D}_k(f), \widetilde{D}_k(g)\right](h)\\
&= \left[\sum_{i\in Y} f_i G_i, \sum_{j\in Y} g_j G_j\right](h)\\
&= \sum_{i,j\in Y}\left(f_i G_i\left(g_j G_j(h)\right) - (-1)^{\theta\mu} g_j G_j\left(f_i G_i(h)\right)\right)\\
&= \sum_{i,j\in Y}\left(f_i G_i\left(g_j\right)G_j(h) - (-1)^{\theta\mu+\theta+1} g_j f_i G_i\left(G_j(h)\right)\right.\\
&\quad \left. - (-1)^{\theta\mu} g_j G_j\left(f_i\right)G_i(h) + (-1)^{\mu+1} f_i g_j G_j\left(G_i(h)\right)\right)\\
&= \sum_{i,j\in Y}\left(f_i G_i\left(g_j\right)G_j - (-1)^{\theta\mu} g_j G_j\left(f_i\right)G_i\right)(h)\\
&\quad + (-1)^{\mu+1}\sum_{i,j\in Y} f_i g_j\left[G_i, G_j\right](h)\\
&= \sum_{i,j\in Y}\left(f_i G_i\left(g_j\right)G_j - (-1)^{\theta\mu} g_j G_j\left(f_i\right)G_i\right)(h)\\
&\quad + (-1)^{\mu+1}\sum_{i,j\in Y}\delta_{ij} f_i g_j G_n(h).
\end{aligned} \tag{14}$$

It follows that

$$
\begin{aligned}
&[\widetilde{D}_k(f), \widetilde{D}_k(g)] \\
&= \left[\sum_{i\in Y} f_i G_i, \sum_{j\in Y} g_j G_j\right] \\
&= \sum_{i,j\in Y}\left(f_i G_i\left(g_j\right)G_j - (-1)^{\theta\mu} g_j G_j(f_i)G_i\right) - (-1)^{\mu}\sum_{i,j\in Y}\delta_{ij} f_i g_j G_n.
\end{aligned} \tag{15}
$$

For all $j \in J$, we have

$$
\begin{aligned}
h_j &= \sum_{i\in J} f_i G_i\left(g_j\right) - (-1)^{\theta\mu}\sum_{i\in J} g_i G_i\left(f_j\right) + fG_n\left(g_j\right) - (-1)^{\theta\mu} gG_n\left(f_j\right) \\
&= -\sum_{i\in J} f_i G_j(g_i) + (-1)^{\theta\mu}\sum_{i\in J} g_i G_j(f_i) + (-1)^{\mu} f_j G_n(g) - (-1)^{\theta\mu+\theta} g_j G_n(f) \\
&\quad + fG_n\left(g_j\right) - (-1)^{\theta\mu} gG_n\left(f_j\right) \\
&= (-1)^{\theta}\sum_{i\in J}\left(G_j(f_i)g_i + (-1)^{\theta+1} f_i G_j(g_i)\right) + (-1)^{\theta+\mu} G_j(f)G_n(g) \\
&\quad - (-1)^{\theta\mu+\theta+\mu} G_j(g)G_n(f) + fG_n\left(g_j\right) - (-1)^{\theta\mu} gG_n\left(f_j\right) \\
&= (-1)^{\theta} G_j\left(\sum_{i\in J} f_i g_i\right) + (-1)^{\theta+\mu} G_j(f)G_n(g) - (-1)^{\theta+\mu+\theta} G_n(f)G_j(g) \\
&\quad + fG_n\left((-1)^{\mu}G_j(g)\right) - (-1)^{\theta\mu} gG_n\left((-1)^{\theta}G_j(f)\right) \\
&= (-1)^{\theta} G_j\left(\sum_{i\in J}(-1)^{\theta}G_i(f)(-1)^{\mu}G_i(g)\right) + (-1)^{\theta+\mu} G_j\left(fG_n(g) - G_n(f)g\right) \\
&= (-1)^{\theta+\mu} G_j\left(\sum_{i\in J}(-1)^{\theta}G_i(f)G_i(g) + fG_n(g) - G_n(f)g\right) \\
&= (-1)^{\theta+\mu} G_j\left(\widetilde{D}_k(f)(g) - G_n(f)(g)\right) \\
&= (-1)^{\theta+\mu} G_j\left(\langle f, g\rangle\right).
\end{aligned} \tag{16}
$$

Then, h_j exactly equals to the coefficient of G_j in $\widetilde{D}_k(\langle f, g\rangle)$.

