Hindawi
Journal of Mathematics
Volume 2021, Article ID 7570145, 18 pages
https://doi.org/10.1155/2021/7570145

Research Article

Some Applications of New Complex Function Space Constructed by Different Weights and Exponents

Awad A. Bakery [1,2] **and Elsayed A. E. Mohamed** [1,3]

[1] University of Jeddah, College of Science and Arts at Khulis, Department of Mathematics, Jeddah, Saudi Arabia
[2] Department of Mathematics, Faculty of Science, Ain Shams University, Abbassia, Egypt
[3] Department of Mathematics, Faculty of Education, Alzaeim Alazhari University, Khartoum, Sudan

Correspondence should be addressed to Elsayed A. E. Mohamed; elsayed.bayen2020@yahoo.com

Received 9 July 2021; Accepted 1 September 2021; Published 22 September 2021

Academic Editor: Adam Lecko

In this article, we develop and study a new complex function space formed by varying the weights and exponents under a definite function. We investigate the geometric and topological characteristics of mapping ideals created using s-numbers and this complex function space. Also, the action of shift mappings on this complex function space has been discussed. Finally, we introduced an extension of Caristi's fixed point theorem on it.

1. Introduction

Numerous researchers are attempting to extend the Banach fixed point theorem [1] in a realistic manner. Kannan [2] recognized a subclass of mappings that execute the same fixed point operations as contractions but are not continuous. Ghoncheh [3] pioneered the study of Kannan mappings in modular vector spaces. Lebesgue spaces with variable exponents, $L_{(r)}$, include Nakano sequence spaces. Across the second half of the twentieth century, it was thought that these variable exponent spaces offered an adequate framework for the mathematical components of a variety of problems for which the traditional Lebesgue spaces were inadequate. Due to the importance of these areas and their consequences, they have developed a reputation as an effective instrument for resolving a wide variety of problems; presently, the study of $L_{(r)}(\Omega)$ spaces is a developing field of research, with implications reaching across a broad range of mathematical disciplines [4]. The investigation of variable exponent Lebesgue spaces was accelerated further by the mathematical description of non-Newtonian fluid hydrodynamics [5, 6]. Non-Newtonian fluids, also known as electrorheological fluids, have a wide range of applications in a number of fields ranging from military science to civil engineering to orthopedics and beyond. Mapping ideal theory has a diverse range of applications in Banach space geometry, fixed point theory, spectral theory, and other areas of mathematics, as well as other fields of knowledge (for further information, see [7–13]). Bakery and Mohamed [14] studied the notion of a pre-quasi norm on Nakano sequence space with a variable exponent in the range $(0,1]$. They explored the conditions under which it generates pre-quasi Banach and closed space when endowed with a particular pre-quasi norm as well as the Fatou property of various pre-quasi norms on it. Additionally, they showed the existence of a fixed point for Kannan pre-quasi norm contraction mappings on it as well as on the pre-quasi Banach operator ideal formed from this sequence space's s-numbers. In [15], they investigated some fixed points results of Kannan non-expansive mappings on generalized Cesàro backward difference sequence space of non-absolute type.

We will mark the complex and non-negative integers as $\mathbb{C}$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$, respectively. By $\mathbb{C}^{\mathbb{C}}$, we denote the space of all complex functions with complex variable. Assuming that $r = (r_y)_{y\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$, Bakery and El Dewaik [16] defined the following function space:

$$(\mathscr{H}_w(r_v))_\psi = \left\{ h \in \mathbb{C}^{\mathbb{C}}: h(x) = \sum_{y=0}^{\infty} \widehat{h_y} x^y \in \mathbb{C}; \text{and } \psi(\varpi h) < \infty, \text{ for some } \varpi > 0 \right\}, \tag{1}$$

where

$$\psi(h) = \sum_{y=0}^{\infty} \left| \frac{\widehat{h_y}}{y+1} \right|^{r_y}. \tag{2}$$

They studied several of the topological and geometric properties for $(\mathscr{H}_w((r_v)))_\psi$ and even a pre-quasi ideal construction based on the $(\mathscr{H}_w((r_v)))_\psi$ and s-numbers. Upper bounds for s-numbers of infinite series of the weighted v-th power forward shift operator on $(\mathscr{H}_w((r_v)))_\psi$ were also introduced for some entire functions. Further, they evaluated Caristi's fixed point theorem in $(\mathscr{H}_w((r_v)))_\psi$. For extra information on formal power series spaces and their behaviors, see [17–20]. We denote the space of every, finite rank, approximable, and compact bounded linear mappings from a Banach space X into a Banach space Y by $L(X,Y)$, $F(X,Y)$, $\Lambda(X,Y)$, and $L_c(X,Y)$, and if $X = Y$, we mark $L(X)$, $F(X)$, $\Lambda(X)$, and $L_c(X)$, respectively. The ideal of all, finite rank, approximable, and compact mappings are denoted by L, F, Λ, and L_c. We will indicate the sequence of s-numbers, approximation numbers, and Kolmogorov numbers for any bounded linear mapping G by $(s_a(G))_{a\in\mathbb{N}}$, $(\alpha_a(G))_{a\in\mathbb{N}}$, and $(d_a(G))_{a\in\mathbb{N}}$. The mapping ideals constructed by the sequence of s-numbers, approximation numbers, and Kolmogorov numbers in sequence space $\mathscr{V}$ are marked by $S_{\mathscr{V}}$, $S_{\mathscr{V}}^{\text{app}}$, and $S_{\mathscr{V}}^{Kol}$. For any Banach spaces X and Y, we will use the following notations.

Notations 1 (see [16])

$$S_{\mathscr{H}} := \{S_{\mathscr{H}}(X,Y)\}, \text{ where } S_{\mathscr{H}}(X,Y) := \left\{ P \in L(X,Y): h_s \in \mathscr{H}, \text{ where, } h_s(x) = \sum_{v=0}^{\infty} s_v(P) x^v \in \mathbb{C} \right\}.$$

$$S_{\mathscr{H}}^{\text{app}} := \{S_{\mathscr{H}}^{\text{app}}(X,Y)\}, \text{ where } S_{\mathscr{H}}^{\text{app}}(X,Y) := \left\{ P \in L(X,Y): h_{\text{app}} \in \mathscr{H}, \text{ where, } h_{\text{app}}(x) = \sum_{v=0}^{\infty} \alpha_v(P) x^v \in \mathbb{C} \right\}.$$

$$S_{\mathscr{H}}^{Kol} := \left\{S_{\mathscr{H}}^{Kol}(X,Y)\right\}, \text{ where } S_{\mathscr{H}}^{Kol}(X,Y) := \left\{ P \in L(X,Y): h_{Kol} \in \mathscr{H}, \text{ where, } h_{Kol}(x) = \sum_{v=0}^{\infty} d_v(P) x^v \in \mathbb{C} \right\}. \tag{3}$$

$$\left(S_{\mathscr{H}_\rho}\right)^\lambda := \left\{ \left(S_{\mathscr{H}_\rho}\right)^\lambda (X,Y) \right\}, \text{ where}$$

$$\left(S_{\mathscr{H}_\rho}\right)^\lambda (X,Y) := \left\{ T \in L(X,Y): h_\lambda \in \mathscr{H}_\rho, \text{ where, } h_\lambda(x) = \sum_{v=0}^{\infty} \lambda_v(P) x^v \in \mathbb{C} \text{ and } \left\| P - \lambda_v(P) I \right\| = 0, \forall v \in \mathbb{N} \right\}.$$

The purpose of this study is straightforward, as follows. In Section 3, we introduce and investigate the complex function space $(\mathbb{H}((b_v),(p_v)))_\rho$ under the definite function ρ. In Section 4, the mapping ideals constructed by s-numbers and $(\mathbb{H}((b_v),(p_v)))_\rho$ are presented. We have studied their geometric and topological properties. Specifically, we explore in Section 5 the upper limits of s-numbers for infinite series of the weighted v-th power forward and backward shift mapping on $(\mathbb{H}((b_v),(p_v)))_\rho$ and their applications to various entire functions. Finally, in Section 6, we present an extension of Caristi's fixed point theorem in $(\mathbb{H}((b_v),(p_v)))_\rho$.

2. Definitions and Preliminaries

Let $\mathbb{R}^{\mathbb{N}}$, ℓ_∞, ℓ^r, and c_0 denote the spaces of each, bounded, r-absolutely summable, and null sequences of real numbers, respectively.

Definition 1 (see [16]). The function space $\mathscr{H} = \left\{ h \in \mathbb{C}^{\mathbb{C}}: h(y) = \sum_{v=0}^{\infty} \widehat{h_v} y^v \right\}$ is called a special space of formal power series (or in short ssfps), if it shows the following settings:

(1) $e^{(b)} \in \mathscr{H}$, for all $b \in \mathbb{N}$, where $e^{(b)}(y) = \sum_{v=0}^{\infty} \widehat{e_v^{(b)}} y^v = y^b$.

(2) If $g \in \mathscr{H}$ and $|\widehat{h_v}| t \le n |q \widehat{g_v}|$, for all $v \in \mathbb{N}$, one has $h \in \mathscr{H}$.

(3) Suppose $h \in \mathscr{H}$, then $h_{[.]} \in \mathscr{H}$, where $h_{[.]}(y) = \sum_{b=0}^{\infty} \widehat{h_{[b/2]}} y^b$ and $[b/2]$ marks the integral part of $[b/2]$.

Theorem 1 (see [16]). *$S_{\mathscr{H}}$ is a mapping ideal, when $\mathscr{H}$ is a ssfps.*

We denote the space of finite formal power series by $\mathfrak{F}$, i.e., if $h \in \mathfrak{F}$, one has $k \in \mathbb{N}$ with $h(y) = \sum_{v=0}^{k} \widehat{h}_v y^v$. Also, θ indicates the zero function of $\mathscr{H}$.

Definition 2 (see [16]). A subspace $\mathscr{H}_\rho$ of the ssfps is said to be a pre-quasi normed ssfps, if there is a function $\rho: \mathscr{H} \longrightarrow [0,\infty)$ which verifies the next conditions:

(i) For $h \in \mathscr{H}$, we have $\rho(h) \geq 0$ and $h = \theta \Longleftrightarrow \rho(h) = 0$.
(ii) Suppose $h \in \mathscr{H}$ and $\lambda \in \mathbb{C}$, then there are $l \geq 1$ with $\rho(\lambda h) \leq |\lambda| l \rho(h)$.
(iii) Let $f, g \in \mathscr{H}$; then, there are $K \geq 1$ such that $\rho(f+g) \leq K(\rho(f) + \rho(g))\mathbb{Z}$.

Recall that if the space $\mathscr{H}_\rho$ is complete, then $\mathscr{H}_\rho$ is called a pre-quasi Banach ssfps.

Definition 3 (see [16]). A subspace $\mathscr{H}_\rho$ of the ssfps is called a pre-modular ssfps, if there is a function $\rho: \mathscr{H} \longrightarrow [0,\infty)$ which verifies the next conditions:

(i) For $h \in \mathscr{H}$, we have $\rho(h) \geq 0$ and $h = \theta \Longleftrightarrow \rho(h) = 0$.
(ii) Suppose $h \in \mathscr{H}$ and $\lambda \in \mathbb{C}$, then there are $l \geq 1$ with $\rho(\lambda h) \leq |\lambda| l \rho(h)$.
(iii) Let $f, g \in \mathscr{H}$; then, there are $K \geq 1$ such that $\rho(f+g) \leq K((\rho(f) + \rho(g))$.
(iv) Suppose $|\widehat{f}_b| t \leq n|q\widehat{g}_b|$, for every $b \in \mathbb{N}$; then, $\rho(f) \leq \rho(g)\mathbb{F}$.
(v) There are $K_0 \geq 1$ so that $\rho(f) \leq \rho(f_{[.]}) \leq K_0 \rho(f)$.
(vi) $\overline{\mathfrak{F}} = \mathscr{H}_\rho$.
(vii) One has $\xi > 0$ with $\rho(\lambda e^{(0)}) \geq \xi |\lambda| \rho(e^{(0)})$, where $\lambda \in \mathbb{C}$.

Theorem 2 (see [16]). *Every pre-modular ssfps $\mathscr{H}_\rho$ is a pre-quasi normed ssfps.*

Definition 4 (see [21]). A function $s: L(X,Y) \longrightarrow [0,\infty)^{\mathbb{N}}$ is called an s-number, if the sequence $(s_b(B))_{a=0}^{\infty}$, for any $B \in L(X,Y)$, satisfies the following setup:

(a) If $B \in L(X,Y)$, then $\|B\| = s_0(B) \geq s_1(B) \geq s_2(B) \geq \ldots \geq 0$.
(b) $s_{b+a-1}(B_1 + B_2) \leq s_b(B_1) + s_a(B_2)$, for every $B_1, B_2 \in L(X,Y)$, b, $a \in \mathbb{N}$.
(c) The inequality $s_a(AB\,D) \leq \|A\| s_a(B) \| \geq \|$ holds, if $D \in L(X_0, X)$, $B \in L(X,Y)$, and $A \in L(Y, Y_0)$; suppose that X_0 and Y_0 are any two Banach spaces.
(d) For $A \in L(X,Y)$ and $\lambda \in \mathbb{R}$, then $s_a(\lambda A) = |\lambda| s_a(A)$.
(e) Suppose $\operatorname{rank}(A) \leq b$; then, $s_b(A) = 0$, whenever $A \in L(X,Y)$,
(f) Assume that I_b represents the unit map on the b-dimensional Hilbert space ℓ_2^b; then, $s_{r \geq b}(I_b) = 0$ or $s_{r<b}(I_b) = 1$.

