Research Article

Construction of Type 2 Poly-Changhee Polynomials and Its Applications

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In this paper, we introduce type 2 poly-Changhee polynomials by using the polyexponential function. We derive some explicit expressions and identities for these polynomials, and we also prove some relationships between poly-Changhee polynomials and Stirling numbers of the first and second kind. Also, we introduce the unipoly-Changhee polynomials by employing unipoly function and give multifarious properties. Furthermore, we provide a correlation between the unipoly-Changhee polynomials and the classical Changhee polynomials.

1. Introduction

Special polynomials and their generating functions have vital roles in several branches of arithmetic, probability, statistics, mathematical physics, and additionally engineering. Since polynomials are appropriate for applying well-known operations like by-product and integral, polynomials are helpful to check real-world issues within the said areas. As an example, generating functions for special polynomials with their congruousness properties, repetition relations, process formulae, and regular add involving these polynomials are studied in recent years (see [1–4]).

For \( j \geq 0 \), the Stirling numbers of the first kind are defined by the following (see [1, 2, 5–14]):

\[
(\xi)_{j} = \sum_{l=0}^{j} S_{1}(j, l)\xi^{l},
\]

(1)

where \((\xi)_{0} = 1\) and \((\xi)_{j} = \xi(\xi - 1), \ldots, (\xi - j + 1), (j \geq 1)\). From (1), it is easy to see that (see [4, 15–22])

\[
\frac{1}{r!}(\log(1 + z))^{r} = \sum_{j=r}^{\infty} S_{1}(j, r)\frac{z^{j}}{j!}, \quad (r \geq 0).
\]

(2)

For \( j \geq 0 \), the Stirling numbers of the second kind are defined by the following (see [3, 11–16]):

\[
\xi^{j} = \sum_{l=0}^{j} S_{2}(j, l)(\xi)_{l};
\]

(3)

From (3), we see that

\[
\frac{1}{r!}(e^{z} - 1)^{r} = \sum_{j=r}^{\infty} S_{2}(j, r)\frac{z^{j}}{j!}, \quad (r \geq 0).
\]

(4)
The Bernoulli $B_j(\xi)$, Euler $E_j(\xi)$, and Genocchi $G_j(\xi)$ polynomials are defined by the following (see [1, 6, 7]):

$$\frac{z}{e^z - 1} \xi^j \left|\xi\right| < \pi,$$

$$\frac{2z}{e^z + 1} \xi^j \left|\xi\right| < \pi,$$

respectively.

Let the Changhee polynomials $Ch_j(\xi)$ be given by the following (see [2, 13, 14, 16]):

$$\int_{\mathbb{C}} (1 + z)^{\xi-\eta} \mu_{-1}(\eta) = \frac{2}{2 + z} (1 + z)^{\xi} = \sum_{j=0}^{\infty} Ch_j(\xi) \frac{z^j}{j!}.$$

In the case when $\xi = 0$, $Ch_j = Ch_j(0)$ are called the Changhee numbers.

Moreover, we have the following (see [13]):

$$Ch_j(\xi) = \sum_{j=0}^{j} E_j(\xi) S_j(j, l),$$

$$E_j(\xi) = \sum_{j=0}^{j} Ch_j(\xi) S_j(j, l), \quad (j \geq 0).$$

The polyexponential function as an inverse to the polylogarithm function is defined by Kim and Kim [12] to be

$$\bar{E}_k(\xi) = \sum_{j=1}^{\infty} \frac{\xi^j}{(j - 1)!j!} (k \in \mathbb{Z}).$$

We note that

$$\bar{E}_1(\xi) = \sum_{j=1}^{\infty} \frac{\xi^j}{j!} = e^\xi - 1.$$

In 2019, Kim and Kim [12] introduced the poly-Bernoulli polynomials which are defined by

$$\bar{B}_j(k)(\xi) \left|\xi\right| < 1,$$

Letting $\xi = 0$, $B_j(k)(0)$ are called the poly-Bernoulli numbers.

For $k \in \mathbb{Z}$, the polylogarithm function is defined by the following (see [5, 21]):

$$\bar{L}_j(\xi) = \sum_{j=1}^{\infty} \frac{\xi^j}{j} (\left|\xi\right| < 1).$$

Note that

$$\bar{L}_1(\xi) = \sum_{j=1}^{\infty} \frac{\xi^j}{j} = -\log(1 - \xi).$$

Lee et al. [22] introduced the type 2 poly-Euler polynomials which are given by

$$\bar{E}_j(\xi) \left|\xi\right| < \pi,$$

respectively.