In addition,

$$
\begin{aligned}
h_n &= \sum_{i\in J} f_i G_i(g) - (-1)^{\theta\mu}\sum_{i\in J} g_i G_i(f) \\
&\quad + fG_n(g) - (-1)^{\theta\mu} gG_n(f) - (-1)^{\mu}\sum_{i\in J} f_i g_i \\
&= \sum_{i\in J}(-1)^{\theta}G_i(f)G_i(g) - (-1)^{\theta\mu}\sum_{i\in J}(-1)^{\mu}G_i(g)G_i(f) \\
&\quad + fG_n(g) - (-1)^{\theta\mu} gG_n(f) - (-1)^{\mu}\sum_{i\in J}(-1)^{\theta}G_i(f)(-1)^{\mu}G_i(g) \\
&= -(-1)^{\theta\mu+\mu+(\theta+1)(\mu+1)}\sum_{i\in J} G_i(f)G_i(g) + fG_n(g) - G_n(f)g \\
&= \sum_{i\in J}(-1)^{\theta}G_i(f)G_i(g) + fG_n(g) - G_n(f)g \\
&= \widetilde{D}_k(f)(g) - G_n(f)(g) \\
&= \langle f, g\rangle.
\end{aligned} \tag{17}
$$

Then, h_n exactly equals to the coefficient of G_n in $\widetilde{D}_k(\langle f,g\rangle)$.

Therefore, $[\widetilde{D}_k(f),\widetilde{D}_k(g)]=\widetilde{D}_k(\langle f,g\rangle)$.

An immediate corollary of this proposition is the following. □

Corollary 1. *$\overline{K}(n,m)$ is a subalgebra of $W(n,m)$.*

Next, we give another way to express $\overline{K}(n,m)$. We still denote the linear map from $\mathscr{U}$ to $\overline{K}(n,m)$ by $\widetilde{D}_k$. Namely,

$$\widetilde{D}_k\colon \mathscr{U}\longrightarrow \overline{K}(n,m). \tag{18}$$

Then, we prove the following proposition.

Proposition 3. *$\overline{K}(n,m)\cong\mathscr{U}$.*

Proof. Let $f\in\operatorname{Ker}\widetilde{D}_k$. We obtain

$$\begin{aligned} &f_i=(-1)^{\deg f}\left(x_ix_nD_n(f)+D_i(f)\right)=0,\quad \forall i\in J,\\ &f_n=2f-\sum_{i\in J}x_iD_i(f)=0,\\ &f=\frac{1}{2}\sum_{i\in J}x_iD_i(f)=-\frac{1}{2}\sum_{i\in J}x_ix_ix_nD_n(f)=0. \end{aligned} \tag{19}$$

Therefore, $\operatorname{Ker}\widetilde{D}_k=0$. Then, $\widetilde{D}_k$ is injective.

We define an operator $[,]$ in $\mathscr{U}$. For any $f,g\in\mathscr{U}$, we have

$$[f,g]=\widetilde{D}_k(f)(g)-G_n(f)(g). \tag{20}$$

By Proposition 2, we have

$$\widetilde{D}_k([f,g])=[\widetilde{D}_k(f),\widetilde{D}_k(g)]. \tag{21}$$

Note that $\widetilde{D}_k$ is injective. Therefore, it can be concluded that $\mathscr{U}$ is a Lie superalgebra about the operator $[,]$ (see [1]). It follows from equation (21) that $\overline{K}(n,m)\cong\mathscr{U}$ as Lie superalgebras.

