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Research Article

Minimum Partition of an r – Independence System

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Graph partitioning has been studied in the discipline between computer science and applied mathematics. It is a technique to distribute the whole graph data as a disjoint subset to a different device. The minimum graph partition problem with respect to an independence system of a graph has been studied in this paper. The considered independence system consists of one of the independent sets defined by Boutin. We solve the minimum partition problem in path graphs, cycle graphs, and wheel graphs. We supply a relation of twin vertices of a graph with its independence system. We see that a maximal independent set is not always a minimal set in some situations. We also provide realizations about the maximum cardinality of a minimum partition of the independence system. Furthermore, we study the comparison of the metric dimension problem of a graph with the minimum partition problem of that graph.

1. Introduction

An abstract idea of representing any objects which are connected to each other in a form of relation is a graph. In this representation, the object is called as a vertex and their relation denotes as an edge. Partition of a graph is the distribution of the whole graph data into disjoint subsets to different devices. The need of distributing huge graph data set is to process data efficiently and faster process of any graph related applications. Where graph partitioning is essential and applicable are given as follows:

(1) Complex networks which include biological networks (in solving biological interaction problem in a huge a biological network), social networks (Facebook, Twitter, and LinkedIn etc., and graph partitioning technology is used to process user query efficiently, as replying a query in a distributed manner is very handy and effective) [1], and transportation networks (graph partitioning can speed up and could be effective in planning a route by using a GPS (global positioning system) tool in the digital era).
(2) PageRank, which is an application used to compute the rank of web rank from web network.
(3) VLSI design: Very large-scale integration (VLSI) system is one of the graph partitioning problems in order to reduce the connection between circuits in designing VLSI. The main objective of this partitioning is to reduce the VLSI design complexity by splitting them into a smaller component.
(4) Image processing: Graph partitioning is one of the most attractive tools to split into several components of a picture, where pixels are denoted by vertices and if there are similarities between pixels are represented as edges [2].

Inspired by these interesting applications of graph partition, we consider a graph partition in the context of resolving set of a graph, which is a well-known parameter in graph theory and having remarkable application in network discovery and verification.

A *set system* is a finite set S together with a family $\mathscr{F}$ of subsets of S and is denoted by the pair $(S, \mathscr{F})$. A set system $(S, \mathscr{P})$ is said to be an independence system if for every subset X of S possessing property $\mathscr{P}$, each proper subset of X also possesses the property $\mathscr{P}$, i.e., for each $X \subset S$ such that $X \in \mathscr{P}$, $Y \in \mathscr{P}$ for all $Y \subset X$. Actually, in an independence system $(S, \mathscr{P})$, $\mathscr{P}$ identified with the family of subsets of S possessing the property $\mathscr{P}$. A subset X of S which possess the property $\mathscr{P}$ is said to be an *independent set* and *dependent set* otherwise. The *chromatic number* of $(S, \mathscr{P})$ is the smallest natural number n such that S can be partitioned into n independent sets and is denoted by $\chi(S, \mathscr{P})$. Clearly, a partition of S into n independent sets of $(S, \mathscr{P})$ can be identified by a coloring $\lambda: S \longrightarrow \{1, 2, \ldots, n\}$ of S such that for each color $c \in \{1, 2, \ldots, n\}$, the color class $\{s \in S, \lambda(s) = c\}$ has the property $\mathscr{P}$, and vice versa. The coloring λ of S is called a $\mathscr{P}$*−coloring* of S. Thus, $\chi(S, \mathscr{P})$ is the least number of colors required by a $\mathscr{P}$−coloring of S and is also called the $\mathscr{P}$−chromatic number of S [3].

The $\mathscr{P}$−chromatic number $\chi(S, \mathscr{P})$ has been extensively studied by various graph theorists. Remarkable work has been done when S is V or E for a graph G having vertex set V and edge set E, and $\mathscr{P}$ is a hereditary graphical property. For example, if $\mathscr{P}$ is the property $\mathscr{I}$ of being a vertex independent set, then $\chi(V, \mathscr{I})$ is the ordinary chromatic number of G; if $\mathscr{P}$ is the property $\mathscr{E}$ of being an edge independent set, then $\chi(E, \mathscr{E})$ is the edge chromatic number of G; if $\mathscr{P}$ is the property $\mathscr{F}$ of being a forest, then $\chi(E, \mathscr{F})$ is the arboricity of G. In the next section, we consider $\mathscr{P}$ as the property $\mathscr{R}$ of being a resolving set for G and define the $\mathscr{R}$−chromatic number of G associated with an r−independence system $(V, \mathscr{R})$.

2. r – Independence System

Hereafter, we consider nontrivial, simple, and connected graph G with vertex set V and edge set E. We denote two adjacent vertices u and v in G by $u \sim v$ and nonadjacent vertices by $u \nsim v$. The distance $d: V \times V \longrightarrow \mathbb{Z}^+ \cup \{0\}$ is the length of a shortest path between two vertices in the pair $(u, v) \in V \times V$ and is denoted by $d(u, v)$. The maximum distance between the vertices of G is called the *diameter* of G, denoted by $\operatorname{diam}(G)$. Two vertices u and v in G are *antipodal or diametral* if $d(u, v) = \operatorname{diam}(G)$; otherwise, they are nonantipodal.

Let G be a graph. For any vertex v of G, the *metric code* or *code* of v with respect to an ordered k−subset $W = \{w_1, w_2, \ldots, w_k\}$ of V is defined as

$$c_W(v) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k)). \tag{1}$$

An ordered k−subset W of V is a *resolving set* for G if $c_W(u) \neq c_W(v)$ for every pair of vertices $(u, v) \in V \times V$. The cardinality of a minimum resolving set for G is called the *metric dimension* of G, denoted by $\dim(G)$ or $\beta(G)$. A resolving set for G of cardinality $\dim(G)$ is called a *metric basis* or a *basis* of G [4–8]. In [9], it was found and, in [6], an explicit construction was given that finding the metric dimension of a graph is NP-hard. The concept of a resolving set, other than graph theory, is applied in many other areas such as coin-weighing problems [10], network discovery and verification [2], strategies for mastermind games [11], pharmaceutical chemistry [12], robot navigation [6], connected joins in graphs and combinatorial optimization [13], and sonar and coast guard Loran [8].

A subset S of the vertex set V of a graph G is an r*−independent set* if no proper subset of S is a resolving set for G. We denote $\mathscr{R}$ as the property of being an r−independent set. That is, a subset S of V possesses the property $\mathscr{R}$ if and only if S is an r−independent set. This concept was firstly introduced by Boutin and used the term *res*-independent set [14]. For simplicity, we use the term r−independent set rather than *res*-independent set. A family of subsets of V possessing the property $\mathscr{R}$ is defined as

$$(V, \mathscr{R}) = \{S \subset V \mid S \text{ possesses the property } \mathscr{R}\}. \tag{2}$$

Thus, we have a set system $(V, \mathscr{R})$ consisting of those subsets of V which are possessing the property $\mathscr{R}$ and is called the r−independence system. All the subsets possessing the property $\mathscr{R}$ may or may not be resolving. This was an error made by Boutin in [14], and we rectified it in [15].

