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Research Article

Composition Formulae for the k-Fractional Calculus Operator with the S-Function

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In this study, the S-function is applied to Saigo's k-fractional order integral and derivative operators involving the k-hypergeometric function in the kernel; outcomes are described in terms of the k-Wright function, which is used to represent image formulas of integral transformations such as the beta transform. Several special cases, such as the fractional calculus operator and the S-function, are also listed.

1. Introduction and Preliminaries

Fractional calculus was first introduced in 1695, but only in the last two decades have researchers been able to use it efficiently due to the availability of computing tools. Significant uses of fractional calculus have been discovered by scholars in engineering and science. In literature, many applications of fractional calculus are available in astrophysics, biosignal processing, fluid dynamics, nonlinear control theory, and stochastic dynamical system. Furthermore, research studies in the field of applied science [1, 2], and on the application of fractional calculus in real-world problems [3, 4], have recently been published. A number of researchers [5–15] have also investigated the structure, implementations, and various directions of extensions of the fractional integration and differentiation in detail. A detailed description of such fractional calculus operators, as well as their characterization and application, can be found in research monographs [16, 17].

Recently, a series of research publications with respect to generalized classical fractional calculus operators was published. Mubeen and Habibullah [18] broughtout k-fractional order integral of the Riemann–Liouville version and its applications. Dorrego [19] introduced an alternative definition for the k-Riemann–Liouville fractional derivative.

Gupta and Parihar [20] introduced the left and right sides of Saigo k-fractional integration and differentiation operators connected with the k-Gauss hypergeometric function which are as follows:

$$\left(I_{0+,k}^{\vartheta,\varsigma,\gamma} f\right)(x) = \frac{x^{(-\vartheta-\varsigma)/k}}{k\Gamma_k(\vartheta)} \int_0^x (x-t)^{(\vartheta/k)-1} {}_2F_{1,k}\left((\vartheta+\varsigma,k),(-\gamma,k);(\vartheta,k);\left(1-\frac{t}{x}\right)\right) f(t)\mathrm{d}t; \quad (\Re(\vartheta)>0, k>0), \tag{1}$$

$$\left(I_{-,k}^{\vartheta,\varsigma,\gamma} f\right)(x) = \frac{1}{k\Gamma_k(\vartheta)} \int_x^\infty (t-x)^{(\vartheta/k)-1} t^{(-\vartheta-\varsigma)/k} {}_2F_{1,k}\left((\vartheta+\varsigma,k),(-\gamma,k);(\vartheta,k);\left(1-\frac{x}{t}\right)\right) f(t)\mathrm{d}t; \quad (\Re(\vartheta)>0, k>0). \tag{2}$$

Mubeen and Habibullah [18] defined ${}_2F_{1,k}((\vartheta,k),(\varsigma,k);(\gamma,k);x)$, i.e., the k-Gauss hypergeometric function for $x\in\mathbb{C},|x|<1,\Re(\gamma)>\Re(\varsigma)>0$:

$$ {}_2F_{1,k}((\vartheta,k),(\varsigma,k);(\gamma,k);x)=\sum_{n=0}^{\infty}\frac{(\vartheta)_{n,k}(\varsigma)_{n,k}x^n}{(\gamma)_{n,k}n!}. \quad (3)$$

Equations (1) and (2) are the left and right sides of fractional differential operators involving k-Gauss hypergeometric function, respectively:

$$\begin{aligned}\left(D_{0+,k}^{\vartheta,\varsigma,\gamma}f\right)(x)&=\left(\frac{d}{dx}\right)^n\left(I_{0+,k}^{-\vartheta+n,-\varsigma-n,\vartheta+\gamma-n}f\right)(x);\Re(\vartheta)>0,k>0;n=[\Re(\vartheta)+1]\\&=\left(\frac{d}{dx}\right)^n\frac{x^{\vartheta+\varsigma/k}}{k\Gamma_k(-\vartheta+n)}\int_0^x(x-t)^{-\vartheta/k+n-1}\times{}_2F_{1,k}\left((-\vartheta-\varsigma,k),(-\gamma-\vartheta+n,k);(-\vartheta+n,k);\left(1-\frac{t}{x}\right)\right)f(t)dt,\end{aligned} \quad (4)$$

$$\begin{aligned}\left(D_{-,k}^{\vartheta,\varsigma,\gamma}f\right)(x)&=\left(-\frac{d}{dx}\right)^n\left(I_{-,k}^{-\vartheta+n,-\varsigma-n,\vartheta+\gamma}f\right)(x);\Re(\vartheta)>0,k>0;n=[\Re(\vartheta)+1]\\&=\left(-\frac{d}{dx}\right)^n\frac{1}{k\Gamma_k(-\vartheta+n)}\int_x^\infty(t-x)^{-\vartheta+n/k-1}t^{\vartheta+\varsigma/k}\times{}_2F_{1,k}\left((-\vartheta-\varsigma,k),(-\gamma-\vartheta,k);(-\vartheta+n,k);\left(1-\frac{x}{t}\right)\right)f(t)dt,\end{aligned} \quad (5)$$

where $x>0,\vartheta\in\mathbb{C},\Re(\vartheta)>0,k>0$ and $[\Re(\vartheta)]$ is the integer part of $\Re(\vartheta)$.

Remark 1. When we set $k=1$ in equations, operators (1), (2), (4), and (5) reduce into Saigo's fractional integral and derivative operators, as stated in [9], respectively.

We consider the following basic results for our study.

Lemma 1 (see p. 497, equation 4.2, in [20]). *Let* $\vartheta,\varsigma,\gamma,\varepsilon\in\mathbb{C},\Re(\varepsilon)>\max[0,\Re(\varsigma-\gamma)]$; *then,*

$$\left(I_{0+,k}^{\vartheta,\varsigma,\gamma}t^{(\varepsilon/k)-1}\right)(x)=\sum_{n=0}^{\infty}k^n\frac{\Gamma_k(\varepsilon)\Gamma_k(\varepsilon-\varsigma+\gamma)}{\Gamma_k(\varepsilon-\varsigma)\Gamma_k(\varepsilon+\vartheta+\gamma)}x^{(\varepsilon-\varsigma/k)-1}. \quad (6)$$