Since $\overline{K}(n,m)\cong\mathscr{U}$, we use $\overline{K}(n,m)$ instead of $\mathscr{U}$ defined by equation (20). By equations (5) and (6) and Proposition 2, for $f,g\in\overline{K}(n,m)$, we have

$$\begin{aligned} [f,g]&=\widetilde{D}_k(f)(g)-G_n(f)(g)\\ &=\sum_{i\in J}(-1)^{\deg f}G_i(f)G_i(g)+fG_n(g)-G_n(f)(g)\\ &=\sum_{i\in J}(-1)^{\deg f}D_i(f)D_i(g)\\ &\quad+\sum_{i\in J}(-1)^{\deg f}D_i(f)x_ix_nD_n(g)+2fx_nD_n(g)\\ &\quad+\sum_{i\in J}(-1)^{\deg f}x_ix_nD_n(f)D_i(g)-2x_nD_n(f)g\\ &\quad+\sum_{i\in J}(-1)^{\deg f}x_ix_nD_n(f)x_ix_nD_n(g)\\ &=\left(2f-\sum_{i\in J}x_iD_i(f)\right)x_nD_n(g)\\ &\quad-(-1)^{\deg(f)\deg(g)}\left(2g-\sum_{i\in J}x_iD_i(g)\right)x_nD_n(f)\\ &\quad+\sum_{i\in J}(-1)^{\deg f}D_i(f)D_i(g). \end{aligned} \tag{22}$$

□

Definition 2. $K(n,m)=[\overline{K}(n,m),\overline{K}(n,m)]$.

Proposition 4. *$K(n,m)=\left\{x^uy^\lambda \mid x^uy^\lambda\in\mathscr{U}, x^uy^\lambda\neq x^{\widehat{u}}y^\lambda\right\}$, where $\widehat{u}=\langle 1,\ldots,n-1\rangle$.*

Proof. Let $\widetilde{K}(n,m)=\left\{x^uy^\lambda \mid x^uy^\lambda\in\mathscr{U}, x^uy^\lambda\neq x^{\widehat{u}}y^\lambda\right\}$. It suffices to prove that $K(n,m)=\widetilde{K}(n,m)$. For any $f\in\widetilde{K}(n,m)$, let $f=\sum_t\sum_{\lambda\in G}c_{t\lambda}x^{u_t}x_n^{\delta_t}y^\lambda$, where $c_{t\lambda}\in\mathbb{F}$, $u_t=\{t_1,\ldots,t_s\}$ $(t_j\in J, j=1,\ldots,s)$. If generator x_n is contained in $x^{u_t}x_n^{\delta_t}y^\lambda$, then $\delta_t=1$. Otherwise, $\delta_t=0$.

Firstly, we prove $K(n,m)\supseteq\widetilde{K}(n,m)$:

(i) If $|u_t|<n-1$, there exist $t_r\notin\{u_t\}$ such that

$$\begin{aligned} &\left[x_{t_r},x_{t_r}x^{u_t}x_n^{\delta_t}y^\lambda\right]\\ &=\left(2x_{t_r}-\sum_{i\in J}x_iD_i\left(x_{t_r}\right)\right)x_nD_n\left(x_{t_r}x^{u_t}x_n^{\delta_t}y^\lambda\right)\\ &\quad-(-1)^{\deg\left(x_{t_r}\right)\deg\left(x_{t_r}x^{u_t}x_n^{\delta_t}\right)}\left(2x_{t_r}x^{u_t}x_n^{\delta_t}y^\lambda-\sum_{i\in J}x_iD_i\left(x_{t_r}x^{u_t}x_n^{\delta_t}y^\lambda\right)\right)x_nD_n\left(x_{t_r}\right)\\ &\quad+\sum_{i\in J}(-1)^{\deg x_{t_r}}D_i\left(x_{t_r}\right)D_i\left(x_{t_r}x^{u_t}x_n^{\delta_t}y^\lambda\right)\\ &=x_{t_r}x_{t_r}x^{u_t}x_ny^\lambda-(-1)^{\tau\left(u_t',t_r\right)}x^{u_t}x_n^{\delta_t}y^\lambda\\ &=-(-1)^{\tau\left(u_t',t_r\right)}x^{u_t}x_n^{\delta_t}y^\lambda.\ \left(\{u_t'\}=\{t_r,u_t,n\}\right). \end{aligned} \tag{23}$$

Therefore, $x^{u_t}x_n^{\delta_t}y^{\lambda} \in [\overline{K}(n,m), \overline{K}(n,m)] = K(n,m)$.