In the case when $\xi = 0$, $E_j(k)(0)$ are called the type 2 poly-Euler numbers.

The Dahee polynomials are defined by the following (see [7, 16]):

$$\frac{\log(1 + z)}{z} \left(1 + z\right)^\xi = \sum_{j=0}^{\infty} D_j(\xi) \frac{z^j}{j!}.$$

When $\xi = 0$, $D_j = D_j(0)$ are called the Dahee numbers.

The following paper is as follows. In Section 2, we introduce type 2 poly-Changhee polynomials and numbers and derive some identities of these polynomials. We derive some recurrence relations and relationships between Bernoulli number, Euler numbers, and Daehee numbers. In Section 3, we introduce unipoly-Changhee polynomials and investigate some identities of these polynomials.

### 2. Type 2 Poly-Changhee Numbers and Polynomials

In this section, we define type 2 poly-Changhee polynomials by using the polyexponential functions and represent the usual Changhee polynomials (more precisely, the values of Changhee polynomials at 1) when $k = 1$. At the same time, we give explicit expressions and identities involving poly-Changhee polynomials.

For $k \in \mathbb{Z}$, we define type 2 poly-Changhee polynomials by means of the following exponential generating function (in a suitable neighborhood of $z = 0$) including the polyexponential function given as follows:

$$\frac{\bar{E}_k(\log(1 + z))}{(1 + z)^\xi} = \sum_{j=0}^{\infty} \bar{Ch}_j^{(k)}(\xi) \frac{z^j}{j!}.$$

At the point $\xi = 0$, $\bar{Ch}_j^{(k)}(0) = \bar{Ch}_j^{(k)}(0)$ are called type 2 poly-Changhee numbers.

For $k = 1$, by using (9) and (16), we see that

$$\bar{E}_j \left(\log(1 + z)\right) \left(1 + z\right)^\xi = \frac{2}{2 + z} \left(1 + z\right)^\xi$$

$$= \sum_{j=0}^{\infty} \bar{Ch}_j(\xi) \frac{z^j}{j!}, \quad (j \geq 0),$$

where $\bar{Ch}_j(\xi)$ are called the Changhee polynomials (see equation (7)).

**Theorem 1.** Let $j$ be the nonnegative number and $k \in \mathbb{Z}$. Then,

$$\bar{Ch}_j^{(k)}(\xi) = \sum_{l=0}^{j} \binom{j}{l} \sum_{i=0}^{l} \frac{1}{(i + 1)^{k-1}} \frac{S_{l}(l + 1, i + 1)2^l}{l + 1} Ch_{j-l}(\xi).$$

**Proof.** By (16), we have
\[
\frac{\text{Ei}_k(\log(1 + 2z))}{z(2 + z)}(1 + z)^k = \frac{1}{z(2 + z)}(1 + z)^k \sum_{i=0}^{\infty} \frac{1}{(i + 1)^{k+1}} \sum_{l=0}^{\infty} S_1(l, i + 1) \frac{2^{l-1}}{l!}
\]
\[
= \frac{2}{2 + z}(1 + z)^k \sum_{i=0}^{\infty} \frac{1}{(i + 1)^{k+1}} \sum_{l=0}^{\infty} S_1(l + 1, i + 1) \frac{2^{l-1}}{l!}
\]
\[
= \sum_{j=0}^{\infty} C(j) \frac{z^j}{j!} \sum_{i=0}^{\infty} \frac{1}{(i + 1)^{k+1}} \sum_{l=0}^{\infty} S_1(l + 1, i + 1) \frac{2^{l-1}}{l!} C(j-1)(\xi)
\]
\[
= \sum_{j=0}^{\infty} \sum_{l=0}^{j} \frac{j!}{l!} \sum_{i=0}^{\infty} \frac{1}{(i + 1)^{k+1}} \sum_{l=0}^{\infty} S_1(l + 1, i + 1) \frac{2^{l-1}}{l!} C(j-1)(\xi) \frac{z^j}{j!},
\]
(19)

where \(B_j^{(\alpha)}(\xi)\) are the Bernoulli polynomials of order \(\alpha\) by
\[
\left(\frac{z}{e^z - 1}\right)^{\alpha} e^{\xi} = \sum_{j=0}^{\infty} B_j^{(\alpha)}(\xi) \frac{z^j}{j!}
\]
(22)