The following are some instances of s-numbers:

(i) The k-th approximation number, $\alpha_k(A)$, is presented as

$$\alpha_k(A) = \inf\{\|A - B\|: B \in L(X,Y) \text{ and } \operatorname{rank}(B) \leq k\}. \tag{4}$$

(ii) The k-th Kolmogorov number, $d_k(A)$, is presented as

$$d_k(A) = \inf_{\dim(Y)} \leq k \sup_{\|u\| \leq 1} \inf_{v \in Y} \|Au - v\|. \tag{5}$$

Lemma 1 (see [7]). *Assume that $B \in L(X,Y)$ and $B \notin \Lambda(X,Y)$, and we have maps $D \in L(X)$ and $M \in L(Y)$ with $MB\,De_b = e_b$, for each $b \in \mathbb{N}$.*

Definition 5 (see [7]). A Banach space Y is named simple if $L(Y)$ contains one and only one non-trivial closed ideal.

Theorem 3 (see [7]). *Suppose Z is a Banach space with* $\dim(Z) = \infty$*, and we have*

$$F(Z) \subsetneqq \Lambda(Z) \subsetneqq L_c(Z) \subsetneqq L(Z). \tag{6}$$

Definition 6 (see [7]). A class $\mathscr{U} \subseteq L$ is said to be a mapping ideal if every component $\mathscr{U}(X,Y) = \mathscr{U} \cap L(X,Y)$ satisfies the next setups:

(i) $F \subseteq \mathscr{U}$.
(ii) $\mathscr{U}(X,Y)$ is linear space on $\mathbb{R}$.
(iii) Assume $D \in L(X_0, X)$, $B \in \mathscr{U}(X,Y)$, and $A \in L(Y, Y_0)$; then, $AB\,D \in \mathscr{U}(X_0, Y_0)$.

Definition 7 (see [10]). A function $g: \mathscr{U} \longrightarrow [0,\infty)$ is called a pre-quasi norm on the ideal $\mathscr{U}$ if it satisfies the following setups:

(1) Suppose $B \in L(X,Y)$, $g(B) \geq 0$, and $g(B) = 0 \Longleftrightarrow B = 0$.
(2) There is $M \geq 1$ with $g(vA) \leq M|v| g(A)$, for all $v \in \mathbb{C}$ and $A \in \mathscr{U}(X,Y)$.
(3) One has $K \geq 1$ so that $g(A_1 + A_2) \leq K[g(A_1) + g(A_2)]$, for every $A_1, A_2 \in \mathscr{U}(X,Y)$.
(4) We get $C \geq 1$ so that if $A \in L(X_0, X)$, $B \in \mathscr{U}(X,Y)$, and $D \in L(Y, Y_0)$, then $g(DB\,A) \leq C\|D\| g(B) \|A\|$, where X_0 and Y_0 are normed spaces.

Theorem 4 (see [10]). *Every quasi norm is a pre-quasi norm on the same ideal.*

With finite non-zero coordinates, we denote the space of every sequence by $\mathscr{F}$.

Theorem 5 (see [22]). *Suppose s−type $\mathscr{V}_v := \{f = (s_r(T)) \in \mathbb{R}^{\mathbb{N}}: T \in L(X,Y)$ and $v(f) < \infty\}$. If $S_{\mathscr{V}_v}$ is a mapping ideal, we have*

(1) $\mathscr{F} \subset s-type\ \mathscr{V}_{v}$.
(2) Assume $(s_r(T_1))_{r=0}^{\infty} \in s-type\ \mathscr{V}_{v}$ and $(s_r(T_2))_{r=0}^{\infty} \in s-type\ \mathscr{V}_{v}$; then, $(s_r(T_1+T_2))_{r=0}^{\infty} \in s-type\ \mathscr{V}_{v}$.
(3) If $\lambda \in \mathbb{C}$ and $(s_r(T))_{r=0}^{\infty} \in s-type\ \mathscr{V}_{v}$, then $|\lambda|(s_r(T))_{r=0}^{\infty} \in s-type\ \mathscr{V}_{v}$.
(4) $\mathscr{V}_{v}$ is solid, i.e., if $(s_x(J))_{x=0}^{\infty} \in s-type\ \mathscr{V}_{v}$ and $s_x(H) \le s_x(J)$, for all $x \in \mathbb{N}$ and $H, J \in L(X,Y)$, then $(s_x(H))_{x=0}^{\infty} \in s-type\ \mathscr{V}_{v}$.

By card $(\mathfrak{G})$, we denote the number of elements of $\mathfrak{G}$.

Lemma 2 (see [23]). *Suppose $\{\xi_i\}_{i\in\Psi}$ is a bounded family of $\mathbb{R}$. Hence,*

$$\inf_{\mathrm{card}(\mathfrak{G})=b} \sup_{i\notin\mathfrak{G}} \xi_i = \sup_{\mathrm{card}(\mathfrak{G})=b+1} \inf_{i\in\mathfrak{G}} \xi_i. \tag{7}$$

We will apply the next inequality [24]. For all $(r_a), (t_a) \in \mathbb{C}^{\mathbb{N}}$ and $(q_a) \in (0,\infty)^{\mathbb{N}}$, we have

$$|r_a + t_a|^{q_a} \le K\left(|r_a|^{q_a} + |t_a|^{q_a}\right), \tag{8}$$

where $K = \max\{1, 2^{\varpi_q - 1}\}$ and $\varpi_q = \max\{1, \sup_a q_a\}$.

Definition 8 (see [16]). Assume $\mathscr{H}_{\rho}$ is a pre-quasi normed ssfps. A mapping $V_y\colon \mathscr{H}_{\rho} \longrightarrow \mathscr{H}_{\rho}$ is called forward shift, if $V_y h = yh$, for all $h \in \mathscr{H}_{\rho}$, where $V_y h(y) = \sum_{v=0}^{\infty} \widehat{h}_v y^{v+1} \in \mathbb{C}$ and $\rho(V_y h) < \infty$.

Definition 9 (see [16]). Suppose $\mathscr{H}_{\rho}$ is a pre-quasi normed ssfps. A mapping $B_y\colon \mathscr{H}_{\rho} \longrightarrow \mathscr{H}_{\rho}$ is called backward shift, if $B_y h(y) = h(y) - h(0)/y$, for all $h \in \mathscr{H}_{\rho}$, where $B_y h(y) = \sum_{v=0}^{\infty} \widehat{h_{v+1}} y^{v} \in \mathbb{C}$ and $\rho(B_y h) < \infty$.

Definition 10 (see [20]). If $g(y) = \sum_{m=0}^{\infty} a_m y^m$, then $V_{g(y)}(h(y)) := (\sum_{m=0}^{\infty} a_m V_y^m)(h(y))$.

Definition 11 (see [20]). If $g(y) = \sum_{m=0}^{\infty} a_m y^m$, then $B_{g(y)}(h(y)) := (\sum_{m=0}^{\infty} a_m B_y^m)(h(y))$.

3. Pre-Modular ssfps

This section contains the space's definition $(\mathbb{H}((b_n),(p_n)))_{\rho}$ under the function ρ, where $\rho(h) = \sum_{v=0}^{\infty} |b_v \widehat{h}_v|^{p_v}$, for all $h \in \mathbb{H}((b_n),(p_n))$. We offer enough setups on $(\mathbb{H}((b_n),(p_n)))_{\rho}$ to become pre-modular ssfps, which implies that $(\mathbb{H}((b_n),(p_n)))_{\rho}$ is a pre-quasi Banach ssfps.

Let $p = (p_v)_{v\in\mathbb{N}}, (b_v)_{v\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$, and we define the following function space:

$$\mathbb{H}((b_v),(p_v)) = \left\{h \in \mathbb{C}^{\mathbb{C}}\colon h(y) = \sum\nolimits_{v=0}^{\infty} \widehat{h}_v y^v \text{ and } \rho(\gamma h) < \infty, \text{ for some } \gamma > 0\right\}. \tag{9}$$

Theorem 6. *If $(p_v) \in \ell_{\infty}$, then*

$$\mathbb{H}((b_v),(p_v)) = \left\{h \in \mathbb{C}^{\mathbb{C}}\colon h(y) = \sum\nolimits_{v=0}^{\infty} \widehat{h}_v y^v \text{ and } \rho(\gamma h) < \infty, \text{ for all } \gamma > 0\right\}. \tag{10}$$

Proof.

$$\begin{aligned}
\mathbb{H}((b_v),(p_v)) &= \left\{h \in \mathbb{C}^{\mathbb{C}}\colon h(y) = \sum_{v=0}^{\infty} \widehat{h}_v y^v \text{ and } \rho(\gamma h) < \infty, \text{ for some } \gamma > 0\right\} \\
&= \left\{h \in \mathbb{C}^{\mathbb{C}}\colon h(y) = \sum_{v=0}^{\infty} \widehat{h}_v y^v \text{ and } \sum_{v=0}^{\infty} |\gamma b_v \widehat{h}_v|^{p_v} < \infty, \text{ for some } \gamma > 0\right\} \\
&= \left\{h \in \mathbb{C}^{\mathbb{C}}\colon h(y) = \sum_{v=0}^{\infty} \widehat{h}_v y^v \text{ and } \inf_v |\gamma|^{p_v} \sum_{v=0}^{\infty} |b_v \widehat{h}_v|^{p_v} < \infty, \text{ for some } \gamma > 0\right\} \\
&= \left\{h \in \mathbb{C}^{\mathbb{C}}\colon h(y) = \sum_{v=0}^{\infty} \widehat{h}_v y^v \text{ and } \sum_{v=0}^{\infty} |b_v \widehat{h}_v|^{p_v} < \infty, \text{ for any } \gamma > 0\right\} \\
&= \left\{h \in \mathbb{C}^{\mathbb{C}}\colon h(y) = \sum_{v=0}^{\infty} \widehat{h}_v y^v \text{ and } \rho(\gamma h) < \infty, \text{ for any } \gamma > 0\right\}.
\end{aligned} \tag{11}$$

□

Hereafter, we will denote the space of all monotonic decreasing and monotonic increasing sequences of positive reals by $md_{\searrow}$ and $mi_{\nearrow}$, respectively.

Theorem 7. *$\mathbb{H}((b_n),(p_n))$ is a ssfps, if it verifies the next setups:*

(a1) $(p_n)\in mi_{\nearrow}\cap\ell_{\infty}$.

(a2) $(b_n)\in md_{\searrow}$, or $(b_n)\in mi_{\nearrow}$ with $C\geq 1$ so that $b_{2n+1}\leq Cb_n$.

Proof

(1-i) Assume $f,g\in\mathbb{H}((b_n),(p_n))$; then, $f(z)=\sum_{n=0}^{\infty}\widehat{f}_n z^n\in\mathbb{C}$and $g(z)=\sum_{n=0}^{\infty}\widehat{g}_n z^n\in\mathbb{C}$. We have $(f+g)(z)=\sum_{n=0}^{\infty}(\widehat{f}_n+\widehat{g}_n)z^n\in\mathbb{C}$. Since (p_n) is bounded, we get

$$\sum_{n=0}^{\infty}\left|b_n\widehat{f}_n+b_n\widehat{g}_n\right|^{p_n}\leq K\left(\sum_{n=0}^{\infty}\left|b_n\widehat{f}_n\right|^{p_n}+\sum_{n=0}^{\infty}\left|b_n\widehat{g}_n\right|^{p_n}\right)<\infty, \tag{12}$$

and then $f+g\in\mathbb{H}((b_n),(p_n))$.

(1-ii) Let $\lambda\in\mathbb{C}$ and $f\in\mathbb{H}((b_v),(p_v))$. We have $(\lambda f)(z)=\sum_{v=0}^{\infty}\lambda\widehat{f}_v z^v\in\mathbb{C}$. Since (p_v) is bounded, we have

$$\sum_{v=0}^{\infty}|\lambda b_v\widehat{f}_v|^{p_v}\leq\sup_v|\lambda|^{p_v}\sum_{v=0}^{\infty}|b_v\widehat{f}_v|^{p_v}<\infty. \tag{13}$$

Then, $\lambda f\in\mathbb{H}((b_v),(p_v))$. Therefore, by using components (1-i) and (1-ii), $\mathbb{H}((b_v),(p_v))$ is linear. Clearly, $e^{(k)}\in\mathbb{H}((b_v),(p_v))$, for all $k\in\mathbb{N}$, where $e^{(k)}(z)=\sum_{v=0}^{\infty}\widehat{e_v^{(k)}}z^v=z^k$ and $\sum_{v=0}^{\infty}|b_v\widehat{e_v^{(k)}}|^{p_v}=b_k^{p_k}$.

(2) Let $|\widehat{f}_v|t\leq n|q\widehat{g}_v|$, for all $v\in\mathbb{N}$ and $g\in\mathbb{H}((b_v),(p_v))$. Then, $g(z)=\sum_{v=0}^{\infty}\widehat{g}_v z^v\in\mathbb{C}$. Since $b_v>0$, for all $v\in\mathbb{N}$, then

$$\sum_{v=0}^{\infty}\left|b_v\widehat{f}_v\right|^{p_v}\leq\sum_{v=0}^{\infty}\left|b_v\widehat{g}_v\right|^{p_v}<\infty. \tag{14}$$

Hence, $f(z)=\sum_{v=0}^{\infty}\widehat{f}_v z^v\in\mathbb{C}$ and $\rho(f)<\infty$. Therefore, $f\in\mathbb{H}((b_v),(p_v))$.