Remark 1. For a set $S \subset V$, the following assertions are equivalent:

(1) $S \in (V, \mathscr{R})$
(2) S possesses the property $\mathscr{R}$
(3) S is an r−independent set

Remark 2. Let G be a graph with vertex set V of order n. Then,

(1) $\{v\} \in (V, \mathscr{R})$ for each $v \in V$ obviously
(2) $V \notin (V, \mathscr{R})$, because every $(n-1)$−subset of V is a resolving set for G
(3) a minimal resolving set for G is a maximal r−independent set, but converse is not always true [15]

2.1. Minimum Partition Problem. For a connected graph $G = (V, E)$ and the r-independence system $(V, \mathscr{R})$, the minimum partition problem is to make a partition of V into the minimum number of subsets possessing the property $\mathscr{R}$.

The least natural number k, such that V can be partitioned into k subsets possessing the property $\mathscr{R}$, is called the *resolving chromatic number* of G associated with $(V, \mathscr{R})$, denoted by $\chi_r(V, \mathscr{R})$. A coloring $\lambda: V \longrightarrow \{1, 2, \ldots, k\}$ of V such that for each color $c \in \{1, 2, \ldots, k\}$, the color class $\{v \in V, \lambda(v) = c\}$ possesses the property $\mathscr{R}$ is called an $\mathscr{R}$*−coloring* of V. Thus, $\chi_r(V, \mathscr{R})$ is the least number of colors required by an $\mathscr{R}$−coloring of V and is also called the $\mathscr{R}$−chromatic number of G.

Example 1. Let G be a graph with $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1 \sim v_2, v_2 \sim v_3, v_3 \sim v_4, v_4 \sim v_1\}$. Then only two colors are needed to properly color V, and it follows that the ordinary chromatic number $\chi(G) = 2$ with color classes $\{v_1, v_3\}$ and $\{v_2, v_4\}$. The metric dimension of G is 2 and two nonantipodal vertices of G form a basis of G [4]. Accordingly, no 3−element subset of V possess the property $\mathscr{R}$ and so the r−independence system is

$$(V, \mathscr{R}) = \{\{v_i\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}, \{v_1, v_3\}, \{v_2, v_4\}; 1 \leq i \leq 4\}. \tag{3}$$

The minimum partition of V according to $\mathscr{R}$−coloring of V consists of two 2−element subsets of V from $(V, \mathscr{R})$, and hence, $\chi_r(V, \mathscr{R}) = 2$.

In the above example, we obtained that the chromatic number and $\mathscr{R}$−chromatic number of a graph G are same. But, it is not necessary that these numbers are always same.

Example 2. Let G be a graph with $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1 \sim v_2, v_2 \sim v_3, v_3 \sim v_4\}$. Then only two colors are needed to properly color V, and it follows that the ordinary chromatic number $\chi(G) = 2$ with color classes $\{v_1, v_3\}$ and $\{v_2, v_4\}$. The metric dimension of G is 1, and $\{v_1\}, \{v_4\}$ are the only two bases of G [4, 6]. Accordingly, each 2−element subset of V is a resolving set for G, and so no 3−element subset of V possess the property $\mathscr{R}$. Thus, the r−independence system is

$$(V, \mathscr{R}) = \{\{v_i\}, \{v_2, v_3\}; 1 \leq i \leq 4\}. \tag{4}$$

The minimum partition of V according to $\mathscr{R}$−coloring of V consists of two bases sets and the set $\{v_2, v_3\}$ from $(V, \mathscr{R})$. Hence, $\chi_r(V, \mathscr{R}) = 3$.

It is observed, from Examples 1 and 2, that $\chi(G) \leq \chi_r(V, \mathscr{R})$. But, it is not true generally as, in the next example, we have a have a graph G such that $\chi(G) > \chi_r(V, \mathscr{R})$.

Example 3. Let G be a graph with $V = \{v_1, v_2, v_3\}$ and $E = \{v_1 \sim v_2, v_2 \sim v_3, v_3 \sim v_1\}$. Then, three colors are required to properly color V, and it follows that the ordinary chromatic number $\chi(G) = 3$ with color classes $\{v_1\}, \{v_2\}$ and $\{v_3\}$. The metric dimension of G is 2 and any two vertices of G can form a basis of G [4]. Accordingly, the 3−element set V does not possess the property $\mathscr{R}$ and so the r−independence system is

$$(V, \mathscr{R}) = \{\{v_i\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_1\}\}; 1 \leq i \leq 3\}. \tag{5}$$

The minimum partition of V according to $\mathscr{R}$−coloring of V consists of one singleton set $\{v\}$ and one 2−element set $V - \{v\}$ from $(V, \mathscr{R})$, and hence $\chi_r(V, \mathscr{R}) = 2$.

Example 4. Let G be a graph with $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1 \sim v_2, v_2 \sim v_3, v_3 \sim v_4, v_3 \sim v_5\}$. Then two colors are required to properly color V, and it follows that the ordinary chromatic number $\chi(G) = 2$ with color classes $\{v_2\}, \{v_4\}, \{v_5\}$ and $\{v_1\}, \{v_3\}$. The metric dimension of G is 2 and $\{v_4, v_5\}$ is a set of basis of G [4]. But the set of three elements $\{v_1, v_2, v_3\}$ which is not resolving set of G possess the property $\mathscr{R}$ and so the r−independence system is

$$(V, \mathscr{R}) = \{\{v_i\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_1, v_2, v_3\}\}; 1 \leq i \leq 5\}. \tag{6}$$

The minimum partition of V according to $\mathscr{R}$−coloring of V consists of one 2−element set and one 3−element set $V - \{v\}$ from $(V, \mathscr{R})$, and hence $\chi_r(V, \mathscr{R}) = 2$.

3. Three Well-Known Families

In this section, we consider families of path graphs, cycle graphs, and wheel graphs and solve the minimum partition problem for each family.

3.1. Path Graphs. A path graph P_n, for $n \geq 2$, has vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_i \sim v_{i+1}; 1 \leq i \leq n-1\}$. The following result describes which subset of the vertex set of a path graph possesses the property $\mathscr{R}$.

Lemma 1. *No 3−element subset of the vertex set V of a path graph G possess the property $\mathscr{R}$.*

Proof. As $\dim(G) = 1$ and only each end vertex of G forms a basis of G [6], every 2−element subset of V is a resolving set for G. Consequently, no 3−element subset of V possess the property $\mathscr{R}$. □

The next result investigates the number of subsets of the vertex set of a path graph possessing the property $\mathscr{R}$.

