Research Article

An Equivalent Condition and Some Properties of Strong J -Symmetric Ring

Shun Xu

School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China

Correspondence should be addressed to Shun Xu; xushun@mail.ustc.edu.cn

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Let $J(R)$ denote the Jacobson radical of a ring R . We say that ring R is strong J -symmetric if, for any $a, b, c \in R$, $abc \in J(R)$ implies $bac \in J(R)$. If ring R is strong J -symmetric, then it is proved that $R[x]/(x^n)$ is strong J -symmetric for any $n \geq 2$. If R and S are rings and ${}_R W_S$ is a (R, S) -bimodule, $E = T(R, S, W) = \begin{pmatrix} R & W \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & w \\ 0 & s \end{pmatrix} \mid r \in R, w \in W, s \in S \right\}$, then it is proved that R and S are J -symmetric if and only if E is J -symmetric. It is also proved that R and S are strong J -symmetric if and only if E is strong J -symmetric.

1. Introduction

R is a ring with an identity element. The symbols $J(R)$, $N(R)$, $Z(R)$, and $E(R)$, respectively, stand for the Jacobson radical, the set of all nilpotent elements, the set of all central elements, and the set of all idempotent elements of R . We also denote the set $\{a \in R: a^2 = 0\}$ by $N_2(R)$ and denote the ring of $n \times n$ upper triangular matrix over R by $T_n(R)$. R is left quasi-duo if every maximal left ideal of R is an ideal.

Various generalizations of symmetric rings have been carried out by many authors over the last several years. R is symmetric [1] if $abc = 0$ implies $acb = 0$, for all $a, b, c \in R$. The study of symmetric rings also can be found in [2]. R is weak symmetric [3] if, for any $a, b, c \in R$, $abc \in N(R)$ implies $acb \in N(R)$. R is central symmetric [4] if, for any $a, b, c \in R$, $abc = 0$ implies $bac \in Z(R)$. R is generalized weakly symmetric (GWS) [5] if, for any $a, b, c \in R$, $abc = 0$ implies $bac \in N(R)$. It follows that the class of GWS rings contains the class of weak symmetric rings. Moreover, it is known that central symmetric rings are GWS [5]. Let e be an idempotent of R ; R is called an e -symmetric ring [6] if $abc = 0$ implies $acbe = 0$, for all $a, b, c \in R$. And, Wei gives some properties of e -symmetric rings in [7].

As a generalization of symmetric rings, Kwak et al. [8] extended the concept of symmetric rings to ideal-symmetric rings. A ring R is called ideal symmetric if $ABC = 0$ implies $ACB = 0$ for all ideals A, B, C of R . Chakraborty and Das called a ring R right (respectively, left) nil-symmetric [9] if whenever $abc = 0$ (respectively, $cab = 0$), for every $a, b \in N(R)$ and every $c \in R$, then $acb = 0$, and the ring R is nil-symmetric if it is both right and left nil-symmetric. In [10], a ring R is called weak right (respectively, left) nil-symmetric if $abc = 0$, for all $a, b, c \in N(R)$ implies $acb = 0$ (respectively, $bac = 0$), and the ring R is called weak nil-symmetric if it is both weak left nil-symmetric and weak right nil-symmetric. A ring R will be said to be right (respectively, left) nil-ideal symmetric [11] if $IJK = 0$ implies $IKJ = 0$ (respectively, $JIK = 0$) for nil ideals I, J, K of R . Lambek called a right ideal I of a ring R symmetric if $rst \in I$ implies $rts \in I$, for all $rst \in I$. If the zero ideal is symmetric then R is usually called symmetric; while Anderson and Camillo [12] used the term ZC_3 for this concept, Han call an ideal N of a ring R ideal symmetric [13] if $ABC \subseteq N$ implies $ACB \subseteq N$, for any ideals A, B, C in R . Note that if the zero ideal of a ring R is ideal symmetric, then R is ideal symmetric.

In this paper, we introduce a kind of new symmetrical ring, which is strong J -symmetric. A ring R is strong J -symmetric if, for any $a, b, c \in R$, $abc = 0$ implies $bac \in J(R)$.