(3) Let $f\in\mathbb{H}((b_v),(p_v))$, (b_v) be an increasing sequence, and there exists $C>0$ such that $b_{2v+1}\leq Cb_v$ and (p_v) is increasing. Therefore, $f(z)=\sum_{v=0}^{\infty}\widehat{f}_v z^v\in\mathbb{C}$ and $\rho(f)<\infty$. One has

$$\begin{aligned}\rho\left(f_{[.]}\right)&=\sum_{v=0}^{\infty}\left|b_v\widehat{f_{v/2}}\right|^{p_v}=\sum_{v=0}^{\infty}\left|b_{2v}\widehat{f}_v\right|^{p_{2v}}+\sum_{v=0}^{\infty}\left|b_{2v+1}\widehat{f}_v\right|^{p_{2v+1}}\\&\leq\sum_{v=0}^{\infty}\left|b_{2v}\widehat{f}_v\right|^{p_v}+\sum_{v=0}^{\infty}\left|b_{2v+1}\widehat{f}_v\right|^{p_v}\\&\leq\max\{1,2\sup_v C^{p_v}\}\rho(f).\end{aligned} \tag{15}$$

This implies that $f_{[.]}(z)=\sum_{v=0}^{\infty}\widehat{f_{[v/2]}}z^v\in\mathbb{C}$ and $\rho(f_{[.]})<\infty$. Hence, $f_{[.]}\in\mathbb{H}((b_v),(p_v))$. □

Theorem 8. *Let conditions (a1) and (a2) be satisfied; then, the space $(\mathbb{H}((b_n),(p_n)))_\rho$ is a pre-modular Banach ssfps.*

Proof

(i) Evidently, for all $f\in\mathbb{H}((b_n),(p_n))$, then $\rho(f)\geq 0$ and $\rho(f)=0\Longleftrightarrow f=\theta$.

(ii) We have $l=\max\{1,\sup_n|\eta|^{p_n-1}\}\geq 1$, for all $\eta\in\mathbb{R}\setminus\{0\}$ and $l\geq 1$, for $\eta=0$ such that

$$\rho(\eta f)=\sum_{n=0}^{\infty}\left|\eta b_n\widehat{f}_n\right|^{p_n}\leq\sup_n|\eta|^{p_n}\sum_{n=0}^{\infty}\left|b_n\widehat{f}_n\right|^{p_n}\leq l|\eta|\rho(f), \tag{16}$$

for every $f\in\mathbb{H}((b_n),(p_n))$.

(iii) For some $K=\max\{1,2^{\sup_n p_n-1}\}$, we obtain

$$\rho(f+g)=\sum_{n=0}^{\infty}\left|b_n\left(\widehat{f}_n+\widehat{g}_n\right)\right|^{p_n}\leq K(\rho(f)+\rho(g)), \tag{17}$$

for all $f,g\in\mathbb{H}(b_n),(p_n)$.

(iv) It is clear from the proof part (2) of Theorem 7.

(v) From the proof part (3) of Theorem 7, we have that $K_0=\max\{1,2\sup_n C^{p_n}\}\geq 1$.

(vi) It is apparent that $\overline{\mathfrak{F}}=\mathbb{H}((b_n),(p_n))$.

(vii) There is ζ with $0<\zeta\leq\eta^{p_0-1}$ such that $\rho(\eta e^{(0)})\geq\zeta|\eta|\rho(e^{(0)})$, for each $\eta\neq 0$ and $\zeta>0$, if $\eta=0$.

Therefore, the space $(\mathbb{H}((b_v),(p_v)))_\rho$ is pre-modular ssfps. To show that $(\mathbb{H}((b_v),(p_v)))_\rho$ is a pre-modular Banach ssfps, suppose $f^{(n)}$ is a Cauchy sequence in $(\mathbb{H}((b_v),(p_v)))_\rho$; then, for all $\varepsilon\in(0,1)$, there is $n_0\in\mathbb{N}$ such that, for all $n,m\geq n_0$, we get

$$\rho\left(f^{(n)}-f^{(m)}\right)=\sum_{v=0}^{\infty}\left|b_v\left(\widehat{f_v^{(n)}}-\widehat{f_v^{(m)}}\right)\right|^{p_v}<\varepsilon^{\varpi_p}. \tag{18}$$

For $n,m\geq n_0$ and $v\in\mathbb{N}$, we obtain

$$|\widehat{f_v^{(n)}}-\widehat{f_v^{(m)}}|<\varepsilon. \tag{19}$$

Hence, $(\widehat{f_v^{(m)}})$ is a Cauchy sequence in $\mathbb{C}$, for fixed $v\in\mathbb{N}$, so $\lim_{m\longrightarrow\infty}\widehat{f_v^{(m)}}=\widehat{f_v^{(0)}}$, for fixed $v\in\mathbb{N}$. Therefore, $\rho(f^{(n)}-f^{(0)})<\varepsilon^{\varpi_p}$, for each $n\geq n_0$. Finally, to explain that $f^{(0)}\in\mathbb{H}((b_v),(p_v))$, we have

$$\begin{aligned}\rho\left(f^{(0)}\right)=\rho\left(f^{(0)}-f^{(n)}+f^{(n)}\right)&\leq K\left(\rho\left(f^{(n)}-f^{(0)}\right)\right.\\&\left.+\rho\left(f^{(n)}\right)\right)<\infty.\end{aligned} \tag{20}$$

So, $f^{(0)}\in\mathbb{H}((b_v),(p_v))$. This implies that $(\mathbb{H}((b_v),(p_v)))_\rho$ is a pre-modular Banach ssfps. □

Taking into consideration Theorem 2, we put forward the following theorem.

Theorem 9. *Let conditions (a1) and (a2) be satisfied. Then, the space* $(\mathbb{H}((b_n),(p_n)))_\rho$ *is a pre-quasi Banach sspfs.*

Theorem 10. *Let conditions (a1) and (a2) be satisfied. Then, the space* $(\mathbb{H}((b_n),(p_n)))_\rho$ *is a pre-quasi closed sspfs.*

Proof. Assume that the setups are verified. From Theorem 9, the space $(\mathbb{H}((b_n),(p_n)))_\rho$ is a pre-quasi normed sspfs. To show that $(\mathbb{H}((b_n),(p_n)))_\rho$ is a pre-quasi closed sspfs, assume $\{h^{(m)}\}_{m=0}^{\infty} \in (\mathbb{H}((b_n),(p_n)))_\rho$ and $\lim_{m\longrightarrow\infty}\rho(h^{(m)} - h^{(0)}) = 0$; then, for every $\varepsilon \in (0,1)$, there is $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, one has

$$\varepsilon > \rho\left(h^{(m)} - h^{(0)}\right) = \left[\sum_{a=0}^{\infty}\left|b_a\left(\widehat{h_a^{(m)}} - \widehat{h_a^{(0)}}\right)\right|^{p_a}\right]^{1/\varpi_p}. \tag{21}$$

Hence, for $m \geq m_0$ and $a \in \mathbb{N}$, we get

$$\left|\widehat{h_a^{(m)}} - \widehat{h_a^{(0)}}\right| < \varepsilon. \tag{22}$$

So, $(\widehat{h_a^{(m)}})$ is a convergent sequence in $\mathbb{C}$, for fixed $a \in \mathbb{N}$. Therefore, $\lim_{m\longrightarrow\infty}\widehat{h_a^{(m)}} = \widehat{h_a^{(0)}}$, for fixed $a \in \mathbb{N}$. Finally to prove that $h^{(0)} \in (\mathbb{H}((b_n),(p_n)))_\rho$, we consider

$$\begin{aligned}\rho\left(h^{(0)}\right) = \rho\left(h^{(0)} - h^{(m)} + h^{(m)}\right) \leq K\left(\left(\rho\left(h^{(m)} - h^{(0)}\right)\right)\right. \\ \left. + \rho\left(h^{(m)}\right)\right) < \infty,\end{aligned} \tag{23}$$

so $h^{(0)} \in (\mathbb{H}((b_n),(p_n)))_\rho$. This finishes the proof. □

4. Pre-Quasi Ideal

In this section, the mapping ideals constructed by s-numbers and $((\mathbb{H}(b_n),(p_n)))_\rho$ are presented. We have studied their geometric and topological structures. We will use the notation for $B \in S_{\mathbb{H}((b_v),(p_v))_\rho}$, that is, $g(B) = \rho(f_s)$, $f_s(z) = \sum_{v=0}^{\infty} s_v(B)z^v \in \mathbb{C}$, and $\rho(f_s) = \sum_{v=0}^{\infty}(b_v s_v(B))^{p_v}$, for every $f_s \in \mathbb{H}((b_v),(p_v))_\rho$.

In view of Theorems 1 and 7, we conclude the next theorem.

Theorem 11. *Let conditions (a1) and (a2) be satisfied. Then,* $S_{\mathbb{H}((b_n),(p_n))}$ *is a mapping ideal.*

4.1. Ideal of Finite Rank Mappings. In this section, enough setups (not necessary) on $\mathbb{H}((b_n),(p_n))_\rho$ so that F is dense in $S_{\mathbb{H}((b_n),(p_n))_\rho}$ are investigated. This explains the non-linearity of the s–type $\mathbb{H}((b_n),(p_n))_\rho$ spaces (Rhoades open problem [25]).

Example 1. The sequence $(b_n) = (n + 1/n + 2)_{n\in\mathbb{N}}$ satisfies $(b_n) \in mi_{\nearrow}$ and $b_{2n+1} \leq Cb_n$, for some $C \geq 2$.

Theorem 12. $\overline{F(X,Y)} = S_{\mathbb{H}((b_n),(p_n))_\rho}(X,Y)$, *whenever conditions (a1) and (a2) are satisfied.*

Proof. It is clear that $\overline{F(X,Y)} \subset S_{\mathbb{H}((b_n),(p_n))_\rho}(X,Y)$, since the space $S_{\mathbb{H}((b_n),(p_n))_\rho}$ is a mapping ideal. Currently, we substantiate that $S_{\mathbb{H}((b_n),(p_n))_\rho}(X,Y) \subseteq \overline{F(X,Y)}$. On taking $T \in S_{\mathbb{H}((b_n),(p_n))_\rho}(X,Y)$, $f_s \in \mathbb{H}((b_n),(p_n))_\rho$, with $f_s(z) = \sum_{n=0}^{\infty} s_n(T)z^n \in \mathbb{C}$. Hence, $\rho(f_s) < \infty$, and assume $\varepsilon \in (0,1)$, so there is $m \in \mathbb{N} - \{0\}$ such that $\rho(f_s - \sum_{n=0}^{m-1} e^{(n)}) < \varepsilon/4C^2$, for some $C \geq 1$. While $(s_n(T))_{n\in\mathbb{N}}$ is decreasing, we get

$$\begin{aligned}\sum_{n=m+1}^{2m}\left(b_n s_{2m}(T)\right)^{p_n} &\leq \sum_{n=m+1}^{2m}\left(b_n s_n(T)\right)^{p_n} \\ &\leq \sum_{n=m}^{\infty}\left(b_n s_n(T)\right)^{p_n} < \frac{\varepsilon}{4C^2}.\end{aligned} \tag{24}$$

Hence, there exist $A \in F_{2m}(X,Y)$, $\text{rank}(A) \leq 2m$, and

$$\sum_{n=2m+1}^{3m}\left(b_n\|T - A\|\right)^{p_n} \leq \sum_{n=m+1}^{2m}\left(b_n\|T - A\|\right)^{p_n} < \frac{\varepsilon}{4C^2}. \tag{25}$$

Since (p_n) is bounded,

$$\sum_{n=0}^{m}\left(b_n\|T - A\|\right)^{p_n} < \frac{\varepsilon}{4C^2}. \tag{26}$$

Let (b_n) be monotonically increasing such that there exists a constant $C \geq 1$ for which $b_{2n+1} \leq Cb_n$. Then, we have for $n \geq m$ that

$$b_{2m+n} \leq b_{2m+2n+1} \leq Cb_{m+n} \leq Cb_{2n} \leq Cb_{2n+1} \leq C^2 b_n. \tag{27}$$

Since $T - A \in S_{\mathbb{H}((b_n),(p_n))_\rho}(X,Y)$, then $h_s \in \mathbb{H}((b_n),(p_n))_\rho$, where $h_s(z) := \sum_{n=0}^{\infty} s_n(T - A)z^n \in \mathbb{C}$. Since (p_n) is increasing, inequalities (2)–(5) give

$$\begin{aligned}d(T,A) = \rho\left(h_s\right) &= \sum_{n=0}^{3m-1}\left(b_n s_n(T - A)\right)^{p_n} + \sum_{n=3m}^{\infty}\left(b_n s_n(T - A)\right)^{p_n} \\ &\leq \sum_{n=0}^{3m}\left(b_n\|T - A\|\right)^{p_n} + \sum_{n=m}^{\infty}\left(b_{n+2m}s_{n+2m}(T - A)\right)^{p_{n+2m}} \\ &\leq 3\sum_{n=0}^{m}\left(b_n\|T - A\|\right)^{p_n} + C^{2\sup_n p_n}\sum_{n=m}^{\infty}\left(b_n s_n(T)\right)^{p_n} < \varepsilon.\end{aligned} \tag{28}$$

Since $I_8 \in S_{(\mathbb{H}(n+1),(1/n+1))_\rho}(X,Y)$ which gives a counter example of the converse statement, this finishes the proof. □

According to Theorem 12, if (a1) and (a2) are fulfilled, then every compact mapping is represented by finite rank mappings; however, the reverse is not necessarily true.