Lemma 2. *For all $n \geq 2$, if G is a path graph with vertex set V, then $|(V, \mathscr{R})| = (1/2)(n^2 - 3n + 6)$.*

Proof. According to Lemma 1, each singleton subset of V as well as each 2−element subset of $V - \{v_1, v_n\}$ possesses the property $\mathscr{R}$. It follows that for $n = 2, 3$, $(V, \mathscr{R}) = \{\{v_i\}; 1 \leq i \leq n\}$, and for all $n > 3$,

$$(V, \mathscr{R}) = \{\{v_i\}; 1 \leq i \leq n\} \cup V_2, \tag{7}$$

where V_2 denotes the collection of all the $\binom{n-2}{2}$ and 2−element subsets of $V-\{v_1, v_n\}$. Hence,

$$|(V,\mathscr{R})| = n + \binom{n-2}{2} = \frac{1}{2}\left(n^2 - 3n + 6\right). \tag{8}$$

□

The following result solves the minimum partition problem for a path graph.

Theorem 1. *For all $n \geq 2$, the vertex set V of a path graph can be partitioned into $\lfloor (n+3)/2 \rfloor$ minimum number of subsets possessing the property $\mathscr{R}$.*

Proof. Let the color classes, due to a coloring $\lambda\colon V \longrightarrow \{1, 2, \ldots, \lfloor (n+3)/2 \rfloor\}$ of V, are

When n is even, then $C_1 = \{v_1\}, C_2 = \{v_n\}$ and $C_{i+1} = \{v_i, v_{n-i+1}\}$; $2 \leq i \leq ((n-2)/2)$

When n is odd, then $C_1 = \{v_1\}, C_2 = \{v_{((n+1)/2)}\}, C_3 = \{v_n\}$ and $C_{i+2} = \{v_i, v_{n-i+1}\}$; $2 \leq i \leq ((n-3)/2)$

In both cases, all these color classes are lying in $(V,\mathscr{R})$, by Lemma 2. It follows that λ is an $\mathscr{R}$−coloring of V, and these color classes define a partition of V into the sets possessing the property $\mathscr{R}$. Further, any partition of V of cardinality less than $\lfloor (n+3)/2 \rfloor$ contains at least one 2−element or 3−element subset S of V such that $S \in (V,\mathscr{R})$. Thus, a minimum partition of V has $\lfloor (n+3)/2 \rfloor$ subsets of V possessing the property $\mathscr{R}$. □

3.2. Cycle Graphs. A cycle graph C_n, for $n \geq 3$, has vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_i \sim v_{i+1}, v_n \sim v_1; 1 \leq i \leq n-1\}$. In the next result, we investigate which subset of the vertex set of a cycle graph possesses the property $\mathscr{R}$ is.

Lemma 3. *A subset of the vertex set V of a cycle graph G which possess the property $\mathscr{R}$ is a singleton set or a 2−element set.*

Proof. In [4, 5], it was shown that $\dim(G) = 2$ and any two nonantipodal vertices of G form a basis of G. Further note that, a 3−element subset S of V, whether containing two antipodal vertices or not, is not an r−independent set. It completes the proof. □

The number of subsets of the vertex set of a cycle graph possessing the property $\mathscr{R}$ is counted in the following result.

Lemma 4. *For all $n \geq 3$, if G is a cycle graph with vertex set V, then $|(V,\mathscr{R})| = (1/2)(n^2 + n)$.*

Proof. Lemma 3 yields that each singleton subset of V as well as each 2−element subset of V possesses the property $\mathscr{R}$. It follows that for all $n \geq 3$,

$$(V,\mathscr{R}) = \{\{v_i\}; 1 \leq i \leq n\} \cup V_2, \tag{9}$$

where V_2 denotes the collection of all the $\binom{n}{2}$, 2−element subsets of V. Hence,

$$|(V,\mathscr{R})| = n + \binom{n}{2} = \left(\frac{1}{2}\right)\left(n^2 + n\right). \tag{10}$$

□

The minimum partition problem for a cycle graph is solved in the following result.

Theorem 2. *For all $n \geq 3$, the vertex set V of a cycle graph can be partitioned into $\lceil n/2 \rceil$ minimum number of subsets possessing the property $\mathscr{R}$.*

Proof. Let the color classes, due to a coloring $\lambda\colon V \longrightarrow \{1, 2, \ldots, \lceil n/2 \rceil\}$ of V, are as follows:

When n is even, then $C_i = \{v_i, v_{n-i+1}\}$; $1 \leq i \leq (n/2)$.

When n is odd, then $C_1 = \{v_1\}$ and $C_i = \{v_i, v_{n-i+2}\}$; $2 \leq i \leq ((n+1)/2)$.

In both the cases, all these color classes are lying in $(V,\mathscr{R})$, by Lemma 4. It follows that λ is an $\mathscr{R}$−coloring of V, and these color classes define a partition of V into the sets possessing the property $\mathscr{R}$. Further, any partition of V of cardinality less than $\lceil n/2 \rceil$ contains at least one 3−element subset S of V such that $S \in (V,\mathscr{R})$. Thus, a minimum partition of V has $\lceil n/2 \rceil$ subsets of V possessing the property $\mathscr{R}$. □

3.3. Wheel Graphs. For $n \geq 3$, let $C_n\colon v_1 \sim v_2 \sim \cdots \sim v_n \sim v_1$ be a cycle and K_1 be the trivial graph with vertex v. Then a wheel graph is the sum $W_n = K_1 + C_n$ with vertex set $V = \{v, v_i; 1 \leq i \leq n\}$ and edge set $E = \{v \sim v_i; 1 \leq i \leq n\} \cup E(C_n)$. For fixed i; $1 \leq i \leq n$, let a path $P\colon v_i \sim v_{i+1} \sim \cdots \sim v_{i+n-5}$ of order $n-4$ on the cycle C_n of W_n, where the indices greater than n or less than zero will be take modulo n. The following result describes the sets in W_n possessing the property $\mathscr{R}$ for $n \geq 8$.

Remark 3. For $n \geq 8$, let W_n be a wheel graph with vertex set u. Let V be set of any $\{v_i, i = 1, \ldots, n-4\}$ consecutive vertices of wheel graph, then V is a maximal independent set which is not a minimal resolving set.

Lemma 5. *For $n \geq 8$, let W_n be a wheel graph with vertex set V. Then,*

(1) *every k−element subset of the set $S = V(P) \cup \{v\}$ belongs to $(V,\mathscr{R})$ for $1 \leq k \leq n-3$*

(2) *every k−element subset of the set $V - S$ belongs to $(V,\mathscr{R})$ for $1 \leq k \leq 4$*

Proof

(1) For fixed i; $1 \leq i \leq n$, $S = \{v, v_j; i \leq j \leq i+n-5\}$. Since $d(v_{i+n-2}, v) = 1 = d(v_{i+n-3}, v)$ and $d(v_{i+n-2}, u) = 1 = d(v_{i+n-3}, u)$ for each $u \in V(P)$, $c_W(v_{i+n-2}) = c_W(v_{i+n-3})$ for any $W \subseteq S$. It follows the required result.