(ii) If $|u_t| = n-1$, $\delta_t = 1$, then $x^{u_t}x_n^{\delta_t}y^{\lambda} = x^{\widehat{u}}x_n y^{\lambda}$. Therefore,

$$\left[y^{\lambda}, x^{\widehat{u}}x_n\right] = (-1)^{n-1} \cdot 2x^{\widehat{u}}x_n y^{\lambda}. \tag{24}$$

According to (i) and (ii), $x^{u_t}x_n^{\delta_t}y^{\lambda} \in [\overline{K}(n,m), \overline{K}(n,m)] = K(n,m)$. Namely, for all $f \in \widetilde{K}(n,m)$, we have $f \in K(n,m)$. Therefore, $K(n,m) \supseteq \widetilde{K}(n,m)$.

Secondly, we prove $K(n,m) \subseteq \widetilde{K}(n,m)$:

Note that

$$\dim(\overline{K}(n,m)) = 2^n p^m, \dim(\widetilde{K}(n,m)) = (2^n - 1)p^m,$$
$$\dim(\overline{K}(n,m)) - \dim(\widetilde{K}(n,m)) = p^m. \tag{25}$$

Therefore, it suffices to prove that $x^{\widehat{u}}y^{\lambda} \notin [\overline{K}(n,m), \overline{K}(n,m)]$. Without loss of generality, suppose that there exist $x^{u_1}x_n^{\delta_1}y^{\eta}, x^{u_2}x_n^{\delta_2}y^{\mu} \in \overline{K}(n,m)$ such that $[x^{u_1}x_n^{\delta_1}y^{\eta}, x^{u_2}x_n^{\delta_2}y^{\mu}] = x^{\widehat{u}}y^{\lambda}$, where $\lambda = \eta + \mu$.

If $\delta_1 = \delta_2 = 1$, then

$$\begin{aligned}\left[x^{u_1}x_n^{\delta_1}y^{\eta}, x^{u_2}x_n^{\delta_2}y^{\mu}\right] &= \left(2x^{u_1}x_n^{\delta_1}y^{\eta} - \sum_{i\in J} x_i D_i\left(x^{u_1}x_n^{\delta_1}y^{\eta}\right)\right)x_n D_n\left(x^{u_2}x_n^{\delta_2}y^{\mu}\right)\\ &\quad - (-1)^{\deg\left(x^{u_1}x_n^{\delta_1}\right)\deg\left(x^{u_2}x_n^{\delta_2}\right)}\left(2x^{u_2}x_n^{\delta_2}y^{\mu} - \sum_{i\in J} x_i D_i\left(x^{u_2}x_n^{\delta_2}y^{\mu}\right)\right)x_n D_n\left(x^{u_1}x_n^{\delta_1}y^{\eta}\right)\\ &\quad + \sum_{i\in J}(-1)^{\deg x^{u_1}x_n^{\delta_1}} D_i\left(x^{u_1}x_n^{\delta_1}y^{\eta}\right)D_i\left(x^{u_2}x_n^{\delta_2}y^{\mu}\right) = 0.\end{aligned} \tag{26}$$

If $\delta_1 = 1, \delta_2 = 0$, then $[x^{u_1}x_n^{\delta_1}y^{\eta}, x^{u_2}x_n^{\delta_2}y^{\mu}] = c_u x^u x_n y^{\lambda}$, where $c_u \in \mathbb{F}, x^u \in A(n-1)$.

Therefore, $\delta_1 = \delta_2 = 0$. Namely,

$$\begin{aligned}&\left[x^{u_1}x_n^{\delta_1}y^{\eta}, x^{u_2}x_n^{\delta_2}y^{\mu}\right]\\ &= \left[x^{u_1}y^{\eta}, x^{u_2}y^{\mu}\right]\\ &= (-1)^{\deg x^{u_1}}\sum_{i\in J}(-1)^{\tau(u_1,i)+\tau(u_2,i)}x^{u_1+u_2-2\langle i\rangle}y^{\lambda}.\end{aligned} \tag{27}$$

By the definition of the Grassmann algebra, for all $u_1, u_2 \in B(n-1)$, we have

$$\sum_{i\in J}(-1)^{\tau(u_1,i)+\tau(u_2,i)}x^{u_1+u_2-2\langle i\rangle}y^{\lambda} \neq x^{\widehat{u}}y^{\lambda}. \tag{28}$$

It follows that $x^{\widehat{u}}y^{\lambda} \notin [\overline{K}(n,m), \overline{K}(n,m)]$. Then, $K(n,m) = [\overline{K}(n,m), \overline{K}(n,m)] \subseteq \widetilde{K}(n,m)$.

