Research Article

Double Laplace Decomposition Method and Finite Difference Method of Time-fractional Schrödinger Pseudoparabolic Partial Differential Equation with Caputo Derivative

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In this paper, an initial-boundary value problem for a one-dimensional linear time-dependent fractional Schrödinger pseudoparabolic partial differential equation with Caputo derivative of order $\alpha \in (0, 1]$ is being considered. Two strong numerical methods are employed to acquire the solutions to the problem. The first method used is the double Laplace decomposition method where closed-form solutions are obtained for any $\alpha \in (0, 1]$. As the second method, the implicit finite difference scheme is applied to obtain the approximate solutions. To clarify the performance of these two methods, numerical results are presented. The stability of the problem is also investigated.

1. Introduction

The initial-boundary value problem for one-dimensional pseudoparabolic partial differential equation (PPPDE) is considered as follows:

$$u_t(x, t) = \lambda u_{xxt}(x, t) + k u_{xx}(x, t), \quad (1)$$

with the same initial-boundary conditions mentioned for one-dimensional parabolic partial differential equation [1]:

$$u_t(x, t) = k u_{xx}(x, t). \quad (2)$$

Equation (1) has been established considering the hydrostatic overflow pressure within a portion of mud during unification, where the parameter λ is a composite soil property with the dimensions of viscosity [2]. With regard to

the assumption, if the resistance to compression is elastic (proportional to the rate of compression), then equation (1) will result with $\lambda > 0$. On the contrary, equation (2) is led by the classic Terzaghi hypothesis, which presumes that any increase in hydrostatic overflow pressure is proportional to an increase in the ratio of pore volume to solids' volume in mud [3].

PPPDEs appear in several fields of mechanics and physics. They are used to investigate homogeneous fluid flow in fractured rocks, quasi-stationary paths in semiconductors, thermodynamic and transportation phenomena, and different physical systems (see, e.g., [4–6]).

In the next step, the initial-boundary value problem for one-dimensional linear time-dependent Schrödinger pseudoparabolic partial differential equation (SPPPDE) in the domain $[0, L] \times [0, T]$ is considered as follows:

$$\begin{cases} iu_t(x, t) = \lambda u_{xxt}(x, t) + u_{xx}(x, t) + f(x, t), & (x, t) \in (0, L) \times (0, T], \\ u(x, 0) = \phi(x), & x \in [0, L], \\ u(0, t) = u(L, t) = 0, & t \in [0, T], \end{cases} \quad (3)$$

where $i = \sqrt{-1}$ is an imaginary number, $\lambda > 0$, and f and ϕ are known sufficiently smooth functions that may be complex-valued.

Since Schrödinger found his famous equation in 1926, researchers have examined the counterintuitive laws of nature for microscopic systems [7]. The time-dependent Schrödinger partial differential equation (SPDE) is originated from the vector wave equation for the electric field which rules the propagation of electromagnetic waves in an inhomogeneous medium [8]. It arises in many areas such as in quantum physics, chemistry, electromagnetic wave diffusion, underwater acoustics, and design of certain optoelectronic devices [9, 10].

Fractional analysis, which contains derivatives and integrals of the arbitrary order, is the generalized form of classical analysis. Recently, its applications in modeling real world phenomena have developed greatly in different fields of science and engineering, inclosing biology, physics, chemistry, medicine, finance, control theory,

nanotechnology, viscoelasticity, anomalous transport, etc. The main reason for this is that the fractional modeling can be more accurate and effective, than the classical versions.

In recent years, great effort has been made to obtain analytic and approximate solutions to fractional partial differential equations (FPDEs), and numerous effective methods have been proposed, for instance, the double Laplace transform method [11], Laplace Adomian decomposition method [12], fractional iteration algorithm method [13], fractional variational iteration method [14], finite difference method [15], finite element method [16], fractional subequation method [17], modified (G'/G) -expansion method [18], $\exp(\psi(\zeta))$ -expansion method [19], and q -homotopy analysis method [20].