(2) Note that $V - S = \{v_{i+n-1}, v_{i+n-2}, v_{i+n-3}, v_{i+n-4}\}$. For any $x, y \in S - \{v, v_i, v_{i+n-5}\}$ and $d(x, w) = 2 = d(y, w)$ for each $w \in V - S$. This implies that $c_W(x) = c_W(y)$ for any $W \subseteq V - S$. Hence, the required result followed. □

For wheel graphs, the following result solves the minimum partition problem.

Theorem 3. *For $n \geq 3$, let W_n be a wheel graph with vertex set V. Then,*

$$\chi_r(V, \mathscr{R}) = \begin{cases} 2, & \text{when } n \neq 5, 6, 7, \\ 3, & \text{when } n = 5, 6, 7. \end{cases} \quad (11)$$

Proof. It can be easily seen that a partition

(i) $\{\{v_1, v_2\}, \{v_3, v\}\}$ is a minimum partition of V having sets possessing the property $\mathscr{R}$ in W_3,

(ii) $\{\{v_1, v_3, v\}, \{v_2, v_4\}\}$ is a minimum partition of V having sets possessing the property $\mathscr{R}$ in W_4,

(iii) $\{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v\}\}$ is a minimum partition of V having sets possessing the property $\mathscr{R}$ in W_5,

(iv) $\{\{v\}, \{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}\}$ is a minimum partition of V having sets possessing the property $\mathscr{R}$ in W_6, and

(v) $\{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}, \{v_7, v\}\}$ is a minimum partition of V having sets possessing the property $\mathscr{R}$ in W_7.

It follows that $\chi_r(V, \mathscr{R})$ is 2 when $n = 3, 4$ and is 3 when $n = 5, 6, 7$.

For all $n \geq 8$, let $\lambda: V \longrightarrow \{1, 2\}$ be a coloring of V and let the corresponding color classes are $C_1 = V(P) \cup \{v\}$, where $P: v_i \sim v_{i+1} \sim \cdots \sim v_{i+n-5}$ for any fixed $1 \leq i \leq n$, and $C_2 = V - C_1$. Then C_1 and C_2 define a partition of V. Also, Lemma 5 yields that both C_1 and C_2 possess the property $\mathscr{R}$. Therefore, λ is an $\mathscr{R}$−coloring of V, and hence $\chi_r(V, \mathscr{R}) = 2$. □

4. Twins and r – Independence

Let v be a vertex of a graph G having vertex set V. Then the *open neighborhood* of v is $N(v) = \{u \in V: u \sim v \text{ in } G\}$ and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. Two distinct vertices u and v of G are *adjacent twins* if $N[u] = N[v]$ and *nonadjacent twins* if $N(u) = N(v)$. Observe that if u, v are adjacent twins, then $u \sim v$ in G and if u, v are nonadjacent twins, then $u \sim v$ in G. Adjacent twins are called *true twins* and nonadjacent twins are called *false twins*. Either u, v are adjacent or nonadjacent twins, they are called *twins*. A vertex v is called *self twin* if neither $N(u) = N(v)$ nor $N[v] = N[u]$, for all $u \in V$. Each self twin in a graph makes a set of *singleton twins*. A set $T \subseteq V$ is called a *twin set* in G if u, v are twins in G for every pair of distinct vertices $u, v \in T$. The next lemma follows from the above definitions [16].

Lemma 6 (see [16]). *If u and v are twins in a graph G, then $d(u, x) = d(v, x)$ for every vertex $x \in V - \{u, v\}$.*

Due to Lemma 6, we have the following remark.

Remark 4

(1) If u and v are twins in a graph G and W is a resolving set for G. Then either $u \in W$ or $v \in W$.

(2) If T is a twin set in a graph G of order $t \geq 2$, then every resolving set for G contains at least $t - 1$ elements of T.

Removal of two twins from the vertex set makes it r−independent as given in the next result.

Lemma 7. *Let T be a twin set of order $t \geq 2$ in a graph G having the vertex set V. Then, for any two elements $u, v \in T$, the set $V - \{u, v\}$ possesses the property $\mathscr{R}$.*

Proof. Since $d(u, x) = d(v, x)$ for all $x \in V - \{u, v\}$, by Lemma 6, so no subset of $V - \{u, v\}$ is a resolving set for G. It follows the result. □

Remarks 2 and 4 yield the following two results.

Lemma 8. *Let T be a twin set of order $t \geq 2$ in a graph G. Then, for each $1 \leq k \leq t - 1$, any k−element subset S of T and the set $T - S$ both possess the property $\mathscr{R}$.*

Proof. If $t = 2$, then each subset S of T of order less that $|T|$ as well as the set $T - S$ both are singleton, and so Remark 2(1) yields the result. If $t \geq 3$, then as $t - 1$ vertices of T must belong to any resolving set for G, by Remark 4(2), so no subset of T of order $\leq t - 2$ is not a resolving set, because there are at least two twins are remained in T form that one of them must belong to a resolving set for G, by Remark 4(1). It follows that any k−element subset S of T possesses the property $\mathscr{R}$ for each $1 \leq k \leq t - 1$. Further, since the set $T - S$ is either singleton or contains at least more than one twins from T, no subset of $T - S$ is a resolving set for G. Thus, it must possess the property $\mathscr{R}$. □

Lemma 8 can be generalized with the similar proof when a graph G has more than one twin sets of order at least two, and this generalization is stated in the following result.

Theorem 4. *For $l \geq 2$, let $T_1, T_2, \ldots, T_l$ are twin sets in a graph G of orders $t_1, t_2, \ldots, t_l$, respectively, where each $t_i \geq 2$. If S_i is a k_i−element subset of T_i for $1 \leq k_i \leq t_i - 1$, then*

(1) each S_i possesses the property $\mathscr{R}$,

(2) $\cup_{i=1}^{l} S_i$ possesses the property $\mathscr{R}$, and

(3) $\cup_{i=1}^{l} T_i - \cup_{i=1}^{l} S_i$ possesses the property $\mathscr{R}$.

Remark 5. Let $\mathcal{G}$ be a family of graph and $G \in \mathcal{G}$. For $l \geq 2$, let $T_1, T_2, \ldots, T_l$ are twin sets in a graph G of orders $t_1, t_2, \ldots, t_l$, respectively, where each $t_i \geq 2$. Let a nonempty set S is union of singleton twin sets in G, and let $\{v_1, v_2\}$ belong to any one of $T_1, T_2, \ldots, T_l$, then $S \cup T_i - \{v_1, v_2\}$ is a maximal independent set which is not minimal resolving set.

The following result states the relationship between twins and $r-$independence.