**Theorem 2.** Let \(j\) be the nonnegative number. Then,
\[
\sum_{i=0}^{\infty} \frac{\text{Ei}_k(\log(1 + 2z))}{z(2 + z)}(1 + z)^k = \frac{2}{(1 + 2\xi)(\log(1 + 2\xi))} \sum_{j=1}^{\infty} \frac{(\log(1 + 2\xi))^j}{(j - 1)! j^{k-1}}
\]
(24)

For \(k \geq 1\), using equations (21) and (24), we find

**Corollary 1.** Let \(j\) be the nonnegative number. Then,
\[
C(j)(\xi) = \sum_{l=0}^{j} \left(\begin{array}{c} j \\ l \end{array}\right) \sum_{i=0}^{\infty} \frac{1}{(i + 1)^{k+1}} \sum_{l=0}^{\infty} S_1(l + 1, i + 1) \frac{2^{l-1}}{l!} C(j-1)(\xi)
\]
(20)

The higher-order Bernoulli polynomials are defined by the following (see [11]):
\[
\left(\frac{z}{\log(1 + z)}\right)^{r-1} (1 + z)^{\xi - 1} = \sum_{j=0}^{\infty} B_j^{(\alpha-r)}(\xi) \frac{z^j}{j!}, \quad (r \in \mathbb{C}),
\]
(21)
Let \( \text{Corollary 2.} \)

For \( j \geq 0 \), we have

\[
\text{Ch}_j^{(2)} (\xi + \eta) = \sum_{i=0}^{j} \binom{j}{i} \text{Ch}_j^{(k)} (\xi) (\eta)_i.
\]  

(28)

\( \text{Proof.} \) Considering (16), we have

\[ 2 \text{Ch}_j^{(k)} + j \text{Ch}_j^{(k)} = z^{j+1} \sum_{i=1}^{j+1} S_i (j+1, i) \frac{1}{j+1}. \]  

(30)
Theorem 6. Let
\[ \text{Ch}_j^{(k)}(x) = \sum_{r=0}^{j} \binom{j}{r} \xi^r S_j(i,r) \text{Ch}_{j-r}^{(k)}. \]

Proof. Equation (16) can be written as
\[
\frac{E_k((1 + 2z))}{z} = (2 + z) \sum_{j=0}^{\infty} \frac{Ch_j^{(k)} z^j}{j!}
\]
\[= 2 \sum_{j=0}^{\infty} \frac{Ch_j^{(k)} z^j}{j!} + \sum_{j=0}^{\infty} Ch_j^{(k)} \frac{z^{j+1}}{j!}
\]
\[= 2 \sum_{j=0}^{\infty} \frac{Ch_j^{(k)} z^j}{j!} + j \sum_{j=0}^{\infty} \frac{Ch_{j-1}^{(k)} z^j}{j!}
\]
\[= \sum_{j=1}^{\infty} \left(2Ch_j^{(k)} + jCh_{j-1}^{(k)} \right) \frac{z^j}{j!}
\]
\[\text{On the other hand, we have}
\]
\[
\frac{E_k((1 + 2z))}{z} = \frac{1}{z} \sum_{i=1}^{\infty} \frac{(\log(1 + 2z))^i}{(i-1)! z^i}
\]
\[= \frac{1}{z} \sum_{i=1}^{\infty} \frac{(\log(1 + 2z))^i}{(i-1)!}
\]
\[= \frac{1}{z} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{S_i(j+1,i) z^{j+1}}{j!}
\]
\[= \sum_{j=1}^{\infty} \frac{S_j(j+1,i) z^j}{j+1}
\]
Therefore, by (31) and (32), we obtain the result.

Theorem 5. For \( j \geq 0 \), we have
\[ \text{Ch}_j^{(k)}(1) + \text{Ch}_j^{(k)} = \frac{1}{j+1} \sum_{i=1}^{j+1} S_i(j+1,i) z^{j+1}. \]

Proof. By (16), we see that
\[
\sum_{j=1}^{\infty} \left[ \text{Ch}_j^{(k)}(1) + \text{Ch}_j^{(k)} \right] \frac{z^j}{j!} = \left( \frac{E_k((1 + 2z))}{z} \right) \frac{1}{z}
\]
\[= \sum_{j=1}^{\infty} \frac{S_j(j+1,i) z^{j+1}}{j+1}
\]
which completes the proof of the theorem.