Lemma 2 (see p. 497, equation 4.3, in [20]). *Let* $\vartheta,\varsigma,\gamma,\varepsilon\in\mathbb{C},\Re(\vartheta)>0,k\in\Re^+(0,\infty)$ *and* $\Re(\varepsilon)>\max[\Re(-\varsigma),\Re(-\gamma)]$; *then,*

$$\left(I_{-,k}^{\vartheta,\varsigma,\gamma}t^{-(\varepsilon/k)}\right)(x)=\sum_{n=0}^{\infty}k^n\frac{\Gamma_k(\varepsilon+\varsigma)\Gamma_k(\varepsilon+\gamma)}{\Gamma_k(\varepsilon)\Gamma_k(\varepsilon+\vartheta+\varsigma+\gamma)}x^{-\varepsilon-\varsigma/k}. \quad (7)$$

Lemma 3 (see p. 500, equation 6.2, in [20]). *Let* $\vartheta,\varsigma,\gamma,\varepsilon\in\mathbb{C},n=[\Re(\vartheta)]+1,k\in\Re^+(0,\infty)$ *such that* $\Re(\varepsilon)>\max[0,\Re(-\vartheta-\varsigma-\gamma)]$; *then,*

$$\left(D_{0+,k}^{\vartheta,\varsigma,\gamma}t^{(\varepsilon/k)-1}\right)(x)=\sum_{n=0}^{\infty}k^n\frac{\Gamma_k(\varepsilon)\Gamma_k(\varepsilon+\varsigma+\gamma+\vartheta)}{\Gamma_k(\varepsilon+\gamma)\Gamma_k(\varepsilon+\varsigma+n-nk)}x^{(\varepsilon+\varsigma+n/k)-n-1}. \quad (8)$$

Lemma 4 (see p. 500, equation 6.3, in [20]). *Let* $\vartheta,\varsigma,\gamma,\varepsilon\in\mathbb{C}$ *and* $n=[\Re(\vartheta)]+1,k\in\Re^+$, $\Re(\varepsilon)>\max[\Re(-\vartheta-\gamma),\Re(\varsigma-nk+n)]$; *then,*

$$\left(D_{-,k}^{\vartheta,\varsigma,\gamma}t^{-(\varepsilon/k)}\right)(x)=\sum_{n=0}^{\infty}k^n\frac{\Gamma_k(\varepsilon-\varsigma-n+nk)\Gamma_k(\varepsilon+\vartheta+\gamma)}{\Gamma_k(\varepsilon)\Gamma_k(\varepsilon-\varsigma+\gamma)}x^{(-\varepsilon+\varsigma+n/k)-n}. \quad (9)$$

Recent time, the S-function is defined and studied by Saxena and Daiya [21], which is generalization of k-Mittag-Leffler function, K-function, M-series, Mittag-Leffler function (see [22–25]), as well as its relationships with other special functions. These special functions have recently found essential applications in solving problems in physics, biology, engineering, and applied sciences.

The S-function is defined for $\vartheta', \delta', \gamma' \varepsilon \in \mathbb{C}$, $\mathfrak{R}(\vartheta') > 0$, $k \in \mathfrak{R}$, $\mathfrak{R}(\vartheta') > k\mathfrak{R}(\varepsilon)$, $l_i (i = 1, 2, \ldots, p)$, $m_j (j = 1, 2, \ldots, q)$, and $p < q + 1$ as

$$S_{(p,q)}^{\vartheta',\delta',\gamma',\varepsilon,k}\left[l_1, l_2, \ldots, l_p; m_1, m_2, \ldots, m_q; x\right] = \sum_{n=0}^{\infty} \frac{(l_1)_n \ldots (l_p)_n (\gamma')_{n\varepsilon,k}}{(m_1)_n \ldots (m_q)_n \Gamma_k(n\vartheta' + \delta')} \frac{x^n}{n!}. \tag{10}$$

Here, Díaz and Pariguan [26] introduced the k-Pochhammer symbol and k-gamma function as follows:

$$(\gamma')_{n,k} = \begin{cases} \dfrac{\Gamma_k(\gamma' + nk)}{\Gamma_k(\gamma')}, & k \in \mathfrak{R}, \gamma' \in \mathbb{C}/\{0\}, \\ \gamma'(\gamma' + k) \ldots (\gamma' + (n-1)k), & (n \in \mathbb{N}, \gamma' \in \mathbb{C}), \end{cases} \tag{11}$$

as well as the relationship with the classic Euler's gamma function:

$$\Gamma_k(\gamma') = k^{(\gamma'/k)-1} \Gamma\left(\frac{\gamma'}{k}\right), \tag{12}$$

where $\gamma' \in \mathbb{C}, k \in \mathfrak{R}$, and $n \in \mathbb{N}$. Refer to Romero and Cerutti's papers [27] for more information on the k-Pochhammer symbol, k-special functions, and fractional Fourier transforms.

The following are some significant special cases of the S-function:

(i) For $p = q = 0$, the generalized k-Mittag-Leffler function [28]

$$E_{k,\vartheta',\delta'}^{\gamma',\varepsilon}(x) = S_{(0,0)}^{\vartheta',\delta',\gamma',\varepsilon,k}[-;-;x] = \sum_{n=0}^{\infty} \frac{(\gamma')_{n\varepsilon,k}}{\Gamma_k(n\vartheta' + \delta')} \frac{x^n}{n!}, \mathfrak{R}\left(\left(\frac{\vartheta'}{k}\right) - \varepsilon\right) > p - q. \tag{13}$$

(ii) Again, for $k = \varepsilon = 1$, the S-function is the generalized K-function [29]:

$$\begin{aligned} K_{(p,q)}^{\vartheta',\delta',\gamma'}\left[l_1, l_2, \ldots, l_p; m_1, m_2, \ldots, m_q; x\right] &= S_{(p,q)}^{\vartheta',\delta',\gamma',1,1}\left[l_1, l_2, \ldots, l_p; m_1, m_2, \ldots, m_q; x\right] \\ &= \sum_{n=0}^{\infty} \frac{(l_1)_n \ldots (l_p)_n (\gamma')_n}{(m_1)_n \ldots (m_q)_n \Gamma(n\vartheta' + \delta')} \frac{x^n}{n!}, \quad \mathfrak{R}(\vartheta') > p - q. \end{aligned} \tag{14}$$

(iii) For $\varepsilon = k = \gamma' = 1$, the S-function reduced to generalized M-series [30]:

$$\begin{aligned} M_{(p,q)}^{\vartheta',\delta'}\left[l_1, l_2, \ldots, l_p; m_1, m_2, \ldots, m_q; x\right] &= S_{(p,q)}^{\vartheta',\delta',1,1,1}\left[l_1, l_2, \ldots, l_p; m_1, m_2, \ldots, m_q; x\right] \\ &= \sum_{n=0}^{\infty} \frac{(l_1)_n \ldots (l_p)_n}{(m_1)_n \ldots (m_q)_n \Gamma(n\vartheta' + \delta')} \frac{x^n}{n!}, \quad \mathfrak{R}(\vartheta') > p - q - 1. \end{aligned} \tag{15}$$