In this work, the initial-boundary value problem for one-dimensional linear time-dependent fractional Schrödinger pseudoparabolic partial differential equation (FSPPDE) with Caputo derivative in the domain $[0, L] \times [0, T]$ is presented as follows:

$$\begin{aligned} \{ i {}^c_0 D_t^\alpha u(x, t) &= \lambda u_{xxt}(x, t) + u_{xx}(x, t) + f(x, t), 0 < \alpha \leq 1, \lambda > 0, (x, t) \in (0, L) \times [0, T], u(x, 0) = \phi(x), x \in [0, L], u(0, t) \\ &= u(L, t) = 0, t \in [0, T], \end{aligned} \quad (4)$$

where $i = \sqrt{-1}$ is an imaginary number, f and ϕ are given sufficiently smooth and may possibly be complex-valued

functions, and the Caputo derivative of order α is given as follows:

$${}_0^c D_t^\alpha u(x, t) := \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{(m-\alpha-1)} \frac{\partial^\alpha u(x, \tau)}{\partial \tau^\alpha} d\tau, & m-1 < \alpha \leq m, \\ \frac{\partial^m u(x, t)}{\partial t^m}, & \alpha = m, \end{cases} \quad (5)$$

where $m = [\alpha]$, $u \in C^m$, and Γ denotes Gamma function.

The double Laplace decomposition method will be constructed to solve problem (4) semianalytically. It considers one of the best integral transform utilized by many researchers. Kaabar et al. [21] defined a novel generalized double Laplace transform associated with the Adomian decomposition method and applied it to solve a reformulated nonlinear SPDE with spatiotemporal dispersion. A modified double Laplace transform decomposition method has been employed to solve the nonlinear coupled Hirota equations by Khan et al. [22]. Karatas Akgül et al. [23] utilized Laplace transform to solve the economic models

based on the market balance in the sense of constant proportional Caputo derivative. Jarad and Abdeljawad [24] generalized the Laplace transform to make it viable to generalized fractional integrals and derivatives.

This work is arranged as follows. In Section 2, some necessary definitions and theorems on the double Laplace transform with an application are provided. In Section 3, the implicit finite difference scheme (FDS) for problem (4) is evoked then the stability is proved. In Section 4, we show an application with figures to illustrate the sufficiency as well as the accuracy of the approximate results. In the end, Section 5 is devoted to our conclusions.

2. Double Laplace Decomposition Method

In this section, we give some basic definitions and theorems of the double Laplace transform.

Definition 1. The double Laplace transform of the function $u(x, t)$ is defined by the following double improper integral:

$$L_x L_t [u(x, t)] := \mathcal{L} U(p, s) := \int_0^\infty \int_0^\infty e^{-(px+st)} dt dx := \lim_{m, n \rightarrow \infty} \int_0^m \int_0^m e^{-(px+st)} dt dx, \quad (6)$$

where p and s are complex numbers and $\Re(p) > 0$ and $\Re(s) > 0$.

Definition 2. If a piecewise continuous function $u(x, t)$ on the intervals $(0, X)$ and $(0, T)$ is of the exponential order, that is,

$$|u(x, t)| \leq M e^{e_1 x + e_2 t}, \quad e_1, e_2 > 0, \quad (7)$$

for a positive constant M , then the double Laplace transform of $u(x, t)$ exists for all p and s provided $\Re(p) > e_1$ and $\Re(s) > e_2$.