Theorem 5. *The $\mathcal{R}-$chromatic number of a graph G (except P_3) of order $n \geq 3$ having a nonsingleton twin set is two.*

Proof. Let G be a graph of order $n \geq 4$ with vertex set V, and let T be a twin set in G. Let $\lambda: V \longrightarrow \{1, 2\}$ be a coloring of V, and let the corresponding color classes are $C_1 = V - \{u, v\}$ for any $u, v \in T$ and $C_2 = \{u, v\}$. Then C_1 and C_2 define a partition of V. Also, Lemma 7 implies that $C_1 \in (V, \mathcal{R})$. Further, since no path graph of order more than three has a twin set, so G is not a path graph. It follows that no singleton subset of C_2 is resolving, because a path graph only has a singleton resolving set [4]. Thus, $C_2 \in (V, \mathcal{R})$. Hence, λ is an $\mathcal{R}-$coloring of V, and so $\chi_r(V, \mathcal{R}) = 2$. □

Remark 6

(1) The converse of Theorem 5 is not true generally. Theorem 3 describes that the $\mathcal{R}-$chromatic number of a wheel graph W_n is two, but W_n has no twin class for any $n \geq 8$.

(2) In Theorem 5, if G is P_3, then G has one twin set containing two end vertices. But, the $\mathcal{R}-$chromatic number of G is three, by Theorem 1.

Next, we provide two well-known families of graphs as in the favor of Theorem 5.

4.1. Complete Multipartite Graphs. Let G be a complete multipartite graph with $k \geq 2$ partite sets $V_1, V_2, \ldots, V_k$ of cardinality $m_1, m_2, \ldots, m_k$, respectively, where each $m_i \geq 1$.

(i) If $m_i = 1$ for all $1 \leq i \leq k$, then G is a complete graph K_k having vertex as the twin set.

(ii) If some of m_i is not equal to one. Let us suppose, without loss of generality, that $m_i = 1$ for $1 \leq i \leq l$ and $2 \leq l < k$. Then, G has $k - l + 1$ twin sets $V_{l+1}, V_{l+2}, \ldots, V_k$ and $\cup_{i=1}^{l} V_i$.

(iii) If $k = 2, m_1 = 1$ and $m_2 = 2$, then G is $K_{1,2} \cong P_3$.

As, the $\mathcal{R}-$chromatic number of P_3 is 3, by Theorem 1, so we receive the following consequence from Theorem 5.

Corollary 1. *The $\mathcal{R}-$chromatic number of a complete multipartite graph (which is not $K_{1,2}$) is two.*

Example 5 (Circulant networks).

The family of circulant networks is an important family of graphs, which is useful in the design of local area networks [17]. These networks are the special case of Cayley graphs Cay $(G; S)$ when the group G is Z_n (an additive group of integers modulo n) and $S \subseteq Z_n \backslash \{0\}$ [18]. These graphs are defined as follows: let n, m and $a_1, a_2, \ldots, a_m$ be positive integers, $1 \leq a_i \leq \lfloor n/2 \rfloor$ and $a_i \neq a_j$ for all $1 \leq i < j \leq m$. An undirected graph with the set of vertices $\{v_{i+1}; i \in Z_n\}$, and the set of edges $\{v_j \sim v_{j+a_l}: 1 \leq j \leq n, 1 \leq l \leq m\}$ is called a circulant graph, denoted by $C_n(a_1, a_2, \ldots, a_m)$. The numbers $a_1, a_2, \ldots, a_m$ are called the generators, and we say that the edge $v_j \sim v_{j+a_l}$ is of type a_l. The indices after n will be taken modulo n. The cycle $v_1 \sim v_2 \sim \ldots \sim v_n \sim v_1$ in $C_n(a_1, a_2, \ldots, a_m)$ is called the principal cycle. Consider a class of circulant networks $C_{2n+2}(1, n)$, for $n \geq 1$. Then there are $n + 1$ twin sets $T_i = \{v_i, v_{i+n+1}\}$ for $1 \leq i \leq n + 1$. Thus, as a consequence of Theorem 5, the $\mathcal{R}-$chromatic number of $C_{2n+2}(1, n)$ is two.

5. Some Realizations

Remark 2(3) describes that there is no $n-$element $r-$independent set in a connected graph of order $n \geq 2$. Lemma 7 illustrates that a connected graph G of order $n \geq 3$ having twins (other than self twins) can have an $(n-2)-$element set as an $r-$independent set. In the result to follow, we characterize all the connected graphs of order $n \geq 2$ in which every $(n-1)-$element subset of the vertex set is $r-$independent.

Theorem 6. *Let G be a connected graph of order $n \geq 2$ with vertex set V. Then any $(n-1)-$element subset of V possesses the property $\mathcal{R}$ if and only if G is a complete graph.*

Proof. Suppose that G is a complete graph. Then every two vertices of G are twins and V itself is the twin set in G. So Lemma 8 yields the required result.

Conversely, suppose that any $(n-1)-$element subset, say $S = \{s_1, s_1, \ldots, s_{n-1}\}$, of V possesses the property $\mathcal{R}$. Then S is a resolving set for G, because for any $s_i \in S$, 0 lies at the ith position in the code $c_S(s_i)$, whereas the code $c_S(v)$ of the element $v \in V - S$ has all nonzero coordinates. Further, S is a minimum resolving set for G, because no $k-$element subset of S is a resolving set for any $1 \leq k \leq n-2$, by our supposition. Thus, $\dim(G) = n - 1$. In [4, 6], it was shown that a graph G of order n has $\dim(G) = n - 1$ if and only if G is a complete graph. It completes the proof. □

Since any singleton set is an $r-$independent, so we have the following consequences for a complete graphs.

Corollary 2. *The $\mathcal{R}-$chromatic number of a complete graph is two.*

If G is a connected graph of order $n \geq 2$ with vertex set V, then $2 \leq \chi_r(V, (R)) \leq n$, by Remark 2(3). The next result characterizes all the connected graphs of order $n \geq 2$ having $\mathcal{R}-$chromatic number n.

Theorem 7. *Let G be a connected graph of order $n \geq 2$ with vertex set V. Then, $\chi_r(V, \mathcal{R}) = n$ if and only if G is either $K_2 (\cong P_2)$ or $P_3 (\cong K_{1,2})$.*

Proof. If $G \cong K_2 (\cong P_2)$, then $\chi_r(V, \mathscr{R}) = 2$, by Corollary 2.

If $G \cong P_3 (\cong K_{1,2})$, then $\chi_r(V, \mathscr{R}) = 3$, by Theorem 1.

Conversely, suppose that for a connected graph G of order $n \geq 2$ with vertex set V, we have $\chi_r(V, \mathscr{R}) = n$. Then,

(i) for $n = 2$, the only connected graph is $K_2 (\cong P_2)$ such that $\chi_r(V, \mathscr{R}) = 2 = n$, by Corollary 2,

(ii) for $n = 3$, G is either $C_3 (\cong K_3)$ or $P_3 (\cong K_{1,2})$. Since the $\mathscr{R}$−chromatic number of C_3 is $2 = n - 1$, by Theorem 2, and the $\mathscr{R}$−chromatic number of P_3 is $3 = n$, by Theorem 1, so $G \cong P_3$ in this case.