2. Main Results

Definition 1. A ring R is J -symmetric if, for any $a, b, c \in R$, $abc = 0$ implies $bac \in J(R)$.

Definition 2. A ring R is strong J -symmetric if, for any $a, b, c \in R$, $abc \in J(R)$ implies $bac \in J(R)$.

Let R be a ring. For any $a, b, c \in R$, if $abc = 0$, then we have $0 = abc \in J(R)$. If a ring R is strong J -symmetric, then we have $bac \in J(R)$. Therefore, R is J -symmetric. We obtain

$$\{\text{ring} \cdot R \mid R \cdot \text{is} \cdot \text{strong} \cdot J\text{-symmetric}\} \quad (1)$$

$$\cdot \sqsubseteq \cdot \{\text{ring} \cdot R \mid R \cdot \text{is} \cdot J\text{-symmetric}\}.$$

Therefore, strong J -symmetric ring is “strong.”

Definition 3. A ring R is strong central symmetric if, for any $a, b, c \in R$, $abc \in J(R)$ implies $bac \in Z(R)$.

Definition 4. A ring R is strong generalized weakly symmetric if, for any $a, b, c \in R$, $abc \in J(R)$ implies $bac \in N(R)$.

The following lemma [14] is well known.

Lemma 1. If R is a ring and $x \in R$, then $x \in J(R)$ if and only if, for any $y \in R$, there exists $u \in R$ such that $(1 - xy)u = 1$ and if and only if, for any $y \in R$, there exists $v \in R$ such that $v(1 - yx) = 1$.

The above lemma plays a critical role in proving the properties of strong J -symmetric ring because we use it repeatedly. Next, we introduce three basic properties of strong J -symmetric ring.

Proposition 1. For any ring R , the following statements are equivalent:

- (1) R is strong J -symmetric
- (2) For any $a, b, c \in R$, $abc \in J(R)$ implies $acb \in J(R)$.

Proof

(1) $\Rightarrow$ (2) Let $a, b, c \in R$ such that $abc \in J(R)$. If $acb \notin J(R)$, then there is a left maximal ideal $M \subseteq R$ such that $acb \notin M$. Therefore, we have $M + Racb = R$. There exists $x \in M$ and $y \in R$ such that $x + yacb = 1$. Since $abc \in J(R)$, we have $(ya)bc \in J(R)$. Since $J(R)$ is strong J -symmetric, we have $byac \in J(R)$. Since $(1 - x)^2 = yac(byac)b \in J(R) \subseteq M$, we have $x \in M$. Therefore, $1 \in M$, that is a contradiction.

(2) $\Rightarrow$ (1) Let $a, b, c \in R$ such that $abc \in J(R)$. If $bac \notin J(R)$, then there is left maximal ideal $M \subseteq R$ such that $bac \notin M$. Therefore, we have $M + Rbac = R$. There exists $x \in M$, $y \in R$ such that $x + ybac = 1$. Since $abc \in J(R)$, we have $ab(cy) \in J(R)$. We

have $acyb \in J(R)$ by (2). Since $(1 - x)^2 = yb(acyb)ac \in J(R) \subseteq M$, we have $x \in M$. Therefore, $1 \in M$, that is a contradiction. $\square$

Corollary 1. Let R be a strong J -symmetric ring. For any $a, b, c \in R$, if $abc \in J(R)$, then $acb, bac, bca, cab, cba \in J(R)$.

Proposition 2. Let R be a strong J -symmetric ring. For any left maximal ideal M of R and any $a, b, c \in R$, if $abc \in J(R)$, then $a \in M$ or $bc \in M$.

Proof. For any left maximal ideal M of R , if $a \notin M$, then $M + Ra = R$. There exist $x \in M$ and $y \in R$ such that $x + ya = 1$. We get $(x - 1)bc \in J(R) \subseteq M$. Since $abc \in J(R)$, we have $bc \in M$.