Theorem 1. Let $u \in C^m(\mathbb{R}^+ \times \mathbb{R}^+)$ and $m = \max\{n, k\}$; there exist $M, e_1, e_2 > 0$ such that

$$\left| \frac{\partial^{i+j} u(x, t)}{\partial x^i \partial t^j} \right| \leq M e^{e_1 x + e_2 t}, \quad i = 0, 1, \dots, n, j = 0, 1, \dots, k. \quad (8)$$

Then, the following formulas hold:

$$\begin{aligned} L_x L_t \left[\frac{\partial^n u(x, t)}{\partial x^n} \right] &= p^n U(p, s) - \sum_{i=0}^{n-1} p^{n-1-i} L_t \left[\frac{\partial^i u(0, t)}{\partial x^i} \right], \\ L_x L_t \left[\frac{\partial^k u(x, t)}{\partial t^k} \right] &= s^k U(p, s) - \sum_{j=0}^{k-1} s^{k-1-j} L_x \left[\frac{\partial^j u(x, 0)}{\partial t^j} \right], \\ L_x L_t \left[\frac{\partial^{n+k} u(x, t)}{\partial x^n \partial t^k} \right] &= p^n s^k \left[U(p, s) - \sum_{i=0}^{n-1} p^{-1-i} L_t \left[\frac{\partial^i u(0, t)}{\partial x^i} \right] \right. \\ &\quad \left. - \sum_{j=0}^{k-1} s^{-1-j} L_x \left[\frac{\partial^j u(x, 0)}{\partial t^j} \right] + \sum_{i=0}^{n-1} \sum_{j=0}^{k-1} p^{-1-i} s^{-1-j} \frac{\partial^{i+j} u(0, 0)}{\partial x^i \partial t^j} \right]. \end{aligned} \quad (9)$$

Theorem 2. If the Laplace transforms of u and $(\partial^j u / \partial t^j)$, $j = 0, 1, \dots, k$, exist, then the double Laplace

transform of the partial fractional Caputo derivative with respect to the time variable is defined as

$$L_x L_t [{}_0^c D_t^\alpha u(x, t)] = s^\alpha \left[U(p, s) - \sum_{j=0}^{k-1} s^{-j-1} L_x \left[\frac{\partial^j u(x, 0)}{\partial t^j} \right] \right]. \quad (10)$$

2.1. Application of the Method. We consider the following one-dimensional linear time-dependent FSPPPDE in terms of Caputo derivative:

$${}_0^c D_t^\alpha u(x, t) = \lambda u_{xxt}(x, t) + u_{xx}(x, t) + f(x, t), \quad 0 < \alpha \leq 1, \lambda > 0, x > 0, t > 0, \quad (11)$$

where

$$f(x, t) = [1 + t^\alpha + \lambda \alpha t^{\alpha-1} + i\Gamma(\alpha + 1)] \sin(x), \quad (12)$$

subject to the initial condition,

$$u(x, 0) = \sin(x), \quad (13)$$

and boundary conditions,

$$u(0, t) = u(\pi, t) = 0. \quad (14)$$

Applying the double Laplace transform with respect to x and t to equation (11) yields

$$L_x L_t [u(x, t)] = \left[\frac{1}{s} + \frac{1}{is^{\alpha+1}} + \frac{\Gamma(\alpha + 1)}{s^{\alpha+1}} + \frac{\Gamma(\alpha + 1)}{is^{2\alpha+1}} + \frac{\lambda \Gamma(\alpha + 1)}{is^{2\alpha}} \right] \frac{1}{p^2 + 1} + \frac{1}{is^\alpha} L_x L_t [\lambda u_{txx}(x, t) + u_{xx}(x, t)]. \quad (15)$$

Taking the inverse double Laplace transform to both sides of equation (15), it results in

$$u(x, t) = \left[1 + \frac{1}{i\Gamma(\alpha + 1)} t^\alpha + t^\alpha + \frac{\Gamma(\alpha + 1)}{i\Gamma(2\alpha + 1)} t^{2\alpha} + \frac{\lambda \Gamma(\alpha + 1)}{i\Gamma(2\alpha)} t^{2\alpha-1} \right] \sin(x) + L_x^{-1} L_t^{-1} \left[\frac{1}{is^\alpha} L_x L_t [\lambda u_{txx}(x, t) + u_{xx}(x, t)] \right]. \quad (16)$$