Theorem 6. Let \( j \) be the nonnegative number. Then,
\[ \text{Ch}_j^{(k)}(\xi) = \sum_{i=0}^{j} \binom{j}{i} \xi^i S_j(i,r) \text{Ch}_{j-r}^{(k)}. \]

Proof. Replacing \( z \) by \( e^z - 1 \) in (16), we get
\[ \sum_{j=0}^{\infty} \text{Ch}_j^{(k)}(\xi) \frac{z^j}{j!} = \left( \frac{E_k((1 + 2z))}{z} \right) \frac{1}{z}
\]
\[= \sum_{j=0}^{\infty} \left( \sum_{i=0}^{j} \binom{j}{i} \text{Ch}_{j-i}^{(k)} \right) \frac{z^j}{j!}
\]
\[= \sum_{j=0}^{\infty} \frac{1}{j!} \left( \sum_{i=0}^{j} \binom{j}{i} \xi^i S_j(i,r) \text{Ch}_{j-i}^{(k)} \right) \frac{z^j}{j!}
\]
By (16) and (36), we obtain the result.

Theorem 7. Let \( j \) be the nonnegative number. Then,
\[ \sum_{r=0}^{j} \text{Ch}_j^{(k)}(r) = \sum_{p=0}^{j} \sum_{q=0}^{p} \binom{p}{q} \binom{j}{p} B_{p-q} E_q (j-p+1)^k
\]

Proof. Replacing \( z \) by \( e^z - 1 \) in (16), we get
From (16), we have

\[ \sum_{r=0}^{\infty} C_{r}^{(k)} \frac{(e^{z} - 1)^{r}}{r!} = \frac{2E_{i}(z)}{(e^{z} - 1)(e^{z} + 1)} \]

\[ \sum_{r=0}^{\infty} C_{r}^{(k)}, \sum_{j=r}^{\infty} S_{2}(j, r) \frac{z^{j}}{j!} = -\frac{2}{e^{z} + 1} \frac{z^{r} - 1}{z} \] (38)

Therefore, by (38) and (39), we get the result. □

**Theorem 8.** Let \( j \) be nonnegative number. Then,

\[ C_{j}^{(k)}(\xi) = \sum_{i=0}^{j-1} \binom{j}{i} E_{i}(\xi) S_{1}(j-i, s) \sum_{r=0}^{\infty} \frac{1}{(r+1)^{k+1}} \frac{z^{i}}{i!} \]

\[ \sum_{j=0}^{\infty} C_{j}^{(k)}(\xi) \frac{z^{j}}{j!} = \frac{E_{i}(\log(1 + 2z))}{z(2 + z)} (1 + z)^{i}, \] (40)

**Proof.** From (16), we have

\[ \sum_{j=0}^{\infty} C_{j}^{(k)}(\xi) \frac{z^{j}}{j!} = \frac{E_{i}(\log(1 + 2z))}{z(2 + z)} (1 + z)^{i}, \] (41)

\[ = \frac{1}{z(2 + z)} (1 + z)^{i} \sum_{r=0}^{\infty} \frac{(\log(1 + 2z))^{r+1}}{r!(r+1)^{k+1}} \]

\[ = \frac{1}{z(2 + z)} (1 + z)^{i} \sum_{r=0}^{\infty} \frac{(\log(1 + 2z))^{r+1}}{r!(r+1)^{k+1}} \]

\[ = \frac{2}{z + 2}(1 + z)^{i} \sum_{i=0}^{j} \frac{1}{(r+1)^{k+1}} \frac{z^{i}}{i!} \]

\[ = e^{\log(1+z)} \sum_{i=0}^{j} \frac{1}{(r+1)^{k+1}} \frac{z^{i}}{i!} \] (42)

\[ \left( \sum_{j=0}^{\infty} \sum_{i=0}^{j} \frac{1}{(r+1)^{k+1}} \frac{z^{i}}{i!} \right) \left( \sum_{i=0}^{j} \frac{1}{(r+1)^{k+1}} \frac{z^{i}}{i!} \right) \]

\[ \left( \sum_{j=0}^{\infty} \binom{j}{i} E_{i}(\xi) S_{1}(j-i, s) \frac{z^{j}}{j!} \right) \left( \sum_{i=0}^{j} \frac{1}{(r+1)^{k+1}} \frac{z^{i}}{i!} \right) \]

\[ = \sum_{j=0}^{\infty} \binom{j}{i} E_{i}(\xi) S_{1}(j-i, s) \frac{z^{j}}{j!} \sum_{r=0}^{\infty} \frac{1}{(r+1)^{k+1}} \frac{z^{i}}{i!} \]
Thus, by (41) and (42), we complete the proof. □

3. Type 2 Unipoly-Changhee Numbers and Polynomials

Recently, Kim and Kim [12] introduced the unipoly function $u_k(\xi|p)$ by

$$u_k(\xi|p) = \sum_{j=1}^{\infty} p(j) \frac{\xi^j}{j^k}, \quad (k \in \mathbb{Z}),$$  \hspace{1cm} (43)

where $u_k(\xi|p)$ is attached to polynomials $p(\xi)$ and $p$ is any arithmetic function which is real or complex.