Let R be a strong J -symmetric ring. For any $e \in E(R)$ and any left maximal ideal M of R , since $0 = e^2 - e = e(1 - e) = e(1 - e) \cdot 1$, we get $e \in M$ or $1 - e \in M$ from Proposition 2.

For any $a \in N_2(R)$, we have $0 = a^2 = a \cdot a \cdot 1$. We get $a \in J(R)$ from Proposition 2. Therefore, $N_2(R) \subseteq J(R)$. $\square$

Proposition 3. Let R be a left quasi-duo ring. For any $a, b, c \in R$, if $abc \in J(R)$, then $a \in M$ or $b \in M$ or $c \in M$ for any maximal left ideal M of R .

Proof. Let M be a maximal left ideal of R . If $a \notin M$, then $M + Ra = R$, which implies that $x + ya = 1$, for some $x \in M$, $y \in R$. Therefore, $xbc = bc - yabc$. When R is left quasi-duo and $x \in M$, we get $bc - yabc \in M$. Since $yabc \in J(R) \subseteq M$, we have $bc \in M$. If $b \notin M$, then $M + Rb = R$ yields $u + vb = 1$, for some $u \in M$, $v \in R$. Since $1 - vb \in M$, we have $(1 - vb)c \in M$. Therefore, we obtain $c \in M$ by using $bc \in M$.

We want to know which types of rings are strong J -symmetric rings. From the following three propositions, we can see that left quasi-duo ring, strong central symmetric ring, and strong generalized weakly symmetric ring are strong J -symmetric rings. $\square$

Proposition 4. A left quasi-duo ring is strong J -symmetric.

Proof. Let R be a left quasi-duo ring and $abc \in J(R)$ and M be a maximal left ideal of R . It follows from Proposition 3 that $a \in M$ or $b \in M$ or $c \in M$. As R is left quasi-duo, we get $bac \in M$. Therefore, $bac \in J(R)$, which proves that R is strong J -symmetric. $\square$

Proposition 5. Strong central symmetric rings are strong J -symmetric.

Proof. Let R be a strong central symmetric ring which is not strong J -symmetric. There exists $a, b, c \in R$ such that $abc \in J(R)$, but $bac \notin J(R)$ so that there exists a maximal left ideal $M \subseteq R$ such that $1 = x + ybac$, for some $x \in M$ and $y \in R$. Now, for any $r_1, r_2 \in R$, we have $(ab)(cr_1)1 \in J(R)$ and $(r_2a)bc \in J(R)$. Hence, cr_1ab and $br_2ac \in Z(R)$. Therefore, we have

$$\begin{aligned}
(1-x)^4 &= (ybac)^4 = ybacyba(cybac)ybac = ybacyba(baccy)ybac \\
&= ybacybabacc(yybacc) = ybacybabacc(bacyy) \\
&= y(b(acybab)ac)cbacyy = ycba(b(acybab)ac)cyy \\
&= ycbaba(c(yb)ab)accyy = ycbab(c(yb)ab)aaccyy \\
&= ycb(abc)ybabaaccyy = 0.
\end{aligned} \tag{2}$$

This leads to $1 \in M$, which is a contradiction. Hence, R is strong J -symmetric. $\square$

Proposition 6. Strong generalized weakly symmetric rings are strong J -symmetric

Proof. Let R be a strong generalized weakly symmetric ring and $abc \in J(R)$. If R is not strong J -symmetric, then there exists a maximal left ideal M of R such that $1 = x + ybac$, for some $x \in M$ and $y \in R$. As R is strong generalized weakly symmetric and $abc \in J(R)$ so that $bacy \in N(R)$. We have $(bacy)^k = 0$ for some positive integer k . Therefore, we have

$$(1-x)^{k+1} = (ybac)^{k+1} = y(bacy)^k bac = 0 \in M. \tag{3}$$

This together with $x \in M$ implies that $1 \in M$, which is a contradiction. Hence, R is strong J -symmetric.

We want to know if a ring is a strong J -symmetric ring. From the next three propositions, we can know whether a ring is strong J -symmetric by the structure of quotient ring. $\square$

Proposition 7. If $R/J(R)$ is symmetric, then R is strong J -symmetric.