For our purpose, we recall the definition of generalized k-Wright function ${}_p\Psi_q^k(x)$, defined by Gehlot and Prajapati [31], for $k \in \mathbb{R}^+; x, a_i, b_j \in \mathbb{C}$, $\vartheta_i, \varsigma_j \in \mathfrak{R}\, (\vartheta_i, \varsigma_j \neq 0; i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$ and $(a_i + \vartheta_i n), (b_j + \varsigma_j n) \in \mathbb{C} \backslash kz^-$, as

$${}_p\Psi_q^k(x) = {}_p\Psi_q^k\left[\begin{matrix} (a_i, \vartheta_i)_{1,p} \\ \\ (b_j, \varsigma_j)_{1,q} \end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma_k(a_i + \vartheta_i n)}{\prod_{j=1}^{q} \Gamma_k(b_j + \varsigma_j n)} \frac{(x)^n}{n!}, \tag{16}$$

which satisfies the condition

$$\sum_{j=1}^{q}\frac{\varsigma_j}{k}-\sum_{i=1}^{p}\frac{\vartheta_i}{k}>-1. \tag{17}$$

2. Saigo k-Fractional Integration in Terms of k-Wright Function

In this section, the results are displayed based on the k-fractional integrals associated with the S-function.

Theorem 1. *Let $\vartheta,\varsigma,\gamma,\vartheta',\varsigma',\gamma',\varepsilon,\varepsilon\in\mathbb{C}; k\in\Re^{+}, c\in\Re$ and $\nu>0$, such that $\Re(\vartheta)>0$, $\Re(\varepsilon)>\max[0,\Re(\varsigma-\gamma)]$, $\Re(\varepsilon+\gamma-\varsigma)>0$, $a_i\,(i=1,2,\ldots,p), b_j\,(j=1,2,\ldots,q)$, $\Re(\vartheta')>k\Re(\varepsilon)$; $p<q+1$. If condition (17) is satisfied and $I_{0+,k}^{\vartheta,\varsigma,\gamma}$ is the left-sided integral operator of the generalized k-fractional integration associated with S-function, then (18) holds true:*

$$\left(I_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon/k)-1}S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{\nu/k}\right)\right)\right)(x)$$
$$=\frac{x^{(\varepsilon-\varsigma/k)-1}}{\Gamma_k(\gamma')}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}k^{\sum_{j=1}^{q}b_j-\sum_{i=1}^{p}a_i}\times{}_{p+3}\Psi_{q+3}^{k}\left[\begin{matrix}(a_1k,k)\ldots(a_pk,k),(\gamma',\varepsilon k),(\varepsilon,\nu),(\varepsilon+\gamma-\varsigma,\nu),\\(b_1k,k)\ldots(b_qk,k),(\varsigma',\vartheta'),(\varepsilon-\varsigma,\nu),(\varepsilon+\vartheta+\gamma,\nu),\end{matrix}\;kcx^{\nu/k}\right]. \tag{18}$$

Proof. We indicate the R.H.S. of equation (18) by I_1; invoking equation (10), we have

$$\begin{aligned}I_1&=I_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon/k)-1}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n(\gamma')_{n\varepsilon,k}}{(b_1)_n\ldots(b_q)_n\Gamma_k(\varsigma'+\vartheta'n)}\frac{\left(ct^{\nu/k}\right)^n}{n!}\right)(x)\\&=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n(\gamma')_{n\varepsilon,k}}{(b_1)_n\ldots(b_q)_n\Gamma_k(\varsigma'+\vartheta'n)}\frac{c^n}{n!}I_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon+\nu n/k)-1}\right)(x).\end{aligned} \tag{19}$$

Now, applying equation (6) and (11), we obtain

$$I_1=\frac{x^{(\varepsilon-\varsigma/k)-1}}{\Gamma_k(\gamma')}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n\Gamma_k(\gamma'+n\varepsilon k)\Gamma_k(\varepsilon+\nu n)\Gamma_k(\varepsilon+\gamma-\varsigma+\nu n)}{(b_1)_n\ldots(b_q)_n\Gamma_k(\varsigma'+\vartheta'n)\Gamma_k(\varepsilon-\varsigma+\nu n)\Gamma_k(\varepsilon+\vartheta+\gamma+\nu n)}\frac{\left(kcx^{\nu/k}\right)^n}{n!}. \tag{20}$$

Using (12) and some important simplifications on the above equation, we obtain

$$\begin{aligned}I_1&=\frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}\frac{x^{(\varepsilon-\varsigma/k)-1}}{\Gamma_k(\gamma')}k^{(b_1+\cdots+b_q)-(a_1+\cdots+a_p)}\\&\times\sum_{n=0}^{\infty}\frac{\Gamma_k(a_1k+nk)\ldots\Gamma_k(a_pk+nk)\Gamma_k(\gamma'+n\varepsilon k)\Gamma_k(\varepsilon+\nu n)\Gamma_k(\varepsilon+\gamma-\varsigma+\nu n)}{\Gamma_k(b_1k+nk)\ldots\Gamma_k(b_qk+nk)\Gamma_k(\varsigma'+\vartheta'n)\Gamma_k(\varepsilon-\varsigma+\nu n)\Gamma_k(\varepsilon+\vartheta+\gamma+\nu n)}\frac{\left(kcx^{\nu/k}\right)^n}{n!}.\end{aligned} \tag{21}$$

Interpreting the definition of Wright hypergeometric function (16) on the above equation, we arrive at the desired result (18). □

Theorem 2. *Let $\vartheta,\varsigma,\gamma,\vartheta',\varsigma',\gamma',\varepsilon,\varepsilon\in\mathbb{C}; k\in\Re^{+}, c\in\Re$, and $\nu>0$, such that $\Re(\vartheta)>0$, $\Re(\vartheta')>0$, and $\Re(\varepsilon+\vartheta)>\max[-\Re(\varsigma),-\Re(\gamma)]$, with $\Re(\varsigma)\neq\Re(\gamma)$, $a_i\,(i=1,2,$*