4.2. Closed and Banach. In this part, we have investigated the sufficient conditions on $\mathbb{H}((b_v),(p_v))_\rho$ such that the pre-quasi mapping ideal $S_{\mathbb{H}((b_v),(p_v))_\rho}$ is Banach and closed.

Theorem 13. *If X and Y are Banach spaces and conditions (a1) and (a2) are satisfied, then the function $g(B)=\rho(f_s)$ is a pre-quasi norm on $S_{\mathbb{H}((b_v),(p_v))_\rho}$.*

Proof. Suppose the conditions are verified, so g verifies the next setups:

(1) Let $B \in S_{\mathbb{H}((b_v),(p_v))_\rho}(X,Y)$, then we have $g(B)=\rho(f_s)\geq 0$, and it is clear that $g(B)=\rho(f_s)=0$, if and only if, $s_v(B)=0$, for all $v \in \mathrm{N}$, if and only if, $B=0$.

(2) We have $l\geq 1$ with $g(\lambda B)=\rho(\lambda f_s)\leq l|\lambda|\rho(f_s)=l|\lambda|g(B)$, for every $B\in S_{\mathbb{H}((b_v),(p_v))_\rho}(X,Y)$ and $\lambda\in\mathbb{C}$.

(3) One has $KK_0\geq 1$ for $B_1,B_2\in S_{\mathbb{H}((b_v),(p_v))_\rho}(X,Y)$. Therefore, $f1_s(z)=\sum_{v=0}^{\infty}s_v(B_1)z^v\in\mathbb{C}$ and $f2_s(z)=\sum_{v=0}^{\infty}s_v(B_2)z^v\in\mathbb{C}$. Therefore, for $h_s(z)=:\sum_{v=0}^{\infty}s_v(B_1+B_2)z^v$, one can see that

$$\begin{aligned} g(B_1+B_2)&=\rho(h_s)\leq\rho\Big((f1_s)_{[.]}+(f2_s)_{[.]}\Big)\leq K\Big(\rho(f1_s)_{[.]}\Big)\\ &\quad+\rho\Big((f2_s)_{[.]}\Big)\leq KK_0(g(B_1)+g(B_2)). \end{aligned}\tag{29}$$

(4) We have $C\geq 1$; suppose $A\in L(X_0,X)$, $B\in S_{\mathbb{H}((b_v),(p_v))_\rho}(X,Y)$, and $D\in L(Y,Y_0)$. Therefore, $f_s(z)=\sum_{v=0}^{\infty}s_v(B)z^v\in\mathbb{C}$. Then, for $h_s(z)=:\sum_{v=0}^{\infty}s_v(DB\,A)z^v$, one can see that

$$g(DB\,A)=\rho(h_s)\leq\rho(\|A\|\|D\|f_s)\leq C\,\|A\|g(B)\,\|D\|.\tag{30}$$

□

Theorem 14. *If X and Y are Banach spaces and conditions (a1) and (a2) are satisfied, then $(S_{\mathbb{H}((b_v),(p_v))_\rho},g)$ is a pre-quasi Banach mapping ideal.*

Proof. Suppose the conditions are verified, then the function $g(B)=\rho(f_s)$ is a pre-quasi norm on $S_{\mathbb{H}((b_v),(p_v))_\rho}$. Let (B_m) be a Cauchy sequence in $S_{\mathbb{H}((b_v),(p_v))_\rho}(X,Y)$. Therefore, $f_s^{(m)}\in\mathbb{H}((b_v),(p_v))_\rho\in\mathbb{C}$ and $f_s^{(m)}(z)=\sum_{v=0}^{\infty}s_v(B_m)z^v\in\mathbb{C}$. Assume $h_s(z)=:\sum_{v=0}^{\infty}s_v(B_i-B_j)z^n$; then, by using conditions (iv) and (vii) of Definition 3 and since $L(X,Y)\supseteq S_{\mathbb{H}((b_v),(p_v))_\rho}(X,Y)$, we get

$$\begin{aligned} g\big(B_i-B_j\big)&=\rho(h_s)\geq\rho\big(s_0\big(B_i-B_j\big)e^{(0)}\big)\\ &=\rho\Big(\big\|B_i-B_j\big\|e^{(0)}\Big)\geq\xi\,\big\|B_i-B_j\big\|\rho\big(e^{(0)}\big). \end{aligned}\tag{31}$$

Thus, $(B_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $L(X,Y)$. While the space $L(X,Y)$ is a Banach space, there exists $B\in L(X,Y)$ with $\lim_{m\longrightarrow\infty}\|B_m-B\|=0$ and since $f_s^{(m)}\in\mathbb{H}((b_v),(p_v))_\rho$, for each $m\in\mathbb{N}$, using Theorem 13 and the continuity of ρ at θ, we obtain

$$\begin{aligned} g(B)&=g(B-B_m+B_m)\leq KK_0(g(B_m-B)+g(B_m))\\ &=KK_0\rho\left(\|B_m-B\|\sum_{m=0}^{\infty}e^{(m)}\right)+KK_0\rho\big(f_s^{(m)}\big)<\varepsilon. \end{aligned}\tag{32}$$

Thus, we have $f_s\in\mathbb{H}((b_v),(p_v))_\rho$; then, $B\in S_{\mathbb{H}((b_v),(p_v))_\rho}(X,Y)$. □

Theorem 15. *If X and Y are Banach spaces and conditions (a1) and (a2) are satisfied, then $(S_{\mathbb{H}((b_v),(p_v))_\rho},g)$ is a pre-quasi closed mapping ideal.*

Proof. Suppose the conditions are verified; then, the function $g(B)=\rho(f_s)$ is a pre-quasi norm on $S_{\mathbb{H}((b_v),(p_v))_\rho}$. Assume $B_m\in S_{\mathbb{H}((b_v),(p_v))_\rho}(X,Y)$, with $m\in\mathbb{N}$ and $\lim_{m\longrightarrow\infty}g(B_m-B)=0$. Therefore, $f_s^{(m)}\in\mathbb{H}((b_v),(p_v))_\rho\in\mathbb{C}$ and $f_s^{(m)}(z)=\sum_{v=0}^{\infty}s_v(B_m)z^v\in\mathbb{C}$. Suppose $h_s(z)=:\sum_{v=0}^{\infty}s_v(B_i-B_j)z^v$; then, from conditions (iv) and (vii) of Definition 3 and since $L(X,Y)\supseteq S_{\mathbb{H}((b_v),(p_v))_\rho}(X,Y)$, we get

$$\begin{aligned} g\big(B-B_j\big)&=\rho(h_s)\geq\rho\big(s_0\big(B-B_j\big)e^{(0)}\big)=\rho\Big(\big\|B-B_j\big\|e^{(0)}\Big)\\ &\geq\xi\big\|B-B_j\big\|\rho\big(e^{(0)}\big), \end{aligned}\tag{33}$$

and then $(B_m)_{m\in\mathbb{N}}$ is a convergent sequence in $L(X,Y)$. While the space $L(X,Y)$ is a Banach space, there exists $B\in L(X,Y)$ with $\lim_{m\longrightarrow\infty}\|B_m-B\|=0$ and since $f_s^{(m)}\in\mathbb{H}((b_v),(p_v))_\rho$, for each $m\in\mathbb{N}$, using Theorem 13 and the continuity of ρ at θ, one can see that

$$\begin{aligned} g(B)&=g(B-B_m+B_m)\leq KK_0(g(B_m-B)+g(B_m))\\ &=KK_0\rho\left(\|B_m-B\|\sum_{m=0}^{\infty}e^{(m)}\right)+KK_0\rho\big(f_s^{(m)}\big)<\varepsilon, \end{aligned}\tag{34}$$

and we have $f_s\in\mathbb{H}((b_v),(p_v))_\rho$; then, $B\in S_{\mathbb{H}((b_v),(p_v))_\rho}(X,Y)$. □

We deduce the following characteristics of the $s-$type $\mathbb{H}((b_v),(p_v))_\rho$ using Theorem 5.

Theorem 16. *For $s-$type $\mathbb{H}((b_v),(p_v))_\rho:=\Big\{(s_n(T))\in\mathbb{R}^{\mathbb{N}}:\ T\in S_{\mathbb{H}((b_v),(p_v))_\rho}(X,Y)\Big\}$, the following holds:*

(1) We have $s-$type $\mathbb{H}((b_v),(p_v))_\rho^{\mathscr{F}}$.

(2) *If* $(s_r(T_1))_{r=0}^{\infty} \in s$*−type* $\mathbb{H}((b_v),(p_v))_\rho$ *and* $(s_r(T_2))_{r=0}^{\infty} \in s$*−type* $\mathbb{H}((b_v),(p_v))_\rho$*, then* $(s_r(T_1+T_2))_{r=0}^{\infty} \in s$*−type* $\mathbb{H}((b_n),(p_n))_\rho$*.*

(3) *For all* $\lambda \in \mathbb{C}$ *and* $(s_r(T))_{r=0}^{\infty} \in s$*−type* $\mathbb{H}((b_n),(p_n))_\rho$*, then* $|\lambda|(s_r(T))_{r=0}^{\infty} \in s$*−type* $\mathbb{H}((b_n),(p_n))_\rho$*.*

(4) *The* s*−type* $\mathbb{H}((b_n),(p_n))_\rho$ *is solid.*

4.3. Smallness. We give here some inclusion relations concerning the space $S_{\mathbb{H}((b_n),(p_n))_\rho}$ for different (b_n) and (p_n).

Theorem 17. *If* X *and* Y *are Banach spaces with* $\dim(X)=\dim(Y)=\infty$, $0<p_n<q_n$, $0<a_n<b_n$*, for all* $n\in\mathbb{N}$*, and setups (a1) and (a2) are satisfied, it is true that*

$$S_{\mathbb{H}((b_n),(p_n))_\rho}(X,Y) \subset \neq S_{\mathbb{H}((a_n),(q_n))_\rho}(X,Y) \subset \neq L(X,Y). \tag{35}$$

Proof. Suppose $T\in S_{\mathbb{H}((b_n),(p_n))_\rho}(X,Y)$. Therefore, $f_s \in \mathbb{H}((b_n),(p_n))_\rho$ and $f_s(z)=\sum_{n=0}^{\infty} s_n(T)z^n \in \mathbb{C}$. One can see that

$$\sum_{n=0}^{\infty}(a_n s_n(T))^{q_n} < \sum_{n=0}^{\infty}(b_n s_n(T))^{p_n} < \infty, \tag{36}$$

hence $T\in S_{\mathbb{H}((a_n),(q_n))_\rho}(X,Y)$. Next, if we take T with $s_n(T)=((n+1)^{-1/p_n}/b_n)$, then $T\notin S_{\mathbb{H}((b_n),(p_n))_\rho}(X,Y)$ and $T\in S_{\mathbb{H}((a_n),(q_n))_\rho}(X,Y)$. Clearly, $S_{\mathbb{H}((a_n),(q_n))_\rho}(X,Y)\subset L(X,Y)$. By choosing T with $s_n(T)=((n+1)^{-1/q_n}/a_n)$, then $T\notin S_{\mathbb{H}((a_n),(q_n))_\rho}(X,Y)$ and $T\in L(X,Y)$. This finishes the proof. □

In this part, we investigate the setups for which $S^{\text{app}}_{\mathbb{H}((b_n),(p_n))_\rho}$ is small.