Putting $k = 1$, (43) to get (see [5])

$$u_k(x|1) = \sum_{j=1}^{\infty} \frac{\xi^j}{j} = \text{Li}_k(\xi),$$  \hspace{1cm} (44)

and it is called the polylogarithm function.

By using (7) and (43), we consider the unipoly-Changhee polynomials attached to polynomials $p(\xi)$ by

$$\frac{1}{z(2+z)} u_k(\log(1+2z)|p) (1+z)^{z} = \sum_{j=0}^{\infty} Ch_{j,p}^{(k)}(\xi) \frac{z^j}{j!}.$$

Thus, we complete the proof.

$\Box$

Theorem 9. Let $j$ be the nonnegative number. Then,

$$Ch_{j,1}^{(k)}(\xi) = Ch_{j}^{(k)}.$$

$\Box$

Proof. Taking $p(j) = 1/\Gamma(j)$ in (45), we have

$$\sum_{j=0}^{\infty} Ch_{j,1}^{(k)}(\xi) \frac{z^j}{j!} = \frac{1}{z(2+z)} \left( (1+z) u_k(\log(1+2z) \frac{1}{\Gamma}) \right)$$

$$= \frac{1}{z(2+z)} \sum_{r=1}^{\infty} \frac{(\log(1+2z))^r}{r^k (r-1)!}$$

$$= \frac{1}{z(2+z)} \frac{E_k(\log(1+2z))}{\Gamma} = \sum_{j=0}^{\infty} Ch_{j}^{(k)} \frac{z^j}{j!}.$$

$\Box$

Thus, we complete the proof.

Theorem 10. For $j \geq 0$ and $k \in \mathbb{Z}$, we have

$$Ch_{j,p}^{(k)}(\xi) = \sum_{i=0}^{j} \sum_{r=0}^{i} \binom{j}{i} \frac{p(r+1)(r+1)!}{(r+1)^k} \frac{S_{i+1}(i+1,r+1)2^i}{i+1} Ch_{j-i}^{(k)}(\xi).$$

$\Box$

Proof. From (45), we have

$$\sum_{j=0}^{\infty} Ch_{j,p}^{(k)}(\xi) \frac{z^j}{j!} = \frac{1}{z(2+z)} (1+z)^{z} u_k(\log(1+2z)|p)$$

$$= \frac{1}{z(2+z)} (1+z)^{z} \sum_{r=1}^{\infty} \frac{p(r)}{r^k} (\log(1+2z))^r$$

$$= \frac{1}{z(2+z)} (1+z)^{z} \sum_{r=0}^{\infty} \frac{p(r+1)}{(r+1)^k} (\log(1+2z))^{r+1}$$

$$= \frac{1}{z(2+z)} (1+z)^{z} \sum_{r=0}^{\infty} \frac{p(r+1)(r+1)!}{(r+1)^k} \sum_{i=r+1}^{\infty} S_i(i,r+1) \frac{2^i z^i}{i!}$$

$$= \frac{2}{z(2+z)} (1+z)^{z} \sum_{r=0}^{\infty} \frac{p(r+1)(r+1)!}{(r+1)^k} \sum_{i=r+1}^{\infty} S_i(i+1,r+1) \frac{2^i z^i}{i!}$$

$$= \sum_{j=0}^{\infty} Ch_{j}^{(k)}(\xi) \frac{z^j}{j!} \left( \sum_{r=0}^{i} \binom{j}{i} \frac{p(r+1)(r+1)!}{(r+1)^k} \frac{S_i(i+1,r+1)2^i}{i+1} \frac{Ch_{j-i}^{(k)}(\xi)}{j!} \right) \frac{z^j}{j!}.$$
So, the proof is completed.

\[ \text{Corollary 3. Let } j \text{ be the nonnegative number. Then,} \]

\[ Ch_{j,p}^{(k)}(\xi) = \sum_{r=0}^{\infty} \binom{j}{r} Ch_{r,p}(\xi)^{j-r}. \]  