Therefore, $K(n,m) = \{x^u y^{\lambda} | x^u y^{\lambda} \in \mathscr{U}, x^u y^{\lambda} \neq x^{\widehat{u}}y^{\lambda}\}$. □

4. Some Properties of $K(n,m)$

Proposition 5. *$K(n,m)$ does not possess a $\mathbb{Z}$-graded structure as $W(n,m)$.*

Proof. Suppose that $K(n,m)$ has $\mathbb{Z}$-gradation:

$$K(n,m) = \oplus_{i=-r}^{s} K(n,m)_i, \tag{29}$$

where $K(n,m)_i = \text{span}_{\mathbb{F}}\{x^u y^{\lambda} | x^u \in \Lambda(n), \lambda \in G, i = i(u)\}$. Let $y^{\lambda} \in K(n,m)_t, x_n y^{\eta} \in K(n,m)_q$, where $\lambda, \eta \in G$. Suppose that $x_i y^{\mu} \in K(n,m)_l$, for all $i \neq n, \mu \in G$. Since $[x_i y^{\mu}, x_i y^{\mu}] = -y^{2\mu}$, we have $t = 2l$. Since $[y^{\lambda}, x_n y^{\eta}] = 2x_n y^{\lambda+\eta}$, we have $t + q = q$. Then, $t = 2l = 0$. Therefore, $x_i y^{\mu} \in K(n,m)_0$ for all $i \neq n, \mu \in G$. Since $[x_i y^{\mu}, x_n y^{\eta}] = x_i x_n y^{\mu+\eta}$ and $[K(n,m)_0, K(n,m)_q] \subseteq K(n,m)_q$, we obtain $x_i x_n y^{\mu+\eta} \in K(n,m)_q$. For all $x_j \neq x_i, j \neq n, \gamma \in G$, we have $x_j y^{\gamma} \in K(n,m)_0$. Then, $[x_j y^{\gamma}, x_i x_n y^{\mu+\eta}] = x_j x_i x_n y^{\gamma+\mu+\eta} \in K(n,m)_q$, where $\gamma + \mu + \eta \in G$. Following the discussion above, we have $x^u x_n y^{\lambda} \in K(n,m)_q$, where $x^u \in \Lambda(n-1), \lambda \in G$.

On the other hand, let $x_i x_j y^{\lambda} \in K(n,m)_a$, where $i \neq j \neq n, \lambda \in G$. Since $[x_i y^{\mu}, x_i x_j y^{\lambda}] = -(-1)^{\tau(u,i)}x_j y^{\mu+\lambda}$, where $\{u\} = \{i, j\}$, we know $a = 0$. Therefore, $x_i x_j y^{\lambda} \in K(n,m)_0$. For all $x_k y^{\mu} \in K(n,m)_0, k \neq n, i, j$, we have $[x_k y^{\mu}, x_k x_i x_j y^{\lambda}] = -(-1)^{\tau(u,k)}x_i x_j y^{\mu+\lambda}$, where $\{u\} = \{k, i, j\}$. Then, $x_k x_i x_j y^{\lambda} \in K(n,m)_0$. Following the discussion above, we have $x^u y^{\lambda} \in K(n,m)_0$, where $x^u \in \Lambda(n-1), \lambda \in G$.

If $x_n y^{\eta} \in K(n,m)_0$, then $K(n,m) = K(n,m)_0$. If $x_n y^{\eta} \in K(n,m)_1$, then $K(n,m) = K(n,m)_0 \oplus K(n,m)_1$.