Using the decomposition series for $u(x, t)$ into equation (16) leads to

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x, t) &= \left[1 + \frac{1}{i\Gamma(\alpha + 1)} t^\alpha + t^\alpha + \frac{\Gamma(\alpha + 1)}{i\Gamma(2\alpha + 1)} t^{2\alpha} + \frac{\lambda \Gamma(\alpha + 1)}{i\Gamma(2\alpha)} t^{2\alpha-1} \right] \\ &\sin(x) + L_x^{-1} L_t^{-1} \left[\frac{1}{is^\alpha} L_x L_t \left[\lambda \frac{\partial^3}{\partial t \partial x^2} \sum_{n=0}^{\infty} u_n(x, t) + \frac{\partial^2}{\partial x^2} \sum_{n=0}^{\infty} u_n(x, t) \right] \right]. \end{aligned} \quad (17)$$

From equation (17), the following recursive relationship can be obtained:

$$u_0(x, t) = \left[1 - \frac{i}{\Gamma(\alpha + 1)} t^\alpha + t^\alpha - i \frac{\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} t^{2\alpha} - i \lambda \frac{\Gamma(\alpha + 1)}{\Gamma(2\alpha)} t^{2\alpha-1} \right] \sin(x), \quad (18)$$

$$u_{n+1}(x, t) = L_x^{-1} L_t^{-1} \left[\frac{1}{is^\alpha} L_x L_t \left[\lambda \frac{\partial^3}{\partial t \partial x^2} u_n(x, t) + \frac{\partial^2}{\partial x^2} u_n(x, t) \right] \right], \quad n \geq 0. \quad (19)$$

The following formula is reached from equation (19):

$$\begin{aligned}
u_1(x, t) &= \left[\lambda \left(\frac{1 + i\Gamma(\alpha + 1)}{\Gamma(2\alpha)} t^{2\alpha-1} + \frac{\Gamma(\alpha + 1)}{\Gamma(3\alpha)} t^{3\alpha-1} + i\lambda \frac{\Gamma(\alpha + 1)}{\Gamma(3\alpha-1)} t^{3\alpha-2} \right) \right. \\
&\quad \left. + \left(\frac{i}{\Gamma(\alpha + 1)} t^\alpha + \frac{1 + i\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} t^{2\alpha} + i \frac{\Gamma(\alpha + 1)}{\Gamma(3\alpha + 1)} t^{3\alpha} + \lambda \frac{\Gamma(\alpha + 1)}{\Gamma(3\alpha)} t^{3\alpha-1} \right) \right] \sin(x), \\
u_2(x, t) &= \left[\lambda^2 \left(\frac{i - \Gamma(\alpha + 1)}{\Gamma(3\alpha - 1)} t^{3\alpha-2} + i \frac{\Gamma(\alpha + 1)}{\Gamma(4\alpha - 1)} t^{4\alpha-2} - \lambda \frac{\Gamma(\alpha + 1)}{\Gamma(4\alpha - 2)} t^{4\alpha-3} \right) \right. \\
&\quad + \lambda \left(\left(\frac{-1}{\Gamma(2\alpha)} t^{2\alpha-1} + 2 \frac{i - \Gamma(\alpha + 1)}{\Gamma(3\alpha)} t^{3\alpha-1} + (i - 1) \frac{\Gamma(\alpha + 1)}{\Gamma(4\alpha)} t^{4\alpha-1} + \lambda (i - 1) \frac{\Gamma(\alpha + 1)}{\Gamma(4\alpha - 1)} t^{4\alpha-2} \right) \right. \\
&\quad \left. \left. + \left(\left(\frac{-1}{\Gamma(2\alpha + 1)} t^{2\alpha} + \frac{i - \Gamma(\alpha + 1)}{\Gamma(3\alpha + 1)} t^{3\alpha} - \frac{\Gamma(\alpha + 1)}{\Gamma(4\alpha + 1)} t^{4\alpha} + i\lambda \frac{\Gamma(\alpha + 1)}{\Gamma(4\alpha)} t^{4\alpha-1} \right) \right) \right] \sin(x),
\end{aligned} \tag{20}$$

and so on. Hence, the required analytical solution is

$$u(x, t) = (1 + t^\alpha) \sin(x), \tag{21}$$

where the other terms vanish in the limit.