$\ldots, p), b_j\ (j = 1, 2, \ldots, q)$, $\mathfrak{R}(\vartheta') > k\mathfrak{R}(\varepsilon)$, and $p < q + 1$. If condition (17) is satisfied and $I_{-,k}^{\vartheta,\varsigma,\gamma}$ is the right-sided integral operator of the generalized k-fractional integration associated with S-function, then (22) holds true:

$$\left(I_{-,k}^{\vartheta,\varsigma,\gamma}\left(t^{-\vartheta-\varepsilon/k} S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{-\nu/k}\right)\right)\right)(x) = k^{\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i} \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)} \frac{x^{-\vartheta-\varepsilon-\varsigma/k}}{\Gamma_k(\gamma')} \times {}_{p+3}\Psi_{q+3}^{k}\left[\begin{matrix}(a_1k,k)\ldots(a_pk,k),(\gamma',\varepsilon k),(\vartheta+\varepsilon+\varsigma,\nu),(\vartheta+\varepsilon+\gamma,\nu),\\ (b_1k,k)\ldots(b_qk,k),(\varsigma',\vartheta'),(\vartheta+\varepsilon,\nu),(2\vartheta+\varepsilon+\varsigma+\gamma,\nu),\end{matrix} kcx^{-\nu/k}\right]. \tag{22}$$

Proof. The proof is parallel to that of Theorem 1. Therefore, we omit the details. □

The results given in (18) and (22), being very general, can yield a large number of special cases by assigning some suitable values to the involved parameters. Now, we demonstrate some corollaries as follows.

Corollary 1. *If we put $p = q = 0$, then (18) leads to the subsequent result of S-function:*

$$\left(I_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{\varepsilon/k-1} E_{k,\vartheta',\delta'}^{\gamma',\varepsilon}\left(ct^{\nu/k}\right)\right)\right)(x) = \frac{x^{(\varepsilon-\varsigma/k)-1}}{\Gamma_k(\gamma')} \times {}_3\Psi_3^{k}\left[\begin{matrix}(\gamma',\varepsilon k),(\varepsilon,\nu),(\varepsilon+\gamma-\varsigma,\nu),\\ (\varsigma',\vartheta'),(\varepsilon-\varsigma,\nu),(\varepsilon+\vartheta+\gamma,\nu),\end{matrix} kcx^{\nu/k}\right]. \tag{23}$$

Corollary 2. *If $\varepsilon = k = 1$, in (18), we obtain the subsequent result in term of S-function as*

$$\left(I_{0+}^{\vartheta,\varsigma,\gamma}\left(t^{\varepsilon-1} K_{(p,q)}^{\vartheta',\varsigma',\gamma'}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{\nu}\right)\right)\right)(x) = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)} \frac{x^{\varepsilon-\varsigma-1}}{\Gamma(\gamma')} \times {}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),(\gamma',1),(\varepsilon,\nu),(\varepsilon+\gamma-\varsigma,\nu),\\ (b_1,1)\ldots(b_q,1),(\varsigma',\vartheta'),(\varepsilon-\varsigma,\nu),(\varepsilon+\vartheta+\gamma,\nu),\end{matrix} cx^{\nu}\right]. \tag{24}$$

Corollary 3. *If we set $\varepsilon = 1, \gamma' = 1$, and $k = 1$, in equation (18), we obtain the following formula:*

$$\left(I_{0+}^{\vartheta,\varsigma,\gamma}\left(t^{\varepsilon-1} M_{(p,q)}^{\vartheta',\varsigma'}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{\nu}\right)\right)\right)(x) = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)} x^{\varepsilon-\varsigma-1} \times {}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),(\varepsilon,\nu),(\varepsilon+\gamma-\varsigma,\nu),(1,1),\\ (b_1,1)\ldots(b_q,1),(\varsigma',\vartheta'),(\varepsilon-\varsigma,\nu),(\varepsilon+\vartheta+\gamma,\nu),\end{matrix} cx^{\nu}\right]. \tag{25}$$

Corollary 4. *Letting $p = q = 0$ in equation (22), then*

$$\left(I_{-,k}^{\vartheta,\varsigma,\gamma}\left(t^{-\vartheta-\varepsilon/k}E_{k,\vartheta',\delta'}^{\gamma',\varepsilon}\left(ct^{-\nu/k}\right)\right)\right)(x) = \frac{x^{-\vartheta-\varepsilon-\varsigma/k}}{\Gamma_k(\gamma')} \times {}_3\Psi_3^k\left[\begin{matrix}(\gamma',\varepsilon k),(\vartheta+\varepsilon+\varsigma,\nu),(\vartheta+\varepsilon+\gamma,\nu),\\(\varsigma',\vartheta'),(\vartheta+\varepsilon,\nu),(2\vartheta+\varepsilon+\varsigma+\gamma,\nu),\end{matrix} kcx^{-\nu/k}\right]. \tag{26}$$

Corollary 5. *Setting $\varepsilon = 1$, $k = 1$, then equation (22) becomes*

$$\left(I_{-}^{\vartheta,\varsigma,\gamma}\left(t^{-\vartheta-\varepsilon}K_{(p,q)}^{\vartheta',\varsigma',\gamma'}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{-\nu}\right)\right)\right)(x) = \frac{\prod_{j=1}^q\Gamma(b_j)}{\prod_{i=1}^p\Gamma(a_i)}\frac{x^{-\vartheta-\varepsilon-\varsigma}}{\Gamma_k(\gamma')} \times {}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),(\gamma',1),(\vartheta+\varepsilon+\varsigma,\nu),(\vartheta+\varepsilon+\gamma,\nu),\\(b_1,1)\ldots(b_q,1),(\varsigma',\vartheta'),(\vartheta+\varepsilon,\nu),(2\vartheta+\varepsilon+\varsigma+\gamma,\nu),\end{matrix} cx^{-\nu}\right]. \tag{27}$$

Corollary 6. *If we put $\varepsilon = 1, \gamma' = 1$, and $k = 1$ in equation (22), then equation becomes*

$$\left(I_{-}^{\vartheta,\varsigma,\gamma}\left(t^{-\vartheta-\varepsilon}M_{(p,q)}^{\vartheta',\varsigma'}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{-\nu}\right)\right)\right)(x) = \frac{\prod_{j=1}^q\Gamma(b_j)}{\prod_{i=1}^p\Gamma(a_i)}x^{-\vartheta-\varepsilon-\delta} \times {}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),(\vartheta+\varepsilon+\varsigma,\nu),(\vartheta+\varepsilon+\gamma,\nu),(1,1),\\(b_1,1)\ldots(b_q,1),(\varsigma',\vartheta'),(\vartheta+\varepsilon,\nu),(2\vartheta+\varepsilon+\varsigma+\gamma,\nu),\end{matrix} cx^{-\nu}\right]. \tag{28}$$

3. Saigo k-Fractional Differentiation in Terms of k-Wright Function

In this section, the results are displayed based on the k-fractional derivatives associated with the S-function.