Therefore, $K(n,m)$ does not possess a $\mathbb{Z}$-graded structure as $W(n,m)$. □

Lemma 1 (see [15]). *Let $\triangle = \sum_{\lambda\in G}\alpha_{\lambda}y^{\lambda} | \{\sum_{\lambda\in G}\alpha_{\lambda} = 0, \alpha_{\lambda} \in \mathbb{F}\}$. Then, $\triangle$ is an ideal of $\mathbb{T}(m)$ and $\mathbb{T}(m) = \triangle \oplus \mathbb{F}1$.*

Theorem 1. *Let $I_0 = \left\{x^u a | x^u \in \Lambda(n), a \in \triangle, x^u a \neq x^{\widehat{u}}a\right\}$. Then, I_0 is an ideal of $K(n,m)$. Namely, the Lie superalgebra $K(n,m)$ is not simple.*

Proof. Let $x^{u_1}a \in I_0, x^{u_2}b \in K(n,m)$, where $x^{u_1}, x^{u_2} \in \Lambda(n)$, $a \in \triangle, b \in \mathbb{T}(m)$. Then,

$$\begin{aligned}&[x^{u_1}a, x^{u_2}b]\\&= \left(2x^{u_1}a - \sum_{i\in J} x_i D_i(x^{u_1}a)\right) x_n D_n(x^{u_2}b)\\&\quad - (-1)^{\deg(x^{u_1})\deg(x^{u_2})}\left(2x^{u_2}b - \sum_{i\in J} x_i D_i(x^{u_2}b)\right) x_n D_n(x^{u_1}a)\\&\quad + \sum_{i\in J}(-1)^{\deg x^{u_1}} D_i(x^{u_1}a) D_i(x^{u_2}b)\\&= c_u x^u ab,\end{aligned} \tag{30}$$

where $c_u \in \mathbb{F}, x^u \in \Lambda(n)$. Since $\triangle$ is an ideal of $\mathbb{T}(m)$, we have $ab \in \triangle$. Then, $[x^{u_1}a, x^{u_2}b] = c_u x^u ab \in I_0$. Therefore, I_0 is an ideal of $K(n,m)$. Now, we conclude that $K(n,m)$ is not a simple Lie superalgebra.

Let $K(n,m)_0 := K(n,m)/I_0$. Then,

$$K(n,m)_0 = \left\{x^u + I_0 \mid x^u \in \Lambda(n), x^u \neq x^{\widehat{u}}\right\}. \tag{31}$$

Let $\mathscr{K}(n,0) = \left\{x^u | x^u \in \Lambda(n), x^u \neq x^{\widehat{u}}\right\}$. Then, we define an operator $\lfloor , \rfloor$ in $\mathscr{K}(n,0)$. For all $f \in \mathscr{K}(n,0)_\theta, g \in \mathscr{K}(n,0)_\mu$, $\theta, \mu \in \mathbb{Z}_2$, we define $\lfloor f, g\rfloor = \sum_{i\in J}(-1)^\theta D_i(f) D_i(g)$. Then, $\mathscr{K}(n,0)$ is a simple Lie superalgebra. □

Theorem 2. *$K(n,m)_0 \cong \mathscr{K}(n,0)$. Therefore, $K(n,m)$ is an extension of the simple Lie superalgebra $\mathscr{K}(n,0)$ and I_0 is the maximal ideal of $K(n,m)$.*

Proof. We define a linear map σ: $K(n,m)_0 \longrightarrow \mathscr{K}(n,0)$ such that $\sigma(x^u + I_0) = x^u$ for all $x^u + I_0 \in K(n,m)_0$. Obviously, σ is an isomorphism of linear spaces.

In addition, for all $x^{u_1} + I_0, x^{u_2} + I_0 \in K(n,m)_0$, we have

$$\begin{aligned}&\sigma([x^{u_1} + I_0, x^{u_2} + I_0])\\&= \sigma([x^{u_1}, x^{u_2}] + [x^{u_1}, I_0] + [I_0, x^{u_2}] + [I_0, I_0])\\&= \sigma\Bigg(\left(\left(2x^{u_1} - \sum_{i\in J} x_i D_i(x^{u_1})\right) x_n D_n(x^{u_2})\right.\\&\quad - (-1)^{\deg(x^{u_1})\deg(x^{u_2})}\left(2x^{u_2} - \sum_{i\in J} x_i D_i(x^{u_2})\right) x_n D_n(x^{u_1})\\&\quad + \sum_{i\in J}(-1)^{\deg x^{u_1}} D_i(x^{u_1}) D_i(x^{u_2}) + I_0\\&= \sigma(\lfloor x^{u_1}, x^{u_2}\rfloor + I_0)\\&= \lfloor x^{u_1}, x^{u_2}\rfloor\\&= [\sigma(x^{u_1} + I_0), \sigma(x^{u_1} + I_0)].\end{aligned} \tag{32}$$