(ii) for $n \geq 4$, either G has a twin set or no twin set exists in G. In the former case, $\chi_r(V, \mathscr{R}) = 2 \neq n$, by Theorem 5. In the latter case, let $V = \{v_1, v_2, \ldots, v_n\}$. Then, $\chi_r(V, \mathscr{R}) = n$ implies that the minimum partition of the r−independence system $(V, \mathscr{R})$ is $\{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$. It follows that no k−element subset of V belongs to $(V, \mathscr{R})$ for $k \geq 2$. Otherwise, $\chi_r(V, \mathscr{R}) \leq n - 1$. But, in every connected graph of order $n \geq 4$, at least one 2−element subset of V must possess the property $\mathscr{R}$, because singleton resolving sets exist in a path graph only (and in the case of path graph P_n, $(n \geq 4)$, we have 2−element subsets of V in $(V, \mathscr{R})$, by Lemma 2). Therefore, no connected graph G of order $n \geq 4$ exists such that $\chi_r(V, \mathscr{R}) = n$.

From the above three cases, we conclude that G is either $K_2 (\cong P_2)$ or $P_3 (\cong K_{1,2})$. □

From Theorem 7, it concludes that if G is a connected graph of order $n \geq 3$ with vertex set V and $G \cong P_3 (\cong K_{1,2})$, then $2 \leq \chi_r(V, \mathscr{R}) \leq n - 1$. All the connected graphs of order $n \geq 3$ having $\mathscr{R}$−chromatic number $n - 1$ are characterized in the following result.

Theorem 8. *Let G be a connected graph of order $n \geq 3$ with vertex set V. Then, $\chi_r(V, \mathscr{R}) = n - 1$ if and only if G is either $C_3 (\cong K_3)$ or P_4 or P_5.*

Proof. If $G \cong C_3 (\cong K_3)$, then Theorem 2 yields that $\chi_r(V, \mathscr{R}) = n - 1$.

If $G \cong P_4$ or P_5, then $\chi_r(V, \mathscr{R}) = n - 1$, by Theorem 1.

Conversely, suppose that for a connected graph G of order $n \geq 3$ with vertex set V, we have $\chi_r(V, \mathscr{R}) = n - 1$. Then,

(i) for $n = 3$, G is either $P_3 (\cong K_{1,2})$ or $C_3 (\cong K_3)$. $G \cong P_3$ is not true, by Theorem 7. So $G \cong C_3$, by Theorem 2,

(ii) for $n = 4$, $G \cong \{P_4, K_{1,3}, K_4 (\cong W_3), K_4 - e, K_4 - 2e (\cong P_4 \text{ and } C_4), C_4 (\cong K_{2,2})\}$, where $K_4 - e$ and $K_4 - 2e$ can be obtained by deleting one and two edges from K_4, respectively. In this case, except P_4, all the graphs has a twin set and so $\chi_r(V, \mathscr{R}) = 2 \neq n - 1$, by Theorem 5. Thus $G \cong P_4$, by Theorem 1.

(iii) for $n \geq 5$, either G has no twin set or G has a twin set. $\chi_r(V, \mathscr{R}) = 2 \neq n - 1$ in the latter case, by Theorem 5. In the former case, except $G \cong P_n$, a minimum resolving set for G is of at least two order, which implies that every 2−element subset of V belongs to $(V, \mathscr{R})$. It follows that a minimum partition of V according to the r−independence system contains at least two 2−element subsets of V, which implies that $\chi_r(V, \mathscr{R}) \leq n - 2$, a contradiction. However, when $G \cong P_n$, then $\chi_r(V, \mathscr{R}) = 4 = n - 1$ only for $n = 5$ and $\chi_r(V, \mathscr{R}) \leq n - 2$ for $n \geq 6$, by Theorem 1.

From the above three case, we conclude that G is either $C_3 (\cong K_3)$ or P_4 or P_5. □

Theorem 8 concludes that if G is a connected graph of order $n \geq 4$ with vertex set V and $G \cong \{P_4, P_5\}$, then $2 \leq \chi_r(V, \mathscr{R}) \leq n - 2$.

6. Metric Dimension and r – Independence

In this section, we develop a relationship between the metric dimension and $\mathscr{R}$−chromatic number of a connected graph by providing three existing type results.

There exists a connected graph whose metric dimension is different from its $\mathscr{R}$−chromatic number by one.

Theorem 9. *For even $n \geq 4$, there exists a connected graph G with vertex set V such that* $\dim(G) - \chi_r(V, \mathscr{R}) = 1$.

Proof. Let C_n be a cycle graph on even $n \geq 4$ vertices and a path P_2. Then G is a graph obtained by taking the product of C_n and P_2. Let the vertex set of G be $V = \{v_i, u_i; 1 \leq i \leq n\}$, and the edge set is $E = \{v_i \sim v_{i+1}, u_i \sim u_{i+1}, u_j \sim v_j; 1 \leq i \leq n - 1 \wedge 1 \leq j \leq n\}$. The resultant graph G consists of two n cycles: one is outer cycle $v_1 \sim v_2 \sim \cdots \sim v_n \sim v_1$, and the other one is inner cycle $u_1 \sim u_2 \sim \cdots \sim u_n \sim u_1$. It is shown, in [19], that $\dim(G) = 3$. Next, we investigate the $\mathscr{R}$−chromatic number of G with the help of the following five claims:

Claim 1. Every singleton, 2−element and 3−element subset of V possesses the property $\mathscr{R}$. Based on Remark 2(1) and due to $\dim(G) = 3$ [19], this claim is true, because a minimum resolving set for G is of cardinality 3, and so no singleton and 2−element subset of V is a resolving set for G.

Claim 2. For fixed $1 \leq i \leq n$, the sets $X = \{v_i, v_{i+(n/2)}, v\}$ and $Y = \{u_i, u_{i+(n/2)}, v\}$ are minimum resolving sets for G only when $v \in V - \{v_i, v_{i+(n/2)}, u_i, u_{i+(n/2)}\}$. Otherwise, $c_X(v_{i-1}) = c_X(v_{i+1})$ and $c_Y(v_{i-1}) = c_Y(v_{i+1})$.

Claim 3. For fixed $1 \leq i \leq n$, the set $Z = \{v_i, v_{i+(n/2)}, u_i, u_{i+(n/2)}\} \subset V$ possesses the property $\mathscr{R}$. By Claim 2, no subset of Z is a resolving set for G. It follows the required claim.

Claim 4. No r−independent set (other than Z) in G of cardinality greater than 3 contains any of the pairs $(v_i, v_{i+(n/2)})$ and $(u_i, u_{i+(n/2)})$.

Let S be an r−independent set in G of cardinality greater than 3. Suppose, without loss of generality, S contains the pair $(v_i, v_{i+(n/2)})$. Then a subset $\{v_i, v_{i+(n/2)}, v\}$ of S for $v \in S - \{v_i, v_{i+(n/2)}\}$ is a resolving set for G, by Claim 2, which contradicts the r−independence of S. Thus, S cannot contain the pair $(v_i, v_{i+(n/2)})$. Similarly, the pair $(u_i, u_{i+(n/2)})$ will not contained in S.