3. Finite Difference Scheme

The usual notations $x_n = nh, 0 \leq n \leq M$, $t_k = k\delta_t, 0 \leq k \leq N$, and $u(x_n, t_k) \equiv u_n^k$ are used, where u_n^k denotes the numerical solution of the analytical solution of $u(x, t)$ at the grid points (x_n, t_k) .

The numerical approximation based on the Caputo derivative is defined as

$${}_0^c D_t^\alpha u n^{(k)} \approx \sigma_{\alpha, \delta_t} \sum_{j=0}^k d_j^{(\alpha)} (u n^{k-j+1} - u_n^{k-j}) + O(\delta_t^{2-\alpha}), \tag{22}$$

where $\sigma_{\alpha, \delta_t} = \delta_t^{-\alpha} / \Gamma(2 - \alpha)$ and $d_j^{(\alpha)} = (j + 1)^{1-\alpha} - j^{1-\alpha}$.

Since the solution $u \in C^m$ and $m \in \mathbb{N}$, then the following three-layer implicit FDS can be written at any point (x_n, t_k) as follows:

$$\begin{aligned}
(u_{xxt})_n^k &= (u_{txx})_n^k \\
&\approx \frac{1}{h^2} [(u_t)_{n+1}^k - 2(u_t)_n^k + (u_t)_{n-1}^k] + O(h^2) \\
&\approx \frac{1}{\delta_t} \left[\frac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{h^2} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} \right] + O\left(\frac{\delta_t}{h^2} + h^2\right).
\end{aligned} \tag{23}$$

By neglecting the following discretization errors $O(\delta_t^{2-\alpha})$, $O((\delta_t/h^2) + h^2)$, and $O(h^2)$, the implicit FDS construction of problem (4) can be written as

$$\left\{ \begin{aligned} &i\sigma_{\alpha, \delta_t} \sum_{j=0}^k d_j^{(\alpha)} (u_n^{k+1-j} - u_n^{k-j}) \approx \frac{\lambda}{h^2 \delta_t} [(u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}) \\ &\quad - (u_{n+1}^k - 2u_n^k + u_{n-1}^k)] + \frac{1}{h^2} [u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}] + f_n^k, \\ &0 < n < M, 0 < k \leq N, 0 < \alpha \leq 1, \lambda > 0, f_n^k = f(x_n, t_k), \\ &u_n^0 = \phi(x_n), 0 \leq n \leq M, \\ &u_0^k = u_M^k = 0, 0 \leq k \leq N, \end{aligned} \right. \tag{24}$$

and arrange system (24) as

$$\left\{ \begin{array}{l} i\sigma_{\alpha,\delta_t}(u_n^{k+1} - u_n^k) + i\sigma_{\alpha,\delta_t} \sum_{j=1}^k d_j^{(\alpha)}(u_n^{k+1-j} - u_n^{k-j}) \\ \approx \frac{\lambda}{h^2 \delta_t} [(u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}) - (u_{n+1}^k - 2u_n^k + u_{n-1}^k)] \\ + \frac{1}{h^2} [u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}] + f_n^k, \\ 0 < n < M, 0 < k \leq N, 0 < \alpha \leq 1, \lambda > 0, f_n^k = f(x_n, t_k), \\ u_n^0 = \phi(x_n), 0 \leq n \leq M, \\ u_0^k = u_M^k = 0, 0 \leq k \leq N. \end{array} \right. \quad (25)$$