Theorem 3. *Let $\vartheta,\varsigma,\gamma,\vartheta',\varsigma',\gamma',\varepsilon,\varepsilon \in \mathbb{C}$; $k \in \mathfrak{R}^+, c \in \mathfrak{R}$, and $\nu > 0$, such that $\mathfrak{R}(\vartheta) > 0$, $\mathfrak{R}(\vartheta') > 0$, $\mathfrak{R}(\varepsilon) > \max[0, \mathfrak{R}(-\vartheta-\varsigma-\gamma)]$, $\mathfrak{R}(\varepsilon+\gamma+\varsigma) > 0$, $a_i\,(i = 1,2,\ldots,p)$, $b_j\,(j = 1,2,\ldots,q)$, $\mathfrak{R}(\vartheta') > k\mathfrak{R}(\varepsilon)$, and $p < q+1$. If condition (17) is satisfied and $D_{0+,k}^{\vartheta,\varsigma,\gamma}$ is the left-sided differential operator of the generalized k-fractional integration associated with S-function, then (29) holds true:*

$$\left(D_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{\varepsilon/k-1}S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{\nu/k}\right)\right)\right)(x) = \frac{x^{(\varepsilon+\varsigma/k)-1}}{\Gamma_k(\gamma')}\frac{\prod_{j=1}^q\Gamma(b_j)}{\prod_{i=1}^p\Gamma(a_i)}k^{\sum_{j=1}^q b_j-\sum_{i=1}^p a_i} \times {}_{p+3}\Psi_{q+3}^k\left[\begin{matrix}(a_1k,k)\ldots(a_pk,k),(\gamma',\varepsilon k),(\varepsilon,\nu),(\varepsilon+\varsigma+\gamma+\vartheta,\nu),\\(b_1k,k)\ldots(b_qk,k),(\varsigma',\vartheta'),(\varepsilon+\gamma,\nu),(\varepsilon+\delta,1-k+\nu),\end{matrix} cx^{(\nu+1/k)-1}\right]. \tag{29}$$

Proof. For the sake of convenience, let the left-hand side of (29) be denoted by I_2. Using definition (10), we arrive at

$$I_2=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n(\gamma')_{n\varepsilon,k}}{(b_1)_n\ldots(b_q)_n\Gamma_k(\varsigma'+\vartheta'\nu)}\frac{c^n}{n!}D_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon+\nu n/k)-1}\right)(x). \tag{30}$$

Now, applying equation (8) and (11), we obtain

$$I_2=\frac{x^{(\varepsilon+\varsigma/k)-1}}{\Gamma_k(\gamma')}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n\Gamma_k(\gamma'+\varepsilon nk)}{(b_1)_n\ldots(b_q)_n\Gamma_k(\varsigma'+\vartheta' n)}\times\frac{\Gamma_k(\varepsilon+\nu n)\Gamma_k(\varepsilon+\varsigma+\gamma+\vartheta+\nu n)}{\Gamma_k(\varepsilon+\gamma+\nu n)\Gamma_k(\varepsilon+\varsigma+n-nk+\nu n)n!}\left(cx^{(\nu+1/k)-1}\right)^n. \tag{31}$$

Using (12) and simplifications on the above equation, we obtain

$$\begin{aligned}I_2={}&k^{(b_1+\cdots+b_q)-(a_1+\cdots+a_p)}\frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}\frac{x^{(\varepsilon+\varsigma/k)-1}}{\Gamma_k(\gamma')}\sum_{n=0}^{\infty}\frac{\Gamma_k(\gamma'+n\varepsilon k)}{\Gamma_k(\varsigma'+\vartheta' n)}\\&\times\frac{\Gamma_k(a_1k+nk)\ldots\Gamma_k(a_pk+nk)\Gamma_k(\varepsilon+\nu n)\Gamma_k(\varepsilon+\varsigma+\gamma+\vartheta+\nu n)}{\Gamma_k(b_1k+nk)\ldots\Gamma_k(b_qk+nk)\Gamma_k(\varepsilon+\gamma+\nu n)\Gamma_k(\varepsilon+\varsigma+n-nk+\nu n)n!}\left(cx^{(\nu+1/k)-1}\right)^n.\end{aligned} \tag{32}$$

In accordance with (16), we obtain the required result (29). This completed the proof of Theorem 3. □

Theorem 4. *Let* $\vartheta,\varsigma,\gamma,\vartheta',\varsigma',\gamma',\varepsilon,\varepsilon\in\mathbb{C}; k\in\mathfrak{R}^+, c\in\mathfrak{R}$, *and* $\nu>0$, *such that* $\mathfrak{R}(\vartheta)>0$, $\mathfrak{R}(\vartheta')>0$, $\mathfrak{R}(\varepsilon)>\max[\mathfrak{R}(\vartheta+\varsigma)+n-\mathfrak{R}(\gamma)]$, *and* $\mathfrak{R}(\vartheta+\varsigma-\gamma)+n\neq 0$, *where* $n=[\mathfrak{R}(\vartheta)+1]$, $a_i\,(i=1,2,\ldots,p), b_j\,(j=1,2,\ldots,q), \mathfrak{R}(\vartheta')>k\mathfrak{R}(\varepsilon)$, *and* $p<q+1$. *If condition (17) is satisfied and* $D_{-,k}^{\vartheta,\varsigma,\gamma}$ *is the right-sided differential operator of the generalized k-fractional integration associated with S-function, then (33) holds true:*

$$\begin{aligned}&\left(D-,k^{\vartheta,\varsigma,\gamma}\left(t^{\vartheta-\varepsilon/k}S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}(a_1,\ldots,a_p;b_1,\ldots,bq;ct-\nu/k)\right)\right)(x)=\frac{x^{\vartheta-\varepsilon+\varsigma/k}}{\Gamma_k(\gamma')}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}k^{\sum_{j=1}^{q}b_j-\sum_{i=1}^{p}a_i}\\&\times{}_{p+3}\Psi_{q+3}^{k}\left[\begin{matrix}(a_1k,k)\ldots(a_pk,k),(\gamma',\varepsilon k),(\varepsilon-\vartheta-\delta,\nu+k-1),(\varepsilon+\gamma,\nu),\\(b_1k,k)\ldots(b_qk,k),(\varsigma',\vartheta'),(\varepsilon-\vartheta,\nu),(\varepsilon-\vartheta-\varsigma+\gamma,\nu),\end{matrix}\;cx^{(-\nu+1/k)-1}\right].\end{aligned} \tag{33}$$

Proof. The proof is parallel to that of Theorem 3. Therefore, we omit the details. □

The results given in (29) and (33) are reduced as special cases by assigning some suitable values to the involved parameters. Now, we demonstrate some corollaries as follows.