Claim 5. For each $1 \le k \le n$, there is a k−element subset of V possessing the property $\mathscr{R}$. Claim 1 yields the result of $k = 1, 2, 3$. For $k = 4$, we have a set in $(V, \mathscr{R})$, by Claim 3. Next, keeping Claim 4 in mind, let us consider two subset of V of cardinality n as follows: for fixed $1 \le i \le n$,

$$S_{1i} = \left\{u_j, v_l;\ j = i+1, i+2, \ldots, i+\frac{n}{2} \wedge l = i, i-1, \ldots, i-\frac{n}{2}+1\right\},$$
$$S_{2i} = \left\{v_j, u_l;\ j = i+1, i+2, \ldots, i+\frac{n}{2} \wedge l = i, i-1, \ldots, i-\frac{n}{2}+1\right\}, \tag{12}$$

where the indices greater than n or less than or equal to zero will be taken modulo n. Then $d(u_i, x) = d(v_{i+1}, x)$ for all $x \in S_{1i}$ and $d(v_i, y) = d(u_{i+1}, y)$ for all $y \in S_{2i}$.

It follows that $c_W(u_i) = c_W(v_{i+1})$ for any $W \subseteq S_{1i}$ and $c_W(v_i) = c_W(u_{i+1})$ for any $W \subseteq S_{2i}$. Hence, both the S_{1i}, S_{2i} and each subset of any cardinality all are the subsets of V possessing the property $\mathscr{R}$, and of course, they are k−element subsets of V for $1 \le k \le n$.

Now, let $\lambda\colon V \longrightarrow \{1, 2\}$ be a coloring of V, and let the corresponding color classes are, for fixed $1 \le i \le n$,

$$C_1 = \left\{u_j, v_l;\ j = i+1, i+2, \ldots, i+\frac{n}{2} \wedge l = i, i-1, \ldots, i-\frac{n}{2}+1\right\},$$
$$C_2 = \left\{v_j, u_l;\ j = i+1, i+2, \ldots, i+\frac{n}{2} \wedge l = i, i-1, \ldots, i-\frac{n}{2}+1\right\}, \tag{13}$$

where the indices greater than n or less than or equal to zero will be taken modulo n. Then C_1 and C_2 define a partition of V. Also, Claim 5 yields that $C_1, C_2 \in (V, \mathscr{R})$. Hence, λ is an $\mathscr{R}$−coloring of V, and so $\chi_r(V, \mathscr{R}) = 2$. Therefore, $\dim(G) - \chi_r(V, \mathscr{R}) = 1$ for every even value of $n \ge 4$. □

Remark 7. It is not necessary that the difference $\dim(G) - \chi_r(V, \mathscr{R})$ is constant (fixed) always. It can arbitrarily large depending upon the order of the graph. For instance, let G be a wheel graph of order $n \ge 8$, then $\dim(G) = \lfloor (2/5)(n+1) \rfloor$ [20], and $\chi_r(V, \mathscr{R}) = 2$, by Theorem 3. So, it can be seen that the difference $\dim(G) - \chi_r(V, \mathscr{R}) = \lfloor (2/5)(n-4) \rfloor$, which is depending upon n and is not fixed.

The next result shows that there exists a connected graph whose $\mathscr{R}$−chromatic number is different from its metric dimension by one.

Theorem 10. *For odd $n \ge 3$, there exists a connected graph G with vertex set V such that $\chi_r(V, \mathscr{R}) - \dim(G) = 1$.*

Proof. Let a cycle graph C_n on odd $n \ge 3$ vertices and a path P_2. Then G is a graph obtained by taking the product of C_n and P_2. Let the vertex set of G be $V = \{v_i, u_i;\ 1 \le i \le n\}$ and the edge set is $E = \{v_i \sim v_{i+1}, u_i \sim u_{i+1}, u_j \sim v_j;\ 1 \le i \le n-1 \wedge 1 \le j \le n\}$. The resultant graph G consists of two n−cycles: one is outer cycle $v_1 \sim v_2 \sim \cdots \sim v_n \sim v_1$, and the other one is inner cycle $u_1 \sim u_2 \sim \cdots \sim u_n \sim u_1$. Note that $\operatorname{diam}(G) = ((n+1)/2)$, and it was show, in [19], that $\dim(G) = 2$, so a minimum resolving set for G consists of two vertices of G. Next, we investigate the $\mathscr{R}$−chromatic number of G on the base of the following six claims:

Claim 1. Every singleton and 2−element subset of V possesses the property $\mathscr{R}$. Based on Remark 2(1) and due to $\dim(G) = 2$ [19], this claim is true, because a minimum resolving set for G is of cardinality 2, and so no singleton subset of V is a resolving set for G.

Claim 2. A minimum resolving set for G contains both the vertices either from the outer cycle or from the inner cycle of G.

Let W be a minimum resolving set for G. For fixed $1 \le i \le n$, let $W = \{v_i, u_j\}$, where $1 \le j \le n$. Then,

$c_W(v_{i+1}) = c_W(v_{i-1})$ when $j = i$,

$c_W(v_{i+1}) = c_W(u_i)$ when $j = i+1, i+2, \ldots, i+((n-1)/2)$, and

$c_W(v_j) = c_W(u_{j+1})$ when $j = i+((n-1)/2)+1, \ldots, i+n$,

where the indices greater than n or less than zero will be taken modulo n. It follows that W is not a resolving set for G, a contradiction.

Claim 3. For fixed $1 \le i \le n$, a minimum resolving set for G contains both the vertices:

$$x, y \in \left\{v_i, v_{i+((n-1)/2)}, v_{i+((n-1)/2)+1}, u_i, u_{i+((n-1)/2)}, u_{i+((n-1)/2)+1}\right\}, \tag{14}$$

such that $d(x, y) = \operatorname{diam}(G) - 1$.

Let W be a minimum resolving set for G, then W must contain both the vertices from the same cycle of G, by Claim 2. Let $W = \{v_i, v\}$ and $d(v_i, v) < \operatorname{diam}(G) - 1$ (in G, no two vertices u, v belonging to a same cycle have $d(u, v) = \operatorname{diam}(G)$). Then, $v \ne v_{i+((n-1)/2)}, v_{i+((n-1)/2)+1}$ and

$c_W(u_{i+((n-1)/2)-1}) = c_W(v_{i+((n-1)/2)})$ if $v \in \{v_{i+1}, \ldots, v_{i+((n-1)/2)-1}\}$, and

$c_W(v_{i+((n-1)/2)+1}) = c_W(u_{i+((n-1)/2)+2})$ if $v \in \{v_{i+((n-1)/2)+2}, \ldots, v_{i+n}\}$,

where the indices greater than n or less than or equal to zero will be taken modulo n. A contradiction to the fact that W is a resolving set for G.