**Theorem 11.** Let \( j \) be the nonnegative number. Then,

\[ Ch_{j,p}^{(k)}(\xi) = \sum_{r=0}^{\infty} \binom{j}{r} Ch_{r,p}(\xi)^{j-r}. \]  

**Proof.** Using (45), we have

\[ \frac{1}{z(2+z)} u_k (\log(1+2z)|p) \sum_{j=0}^{\infty} \binom{j}{r} \frac{z^j}{j!} \]

\[ = \frac{1}{z(2+z)} u_k (\log(1+2z)|p) \sum_{j=0}^{\infty} \binom{j}{r} \frac{z^j}{j!} \]

\[ = \sum_{r=0}^{\infty} \binom{j}{r} Ch_{r,p}(\xi)^{j-r} \frac{z^j}{j!} \]

\[ = \sum_{r=0}^{\infty} \binom{j}{r} Ch_{r,p}(\xi)^{j-r} \frac{z^j}{j!} \]  

\[ \sum_{j=0}^{\infty} \binom{j}{r} \frac{z^j}{j!} = \frac{1}{z(2+z)} u_k (\log(1+2z)|p) \]

\[ = \frac{1}{z(2+z)} \frac{p(s+1)}{s+1} (\log(1+2z))^{s+1} \]

\[ = \log(1+2z) \sum_{j=0}^{\infty} \frac{p(s+1)}{s+1} \sum_{l=0}^{\infty} S_1(l,m) \frac{z^j}{j!} \]

\[ = \log(1+2z) \sum_{j=0}^{\infty} \frac{p(m+1)!}{(m+1)^k} \sum_{l=0}^{\infty} S_1(l,m) \frac{z^j}{j!} \]

\[ = \sum_{r=0}^{\infty} \binom{j}{r} \frac{z^j}{j!} Ch_{r,p}(\xi)^{j-r} \]

\[ = \sum_{r=0}^{\infty} \binom{j}{r} Ch_{r,p}(\xi)^{j-r} \frac{z^j}{j!} \]  

\[ \sum_{j=0}^{\infty} \frac{p(s+1)!}{s+1} \sum_{l=0}^{\infty} S_1(l,m) \frac{z^j}{j!} \]

\[ = \sum_{j=0}^{\infty} \binom{j}{r} \frac{z^j}{j!} Ch_{r,p}(\xi)^{j-r} \]

\[ = \sum_{r=0}^{\infty} \binom{j}{r} Ch_{r,p}(\xi)^{j-r} \frac{z^j}{j!} \]  

\[ \sum_{j=0}^{\infty} \frac{p(s+1)!}{s+1} \sum_{l=0}^{\infty} S_1(l,m) \frac{z^j}{j!} \]

\[ = \sum_{j=0}^{\infty} \binom{j}{r} \frac{z^j}{j!} Ch_{r,p}(\xi)^{j-r} \]

\[ = \sum_{j=0}^{\infty} \binom{j}{r} Ch_{r,p}(\xi)^{j-r} \frac{z^j}{j!} \]  

\[ \sum_{j=0}^{\infty} \frac{p(s+1)!}{s+1} \sum_{l=0}^{\infty} S_1(l,m) \frac{z^j}{j!} \]

\[ = \sum_{j=0}^{\infty} \binom{j}{r} Ch_{r,p}(\xi)^{j-r} \frac{z^j}{j!} \]  

\[ \sum_{j=0}^{\infty} \frac{p(s+1)!}{s+1} \sum_{l=0}^{\infty} S_1(l,m) \frac{z^j}{j!} \]
Therefore, by (45) and (54), we obtain the result.

4. Conclusion

In the previous sections, we have touched on the problem of recognizing the algebraic structure underlying the poly-Changhee polynomials as given by definition (16). The analysis is aimed at accounting for the wealth of the properties exhibited by these polynomials within the context of the poly-Changhee numbers and polynomials which provide a unifying formalism where the theory of special functions can be framed inherently. Some analogies with the theory of poly-Changhee numbers and polynomials can be recognized and usefully exploited to infer further properties of these polynomials and links with other special functions. Let us stress that the scheme suggested by the following properties of Changhee numbers and polynomials $Ch_n(x)$ studied in detail by Kim and Kim [14] can be applied to connect other special functions of relevance in mathematical physics.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors contributed equally to the manuscript and typed, read, and approved final manuscript.

References