Theorem 18. *If* X *and* Y *are Banach spaces with* $\dim(X)=\dim(Y)=\infty$*, assume that the conditions (a1), (a2), and* $(b_n)\notin \ell^{(p_n)}$ *are satisfied, and hence* $(S^{app}_{\mathbb{H}((b_n),(p_n))_\rho}, g)$ *is small, where* $g(U)=\sum_{j=0}^{\infty}(b_j\alpha_j(U))^{p_j}$*.*

Proof. Let $S^{\text{app}}_{\mathbb{H}((b_n),(p_n))_\rho}(X,Y)=L(X,Y)$. Therefore, one gets $V>0$ so that $g(U)\le V\|U\|$, for every $U\in L(X,Y)$. According to Dvoretzky's theorem [26] with $r\in\mathbb{N}$, there are quotient spaces X/λ_r and subspaces η_r of Y that mapped onto ℓ_2^r by isomorphisms D_r and B_r with $\|D_r\|\|D_r^{-1}\|\le 2$ and $\|B_r\|\|B_r^{-1}\|\le 2$. Let I_r be the identity mapping on ℓ_2^r, ζ_r be the quotient mapping from X onto X/λ_r, and J_r be the natural embedding mapping from η_r into Y. Let h_a, for all $a\in\mathbb{N}$, be the Bernstein numbers [27]; we have then

$$\begin{aligned}1=h_a(I_r)&=h_a\left(B_rB_r^{-1}I_rD_rD_r^{-1}\right)\le\|B_r\|h_a\left(B_r^{-1}I_rD_r\right)\|D_r^{-1}\|,\\&=\|B_r\|h_a\left(J_rB_r^{-1}I_rD_r\right)\|D_r^{-1}\|\le\|B_r\|_r d_a\left(J_rB_r^{-1}I_rD_r\right)\|D_r^{-1}\|\\&=\|B_r\|d_a\left(J_rB_r^{-1}I_rD_r\zeta_r\right)\|D_r^{-1}\|\le\|B_r\|_r\alpha_a\left(J_rB_r^{-1}I_rD_r\zeta_r\right)\|D_r^{-1}\|,\end{aligned}\tag{37}$$

for $0\le j\le r$. We have $l\ge 1$ so that

$$\begin{aligned}b_j^{p_j}&\le\left(\|B_r\|\|D_r^{-1}\|\right)^{p_j}\left(b_j\alpha_j\left(J_rB_r^{-1}I_rD_r\zeta_r\right)\right)^{p_j},\\b_j^{p_j}&\le l\|B_r\|\left(b_j\alpha_j\left(J_rB_r^{-1}I_rD_r\zeta_r\right)\right)^{p_j}\|D_r^{-1}\|,\\\sum_{j=0}^{r}b_j^{p_j}&\le l\|B_r\|\|D_r^{-1}\|\sum_{j=0}^{r}b_j\left(\alpha_j\left(J_rB_r^{-1}I_rD_r\zeta_r\right)\right)^{p_j},\\\sum_{j=0}^{r}b_j^{p_j}&\le l\|B_r\|\|D_r^{-1}\|g\left(J_rB_r^{-1}I_rD_r\zeta_r\right),\\\sum_{j=0}^{r}b_j^{p_j}&\le lV\|B_r\|\|D\|_r^{-1}\|J_rB_r^{-1}I_rD_r\zeta\|_r,\\\sum_{j=0}^{r}b_j^{p_j}&\le lV\|B_r\|\|D_r^{-1}\|\|J_r\|B_r^{-1}\|I\|_r\|D_r\zeta_r\|\\&=lV\|B_r\|\|D_r^{-1}\|\|B_r^{-1}\|\|I_r\|\|D_r\|,\\\sum_{j=0}^{r}b_j^{p_j}&\le 4lV.\end{aligned}\tag{38}$$

As $r\longrightarrow\infty$, then $\sum_{j=0}^{\infty}b_j^{p_j}<\infty$. This contradicts $(b_n)\notin\ell^{(p_n)}$. Therefore, $\dim(X)<\infty$ and $\dim(Y)<\infty$. Hence, the space $S^{\text{app}}_{\mathrm{H}((b_n),(p_n))_\rho}$ is small. □

By the same manner, we can easily conclude the next theorem.

Theorem 19. *If* X *and* Y *are Banach spaces with* $\dim(X)=\dim(Y)=\infty$*, assume that conditions (a1), (a2), and* $(b_n)\notin\ell^{(p_n)}$ *are satisfied, and hence* $(S^{Kol}_{\mathbb{H}((b_n),(p_n))_\rho}, g)$ *is small, where* $g(U)=\sum_{j=0}^{\infty}(b_jd_j(U))^{p_j}$*.*

4.4. Simpleness. We introduce an answer of the next question; for which $\mathbb{H}((b_n),(p_n))_\rho$, is the space $S_{\mathbb{H}((b_n),(p_n))_\rho}$ simple?

Theorem 20. *If* (p_n), (q_n) *verify* $1\le p_n<q_n$ *and* $0<a_n<b_n$*, for all* $n\in\mathbb{N}$*, and the setups (a1), (a2) are satisfied, then*

$$\begin{aligned}&L\left(S_{\mathbb{H}((a_n),(q_n))_\rho}, S_{\mathbb{H}((b_n),(p_n))_\rho}\right)\\&=\Lambda\left(S_{\mathbb{H}((a_n),(q_n))_\rho}, S_{\mathbb{H}((b_n),(p_n))_\rho}\right).\end{aligned}\tag{39}$$

Proof. Suppose there is $T \in L(S_{\mathbb{H}((a_n),(q_n))_\rho}, S_{\mathbb{H}((b_n),(p_n))_\rho})$. and $T \notin \Lambda(S_{\mathbb{H}((a_n),(q_n))_\rho}, S_{\mathbb{H}((b_n),(p_n))_\rho})$. According to Lemma 1, we can find $G \in L(S_{\mathbb{H}((a_n),(q_n))_\rho})$ and $B \in L(S_{\mathbb{H}((b_n),(p_n))_\rho})$ with $BTGI_m = I_m$. For every $m \in \mathbb{N}$, one has

$$\begin{aligned}\|I_m\|_{S_{\mathbb{H}((b_v),(p_v))_\rho}} &= \left(\sum_{v=0}^{\infty}(b_v\alpha_v(I_m))^{p_v}\right)^{(1/\sup p_v)} = \left(\sum_{v=0}^{m-1} b_v\right)^{(1/\sup p_v)} \le \|BTG\|\|I_m\|_{S_{\mathbb{H}((a_v),(q_v))_\rho}} \\ &\le \left(\sum_{v=0}^{\infty}(a_v\alpha_v(I_m))^{q_v}\right)^{(1/\sup q_v)} = \left(\sum_{v=0}^{m-1} a_v\right)^{(1/\sup q_v)}.\end{aligned} \tag{40}$$

This contradicts Theorem 17. □

Corollary 1. *If (p_n), (q_n) verify $1 \le p_n < q_n$ and $0 < a_n < b_n$, for all $n \in \mathbb{N}$, and the setups (a1), (a2) are satisfied, then $L((S_{\mathbb{H}((a_n),(q_n))_\rho}, S_{\mathbb{H}((b_n),(p_n))_\rho})) = L_C((S_{\mathbb{H}((a_n),(q_n))_\rho}, S_{\mathbb{H}((b_n),(p_n))_\rho}))$.*

Proof. It is clear from $\Lambda \subseteq L_c$. □

Theorem 21. *If setups (a1), (a2) are satisfied with $p_0 \ge 1$, then the space $S_{\mathbb{H}((b_n),(p_n))_\rho}$ is simple.*

Proof. Assume $T \in L_C(S_{\mathbb{H}((b_n),(p_n))_\rho})$ and $T \notin \Lambda(S_{\mathbb{H}((b_n),(p_n))_\rho})$. From Lemma 1, one has $G, B \in L(S_{\mathbb{H}((b_n),(p_n))_\rho})$ so that $BTGI_k = I_k$. We have $I_{\mathbb{H}((b_n),(p_n))_\rho} \in L_C(S_{\mathbb{H}((b_n),(p_n))_\rho})$. Therefore, $L(S_{\mathbb{H}((b_n),(p_n))_\rho}) = L_C(S_{\mathbb{H}((b_n),(p_n))_\rho})$. This implies that there is one non-trivial closed ideal $\Lambda(S_{\mathbb{H}((b_n),(p_n))_\rho})$ in $L(S_{\mathbb{H}((b_n),(p_n))_\rho})$. □

4.5. Spectrum. In this part, we expound the sufficient conditions on $\mathbb{H}((b_n),(p_n))_\rho$ such that $(S_{\mathbb{H}((b_n),(p_n))_\rho})^\lambda$ equals $S_{\mathbb{H}((b_n),(p_n))_\rho}$.

Theorem 22. *If X and Y are Banach spaces with $\dim(X) = \dim(Y) = \infty$ and suppose setups (a1), (a2) are satisfied and $\inf_n b_n^{p_n} > 0$, then*

$$\left(S_{\mathbb{H}((b_n),(p_n))_\rho}\right)^\lambda(X,Y) = S_{\mathbb{H}((b_n),(p_n))_\rho}(X,Y). \tag{41}$$

Proof. Let $T \in (S_{\mathbb{H}((b_n),(p_n))_\rho})^\lambda(X,Y)$, and hence $f_\lambda \in \mathbb{H}((b_n),(p_n))_\rho$, where $f_\lambda(z) = \sum_{n=0}^\infty \lambda_n(T)z^n \in \mathbb{C}$ with $\rho(f_\lambda) = \sum_{n=0}^\infty |b_n\lambda_n(T)|^{p_n} < \infty$, and $\|T - \lambda_l(T)I\| = 0$, for all $l \in \mathbb{N}$. We have $T = \lambda_l(T)I$, with $l \in \mathbb{N}$, so $s_l(T) = s_l(\lambda_l(T)I) = |\lambda_l(T)|$, with $l \in \mathbb{N}$. Therefore, $f_s \in \mathbb{H}((b_n),(p_n))_\rho$, so $T \in S_{\mathbb{H}((b_n),(p_n))_\rho}(X,Y)$.

Secondly, assume $T \in S_{\mathbb{H}((b_n),(p_n))_\rho}(X,Y)$. Therefore, $f_s \in \mathbb{H}((b_n),(p_n))_\rho$. Hence, we have

$$\infty > \sum_{r=0}^{\infty}\left|b_r s_r(T)\right|^{p_r} \ge \inf_r b_r^{p_r}\sum_{r=0}^{\infty}\left[s_r(T)\right]^{p_r}. \tag{42}$$

Since $\inf_r b_r^{p_r} > 0$, then $\lim_{r\longrightarrow\infty} s_r(T) = 0$. Assume $\|T - s_r(T)I\|^{-1}$ exists, for every $r \in \mathbb{N}$. Therefore, $(\|T - s_r(T)I\|^{-1})_{r\in\mathbb{N}} \in \ell_\infty$. So, $\lim_{r\longrightarrow\infty}\|T - s_r(T)I\|^{-1} = \|T\|^{-1}$ exists and is bounded. From the pre-quasi mapping ideal of $(S_{\mathbb{H}((b_n),(p_n))_\rho}, g)$, we obtain

$$\begin{aligned} I = TT^{-1} &\in S_{\left(\mathbb{H}((b_r),(p_r))_\rho\right)}(X,Y) \Rightarrow \sum_{r=0}^{\infty} e^{(r)} \\ &\in \mathbb{H}((b_r),(p_r))_\rho \Rightarrow \sum_{r=0}^{\infty} b_r^{p_r} < \infty. \end{aligned} \tag{43}$$

This contradicts $\inf_r b_r^{p_r} > 0$. Therefore, $\|T - s_r(T)I\| = 0$, for every $r \in \mathbb{N}$. This gives $T \in (S_{\mathbb{H}((b_n),(p_n))_\rho})^\lambda(X,Y)$. This provides the proof. □

5. Application of Shift Mappings on $\mathbb{H}((b_r),(p_r))_\rho$

Specifically, we explore the upper limits of s-numbers for infinite series of the weighted r-th power forward and backward shift mapping on $\mathbb{H}((b_r),(p_r))_\rho$ and their applications to various entire functions in this section, where $\rho(f) = [\sum_{r=0}^\infty |b_r\hat{f}_r|^{p_r}]^{(1/\varpi_p)}$, for all $f \in \mathbb{H}((b_r),(p_r))_\rho$.

Theorem 23. *Let conditions (a1) and (a2) be satisfied, $\inf_n b_n \ge 1$, and $\sup_r (b_{r+1}/b_r)^{p_{r+1}/\varpi_p} < \infty$; then, $V_z \in L(\mathbb{H}((b_r),(p_r))_\rho)$ with $\|V_z\| = \sup_r (b_{r+1}/b_r)^{(p_{r+1}/\varpi_p)}$.*

Proof. Assume that the conditions are satisfied. For $f \in \mathbb{H}((b_r),(p_r))_\rho$, since (p_r) is increasing and bounded from above with $p_r > 0$, for all $r \in \mathbb{N}$, then

$$\begin{aligned}\rho(V_z f) = \rho(zf) &= \left[\sum_{r=0}^{\infty}\left|b_{r+1}\hat{f}_r\right|^{p_{r+1}}\right]^{1/\varpi_p} \le \sup_r\left(\frac{b_{r+1}}{b_r}\right)^{p_{r+1}/\varpi_p} \\ &\quad \left[\sum_{r=0}^{\infty}|b_r\hat{f}_r|^{p_{r+1}}\right]^{1/\varpi_p} \\ &\le \sup_r\left(\frac{b_{r+1}}{b_r}\right)^{p_{r+1}/\varpi_p}\rho(f). \end{aligned} \tag{44}$$

This gives $V_z \in L(\mathbb{H}((b_r),(p_r))_\rho)$ with $\|V_z\| \le \sup_r (b_{r+1}/b_r)^{(p_{r+1}/\varpi_p)}$. Since $V_z \in L(\mathbb{H}((b_r),(p_r))_\rho)$, then there is $A > 0$ with $\rho(V_z f) \le A\rho(f)$, for all $f \in \mathbb{H}((b_r),(p_r))_\rho$. Hence, $\rho(V_z e^{(r)}) \le A\rho(e^{(r)})$, and one gets $\sup_r (b_{r+1}/b_r)^{(p_{r+1}/\varpi_p)} \le \|V_z\|$. This completes the proof. □

Theorem 24. *Let conditions (a1) and (a2) be satisfied,* $\sup_n b_n \ge 1$, *and* $\sup_r (b_r/b_{r+1})^{(p_r/\varpi_p)} < \infty$; *then,* $B_z \in L(\mathbb{H}((b_r),(p_r))_\rho)$ *with* $\|B_z\| = \sup_r (b_r/b_{r+1})^{(p_r/\varpi_p)}$.