System (25) can be rewritten as follows:

$$\left\{ \begin{array}{l} \frac{\lambda}{h^2 \delta_t} u_{n+1}^k - \left(\frac{1}{h^2} + \frac{\lambda}{h^2 \delta_t} \right) u_{n+1}^{k+1} - \left(\frac{2\lambda}{h^2 \delta_t} + i\sigma_{\alpha,\delta_t} \right) u_n^k \\ + \left(\frac{2}{h^2} + \frac{2\lambda}{h^2 \delta_t} + i\sigma_{\alpha,\delta_t} \right) u_n^{k+1} + \frac{\lambda}{h^2 \delta_t} u_{n-1}^k - \left(\frac{1}{h^2} + \frac{\lambda}{h^2 \delta_t} \right) u_{n-1}^{k+1} \\ + i\sigma_{\alpha,\delta_t} \sum_{j=1}^k d_j^{(\alpha)}(u_n^{k+1-j} - u_n^{k-j}) \approx f_n^k, \\ 0 < n < M, 0 < k \leq N, 0 < \alpha \leq 1, \lambda > 0, f_n^k = f(x_n, t_k), \\ u_n^0 = \phi(x_n), 0 \leq n \leq M, \\ u_0^k = u_M^k = 0, 0 \leq k \leq N. \end{array} \right. \quad (26)$$

Writing system (26) in a matrix form is demonstrated as

$$Qu_{n+1} + Gu_n + Qu_{n-1} = \eta_n, \quad (27)$$

where $u_n = [u_n^0, u_n^1, \dots, u_n^N]^T$, $\eta_n = [\eta_n^0, \eta_n^1, \dots, \eta_n^N]^T$, and $\eta_n^k = f_n^k = f(x_n, t_k)$, $0 < n < M$, $0 < k < N$. Also, Q and G are two matrices of the size $(N+1) \times (N+1)$ and have the following form:

$$Q = \begin{bmatrix} 0 & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & a & b & 0 & \dots & \dots & 0 \\ 0 & 0 & a & b & 0 & \dots & 0 \\ \vdots & \vdots & & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 0 & a & b \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 \end{bmatrix}, \quad (28)$$

$$G = \begin{bmatrix} 1 & 0 & 0 & \dots & \dots & \dots & 0 \\ -\xi_1 & c + \gamma_1 & d & 0 & \dots & \dots & 0 \\ -\xi_2 & \gamma_2 & c + \gamma_1 & d & \dots & \dots & 0 \\ \vdots & \vdots & & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \vdots \\ -\xi_{k-1} & \gamma_{k-1} & \dots & \dots & \gamma_2 & c + \gamma_1 & d \\ -\xi_k & \gamma_k & \dots & \dots & \dots & \gamma_2 & c + \gamma_1 \end{bmatrix},$$

where $a = (\lambda/h^2 \delta_t)$, $b = -((1/h^2) + (\lambda/h^2 \delta_t))$, $c = -(2\lambda/h^2 \delta_t)$, $d = (2/h^2) + (2\lambda/h^2 \delta_t) + i\sigma_{\alpha, \delta_t}$, $\xi_n = i\sigma_{\alpha, \delta_t} d_n^{(\alpha)}$, and $\gamma_n = \xi_n - \xi_{n-1}$, $1 \leq n \leq k$.

3.1. Stability and Convergence of the FDM. To mutate the two-step FDS (27) into one-step, the modified Gauss elimination method is used to achieve

$$u_n = \theta_{n+1} u_{n+1} + \beta_{n+1}, \quad n = 1, 2, \dots, M-1. \quad (29)$$

By using the given Dirichlet boundary conditions in equation (29), we obtain

$$\theta_1 u_1 + \beta_1 = u_0. \quad (30)$$

Thus, it gives

$$\begin{aligned} \theta_1 &= 0_{(N+1) \times (N+1)}, \\ \beta_1 &= 0_{(N+1) \times 1}. \end{aligned} \quad (31)$$

To determine the matrices θ_{n+1} and β_{n+1} , we substitute equation (29) into equation (27), and we obtain

$$(Q + G\theta_{n+1} + Q\theta_n \theta_{n+1})u_{n+1} + (G\beta_{n+1} + Q\beta_n + Q\theta_n \beta_{n+1}) = \eta_n. \quad (32)$$