Proof. Let $a, b, c \in R$ such that $abc \in J(R)$, so $\overline{abc} = \overline{0}$. As $R/J(R)$ is symmetric, so $\overline{bac} = \overline{0}$ which means that $bac \in J(R)$. Therefore, R is strong J -symmetric. $\square$

Lemma 2. Let R be a ring and I be an ideal of R ; then, $(I + J(R))/I \subseteq J(R/I)$.

Proof. $J(R/I) = (\cap_{I \subseteq m} m)/I$, where m is maximal left ideal of R , $I \subseteq (\cap_{I \subseteq m} m)$, and $J(R) = \cap_m m$, m is maximal left ideal of R , so $J(R) \subseteq (\cap_{I \subseteq m} m)$. Therefore, we have $(I + J(R))/I \subseteq J(R/I)$. $\square$

Proposition 8. If I is a nil ideal of a ring R such that R/I is strong J -symmetric, then R is strong J -symmetric.

Proof. Let $a, b, c \in R$ such that $abc \in J(R)$. From Lemma 2, we have $\overline{abc} \in J(R/I)$ in R/I . Since R/I is strong J -symmetric, we have $\overline{bac} \in J(R/I)$. For any $r \in R$, there exists $t \in R$ such that $1 - t(1 - rbac) \in I \subseteq J(R)$ from Lemma 1. Let $x = 1 - t(1 - rbac)$; since I is nil, there exists positive integer n such that $x^n = 0$. Therefore, we have $(1 + x + \cdots + x^{n-1})(1 - x) = 1$. It follows that $(1 - rbac)$ is left invertible, and hence, $bac \in J(R)$. Therefore, R is strong J -symmetric $\square$

Proposition 9. If I is an ideal of a J -symmetric ring, S and R are subrings of S containing I and $J(R) = J(S)$; then, R/I is strong J -symmetric, which implies R is strong J -symmetric.

Proof. Let $a, b, c \in R$ such that $abc \in J(R)$ in $R \subseteq S$. Since S is strong J -symmetric and $abc \in J(S) = J(R)$, we have $bac \in J(S)$. For any $r \in R \subseteq S$, there exists $s \in S$ such that $s(1 - rbac) = 1$. From Lemma 2, we get $\overline{abc} \in J(R/I)$. Since R/I is J -symmetric, we have $\overline{bac} \in J(R/I)$. Therefore, there exists $t \in R$ such that $(1 - (1 - rbac)t) \in I$. This yields $s - s(1 - rbac)t \in I$, so $s - t \in I \subseteq R$. This implies $s \in R$. Hence, $(1 - rbac)$ is left invertible in R so that $bac \in J(R)$. Therefore, R is strong J -symmetric.

Next, we consider the direct product and J -subdirect product of strong J -symmetric rings. From the next two propositions, we can see that the direct product and J -subdirect product of arbitrary family of strong J -symmetric rings are both strong J -symmetric. $\square$

Proposition 10. Direct product of arbitrary family of strong J -symmetric rings is strong J -symmetric.

Proof. For any arbitrary family of rings $\{R_i : i \in I\}$, we know that $J(\prod_{i \in I} R_i) = \prod_{i \in I} J(R_i)$. Hence, the result easily follows. $\square$

Definition 5. Let $R_i (i \in I)$ be a family of rings and $R = \prod_{i \in I} R_i$ be the direct product of rings R_i with the projections $\rho_i : R \rightarrow R_i$. A ring B is called J -subdirect product of R_i if there is an imbedding $m : B \rightarrow R$ such that the homomorphisms $\rho_i m (i \in I)$ are surjective and $J(R) = J(B)$.

Proposition 11. J -subdirect product of arbitrary family of strong J -symmetric rings is strong J -symmetric.