Proof. Assume the conditions are satisfied. For $f \in \mathbb{H}((b_r),(p_r))_\rho$, since (p_r) is increasing and bounded from above with $p_r > 0$, for all $r \in \mathbb{N}$, then

$$\rho(B_z f) = \left[\sum_{r=0}^{\infty}\left|b_r \widehat{f_{r+1}}\right|^{p_r}\right]^{1/\varpi_p} \le \sup_r\left(\frac{b_r}{b_{r+1}}\right)^{p_r/\varpi_p} \left[\sum_{r=0}^{\infty}\left|b_{r+1}\widehat{f_{r+1}}\right|^{p_r}\right]^{1/\varpi_p} \le \sup_r\left(\frac{b_r}{b_{r+1}}\right)^{p_r/\varpi_p}\rho(f). \tag{45}$$

This gives $B_z \in L(\mathbb{H}((b_r),(p_r))_\rho)$ with $\|B_z\| \le \sup_r (b_r/b_{r+1})^{(p_r/\varpi_p)}$. Since $B_z \in L(\mathbb{H}((b_r),(p_r))_\rho)$, then there is $A > 0$ with $\rho(B_z f) \le A\rho(f)$, for all $f \in \mathbb{H}((b_r),(p_r))_\rho$. Hence, $\rho(B_z e^{(r)}) \le A\rho(e^{(r)})$, and one gets $\sup_r (b_r/b_{r+1})^{p_r/\varpi_p} \le \|B_z\|$. This completes the proof. □

By $\mathbb{U}$, we denote the open unit disc in $\mathbb{C}$.

Theorem 25. *Let conditions (a1) and (a2) be satisfied with* $p_0 \ge 1$. *If* $\limsup \sqrt[r]{b_r^{p_r}} = 1$, *then every function in* $\mathbb{H}((b_r),(p_r))_\rho$ *is analytic on* $\mathbb{U}$. *Furthermore, the convergence in* $\mathbb{H}((b_r),(p_r))_\rho$ *implies the uniform convergence on* $B \subseteq \mathbb{U}$, *where* B *is compact.*

Proof. Let $\limsup \sqrt[r]{b_r^{p_r}} = 1$, and $h \in \mathbb{H}((b_r),(p_r))_\rho$. Then, $h(y) = \sum_{r=0}^{\infty} \hat{h}_r y^r \in \mathbb{C}$, with $y \in \mathbb{C}$ and $\rho(h) = [\sum_{r=0}^{\infty} |b_r \hat{h}_r|^{p_r}]^{1/\varpi_p} < \infty$. Therefore, $\limsup \sqrt[r]{|b_r \hat{h}_r|^{p_r}} < 1$. This gives

$$\limsup \sqrt[r]{\left|\hat{h}_r\right|^{p_r}} < \frac{1}{\limsup \sqrt[r]{\left|b_r\right|^{p_r}}} = 1. \tag{46}$$

As $(p_r) \in mi_{\nearrow} \cap \ell_\infty$, one gets $\limsup \sqrt[r]{|\hat{h}_r|}|y| < |y| < 1$, with $y \in \mathbb{U}$. Hence, $h(y) = \sum_{r=0}^{\infty} \hat{h}_r y^r \in \mathbb{C}$, with $y \in \mathbb{U}$. Assume $h^k(y) \in B$, with $k \in \mathbb{N}$. Suppose $\lim_{k\longrightarrow\infty} \rho(h^k - h) = 0$, where $h \in \mathbb{H}((b_r),(p_r))_\rho$, and we have

$$\left|h^k(y) - h(y)\right| = \left|\sum_{r=0}^{\infty}\left(\hat{h}_r^k - \hat{h}_r\right)y^r\right| \le \left|\sum_{r=0}^{\infty}\left\|\hat{h}_r^k - \hat{h}_r\right\| y^r\right| \le \left[\sum_{n=0}^{\infty}\left|\hat{h}_r^k - \hat{h}_r\right|^{p_r} b_r^{p_r}\right]^{1/\varpi_p} = \left[\sum_{r=0}^{\infty}\frac{|y|^{rq_r}}{b_r^{q_r}}\right]^{1/\varpi_q}\rho\left(h^k - h\right), \left[\sum_{r=0}^{\infty}\frac{|y|^{rq_r}}{b_r^{q_r}}\right]^{1/\varpi_q}, \tag{47}$$

where (q_r) is increasing and bounded with $q_0 \ge 1$ and $(1/p_r) + (1/q_r) = 1$, for all $r \in \mathbb{N}$. Clearly, $\limsup_{r\longrightarrow\infty} (|y|^{q_r}/b_r^{(q_r/r)}) < 1$; then, $\sum_{r=0}^{\infty} |y|^{rq_r}/b_r^{q_r} < \infty$. So, $\lim_{k\longrightarrow\infty} h^k(y) = h(y) \in B$. □

Theorem 26. *If* V_z *is the forward shift mapping on* $\mathbb{H}((b_r),(p_r))_\rho$, *we have*

$$\sup_{\mathrm{card}\xi=r+1}\inf_{k\in\xi}\left(\frac{b_{k+n}}{b_k}\right)^{(p_{k+n}/\varpi_p)}\frac{1}{A_n} \le s_r\left(V_z^n\right) \le \sup_{\mathrm{card}\xi=r+1}\inf_{k\in\xi}\left(\frac{b_{k+n}}{b_k}\right)^{(p_{k+n}/\varpi_p)}, \tag{48}$$

where $A_n = [[\sum_{k=0}^{\infty} |b_k \hat{f}_k|^{p_k}]^{(1/\varpi_p)}/[\sum_{k=0}^{\infty} |b_k \hat{f}_k|^{p_{k+n}}]^{(1/\varpi_p)}]$.

Proof. Let card $\xi = r+1$ and $V_z^n f \in \mathbb{H}((b_r),(p_r))_\rho$, for all $f \in \mathbb{H}((b_r),(p_r))_\rho$, for which $f(y) = \sum_{k=0}^{\infty} \hat{f}_k y^k \in \mathbb{C}$ with $y \in \mathbb{C}$ and $\rho(f) = [\sum_{k=0}^{\infty} |b_k \hat{f}_k|^{p_k}]^{(1/\varpi_p)} < \infty$. Therefore, $V_z^n f(z) = \sum_{k=0}^{\infty} \hat{f}_k z^{k+n}$ and $\rho(V_z^n f) = [\sum_{k=0}^{\infty} |b_{k+n} \hat{f}_k|^{p_{k+n}}]^{(1/\varpi_p)} < \infty$.

Let P_ξ be a mapping on $\mathbb{H}((b_r),(p_r))_\rho$ with rank $P_\xi = r+1$ defined by

$$\left(P_\xi g\right)(z) = P_\xi\left(\sum_{k=0}^{\infty}\hat{f}_k z^{k+n}\right) = \sum_{k\in\xi}\hat{f}_k z^{k+n}. \tag{49}$$

Since $\rho(P_\xi g) = [\sum_{k\in\xi} |b_{k+n}\hat{f}_k|^{p_{k+n}}]^{(1/\varpi_p)} \le [\sum_{k=0}^{\infty} |b_{k+n} \hat{f}_k|^{p_{k+n}}]^{(1/\varpi_p)} = \rho(g)$, this gives $\|P_\xi\| \le 1$. Define a mapping S_z^n by $(S_z^n h)(z) = S_z^n(\sum_{k\in\xi} \hat{f}_k z^{k+n}) = \sum_{k=0}^{\infty} \hat{f}_k z^k$, and we have

$$\rho\left(S_z^n h\right) = \left[\sum_{k=0}^{\infty}\left|b_k\hat{f}_k\right|^{p_k}\right]^{(1/\varpi_p)} \le U_n\left[\sum_{k\in\xi}\left|b_{k+n}\hat{f}_k\right|^{p_{k+n}}\right]^{(1/\varpi_p)} = U_n\rho(g). \tag{50}$$

This implies that $\|S_z^n\| \le U_n$, where $1 \le U_n = [\sum_{k=0}^{\infty} |b_k \hat{f}_k|^{p_k}]^{(1/\varpi_p)}/[\sum_{k\in\xi} |b_{k+n}\hat{f}_k|^{p_{k+n}}]^{(1/\varpi_p)} < \infty$. Then, the identity mapping will be $I_{r+1} = P_\xi V_z^n S_z^n$, and from the definition of s-numbers, we have

$$s_r\left(I_{r+1}\right) = 1 \le \left\|P_\xi\right\| s_r\left(V_z^n\right)\left\|S_z^n\right\| \le s_r\left(V_z^n\right)\left\|S_z^n\right\| \Rightarrow$$

$$s_r\left(V_z^n\right) \ge \frac{1}{\|S\|_z^n} \ge \frac{1}{U_n} = \frac{\left[\sum_{k\in\xi}\left|b_{k+n}\hat{f}_k\right|^{p_{k+n}}\right]^{(1/\varpi_p)}}{\left[\sum_{k=0}^{\infty}\left|b_k\hat{f}_k\right|^{p_k}\right]^{(1/\varpi_p)}} \ge \inf_{k\in\xi}\left(\frac{b_{k+n}}{b_k}\right)^{(p_k+n/\varpi_p)}\frac{1}{A_n}. \tag{51}$$

Since for all card $\xi = r+1$, the last inequality is verified, so one can see that

$$s_r\left(V_z^n\right) \ge \sup_{\mathrm{card}\xi=r+1}\inf_{k\in\xi}\left(\frac{b_{k+n}}{b_k}\right)^{(p_k+n/\varpi_p)}\frac{1}{A_n}. \tag{52}$$

In contrary, let card $\xi = r$, where $\xi \subset \mathbb{N}$. Define the mapping R_z^n as $(R_z^n v)(z) = R_z^n(\sum_{k=0}^{\infty} \hat{f}_k z^k) = \sum_{k\in\xi}\widehat{f_k}$

$z^{k+n}\sum_{k\in\xi}\hat{f}_k z^{k+n}$. From the definition of approximation numbers, we have

$$\begin{aligned} s_r(V_z^n) &\le \alpha_r(V_z^n) \le \|V_z^n - R_z^n\| \le \sup_{|f(z)|\ne 0}\frac{|(V_z^n - R_z^n) f(z)|}{|f(z)|} = \sup_{|f(z)|\ne 0}\frac{\sum_{k\notin\xi}\hat{f}_k z^{k+n}}{|f(z)|} \\ &\le \sup_{|f(z)|\ne 0}\frac{\left[\sum_{k\notin\xi}\left|b_{k+n}\hat{f}_k\right|^{p_{k+n}}\right]^{(1/\varpi_p)}}{|f(z)|} \le \sup_{k\notin\xi}\left(\frac{b_{k+n}}{b_k}\right)^{(p_k+n/\varpi_p)} \end{aligned} \tag{53}$$

Since for all card $\xi = r$, the last inequality holds and by using Lemma 2, one has

$$\sup_{card\xi=r+1}\inf_{k\in\xi}\left(\frac{b_{k+n}}{b_k}\right)^{(p_{k+n}/\varpi_p)}\frac{1}{A_n} \le s_r(V_z^n) \le \inf_{card\xi=r}\sup_{k\notin\xi}\left(\frac{b_{k+n}}{b_k}\right)^{(p_k+n/\varpi_p)} = \sup_{card\xi=r+1}\inf_{k\in\xi}\left(\frac{b_{k+n}}{b_k}\right)^{(p_{k+n}/\varpi_p)}. \tag{54}$$

This completes the proof. □

Theorem 27. *If B_z is the backward shift mapping on $\mathbb{H}((b_r),(p_r))_\rho$, then*

$$\begin{aligned} &\sup_{card\xi=r+1}\inf_{k\in\xi}\left(\frac{b_k}{b_{k+n}}\right)^{(p_k/\varpi_p)}\frac{1}{G_n} \le s_r(B_z^n) \\ &\le \sup_{card\xi=r+1}\inf_{k\in\xi}\left(\frac{b_k}{b_{k+n}}\right)^{(p_k/\varpi_p)}, \end{aligned} \tag{55}$$

where $G_n = [\sum_{k=0}^{\infty}|b_k\hat{f}_k|^{p_k}]^{(1/\varpi_p)}/[[\sum_{k\in\xi}|b_{k+n}\widehat{f_{k+n}}|^{p_k}]^{(1/\varpi_p)}]$.

Proof. Assume card $\xi = r+1$ and $B_z^n f \in \mathbb{H}((b_r),(p_r))_\rho$, for every $f \in \mathbb{H}((b_r),(p_r))_\rho$, where $f(y) = \sum_{k=0}^{\infty}\hat{f}_k y^k \in \mathbb{C}$ with $y \in \mathbb{C}$ and $\rho(f) = [\sum_{k=0}^{\infty}|b_k\hat{f}_k|^{p_k}]^{(1/\varpi_p)} < \infty$. Therefore, $B_z^n f(z) = \sum_{k=0}^{\infty}\widehat{f_{k+n}}z^k$ and $\rho(B_z^n f) = [\sum_{k=0}^{\infty}|b_k\widehat{f_{k+n}}|^{p_k}]^{(1/\varpi_p)} < \infty$.