Therefore, σ is a homomorphism of Lie superalgebras. Then, σ is an isomorphism of Lie superalgebras. We consider the following sequence:

$$0 \xrightarrow{i} I_0 \xrightarrow{\rho} K(n,m) \xrightarrow{\sigma\circ\pi} \mathscr{K}(n,0) \xrightarrow{0} 0. \tag{33}$$

In the above sequence, ρ is the embedded map from I_0 to $K(n,m)$ and π is the natural homomorphism from $K(n,m)$ to $K(n,m)_0$. Obviously, $\rho(I_0) = Ker(\sigma\circ\pi)$. Therefore, the above sequence is an exact sequence. Note that I_0 is an ideal of $K(n,m)$ and $K(n,m) = \mathscr{K}(n,0) \oplus I_0$. Therefore, $K(n,m)$ is an extension of the simple Lie superalgebra $\mathscr{K}(n,0)$.

Let $\mathfrak{U}(n-1, \underline{1})$ denote the divided power algebra over $\mathbb{F}$ with basis $\{x^{(\alpha)} | \alpha = (\ldots, \alpha_i, \ldots) \in \mathbb{N}_0^{n-1}, \alpha_i \leq p-1\}$. Let $\Lambda(n-1, n, \underline{1}) = \mathfrak{U}(n-1, \underline{1}) \otimes \Lambda(n)$. For $i = 1, \ldots, 2n-1$, let $\overline{D}_i$ be the linear transformation of the superalgebra $\Lambda(n-1, n, \underline{1})$ such that

$$\overline{D}_i\left(x^{(\alpha)}x^u\right) = \begin{cases} x^{(\alpha-\varepsilon_i)}x^u, & i = 1, \ldots, n-1, \\ x^{(\alpha)}\left(\dfrac{\partial x^u}{\partial x_i}\right), & i = n, \ldots, 2n-1. \end{cases} \tag{34}$$

Let $W(n-1, n, \underline{1}) = \left\{\sum_{i=1}^{2n-1} f_i \overline{D}_i | f_i \in \Lambda(n-1, n, \underline{1}), i = 1, \ldots, 2n-1\right\}$. Then, we define a linear map D_{KO}: $\Lambda(n-1, n, \underline{1}) \longrightarrow W(n-1, n, \underline{1})$ such that

$$\begin{aligned}D_{KO}(f) = &\sum_{i=1}^{2n-2}\left((-1)^{\mu(i_\prime)\deg f}\overline{D}_{i'}(f) + (-1)^{\deg f}\overline{D}_{2n-1}(f)x_i\right)\overline{D}_i\\&+ \left(\sum_{i=1}^{2n-2} x_i D_i(f) - 2f\right)\overline{D}_{2n-1},\end{aligned} \tag{35}$$

for all $f \in \Lambda(n-1, n, \underline{1})$. Let $KO(n-1, n, \underline{1}) = \text{span}_{\mathbb{F}}\{D_{KO}(f) \mid f \in \Lambda(n-1, n, \underline{1})\}$. Then, $KO(n-1, n, \underline{1})$ is a finite-dimensional odd contact Lie superalgebra (see [19]). □

Proposition 6 (see [15]). *Let $m = n-1$. Then, $\mathbb{T}(n-1)$ is isomorphic to $\mathfrak{U}(n-1, \underline{1})$.*

In [15], we can see that the isomorphism ς: $\mathbb{T}(n-1) \longrightarrow \mathfrak{U}(n-1, \underline{1})$ can be extended to an isomorphism $\widetilde{\varsigma}$: $\mathscr{U} \longrightarrow \Lambda(n-1, n, \underline{1})$.

Theorem 3. *If $m = n-1$, then $K(n,m)$ is isomorphic to a subalgebra of $KO(n-1, n, \underline{1})$.*

Proof. Let $KO_1(n-1, n, \underline{1}) = \{D_{KO}(\widetilde{\varsigma}(f)) | f \in \mathscr{U}\}$. It is a subalgebra of $KO(n-1, n, \underline{1})$. Then, we define a map φ: $K(n, n-1) \longrightarrow KO_1(n-1, n, \underline{1})$ such that $\varphi(\widetilde{D}_k(f)) = D_{KO}(\widetilde{\varsigma}(f))$ for all $\widetilde{D}_k(f) \in K(n, n-1)$. By virtue of Proposition 6, φ is an isomorphism of Lie superalgebras. □

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the Fundamental Research Funds for the Central Universities (No. 2572021BC02) and the National Natural Science Foundation of China (Grant No. 11626056).