Proof. Let R be a J -subdirect product of a family of strong J -symmetric rings $\{R_i\}_{i \in I}$. For each $i \in I$, we have epimorphism $\phi_i : R \rightarrow R_i$. Hence, $\prod_{i \in I} R/\text{Ker}(\phi_i) \cong \prod_{i \in I} R_i$ is strong J -symmetric from Proposition 10. Since the map

$$\begin{aligned}
\Phi : R &\rightarrow \prod_{i \in I} R/\text{Ker}(\phi_i), \\
\Phi(r) &= (r + \text{Ker}(\phi_i))_{i \in I},
\end{aligned} \tag{4}$$

is a monomorphism, we have $R \cong \text{Im}(\Phi) \cong \text{Im}(\Phi)/\text{Ker}(\Phi)$ ($\text{Ker}(\Phi) \cong \bigcap_{i \in I} \text{Ker}(\phi_i)$) which is also a strong J -symmetric. Now, $\Phi(\text{Ker}(\Phi)) \subseteq \text{Im}(\Phi) \subseteq \prod_{i \in I} R/\text{Ker}(\phi_i)$. Therefore, according to Proposition 9, $\text{Im}(\Phi) \cong R$ is a strong J -symmetric. $\square$

Theorem 1. If R is a strong J -symmetric ring, then

- (1) $T_n(R)$ is a strong J -symmetric, for any $n \geq 2$.
- (2) $R[x]/(x^n)$ is a strong J -symmetric, for any $n \geq 2$.
- (3)

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} : a, a_{ij} \in R, 1 \leq i < j \leq n \right\}, \quad (5)$$

is a strong J -symmetric, for any $n \geq 2$.

Proof. Let

$$I = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix} : a_{ij} \in R, 1 \leq i < j \leq n \right\}. \quad (6)$$

Hence, I is a nil ideal of $T_n(R)$ as well as $S_n(R)$.

- (1) $T_n(R)/I$ is isomorphic to direct product of n -copies of R . Therefore, according to Proposition 10 and Proposition 8, $T_n(R)$ is a strong J -symmetric.
- (2) Let

$$M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}_{n \times n}, \quad R[M] = \{a_0 I_n + a_1 M + \cdots + a_{n-1} M^{n-1} \mid a_i \in R, i = 0, \dots, n-1\}. \quad (7)$$

Consider the map

$$\begin{aligned} \varphi: R[x]/(x^n) &\longrightarrow R[M], \\ \bar{x} &\longrightarrow M. \end{aligned} \quad (8)$$

We get $R[x]/(x^n) \cong R[M]$. Consider

$$V_n(R) = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} : a_i \in R, i = 0, 1, 2, \dots, n-1 \right\}. \quad (9)$$

Therefore, we have $R[M] = V_n(R)$ and $R[x]/(x^n) \cong V_n(R)$. Consider

$$K = \left\{ \begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} : a_i \in R, i = 1, 2, \dots, n-1 \right\}. \quad (10)$$

Hence, K is a nil ideal of $V_n(R)$, and $V_n(R)/K \cong R$ is a strong J -symmetric. Therefore, $R[x]/(x^n) \cong V_n(R)$ is a strong J -symmetric by Proposition 8.

- (3) Since $S_n(R)/I \cong R$, it follows that $S_n(R)$ is also strong J -symmetric by Proposition 8.

Let R and S be rings and ${}_R W_S$ be a (R, S) -bimodule. Let

$$\begin{aligned} E &= T(R, S, W) = \begin{pmatrix} R & W \\ 0 & S \end{pmatrix} \\ &= \left\{ \begin{pmatrix} r & w \\ 0 & s \end{pmatrix} \mid r \in R, w \in W, s \in S \right\}. \end{aligned} \quad (11)$$

For any $\begin{pmatrix} r_1 & w_1 \\ 0 & s_1 \end{pmatrix}, \begin{pmatrix} r_2 & w_2 \\ 0 & s_2 \end{pmatrix} \in E$, we define

$$\begin{aligned} \begin{pmatrix} r_1 & w_1 \\ 0 & s_1 \end{pmatrix} + \begin{pmatrix} r_2 & w_2 \\ 0 & s_2 \end{pmatrix} &= \begin{pmatrix} r_1 + r_2 & w_1 + w_2 \\ 0 & s_1 + s_2 \end{pmatrix}, \\ \begin{pmatrix} r_1 & w_1 \\ 0 & s_1 \end{pmatrix} \begin{pmatrix} r_2 & w_2 \\ 0 & s_2 \end{pmatrix} &= \begin{pmatrix} r_1 r_2 & r_1 w_2 + w_1 s_2 \\ 0 & s_1 s_2 \end{pmatrix}. \end{aligned} \quad (12)$$

Therefore, E is a ring.