Corollary 7. *If* $p=q=0$, *then (29) holds the following formula:*

$$\begin{aligned}&\left(D_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon/k)-1}E_{k,\vartheta',\delta'}^{\gamma',\varepsilon}\left(ct^{-\nu/k}\right)\right)\right)(x)=\frac{x^{(\varepsilon+\varsigma/k)-1}}{\Gamma_k(\gamma')}\\&\times{}_3\Psi_3^{k}\left[\begin{matrix}(\gamma',\varepsilon k),(\varepsilon,\nu),(\varepsilon+\varsigma+\gamma+\vartheta,\nu),\\(\varsigma',\vartheta'),(\varepsilon+\gamma,\nu),(\varepsilon+\delta,1-k+\nu),\end{matrix}\;cx^{(\nu+1/k)-1}\right].\end{aligned} \tag{34}$$

Corollary 8. *If we put* $\varepsilon=1$ *and* $k=1$, *then (29) gives the result in term of S-function as follows:*

$$\left(D_{0+}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon/k)-1}K_{(p,q)}^{\vartheta',\varsigma',\gamma'}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{\nu}\right)\right)\right)(x)=\frac{x^{\varepsilon+\varsigma-1}}{\Gamma(\gamma')}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}\times{}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),(\gamma',1),(\varepsilon,\nu),(\varepsilon+\varsigma+\gamma+\vartheta,\nu),\\(b_1,1)\ldots(b_q,1),(\varsigma',\vartheta'),(\varepsilon+\gamma,\nu),(\varepsilon+\delta,\nu),\end{matrix}\;cx^{\nu}\right]. \tag{35}$$

Corollary 9. *If we put* $\varepsilon=1,\gamma'=1$, *and* $k=1$, *in equation (29), then*

$$\left(D_{0+}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon/k)-1}M_{(p,q)}^{\vartheta',\varsigma'}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{\nu}\right)\right)\right)(x)=\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}x^{\varepsilon+\varsigma-1}\times{}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),(\varepsilon,\nu),(\varepsilon+\varsigma+\gamma+\vartheta,\nu),(1,1),\\(b_1,1)\ldots(b_q,1),(\varsigma',\vartheta'),(\varepsilon+\gamma,\nu),(\varepsilon+\delta,\nu),\end{matrix}\;cx^{\nu}\right]. \tag{36}$$

Corollary 10. *If we set* $p=q=0$, *then (33) provides the result as*

$$\left(D_{-,k}^{\vartheta,\varsigma,\gamma}\left(t^{\vartheta-\varepsilon/k}E_{k,\vartheta',\delta'}^{\gamma',\varepsilon}\left(ct^{-\nu/k}\right)\right)\right)(x)=\frac{x^{(\vartheta-\varepsilon+\varsigma/k)-1}}{\Gamma_k(\gamma')}={}_{3}\Psi_{3}^{k}\left[\begin{matrix}(\gamma',k),(\varepsilon-\vartheta-\delta,\nu+k-1),(\varepsilon+\gamma,\nu),\\(\varsigma',\vartheta'),(\varepsilon-\vartheta,\nu),(\varepsilon-\vartheta-\varsigma+\gamma,\nu),\end{matrix}\;cx^{(-\nu+1/k)-1}\right]. \tag{37}$$

Corollary 11. *By letting* $\varepsilon=1$ *and* $k=1$, *in equation (33), then*

$$\left(D_{-}^{\vartheta,\varsigma,\gamma}\left(t^{\vartheta-\varepsilon}K_{(p,q)}^{\vartheta',\varsigma',\gamma'}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{-\nu}\right)\right)\right)(x)=\frac{x^{\vartheta-\varepsilon+\delta}}{\Gamma(\gamma')}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}\times{}_{p+3}\Psi_{q+3}\left[\begin{matrix}(a_1,1)\ldots(a_p,1),(\gamma',1),(\varepsilon-\vartheta-\varsigma,\nu),(\varepsilon+\gamma,\nu),\\(b_1,1)\ldots(b_q,1),(\varsigma',\vartheta'),(\varepsilon-\vartheta,\nu),(\varepsilon-\vartheta-\varsigma+\gamma,\nu),\end{matrix}\;cx^{-\nu}\right]. \tag{38}$$

Corollary 12. *When* $\varepsilon=1,\gamma'=1$, *and* $k=1$, *in equation (33), then equation becomes*

$$\left(D_{-}^{\vartheta,\varsigma,\gamma}\left(t^{\vartheta-\varepsilon}M_{(p,q)}^{\vartheta',\varsigma'}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;ct^{-\nu}\right)\right)\right)(x)=\frac{\prod_{j=1}^{q}\Gamma\left(b_j\right)}{\prod_{i=1}^{p}\Gamma\left(a_i\right)}x^{\vartheta-\varepsilon+\delta}$$
$$\times{}_{p+3}\Psi_{q+3}\left[\begin{array}{c}\left(a_1,1\right)\ldots\left(a_p,1\right),(\varepsilon-\vartheta-\delta,\nu),(\varepsilon+\gamma,\nu),(1,1),\\ \left(b_1,1\right)\ldots\left(b_q,1\right),\left(\varsigma',\vartheta'\right),(\varepsilon-\vartheta,\nu),(\varepsilon-\vartheta-\varsigma+\gamma,\nu),\end{array} cx^{-\nu}\right]. \tag{39}$$

4. Image Formulas Associated with Integral Transforms

In this section, we establish some theorems involving the results obtained in previous sections pertaining with the integral transform. Here, we defined k-beta function as follows.