Hence, equation (32) gives

$$\begin{aligned} Q + G\theta_{n+1} + Q\theta_n \theta_{n+1} &= 0, \\ G\beta_{n+1} + Q\beta_n + Q\alpha_n \beta_{n+1} &= \eta_n. \end{aligned} \quad (33)$$

Finally, the following equalities are acquired:

$$\begin{aligned} \theta_{n+1} &= -(G + Q\theta_n)^{-1}Q, \\ \beta_{n+1} &= (G + Q\alpha_n)^{-1}(\eta_n - Q\beta_n), \end{aligned} \quad (34)$$

where $n = 1, 2, \dots, M-1$.

Stability can be acquired by applying the method of analyzing the eigenvalues of the iteration matrices of the schemes. The maximum of the absolute value of the eigenvalues of the matrix Q has been represented by $\rho(Q)$, which refers to the spectral radius of the given matrix.

Theorem 3. FDS (26) is conditionally stable subject to the following space step limitation:

$$h > \sqrt[3]{\left(\frac{\Gamma(2-\alpha)}{(d_1^{(\alpha)} - d_0^{(\alpha)})}\right)^2 \left(\frac{\delta_t + 4\lambda}{\delta_t^{1-2\alpha}}\right)}. \quad (35)$$

Proof. Using the analytical methods used in [25], we have to prove that $\rho(\theta_n) < 1$, $n = 1, 2, \dots, M$, under condition (35).

(i) Obviously, $\rho(\theta_1) = 0 < 1$.

(ii)

$$\rho(\theta_2) = \rho(-G^{-1}Q) \leq \| -G^{-1}Q \| \leq \| -G^{-1} \| \| Q \|$$

$$= \frac{\| Q \|}{\min_{1 \leq k \leq N-1} (|a_{kk}| - \sum_{m \neq k}^{m=1} |a_{km}|)}$$

$$= \frac{|\lambda/h^2 \delta_t| + |-(1/h^2) + (\lambda/h^2 \delta_t)|}{|(-2\lambda/h^2 \delta_t) + (i\delta t^{-\alpha}/\Gamma(2-\alpha))(d_1^{(\alpha)} - d_0^{(\alpha)})|}$$

$$= \frac{(2\lambda/h^2 \delta_t) + (1/h^2)}{\sqrt{(2\lambda/h^2 \delta_t)^2 + ((\delta t^{-\alpha}/\Gamma(2-\alpha))(d_1^{(\alpha)} - d_0^{(\alpha)})^2}} < 1$$

$$\Leftrightarrow h > \sqrt[3]{\left(\frac{\Gamma(2-\alpha)}{(d_1^{(\alpha)} - d_0^{(\alpha)})}\right)^2 \left(\frac{\delta_t + 4\lambda}{\delta_t^{1-2\alpha}}\right)}. \quad (36)$$

(iii) Assume that $\rho(\theta_n) < 1$ subject to condition (35) and we need to find the value of $\rho(\theta_{n+1}) < 1$. We notice that

FIGURE 1: Profile evolution solution of problem (38) at $N = 210$, $M = 25$, $\alpha = 0.25$, and $\lambda = 2.480$. (a) Numerical simulation. (b) Analytical solution.

$$\theta_{n+1} = \begin{bmatrix} 0 \\ 0 & \frac{\xi^*}{\gamma^* + a\theta_{n_{2,2}}} \\ 0 & * & \frac{\xi^*}{\gamma^* + a\theta_{n_{3,3}}} \\ \vdots & \vdots & \ddots \\ 0 & * & \cdots & \cdots & \frac{\xi^*}{\gamma^* + a\theta_{n_{N+1,N+1}}} \end{bmatrix}, \quad (37)$$

where $\xi^* = (2\lambda/h^2\delta_t) + (1/h^2)$, $\gamma^* = \sqrt{(2\lambda/h^2\delta_t)^2 + ((\delta_t^{-\alpha}/\Gamma(2-\alpha))(d_1^{(\alpha)} - d_0^{(\alpha)}))^2}$, and $a = (\lambda/h^2\delta_t)$.