Whether R and S are strong J -symmetric is closely related to whether E is a strong J -symmetric. Exactly, we know that R and S are J -symmetric if and only if E is a J -symmetric from Theorem 2. We also know that R and S are strong J -symmetric if and only if E is a strong J -symmetric from Theorem 3. $\square$

Lemma 3. For any $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in E$, we have $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in J(E)$ if and only if $a \in J(R)$ and $b \in J(S)$.

Proof. $\Rightarrow$ For any $r \in R$ and $s \in S$, we have $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \in E$. Since $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in J(E)$ and $\begin{pmatrix} u & w \\ 0 & v \end{pmatrix} \in E$, we have

$$\begin{pmatrix} u & w \\ 0 & v \end{pmatrix} \left(\begin{pmatrix} 1_R & 0 \\ 0 & 1_S \end{pmatrix} - \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \right) = \begin{pmatrix} 1_R & 0 \\ 0 & 1_S \end{pmatrix}. \quad (13)$$

So, $u(1 - ra) = 1_R$ and $v(1 - sb) = 1_S$. Therefore, we have $a \in J(R)$ and $b \in J(S)$ by Lemma 1.

$\Leftarrow$ For any $\begin{pmatrix} r & w \\ 0 & s \end{pmatrix} \in E$, since $a \in J(R)$ and $b \in J(S)$, there exist $u \in R$ and $v \in S$ such that

$$\begin{aligned} u(1_R - ra) &= 1_R, \\ v(1_S - sb) &= 1_S. \end{aligned} \quad (14)$$

$$\begin{aligned} &\text{There exists } \begin{pmatrix} u & u(rc + wb)v \\ 0 & v \end{pmatrix} \in E \text{ such that} \\ &\begin{pmatrix} u & u(rc + wb)v \\ 0 & v \end{pmatrix} \left(\begin{pmatrix} 1_R & 0 \\ 0 & 1_S \end{pmatrix} - \begin{pmatrix} r & w \\ 0 & s \end{pmatrix} \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \right) \\ &= \begin{pmatrix} 1_R & 0 \\ 0 & 1_S \end{pmatrix}. \end{aligned} \quad (15)$$

Therefore, we have $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in J(E)$ by Lemma 1. $\square$

Theorem 2. *R and S are J-symmetric if and only if E is J-symmetric*

Proof. $\Rightarrow$ For any

$$\begin{pmatrix} r_1 & w_1 \\ 0 & s_1 \end{pmatrix}, \begin{pmatrix} r_2 & w_2 \\ 0 & s_2 \end{pmatrix}, \begin{pmatrix} r_3 & w_3 \\ 0 & s_3 \end{pmatrix} \in E, \quad (16)$$

if

$$\begin{aligned} &\begin{pmatrix} r_1 & w_1 \\ 0 & s_1 \end{pmatrix} \begin{pmatrix} r_2 & w_2 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} r_3 & w_3 \\ 0 & s_3 \end{pmatrix} \\ &= \begin{pmatrix} r_1 r_2 r_3 & r_1(r_2 w_3 + w_1 s_3) + w_1 s_2 s_3 \\ 0 & s_1 s_2 s_3 \end{pmatrix} = 0, \end{aligned} \quad (17)$$

then $r_1 r_2 r_3 = 0$, $s_1 s_2 s_3 = 0$. Since R and S are J -symmetric, we have $r_2 r_1 r_3 \in J(R)$ and $s_1 s_1 s_3 \in J(S)$. Therefore, we have

$$\begin{aligned} &\begin{pmatrix} r_2 & w_2 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} r_1 & w_1 \\ 0 & s_1 \end{pmatrix} \begin{pmatrix} r_3 & w_3 \\ 0 & s_3 \end{pmatrix} \\ &= \begin{pmatrix} r_2 r_1 r_3 & r_2(r_1 w_3 + w_1 s_3) + w_2 s_1 s_3 \\ 0 & s_2 s_1 s_3 \end{pmatrix} \in J(E), \end{aligned} \quad (18)$$

by Lemma 3. We get E is J -symmetric by Definition 1.