The k-beta function [32] is defined as

$$B_k(g,h)=\frac{1}{k}\int_0^1 z^{(g/k)-1}(1-z)^{(h/k)-1}\mathrm{d}z,\quad g>0,h>0. \tag{40}$$

They have the following important identities:

$$B_k(g,h)=\frac{1}{k}B\left(\frac{g}{k},\frac{h}{k}\right)=\frac{\Gamma_k(g)\Gamma_k(h)}{\Gamma_k(g+h)}. \tag{41}$$

Now, we define k-beta function in the form

$$B_k(f(z);g,h)=\frac{1}{k}\int_0^1 z^{(g/k)-1}(1-z)^{(h/k)-1}f(z)\mathrm{d}z,\quad g>0,h>0. \tag{42}$$

Theorem 5. *Let $\vartheta,\varsigma,\gamma,\vartheta',\varsigma',\varepsilon,\gamma',\varepsilon\in\mathbb{C};k\in\mathfrak{R}^+,c\in\mathfrak{R}$, and $\nu>0$, such that $\mathfrak{R}(\vartheta)>0$, $\mathfrak{R}(\varepsilon)>\max[0,\mathfrak{R}(\varsigma-\gamma)]$, and $\mathfrak{R}(\varepsilon+\gamma-\varsigma)>0$; then, the leading fractional order integral holds true:*

$$B_k\left(\left(I_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon/k)-1}S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;c(zt)^{\nu/k}\right)\right)\right)(x);g,h\right)=\frac{x^{(\varepsilon-\varsigma/k)-1}\Gamma_k(h)}{\Gamma_k(\gamma')}\frac{\prod_{j=1}^{q}\Gamma\left(b_j\right)}{\prod_{i=1}^{p}\Gamma\left(a_i\right)}k^{\sum_{j=1}^{q}b_j-\sum_{i=1}^{p}a_i}$$
$$\times{}_{p+4}\Psi_{q+4}^{k}\left[\begin{array}{c}\left(a_1k,k\right)\ldots\left(a_pk,k\right),\left(\gamma',\varepsilon k\right),(\varepsilon,\nu),(\varepsilon+\gamma-\varsigma,\nu),(g,\nu),\\ \left(b_1k,k\right)\ldots\left(b_qk,k\right),\left(\varsigma',\vartheta'\right),(\varepsilon-\varsigma,\nu),(\varepsilon+\vartheta+\gamma,\nu),(g+h,\nu),\end{array} kcx^{\nu/k}\right]. \tag{43}$$

Proof. Let I_3 be the left-hand side of (43), and using (42), we have

$$I_3=\frac{1}{k}\int_0^1 z^{(g/k)-1}(1-z)^{(h/k)-1}\left(I_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon/k)-1}S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;c(zt)^{\nu/k}\right)\right)\right)(x)\mathrm{d}z, \tag{44}$$

which, using (10) and changing the order of integration and summation, is valid under the conditions of Theorem 1 and yields

$$I_3=\sum_{n=0}^{\infty}\frac{\left(a_1\right)_n\ldots\left(a_p\right)_n\left(\gamma'\right)_{n\varepsilon,k}}{\left(b_1\right)_n\ldots\left(b_q\right)_n\Gamma_k\left(\varsigma'+\vartheta'n\right)}\frac{c^n}{n!}I_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon+\nu n/k)-1}\right)(x)\times\frac{1}{k}\int_0^1 z^{(g+\nu n/k)-1}(1-z)^{(h/k)-1}\mathrm{d}z. \tag{45}$$

From Lemma 1 and substituting (41) in (45), we obtain

$$I_1 = k^{\left(b_1+\cdots+b_q\right)-\left(a_1+\cdots+a_p\right)} \frac{x^{(\varepsilon-\varsigma/k)-1}}{\Gamma_k(\gamma')} \frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)} \sum_{n=0}^{\infty} \frac{\Gamma_k(a_1k+nk)\ldots\Gamma_k(a_pk+nk)}{(b_1k+nk)\ldots\Gamma_k(b_qk+nk)} \times \frac{\Gamma_k(\gamma'+n\varepsilon k)\Gamma_k(\varepsilon+\nu n)\Gamma_k(\varepsilon+\gamma-\varsigma+\nu n)\Gamma_k(g+\nu n)\Gamma_k(h)}{\Gamma_k\Gamma_k(\varsigma'+\vartheta' n)\Gamma_k(\varepsilon-\varsigma+\nu n)\Gamma_k(\varepsilon+\vartheta+\gamma+\nu n)\Gamma_k(g+h+\nu n)} \frac{\left(kcx^{\nu/k}\right)^n}{n!}. \tag{46}$$

Using the definition of (16) in the right-hand side of (46), we arrive at result (43). □

Theorem 6. *Let $\vartheta, \varsigma, \gamma, \vartheta', \varsigma', \gamma', \varepsilon, \varepsilon \in \mathbb{C}; k \in \mathfrak{R}^+, c \in \mathfrak{R}$, and $\nu > 0$, such that $\mathfrak{R}(\vartheta) > 0$, $\mathfrak{R}(\vartheta') > 0$, and $\mathfrak{R}(\varepsilon+\vartheta) > \max[-\mathfrak{R}(\varsigma), -\mathfrak{R}(\gamma)]$, with $\mathfrak{R}(\varsigma) \neq \mathfrak{R}(\gamma)$; then, the following fractional integral holds true:*

$$B_k\left(\left(I_{-,k}^{\vartheta,\varsigma,\gamma}\left(t^{-\vartheta-\varepsilon/k} S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;c(zt)^{-\nu/k}\right)\right)\right)(x);g,h\right) = k^{\sum_{j=1}^q b_j-\sum_{i=1}^p a_i} \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{i=1}^p \Gamma(a_i)} \frac{\Gamma_k(h) x^{-\vartheta-\varepsilon-\varsigma/k}}{\Gamma_k(\gamma')} \times {}_{p+4}\Psi_{q+4}^k\left[\begin{matrix}(a_1k,k)\ldots(a_pk,k),(g,-\nu),(\gamma',\varepsilon k),(\vartheta+\varepsilon+\varsigma,\nu),(\vartheta+\varepsilon+\gamma,\nu), \\ (b_1k,k)\ldots(b_qk,k),(g+h,-\nu),(\varsigma',\vartheta'),(\vartheta+\varepsilon,\nu),(2\vartheta+\varepsilon+\varsigma+\gamma,\nu),\end{matrix} kcx^{-\nu/k}\right]. \tag{47}$$

Proof. The proof is similar of Theorem 5. Therefore, we omit the details. □

Theorem 7. *Let $\vartheta, \varsigma, \gamma, \vartheta', \varsigma', \gamma', \varepsilon, \varepsilon \in \mathbb{C}; k \in \mathfrak{R}^+, c \in \mathfrak{R}$, and $\nu > 0$, such that $\mathfrak{R}(\vartheta) > 0$, $\mathfrak{R}(\vartheta') > 0$, $\mathfrak{R}(\varepsilon) > \max[0, \mathfrak{R}(-\vartheta-\varsigma-\gamma)]$, and $\mathfrak{R}(\varepsilon+\gamma+\varsigma) > 0$; then, the following fractional derivative holds true:*

$$B_k\left(\left(D_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon/k)-1} S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;c(zt)^{\nu/k}\right)\right)\right)(x);g,h\right) = \frac{\Gamma_k(h)}{\Gamma_k(\gamma')} \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{i=1}^p \Gamma(a_i)} x^{(\varepsilon+\varsigma/k)-1} k^{\sum_{j=1}^q b_j-\sum_{i=1}^p a_i} \times {}_{p+4}\Psi_{q+4}^k\left[\begin{matrix}(a_1k,k)\ldots(a_pk,k),(\gamma',\varepsilon k),(\varepsilon,\nu),(\varepsilon+\varsigma+\gamma+\vartheta,\nu),(g,\nu), \\ (b_1k,k)\ldots(b_qk,k),(\varsigma',\vartheta'),(\varepsilon+\gamma,\nu),(g+h,\nu),(\varepsilon+\delta,1-k+\nu),\end{matrix} cx^{(\nu+1/k)-1}\right]. \tag{48}$$