Suppose P_ξ is a mapping on $\mathbb{H}((b_r),(p_r))_\rho$ with rank $P_\xi = r+1$ evident by

$$(P_\xi g)(z) = P_\xi\left(\sum_{k=0}^{\infty}\widehat{f_{k+n}}z^k\right) = \sum_{k\in\xi}\widehat{f_{k+n}}z^k. \tag{56}$$

As $\rho(P_\xi g) = [\sum_{k\in\xi}|b_k\widehat{f_{k+n}}|^{p_k}]^{(1/\varpi_p)} \le [\sum_{k=0}^{\infty}|b_k\widehat{f_{k+n}}|^{p_k}]^{(1/\varpi_p)} = \rho(g)$. This implies that $\|P_\xi\| \le 1$. Define a mapping S_z^n by $(S_z^n h)(z) = S_z^n(\sum_{k\in\xi}\widehat{f_{k+n}}z^k) = \sum_{k=0}^{\infty}\hat{f}_k z^k$, and one gets

$$\rho(S_z^n h) = \left[\sum_{k=0}^{\infty}\left|b_k\hat{f}_k\right|^{p_k}\right]^{(1/\varpi_p)} \le U_n\left[\sum_{k\in\xi}\left|b_k\widehat{f_{k+n}}\right|^{p_k}\right]^{(1/\varpi_p)} = U_n\rho(h). \tag{57}$$

Therefore, $\|S_z^n\| \le U_n$, where $1 \le U_n = [\sum_{k=0}^{\infty}|b_k\hat{f}_k|^{p_k}]^{(1/\varpi_p)}/[\sum_{k\in\xi}|b_k\widehat{f_{k+n}}|^{p_k}]^{(1/\varpi_p)} < \infty$. Hence, the identity mapping will be $I_{r+1} = P_\xi B_z^n S_z^n$, and in view of the definition of s-numbers, one has

$$\begin{aligned} &s_r(I_{r+1}) = 1 \le \|P_\xi\| s_r(B_z^n)\|S_z^n\| \le s_r(B_z^n)\|S_z^n\| \Rightarrow \\ &s_r(B_z^n) \ge \frac{1}{\|S_z^n\|} \ge \frac{1}{U_n} = \frac{\left[\sum_{k\in\xi}\left|b_k\widehat{f_{k+n}}\right|^{p_k}\right]^{(1/\varpi_p)}}{\left[\sum_{k=0}^{\infty}\left|b_k\hat{f}_k\right|^{p_k}\right]^{(1/\varpi_p)}} \\ &\ge \inf_{k\in\xi}\left(\frac{b_k}{b_{k+n}}\right)^{(p_k/\varpi_p)}\frac{1}{G_n}. \end{aligned} \tag{58}$$

Since for every card $\xi = r+1$, the last inequality is confirmed, and one obtains

$$s_r(B_z^n) \ge \sup_{card\xi=r+1}\inf_{k\in\xi}\left(\frac{b_k}{b_{k+n}}\right)^{(p_k/\varpi_p)}\frac{1}{G_n}. \tag{59}$$

In contrary, let card $\xi = r$, where $\xi \subset \mathbb{N}$. Define the mapping R_z^n as $(R_z^n v)(z) = R_z^n(\sum_{k=0}^{\infty}\hat{f}_k z^k) = \sum_{k\in\xi}\widehat{f_{k+n}}z^k$. From the definition of approximation numbers, one gets

$$s_r(B_z^n) \le \alpha_r(B_z^n) \le \|B_z^n - R_z^n\| \le \sup_{|f(z)|\ne 0} \frac{|(B_z^n - R_z^n) f(z)|}{|f(z)|} = \sup_{|f(z)|\ne 0} \frac{\left|\sum_{k\notin\xi} \widehat{f_{k+n}} z^k\right|}{|f(z)|}$$
$$\le \sup_{|f(z)|\ne 0} \frac{\left[\sum_{k\notin\xi} \left|b_k \widehat{f_{k+n}}\right|^{p_k}\right]^{(1/\varpi_p)}}{|f(z)|} \le \sup_{k\notin\xi} \left(\frac{b_k}{b_{k+n}}\right)^{(p_k/\varpi_p)}. \tag{60}$$

Since for all card $\xi = r$, the last inequality holds, and by using Lemma 2, one has

$$\sup_{\text{card}\xi=r+1} \inf_{k\in\xi} \left(\frac{b_k}{b_{k+n}}\right)^{(p_k/\varpi_p)} \frac{1}{G_n} \le s_r(B_z^n)$$
$$\le \inf_{\text{card}\xi=r} \sup_{k\notin\xi} \left(\frac{b_k}{b_{k+n}}\right)^{(p_k/\varpi_p)} = \sup_{\text{card}\xi=r+1} \inf_{k\in\xi} \left(\frac{b_k}{b_{k+n}}\right)^{(p_k/\varpi_p)}. \tag{61}$$

This finishes the proof. □

Theorem 28. *If conditions (a1) and (a2) are satisfied with $p_0 \ge 1$, let $\sum_{m=0}^{\infty} c_m V_z^m$ be a shift mapping on the space $\mathbb{H}((b_r),(p_r))_\rho$ and $(c_m)_{m=0}^{\infty} \in \ell^{((p_m)/\varpi_p)}$; then,*

$$\sup_j \left[\sum_{m=0}^{\infty} |c_m|^{p_{m+j}} \frac{b_{m+j}^{p_{m+j}}}{b_j^{p_j}}\right]^{(1/\varpi_p)} \le \left\|\sum_{m=0}^{\infty} c_m V_z^m\right\|$$
$$\le \sup_{m,j} \left(\frac{b_{m+j}}{b_j}\right)^{(p_{m+j}/\varpi_p)} \sum_{m=0}^{\infty} |c_m|^{(p_m/\varpi_p)}. \tag{62}$$

Proof. For $f \in \mathbb{H}((b_r),(p_r))_\rho$, we have $\sum_{m=0}^{\infty} c_m V_z^m f(z) = \sum_{j=0}^{\infty}\sum_{j=0}^{\infty} c_m \hat{f}_j z^{j+m}$. One has

$$\left\|\sum_{m=0}^{\infty} c_m V_z^m\right\| \ge \frac{\rho\left(\sum_{m=0}^{\infty} c_m V_z^m e^{(j)}\right)}{\rho\left(e^{(j)}\right)} = \left[\frac{\sum_{m=0}^{\infty} |c_m b_{m+j}|^{p_{m+j}}}{b_j^{p_j}}\right]^{(1/\varpi_p)}$$
$$\ge \sup_j \left[\sum_{m=0}^{\infty} |c_m|^{p_{m+j}} \frac{b_{m+j}^{p_{m+j}}}{b_j^{p_j}}\right] 1^{(1/\varpi_p)}. \tag{63}$$

Since ρ satisfies the triangle inequality, we have

$$\left\|\sum_{m=0}^{\infty} c_m V_z^m\right\| = \sup_{\rho(f)\ne 0} \frac{\rho\left(\sum_{m=0}^{\infty} c_m V_z^m f\right)}{\rho(f)} \le \sup_{\rho(f)\ne 0} \frac{\sum_{m=0}^{\infty}\left[\sum_{j=0}^{\infty}\left(|c_m|\left|\hat{f}_j\right| b_{m+j}\right)^{p_{m+j}}\right]^{(1/\varpi_p)}}{\left[\sum_{j=0}^{\infty}\left|\hat{f}_j b_j\right|^{p_j}\right]^{(1/\varpi_p)}}$$
$$\le \sup_{m,j}\left(\frac{b_{m+j}}{b_j}\right)^{(p_{m+j}/\varpi_p)} \frac{\sum_{m=0}^{\infty}\left[\sum_{j=0}^{\infty}\left(|c_m|\left|\hat{f}_j\right| b_j\right)^{p_{m+j}}\right]^{(1/\varpi_p)}}{\left[\sum_{j=0}^{\infty}\left|\hat{f}_j b_j\right|^{p_j}\right]^{(1/\varpi_p)}} \le \sup_{m,j}\left(\frac{b_{m+j}}{b_j}\right)^{(p_{m+j}/\varpi_p)} \sum_{m=0}^{\infty} |c_m|^{(p_m/\varpi_p)}. \tag{64}$$

□

Theorem 29. *If conditions (a1) and (a2) are satisfied with $p_0 \ge 1$, let $\sum_{j=0}^{\infty} c_j B_z^j$ be a shift mapping on the space $\mathbb{H}((b_r),(p_r))_\rho$ and $(c_j)_{j=0}^{\infty} \in \ell^{((p_j)/\varpi_p)}$; then,*

$$\sup_k \left[\sum_{j=0}^{\infty} |c_j|^{p_k} \frac{b_k^{p_k}}{b_{k+j}^{p_{k+j}}}\right]^{1/\varpi_p} \le \left\|\sum_{j=0}^{\infty} c_j B_z^j\right\|$$
$$\le \sup_{j,k}\left(\frac{b_k}{b_{k+j}}\right)^{p_k/\varpi_p} \sum_{j=0}^{\infty} |c_j|^{p_j/\varpi_p}. \tag{65}$$

Proof. Suppose $f \in \mathbb{H}((b_r),(p_r))_\rho$, and one has $\sum_{j=0}^{\infty} c_j B_z^j f(z) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} c_j \widehat{f_{k+j}} z^k$. We have

$$\left\|\sum_{j=0}^{\infty} c_j B_z^j\right\| \ge \frac{\rho\left(\sum_{j=0}^{\infty} c_j B_z^j e^{(k)}\right)}{\rho\left(e^{(k)}\right)} = \left[\frac{\sum_{j=0}^{\infty} |b_{k-j} c_j|^{p_{k-j}}}{b_k^{p_k}}\right]^{(1/\varpi_p)}$$
$$\ge \sup_k \left[\sum_{j=0}^{\infty} |c_j|^{p_k} \frac{b_k^{p_k}}{b_{k+j}^{p_{k+j}}}\right]^{(1/\varpi_p)}. \tag{66}$$

As ρ verifies the triangle inequality, one can see that

$$\begin{aligned}\left\|\sum_{j=0}^{\infty} c_j B_z^j\right\| &= \sup_{\rho(f)\neq 0}\frac{\rho\left(\sum_{j=0}^{\infty} c_j B_z^j f\right)}{\rho(f)} \leq \sup_{\rho(f)\neq 0}\frac{\sum_{j=0}^{\infty}\left[\sum_{k=0}^{\infty}\left(b_k\left|c_j\widehat{f_{k+j}}\right|\right)^{p_k}\right]^{(1/\varpi_p)}}{\left[\sum_{k=0}^{\infty}\left|b_k\widehat{f}_k\right|^{p_k}\right]^{(1/\varpi_p)}}\\ &\leq \sup_{j,k}\left(\frac{b_k}{b_{k+j}}\right)^{(p_k/\varpi_p)}\frac{\sum_{j=0}^{\infty}\left[\sum_{k=0}^{\infty}\left(b_{k+j}\left|c_j\right|\left|\widehat{f_{k+j}}\right|\right)^{p_k}\right]^{(1/\varpi_p)}}{\left[\sum_{k=0}^{\infty}\left|b_k\widehat{f}_k\right|^{p_k}\right]^{(1/\varpi_p)}} \leq \sup_{j,k}\left(\frac{b_k}{b_{k+j}}\right)^{(p_k/\varpi_p)}\sum_{j=0}^{\infty}\left|c_j\right|^{p_j/\varpi_p}.\end{aligned} \tag{67}$$

□

Theorem 30. *If conditions (a1) and (a2) are satisfied with $p_0 \geq 1$, let $\sum_{r=0}^{\infty} c_r V_z^r$ be a shift mapping on $\mathbb{H}((b_r),(p_r))_\rho$; then, the s-numbers of this mapping are given by*

$$\begin{aligned}s_r\left(\sum_{j=0}^{\infty} c_j V_z^j\right) &\leq \sup_{\operatorname{card}\xi=r+1}\inf_{k\in\xi}\sup_j\left(\frac{b_{j+k}}{b_k}\right)^{p_{j+k}/\varpi_p}\\ &\sum_{j=0}^{\infty}\left|c_j\right|^{p_j/\varpi_p}, \quad \text{for all}\left(c_j\right)_{j=0}^{\infty}\in \ell^{(p_j)/\varpi_p}.\end{aligned} \tag{68}$$

Proof. Let $\operatorname{card}\xi = r$, where $\xi\subset\mathbb{N}$. Define the mapping R as $Rf(z) = R\left(\sum_{k=0}^{\infty}\widehat{f}_k z^k\right) = \sum_{k\in\xi}\sum_{j=0}^{k} c_j\widehat{f_{k-j}}z^k$. Since the triangle inequality holds by ρ, we have

$$\begin{aligned}s_r\left(\sum_{j=0}^{\infty} c_j V_z^j\right) &\leq \alpha_r\left(\sum_{j=0}^{\infty} c_j V_z^j\right) \leq \left\|\sum_{j=0}^{\infty} c_j V_z^j - R\right\| \leq \sup_{\rho(f)\neq 0}\frac{\rho\left(\sum_{j=0}^{\infty} c_j V_z^j f - Rf\right)}{\rho(f)}\\ &\leq \sup_{\rho(f)\neq 0}\frac{\sum_{j=0}^{\infty}\left[\sum_{k\notin\xi}\left|c_j\widehat{f}_k b_{k+j}\right|^{p_{k+j}}\right]}{\rho(f)} \leq \sup_{k\notin\xi, j}\left(\frac{b_{j+k}}{b_k}\right)^{(p_{j+k}/\varpi_p)}\sum_{j=0}^{\infty}\left|c_j\right|^{(p_j/\varpi_p)}.\end{aligned} \tag{69}$$

As for all $\operatorname{card}\xi = r$, the last inequality is verified, and one has

$$s_r\left(\sum_{j=0}^{\infty} c_j V_z^j\right) \leq \inf_{\operatorname{card}\xi=r}\sup_{k\notin\xi, j}\left(\frac{b_{j+k}}{b_k}\right)^{p_{j+k}/\varpi_p}\sum_{j=0}^{\infty}\left|c_j\right|^{p_j/\varpi_p} = \sup_{\operatorname{card}\xi=r+1}\inf_{k\in\xi}\sup_j\left(\frac{b_{j+k}}{b_k}\right)^{p_{j+k}/\varpi_p}\sum_{j=0}^{\infty}\left|c_j\right|^{p_j/\varpi_p}. \tag{70}$$