$\Leftarrow$ For any $r_1, r_2, r_3 \in R$ and $s_1, s_2, s_3 \in S$, if $r_1 r_2 r_3 = 0$, $s_1 s_2 s_3 = 0$, then

$$\begin{pmatrix} r_1 & 0 \\ 0 & s_1 \end{pmatrix} \begin{pmatrix} r_2 & 0 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} r_3 & 0 \\ 0 & s_3 \end{pmatrix} = \begin{pmatrix} r_1 r_2 r_3 & 0 \\ 0 & s_1 s_2 s_3 \end{pmatrix} = 0. \quad (19)$$

Since E is J -symmetric, we have

$$\begin{pmatrix} r_2 & 0 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} r_1 & 0 \\ 0 & s_1 \end{pmatrix} \begin{pmatrix} r_3 & 0 \\ 0 & s_3 \end{pmatrix} = \begin{pmatrix} r_2 r_1 r_3 & 0 \\ 0 & s_2 s_1 s_3 \end{pmatrix} \in J(E). \quad (20)$$

Therefore, we have $r_2 r_1 r_3 \in J(R)$ and $s_2 s_1 s_3 \in J(S)$ by Lemma 3. We get R and S are J -symmetric by Definition 1. $\square$

Theorem 3. *R and S are strong J-symmetric if and only if E is a strong J-symmetric.*

Proof. $\Rightarrow$ For any

$$\begin{pmatrix} r_1 & w_1 \\ 0 & s_1 \end{pmatrix}, \begin{pmatrix} r_2 & w_2 \\ 0 & s_2 \end{pmatrix}, \begin{pmatrix} r_3 & w_3 \\ 0 & s_3 \end{pmatrix} \in E, \quad (21)$$

if

$$\begin{aligned} &\begin{pmatrix} r_1 & w_1 \\ 0 & s_1 \end{pmatrix} \begin{pmatrix} r_2 & w_2 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} r_3 & w_3 \\ 0 & s_3 \end{pmatrix} \\ &= \begin{pmatrix} r_1 r_2 r_3 & r_1(r_2 w_3 + w_1 s_3) + w_1 s_2 s_3 \\ 0 & s_1 s_2 s_3 \end{pmatrix} \in J(E), \end{aligned} \quad (22)$$

then $r_1 r_2 r_3 \in J(R)$ and $s_1 s_2 s_3 \in J(S)$ by Lemma 3. Since R and S are strong J -symmetric, we have $r_2 r_1 r_3 \in J(R)$ and $s_2 s_1 s_3 \in J(S)$. Therefore, we have

$$\begin{aligned} &\begin{pmatrix} r_2 & w_2 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} r_1 & w_1 \\ 0 & s_1 \end{pmatrix} \begin{pmatrix} r_3 & w_3 \\ 0 & s_3 \end{pmatrix} \\ &= \begin{pmatrix} r_2 r_1 r_3 & r_2(r_1 w_3 + w_1 s_3) + w_2 s_1 s_3 \\ 0 & s_2 s_1 s_3 \end{pmatrix} \in J(E), \end{aligned} \quad (23)$$

by Lemma 3. E is a strong J -symmetric by Definition 2.