Claim 4. No r−independent set in G of cardinality greater than 2 contains two vertices x, y from a same cycle (outer or inner) of G such that $d(x, y) = \operatorname{diam}(G) - 1$. Otherwise, the set $\{x, y\}$ is such a subset of that r−independent set which is resolving for G, by Claim 3.

Claim 5. For each $1 \leq k \leq n-1$, there is a k−element subset of V possessing the property $\mathscr{R}$.

Claim 1 follows the claim for $k = 1, 2$. Next, keeping Claim 4 in mind, let us consider two subset of V of cardinality $n-1$ as follows: for fixed $1 \leq i \leq n$,

$$
\begin{aligned}
S_{1i} &= \left\{u_j, v_l;\ j = i+1, i+2, \ldots, i+\frac{n-1}{2} \wedge l = i, i-1, \ldots, i-\frac{n-1}{2}+1\right\},\\
S_{2i} &= \left\{v_j, u_l;\ j = i+1, i+2, \ldots, i+\frac{n-1}{2} \wedge l = i, i-1, \ldots, i-\frac{n-1}{2}+1\right\},
\end{aligned}
\tag{15}
$$

where the indices greater than n or less than or equal to zero will be taken modulo n. Then, $d(u_i, x) = d(v_{i+1}, x)$ for all $x \in S_{1i}$ and $d(v_i, y) = d(u_{i+1}, y)$ for all $y \in S_{2i}$.

It follows that $c_W(u_i) = c_W(v_{i+1})$ for any $W \subseteq S_{1i}$ and $c_W(v_i) = c_W(u_{i+1})$ for any $W \subseteq S_{2i}$. Hence, both the set S_{1i}, S_{2i} and each subset of any cardinality all are the subsets of V possessing the property $\mathscr{R}$, and of course, they are k−element subsets of V for $1 \leq k \leq n-1$.

Claim 6. No r−independent set of cardinality greater than $n-1$ exists in G. Otherwise, Claim 4 will be contradicted.

Now, let λ: $V \longrightarrow \{1, 2, 3\}$ be a coloring of V, and let the corresponding color classes are: for fixed $1 \leq i \leq n$, $C_1 = \left\{v_{i+\operatorname{diam}(G)}, u_{i+\operatorname{diam}(G)}\right\}$,

$$
\begin{aligned}
C_2 &= \left\{u_j, v_l;\ j = i+1, i+2, \ldots, i+\frac{n-1}{2} \wedge l = i, i-1, \ldots, i-\frac{n-1}{2}+1\right\},\\
C_3 &= \left\{v_j, u_l;\ j = i+1, i+2, \ldots, i+\frac{n-1}{2} \wedge l = i, i-1, \ldots, i-\frac{n-1}{2}+1\right\},
\end{aligned}
\tag{16}
$$

where the indices greater than n or less than or equal to zero will be taken modulo n. Then C_1, C_2, and C_3 define a partition of V. Also, Claims 1 and 5 yield that $C_1, C_2, C_3 \in (V, \mathscr{R})$. Hence, λ is an $\mathscr{R}$−coloring of V, and so $\chi_r(V, \mathscr{R}) = 3$. Therefore, $\chi_r(V, \mathscr{R}) - \dim(G) = 1$ for every odd value of $n \geq 3$. □

Remark 8. It is not necessary that the difference $\chi_r(V, \mathscr{R}) - \dim(G)$ is constant (fixed) always. It can arbitrarily large by depending upon the order of the graph. For instance, let G be a cycle graph of order $n \geq 3$, then $\dim(G) = 2$ [4] and $\chi_r(V, \mathscr{R}) = \lceil n/2 \rceil$, by Theorem 2. It can be seen that the difference $\chi_r(V, \mathscr{R}) - \dim(G) = \lfloor (n-3)/2 \rfloor$, which is depending upon n and is not fixed.

The following result provides the existence of a connected graph whose metric dimension is equal to its $\mathscr{R}$−chromatic number.

Theorem 11. *For odd $n \geq 3$, there exists a connected graph G with vertex set V such that $\chi_r(V, \mathscr{R}) = \dim(G)$.*

Proof. Let G be a circulant network $C_{2n+1}(1, n)$ for odd $n \geq 3$ with vertex set $V = \{v_{i+1}; i \in Z_n\}$ (defined in Example 5), where the indices will be taken modulo n. The distance between v_i and v_j, $(j \neq i)$ on the principal is denoted by $d^*(u_i, v_j)$ (for instance, in a circulant network $C_n(1, 3)$, $d(v_i, v_{i+3}) = 1$ where as $d^*(v_i, v_{i+3}) = 3$ for any $1 \leq i \leq n$). The metric dimension, $\dim(G)$, of G is 3 as shown in [21]. On the basis of the following three claims, we investigate the $\mathscr{R}$−chromatic number of G.

Claim 1. Every k−element subset of V possesses the property $\mathscr{R}$ for $1 \leq k \leq 3$.

As a minimum resolving set for G is of cardinality three, so no singleton and 2−element subset of V resolves G. It follows the claim.

Claim 2. No r−independent set in G of cardinality greater than three contains two vertices u and v such that $d^*(u, v) = n$.

If S is an r−independent set in G containing two vertices u and v such that $d^*(u,v) = n$, then the set $\{u,v,w\} \subset S$, for any $w \in S - \{u,v\}$, is a resolving set for G, a contradiction.

Claim 3. A maximum r−independent set in G consists of n consecutive vertices form the principal cycle.

Firstly, for fixed $1 \le i \le n$, we show that a set $S = \{v_i, v_{i+1}, \ldots, v_{i+n-1}\} \subset V$ of n consecutive vertices form the principal cycle possesses the property $\mathscr{R}$. Note that, $d(v_{i+n}, s) = d(v_{i-1}, s)$ for all $s \in S$. It follows that $c_W(v_{i+n}) = c_W(v_{i-1})$ for each $W \subseteq S$. Hence, no subset of S is a resolving set for G. Secondly, if an r−independent set S either contains n nonconsecutive vertices or contains more than n vertices (consecutive or nonconsecutive) form the principal cycle, then S must contradict Claim 2.

Now, let $\lambda: V \longrightarrow \{1,2,3\}$ be a coloring of V, and let the corresponding color classes are: for fixed $1 \le i \le n$, $C_1 = \{v_{i+2n}\}$, $C_2 = \{v_i, v_{i+1}, \ldots, v_{i+n-1}\}$, and $C_3 = \{v_{i+n}, v_{i+n+1}, \ldots, v_{i+2n-1}\}$, where the indices will be taken modulo n. Then C_1, C_2 and C_3 define a partition of V. Also, Claims 1 and 3 yield that $C_1, C_2, C_3 \in (V, \mathscr{R})$. Hence, λ is an $\mathscr{R}$−coloring of V, and so $\chi_r(V,\mathscr{R}) = 3$. Therefore, $\chi_r(V,\mathscr{R}) = \dim(G)$ for every odd value of $n \ge 3$. □

Data Availability

All the data and materials used to compute the results are provided in Section 1.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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