Proof. Let I_4 be the left-hand side of (48), and using the definition of Beta transform, we have

$$I_4 = \frac{1}{k}\int_0^1 z^{(g/k)-1}(1-z)^{(h/k)-1} D_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon/k)-1} S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;c(zt)^{\nu/k}\right)\right)(x)\mathrm{d}z, \tag{49}$$

which, using (10) and changing the order of integration and summation, is reasonable under the conditions of Theorem 3 and yields

$$I_4 = \sum_{n=0}^{\infty} \frac{(a_1)_n\ldots(a_p)_n(\gamma')_{n\varepsilon,k}}{(b_1)_n\ldots(b_q)_n\Gamma_k(\varsigma'+\vartheta'\nu)} \frac{c^n}{n!} D_{0+,k}^{\vartheta,\varsigma,\gamma}\left(t^{(\varepsilon+\nu n/k)-1}\right)(x) \times \frac{1}{k}\int_0^1 z^{(g+\nu n/k)-1}(1-z)^{(h/k)-1}\mathrm{d}z. \tag{50}$$

From Lemma 3 and substituting equation (41) in (50), we obtain

$$I_4 = k^{(b_1-a_1)+\cdots+(b_q-a_p)} \frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)} \frac{x^{(\varepsilon+\varsigma/k)-1}}{\Gamma_k(\gamma')} \sum_{n=0}^{\infty} \frac{\Gamma_k(\gamma'+n\varepsilon k)\Gamma_k(a_1k+nk)\ldots}{\Gamma_k(\varsigma'+\vartheta' n)\Gamma_k(b_1k+nk)\ldots} \times \frac{\Gamma_k(a_pk+nk)\Gamma_k(\varepsilon+\nu n)\Gamma_k(\varepsilon+\varsigma+\gamma+\vartheta+\nu n)\Gamma_k(g+\nu n)\Gamma_k(h)}{\Gamma_k(b_qk+nk)\Gamma_k(\varepsilon+\gamma+\nu n)\Gamma_k(\varepsilon+\varsigma+n-nk+\nu n)\Gamma_k(g+h+\nu n)n!}\left(cx^{\nu+1/k-1}\right)^n. \tag{51}$$

Using the definition of (16) in the above equation, we obtain the required result (48). This completed the proof of Theorem 7. □

Theorem 8. *Let $\vartheta, \varsigma, \gamma, \vartheta', \varsigma', \gamma', \varepsilon, \varepsilon \in \mathbb{C}$; $k \in \mathfrak{R}^+, c \in \mathfrak{R}$, and $\nu > 0$, such that $\mathfrak{R}(\vartheta) > 0$, $\mathfrak{R}(\vartheta') > 0$, $\mathfrak{R}(\varepsilon) > \max[\mathfrak{R}(\vartheta+\varsigma) + n - \mathfrak{R}(\gamma)]$, and $\mathfrak{R}(\vartheta+\varsigma-\gamma) + n \neq 0$, where $n = [\mathfrak{R}(\vartheta)+1]$; then, the following fractional derivative holds true:*

$$B_k\left(D_{-,k}^{\vartheta,\varsigma,\gamma}\left(t^{\vartheta-\varepsilon/k} S_{(p,q)}^{\vartheta',\varsigma',\gamma',\varepsilon,k}\left(a_1,\ldots,a_p;b_1,\ldots,b_q;c(zt)^{-\nu/k}\right)\right)\right)(x) = \frac{\Gamma_k(h)}{\Gamma_k(\gamma')} \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)} x^{(\vartheta-\varepsilon+\varsigma/k)-1} k^{\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i} \times {}_{p+4}\Psi_{q+4}^{k}\left[\begin{matrix}(a_1k,k)\ldots(a_pk,k),(\gamma',\varepsilon k),(\varepsilon-\vartheta-\delta,\nu+k-1),(g,-\nu)(\varepsilon+\gamma,\nu),\\(b_1k,k)\ldots(b_qk,k),(\varsigma',\vartheta'),(g+h,-\nu),(\varepsilon-\vartheta,\nu),(\varepsilon-\vartheta-\varsigma+\gamma,\nu),\end{matrix} cx^{(-\nu+1/k)-1}\right]. \tag{52}$$

Proof. The proof is identical to that of Theorem 7. As a result, we exclude the specifics. □

5. Conclusion

The strength of generalized k-fractional calculus operators, also known as general operators by many scholars, is that they generalize classical Riemann-Liouville (R-L) operators and Saigo's fractional calculus operators. For $k \longrightarrow 1$, operators (1) to (5) reduce to Saigo's [9] fractional integral and differentiation operators. If we set $\delta = -\vartheta$, operators (1) to (5) reduce to k-Riemann-Liouville operators as follows:

$$\begin{aligned}\left(I_{0+,k}^{\vartheta,-\vartheta,\gamma} f\right)(x) &= \left(I_{0+,k}^{\vartheta} f\right)(x),\\ \left(I_{-,k}^{\vartheta,-\vartheta,\gamma} f\right)(x) &= \left(I_{-,k}^{\vartheta} f\right)(x),\\ \left(D_{0+,k}^{\vartheta,-\vartheta,\gamma} f\right)(x) &= \left(D_{0+,k}^{\vartheta} f\right)(x),\\ \left(D_{-,k}^{\vartheta,-\vartheta,\gamma} f\right)(x) &= \left(D_{-,k}^{\vartheta} f\right)(x).\end{aligned} \tag{53}$$

On the account of the most general character of the S-function, numerous other interesting special cases of results (18), (22), (29), 2and (33) can be obtained, but for lack of space, they are not represented here.

Data Availability

No data were used to support this study.

Conflicts of Interest

There are no conflicts of interest regarding the publication of this article.

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