$\Leftarrow$ For any $r_1, r_2, r_3 \in R$ and $s_1, s_2, s_3 \in S$, if $r_1 r_2 r_3 \in J(R)$ and $s_1 s_2 s_3 \in J(S)$, then

$$\begin{pmatrix} r_1 & 0 \\ 0 & s_1 \end{pmatrix} \begin{pmatrix} r_2 & 0 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} r_3 & 0 \\ 0 & s_3 \end{pmatrix} = \begin{pmatrix} r_1 r_2 r_3 & 0 \\ 0 & s_1 s_2 s_3 \end{pmatrix} \in J(E), \quad (24)$$

by Lemma 3. Since E is a strong J -symmetric, we have

$$\begin{pmatrix} r_2 & 0 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} r_1 & 0 \\ 0 & s_1 \end{pmatrix} \begin{pmatrix} r_3 & 0 \\ 0 & s_3 \end{pmatrix} = \begin{pmatrix} r_2 r_1 r_3 & 0 \\ 0 & s_2 s_1 s_3 \end{pmatrix} \in J(E). \quad (25)$$

Therefore, we get $r_2 r_1 r_3 \in J(R)$ and $s_2 s_1 s_3 \in J(S)$ by Lemma 3. R and S are strong J -symmetric by Definition 2. $\square$

3. Conclusion

In this paper, we give some basic properties of strong J -symmetrical ring. We determine that three types of rings are strong J -symmetric rings, which are left quasi-duo ring, strong central symmetric ring, and strong generalized weakly symmetric ring. Next, we consider the direct product and J -subdirect product of strong J -symmetric rings. Besides, we give certain quotient ring and subring of polynomial ring and matrix ring with R coefficients, which are strong J -symmetric ring. Finally, we give a necessary and sufficient condition for a ring to be a J -symmetric ring and for a ring to be a strong J -symmetric ring. Of course, there are many other properties of strong J -symmetric rings that are worthy of our discussion.

Data Availability

All data generated or analysed during this study are included in this published article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

- [1] J. Lambek, "On the representation of modules by sheaves of factor modules," *Canadian Mathematical Bulletin*, vol. 14, Article ID 359368, 1971.
- [2] G. Marks, "Reversible and symmetric rings," *Journal of Pure and Applied Algebra*, vol. 174, Article ID 311318, 2002.
- [3] L. Ouyang and H. Chen, "On weak symmetric rings," *Communications in Algebra*, vol. 38, no. 2, pp. 697–713, 2010.
- [4] Kafkas and B. Ungor, "Generalized symmetric rings," *Algebra Discrete Math*, vol. 12, no. 2, pp. 72–84, 2011.
- [5] J. Wei, "Generalized weakly symmetric rings," *Journal of Pure and Applied Algebra*, vol. 218, no. 9, pp. 1594–1603, 2014.
- [6] F. Meng and J. Wei, "e-Symmetric rings," *Communications in Contemporary Mathematics*, vol. 20, no. 3, Article ID 1750039, 2018.
- [7] M. Fanyun and W. E. I. Junchao, "Some properties of e-symmetric rings," *Turkish Journal of Mathematics*, vol. 42, no. 5, 2018.
- [8] V. Camillo, T. Kwak, and Y. Lee, "Ideal-symmetric and semiprime rings," *Communications in Algebra*, vol. 41, no. 12, Article ID 45044519, 2013.
- [9] U. S. Chakraborty and K. Das, "On nil-symmetric rings," *Jurnal Matematika*, vol. 2014, p. 7, Article ID 483784, 2014.
- [10] H. K. Kim, T. K. Kwak, S. I. Lee et al., "A generalization of symmetric ring property," *Bulletin of the Korean Mathematical Society*, vol. 53, no. 5, pp. 1309–1325, 2016.
- [11] T. K. Kwak, Y. Lee, Z. Piao, and Y. J. Seo, "Symmetric ring property on nil ideals," *Journal of Algebra and Its Applications*, vol. 17, no. 01, Article ID 1850013, 2018.
- [12] D. D. Anderson and V. Camillo, "Semigroups and rings whose zero products commute," *Communications in Algebra*, vol. 27, Article ID 28472852, 1999.
- [13] J. Han, Y. Lee, and S. Park, "Rings with ideal-symmetric ideals," *Bulletin of the Korean Mathematical Society*, vol. 54, no. 6, pp. 1913–1925, 2017.
- [14] T. Y. Lam, "A first course in noncommutative rings," *Graduate Texts in Mathematics*, vol. 100, no. 7, 1991.