References

[1] P. Fayet and S. Ferrara, "Supersymmetry," *Physics Reports*, vol. 32, no. 5, pp. 249–334, 1977.

[2] V. Varadarajan, *Supersymmetry for Mathematicians: An Introduction*, American Mathematical Society, Providence, RI, USA, 2004.

[3] V. G. Kac, "Lie superalgebras," *Advances in Mathematics*, vol. 26, no. 1, pp. 8–96, 1977.

[4] M. Scheunert, *The Theory of Lie Superalgebras: An Introduction*, Springer-Verlag, Berlin, Germany, 1979.

[5] V. G. Kac and M. Wakimoto, "Modular invariant representations of infinite-dimensional Lie algebras and superalgebras," *Proceedings of the National Academy of Sciences*, vol. 85, no. 14, pp. 4956–4960, 1988.

[6] V. G. Kac, "Classification of infinite-dimensional simple linearly compact Lie superalgebras," *Advances in Mathematics*, vol. 139, no. 1, pp. 1–55, 1998.

[7] Y. Zhang, "Finite-dimensional Lie superalgebras of Cartan type over fields of prime characteristic," *Chinese Science Bulletin*, vol. 42, no. 9, pp. 720–724, 1997.

[8] D. Leites, "Towards classification of simple finite dimensional modular Lie superalgebras," *Journal of Prime Research in Mathematics*, vol. 3, no. 2, pp. 1–10, 2007.

[9] S. Bouarroudj, P. Grozman, and D. Leites, "Classification of finite dimensional modular Lie superalgebras with indecomposable cartan matrix," *Symmetry, Integrability and Geometry: Methods and Applications*, vol. 5, no. 3, pp. 1–63, 2009.

[10] S. Bouarroudj, A. Lebedev, and D. Leites, "Restricted Lie (super) algebras, classification of simple Lie superalgebras in characteristic 2," *Desalination*, vol. 262, no. 13, pp. 188–195, 2014.

[11] L. Ma and L. Chen, "The classification of modular Lie superalgebras of type M," *Open Mathematics*, vol. 13, no. 1, pp. 251–264, 2015.

[12] J. Yuan, *Some Problems in Structure and Representation Theory of Lie Superalgebras*, Harbin Institute of Technology Press, Harbin, China, 2019, in Chinese.

[13] P. Meyer, "Classification of finite-dimensional Lie superalgebras whose even part is a three-dimensional simple Lie algebra over a field of characteristic not two or three," *Communications in Algebra*, vol. 47, no. 11, pp. 1–22, 2019.

[14] S. Awuti and Y. Zhang, "Modular Lie superalgebra $\overline{W}(n, m)$," *Journal of Northeast Normal University Natural Science Edition*, vol. 40, no. 2, pp. 7–11, 2008, in Chinese.

[15] L. Ren, Q. Mu, and Y. Zhang, "A class of finite-dimensional Lie superalgebras of Hamiltonian type," *Algebra Colloquium*, vol. 18, no. 2, pp. 347–360, 2011.

[16] K. Zheng, Y. Zhang, and W. Song, "The natural filtrations of finite-dimensional modular Lie superalgebras of Witt and Hamiltonian type," *Pacific Journal of Mathematics*, vol. 269, no. 1, pp. 199–218, 2014.

[17] K. Zheng, Y. Zhang, and J. Zhang, "A class of finite dimensional modular Lie superalgebras of special type," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 39, no. 1, pp. 381–390, 2015.

[18] B. Xin, "Lie superalgebra of K (n) type," Master's Thesis, Northeast Normal University, Changchun, China, 2003.

[19] J. Fu, Q. Zhang, and C. Jiang, "The cartan-type modular Lie Superalgebra KO," *Communications in Algebra*, vol. 34, no. 1, pp. 107–128, 2006.