As we know that $\theta_{n_{ij}} = \rho(\theta_n)$ for $2 \leq i \leq N+1$ and from the induction hypothesis, we conclude that $\rho(\theta_{n+1}) < 1$ subject to condition (35). Thus, we acquire the required result by induction. $\square$

4. Numerical Applications

Considering the following initial-boundary value problem for one-dimensional linear time-dependent FSPPDE in the sense of Caputo derivative,

$$\begin{aligned} \{ {}^c_0 D_t^\alpha u(x, t) &= \lambda u_{txx}(x, t) + u_{xx}(x, t) + f(x, t), f(x, t) \\ &= (1 + t^\alpha + \lambda \alpha t^{\alpha-1} + i\Gamma(\alpha + 1)) \sin(x), 0 < \alpha \leq 1, \lambda > 0, 0 < x < \pi, 0 < t \leq 1, u(0, t) = u(\pi, t) = 0, \quad 0 \leq t \leq 1, u(x, 0) = \sin(x), \quad 0 \leq x \leq \pi. \end{aligned} \quad (38)$$

We have shown that the analytical solution of problem (38) is

$$u(x, t) = (1 + t^\alpha) \sin(x). \quad (39)$$

FIGURE 2: Profile evolution solution of problem (38) at $N = 100$, $M = 29$, $\alpha = 0.90$, and $\lambda = 2.412$. (a) Numerical simulation. (b) Analytical solution.

TABLE 1: Comparison between the numerical and analytical solutions for problem (38) with $h = (\pi/M)$ and $\delta_t = (1/N)$, where the chosen values of the initial data satisfy stability condition (35).

N	M	α	λ	Numerical solution	Analytical solution	ϵ
37	15	0.10	1.98	1.9890	1.9883	0.0008
210	25	0.25	2.480	1.9979	1.9961	0.0018
263	16	0.33	2.660	1.9972	2.0000	0.0028
10	8	0.50	2.150	2.0003	2.0000	0.0003
15	15	0.80	2.17	1.9890	1.9850	0.0040
100	29	0.90	2.412	1.9981	1.9971	0.0010

The maximum norm of the error of the numerical solution can be reached with

$$\epsilon = \max_{\substack{n=0,1,\dots,M \\ k=0,1,\dots,N}} |u(x, t) - u(x_n, t_k)|, \quad (40)$$

where $u(x, t)$ and $u(x_n, t_k)$ indicate the analytical and numerical solution, respectively.

Figures 1 and 2 illustrate the comparison between the profile evolution of the analytical and numerical solutions of problem (38). The numerical solution was approximated by using implicit FDS scheme (26), whereas the analytical solution was approximated by using the double Laplace decomposition method.

We observe from Figures 1 and 2 that the numerical solution improves as the value of α approaches 1 (which is the case where the FPDE becomes a classical PDE). This also illustrates how conditional stability (35) of our scheme (26) is affected by the value of α .

To quantitatively show the accuracy and efficiency of our scheme, we ran several simulations with different input parameters to compare the numerical with the analytical solutions. Table 1 shows the error (ϵ) between the two solutions.

5. Conclusion

In this manuscript, the initial-boundary value problem for a one-dimensional linear time-dependent FSPPPDE in the

sense of Caputo derivative of order $\alpha \in (0, 1]$ is presented. Double Laplace decomposition and implicit FDS methods were investigated to obtain the closed-form and numerical solutions of problem (4), respectively. We have proved that implicit FDS (32) of problem (4) is conditionally stable. From the abovementioned table, it can be concluded that the obtained numerical solution has a good agreement with the analytic solution of various $\alpha \in (0, 1]$. The results obtained in this study may be useful in many branches of quantum physics.

In the future work, we hope to extend this model to multidimensional nonlinear time-dependent coupled SPPPDEs with Caputo fractional derivatives and solve them by the finite element method.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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