This completes the proof. □

Theorem 31. *If conditions (a1) and (a2) are satisfied with $p_0 \geq 1$, let $\sum_{j=0}^{\infty} c_j B_z^j$ be a shift mapping on $\mathbb{H}((b_r),(p_r))_\rho$; then, the s-numbers of this mapping are given by*

$$\begin{aligned}s_r\left(\sum_{j=0}^{\infty} c_j B_z^j\right) &\leq \sup_{\operatorname{card}\xi=r+1}\inf_{k\in\xi}\sup_j\left(\frac{b_k}{b_{k+j}}\right)^{p_k/\varpi_p}\\ &\sum_{j=0}^{\infty} c_j^{p_j/\varpi_p}, \quad \text{for all}\left(c_j\right)_{j=0}^{\infty}\in \ell^{(p_j)/\varpi_p}.\end{aligned} \tag{71}$$

Proof. Let card $\xi = r$, where $\xi \subset \mathbb{N}$. Define the mapping R as $Rf(z) = R(\sum_{k=0}^{\infty} \widehat{f}_k z^k) = \sum_{k\in\xi}\sum_{j=0}^{k} c_j \widehat{f_{k-j}} z^k$. Since the triangle inequality holds by ρ, one gets

$$s_r\left(\sum_{j=0}^{\infty} c_j B_z^j\right) \le \alpha_r\left(\sum_{j=0}^{\infty} c_j B_z^j\right) \le \left\|\sum_{j=0}^{\infty} c_j B_z^j - R\right\| \le \sup_{\rho(f)\neq 0} \frac{\rho\left(\sum_{j=0}^{\infty} c_j B_z^j f - Rf\right)}{\rho(f)}$$
$$\le \sup_{\rho(f)\neq 0} \frac{\sum_{j=0}^{\infty}\left[\sum_{k\notin\xi}\left(b_k\left|c_j\right|\left|\widehat{f_{k+j}}\right|\right)^{p_k}\right]^{(1/\varpi_p)}}{\rho(f)} \tag{72}$$
$$\le \sup_{k\notin\xi, j}\left(\frac{b_k}{b_{k+j}}\right)^{(p_k/\varpi_p)} \sum_{j=0}^{\infty}\left|c_j\right|^{(p_j/\varpi_p)}.$$

As for all card $\xi = r$, the last inequality is verified, and one has

$$s_r\left(\sum_{j=0}^{\infty} c_j B_z^j\right) \le \inf_{\text{card}\xi=r}\sup_{k\notin\xi, j}\left(\frac{b_k}{b_{k+j}}\right)^{(p_k/\varpi_p)}\sum_{j=0}^{\infty}\left|c_j\right|^{(p_j/\varpi_p)}$$
$$= \sup_{\text{card}\xi=r+1}\inf_{k\in\xi}\sup_j\left(\frac{b_k}{b_{k+j}}\right)^{(p_k/\varpi_p)}\sum_{j=0}^{\infty}\left|c_j\right|^{(p_j/\varpi_p)}. \tag{73}$$

This completes the proof. □

The following theorems are direct actions of Theorem 30 and Definition 10.

Theorem 32. *If conditions (a1) and (a2) are satisfied with $p_0 \ge 1$, let V_{e^z} be a shift mapping on $\mathbb{H}((b_r), (p_r))_\rho$ and $e^z = \sum_{r=0}^{\infty} z^r/r!$. The upper estimation of the s-numbers of V_{e^z} is given by*

$$s_a(V_{e^z}) \le \sup_{\text{card}\xi=a+1}\inf_{j\in\xi}\sup_r\left(\frac{b_{r+j}}{b_j}\right)^{p_{r+j}/\varpi_p}\sum_{r=0}^{\infty}\left(\frac{1}{r!}\right)^{p_r/\varpi_p}. \tag{74}$$

Theorem 33. *If conditions (a1) and (a2) are satisfied with $p_0 \ge 1$, let $V_{\sin(z)}$ be a shift mapping on $\mathbb{H}((b_r), (p_r))_\rho$ and $\sin(z) = \sum_{m=0}^{\infty}(-1)^m(z^{2m+1}/(2m+1)!)$. The upper estimation of the s-numbers of $V_{\sin(z)}$ is given by*

$$s_a\left(V_{\sin(z)}\right) \le \sup_{\text{card}\xi=a+1}\inf_{j\in\xi}\sup_r\left(\frac{b_{r+j}}{b_j}\right)^{p_{r+j}/\varpi_p}$$
$$\sum_{r=0}^{\infty}\left(\frac{1}{(2r+1)!}\right)^{p_r/\varpi_p}. \tag{75}$$

The following theorems are direct actions of Theorem 31 and Definition 11.

Theorem 34. *If conditions (a1) and (a2) are satisfied with $p_0 \ge 1$, then the mapping B_{e^z} on $\mathbb{H}((b_r), (p_r))_\rho$ holds the following inequality:*

$$s_r(B_{e^z}) \le \sup_{\text{card}\xi=r+1}\inf_{k\in\xi}\sup_m\left(\frac{b_k}{b_{k+m}}\right)^{p_k/\varpi_p}\sum_{m=0}^{\infty}\left(\frac{1}{m!}\right)^{p_m/\varpi_p}. \tag{76}$$

Theorem 35. *If conditions (a1) and (a2) are satisfied with $p_0 \ge 1$ and the mapping $B_{\sin(z)}$ is defined on $\mathbb{H}((b_r), (p_r))_\rho$, then the upper estimation of the s-numbers of $B_{\sin(z)}$ is given by*

$$s_r\left(B_{\sin(z)}\right) \le \sup_{\text{card}\xi=r+1}\inf_{k\in\xi}\sup_m\left(\frac{b_k}{b_{k+m}}\right)^{p_k/\varpi_p}\sum_{m=0}^{\infty}\left(\frac{1}{(2m+1)!}\right)^{p_m/\varpi_p}. \tag{77}$$

6. Caristi's Generalization of Fixed Point Theorem

In modular spaces, the Ekeland variational principle [28] cannot be applied because the modular does not really prove the triangle inequality. In this part, we consider an extension of Caristi's fixed point theorem in $\mathbb{H}((b_r), (p_r))_\rho$ in light of Farkas [28].

Definition 12

(a) The pre-quasi normed ssfps ρ on $\mathbb{H}((b_r), (p_r))_\rho$ is called ρ-convex, if $\rho(\omega v + (1-\omega)t) \le \omega\rho(v) + (1-\omega)\rho(t)$, for each $\omega \in [0,1]$ and $v, t \in \mathbb{H}((b_n), (p_n))_\rho$.

(b) $\{v^{(a)}\}_{a\in\mathbb{N}} \subseteq \mathbb{H}((b_n), (p_n))_\rho$ is ρ-convergent to $v \in \mathbb{H}((b_n), (p_n))_\rho$, if and only if, $\lim_{a\longrightarrow\infty}\rho(v^{(a)} - v) = 0$. If the ρ-limit exists, then it is unique.

(c) $\{v^{(a)}\}_{a\in\mathbb{N}} \subseteq \mathbb{H}((b_n), (p_n))_\rho$ is ρ-Cauchy, when $\lim_{a,b\longrightarrow\infty}\rho(v^{(a)} - v^{(b)}) = 0$.

(d) $\Upsilon \subset \mathbb{H}((b_n), (p_n))_\rho$ is ρ-closed, if for all ρ-converging $\{u^{(a)}\}_{a\in\mathbb{N}} \subset \Upsilon$ to u, and hence $u \in \Upsilon$.

(e) $\Upsilon \subset \mathbb{H}((b_n), (p_n))_\rho$ is ρ-bounded, when $\delta_\rho(\Upsilon) = \sup\{\rho(v-t)\colon v, t \in \Upsilon\} < \infty$.

(f) The ρ-ball of radius $d \geq 0$ and center v, for every $v \in \mathbb{H}((b_n), (p_n))_\rho$, is defined as

$$\mathscr{B}_\rho(v, d) = \left\{t \in \mathbb{H}\left((b_n), (p_n)\right)_\rho\colon \rho(v-t) \leq d\right\}. \quad (78)$$

(g) A pre-quasi normed ssfps ρ on $\mathbb{H}((b_n), (p_n))_\rho$ satisfies the Fatou property, if for any sequence $\{t^{(u)}\} \subseteq \mathbb{H}((b_n), (p_n))_\rho$ with $\lim_{u\longrightarrow\infty}\rho(t^{(u)} - t) = 0$ and any $v \in \mathbb{H}((b_n), (p_n))_\rho$,

$$\rho(v-t) \leq \sup_m \inf_{u\geq m} \rho\left(v - t^{(u)}\right). \quad (79)$$

Consider the fact that the ρ-closedness of the ρ-balls is determined by the Fatou property.

Theorem 36. *Suppose setups (a1) and (a2) are satisfied; then, $\rho(f) = [\sum_{r=0}^{\infty} |b_r \widehat{f}_r|^{p_r}]^{(1/\varpi_p)}$, for all $f \in \mathbb{H}((b_n), (p_n))_\rho$, holds the Fatou property.*

Proof. Assume the setups are fulfilled and $\{f^{(i)}\} \subseteq \mathbb{H}((b_n), (p_n))_\rho$ with $\lim_{i\longrightarrow\infty}\rho(f^{(i)} - f) = 0$. Since the space $\mathbb{H}((b_n), (p_n))_\rho$ is a pre-quasi closed space, then $f \in \mathbb{H}((b_n), (p_n))_\rho$. Then, for any $g \in \mathbb{H}((b_n), (p_n))_\rho$, one can see that

$$\rho(g-f) = \left[\sum_{a=0}^{\infty} \left|b_a\left(\widehat{g}_a - \widehat{f}_a\right)\right|^{p_a}\right]^{1/\varpi_p} \leq \left[\sum_{a=0}^{\infty} \left|b_a\left(\widehat{g}_a - \widehat{f_a^{(i)}}\right)\right|^{p_a}\right]^{1/\varpi_p} + \left[\sum_{a=0}^{\infty} \left|b_a\left(\widehat{f_a^{(i)}} - \widehat{f}_a\right)\right|^{p_a}\right]^{1/\varpi_p} \leq \sup_j \inf_{i\geq j} \rho\left(g - f^{(i)}\right). \quad (80)$$

□

Theorem 37. *The function $\rho(f) = \sum_{r=0}^{\infty} |b_r \widehat{f}_r|^{p_r}$, for all $f \in \mathbb{H}((b_n), (p_n))_\rho$, does not satisfy the Fatou property, if setups (a1) and (a2) are satisfied with $p_0 > 1$.*

Proof. Let the conditions be fulfilled and $\{f^{(i)}\} \subseteq \mathbb{H}((b_n), (p_n))_\rho$ with $\lim_{i\longrightarrow\infty}\rho(f^{(i)} - f) = 0$. Since the space $\mathbb{H}((b_n), (p_n))_\rho$ is a pre-quasi closed space, then $f \in \mathbb{H}((b_n), (p_n))_\rho$. Then, for any $g \in \mathbb{H}((b_n), (p_n))_\rho$, we have

$$\begin{aligned}\rho(g-f) &= \sum_{a=0}^{\infty} \left|b_a\left(\widehat{g}_a - \widehat{f}_a\right)\right|^{p_a} \leq 2^{\sup_a p_a - 1}\left[\sum_{a=0}^{\infty} \left|b_a\left(\widehat{g}_a - \widehat{f_a^{(i)}}\right)\right|^{p_a} + \sum_{a=0}^{\infty} \left|b_a\left(\widehat{f_a^{(i)}} - \widehat{f}_a\right)\right|^{p_a}\right] \\ &\leq 2^{\sup_a p_a - 1} \sup_j \inf_{i\geq j} \rho\left(g - f^{(i)}\right).\end{aligned} \quad (81)$$

Hence, ρ does not satisfy the Fatou property. □

Example 2. The space of functions $\mathbb{H}((a_r), (q_r))_\rho$ is a pre-quasi normed ssfps, not quasi normed ssfps, and not a normed ssfps, where $\delta(h) = [\sum_{r=0}^{\infty} |a_r \widehat{h}_r|^{q_r}]^{(1/\varpi_q)}$, for all $h \in \mathbb{H}((a_r), (q_r))_\rho$.

Example 3. The space of functions $\mathbb{H}((a_r), (q))$, with $0 < q < 1$, is a pre-quasi normed ssfps, quasi normed ssfps, and not a normed ssfps, where $\delta(h) = [\sum_{r=0}^{\infty} |a_r \widehat{h}_r|^{q}]^{(1/q)}$, for each $h \in \mathbb{H}((a_r), (q_r))_\delta$.

Example 4. The space of functions $\mathbb{H}((a_r), (q_r))$ is a pre-quasi normed ssfps, a quasi normed ssfps, and a normed ssfps, where $\delta(h) = \inf\left\{\iota > 0\colon \sum_{r=0}^{\infty} |a_r \widehat{h}_r/\iota|^{q_r} \leq 1\right\}$, for all $h \in \mathbb{H}((a_r), (q_r))_\delta$.

Definition 13. The function $J\colon \mathbb{H}((b_r), (p_r))_\delta \longrightarrow (-\infty, \infty]$ is said to be lower semicontinuous at $h^{(0)} \in \mathbb{H}((b_r), (p_r))_\delta$ if $\sup_{V\in\mathscr{V}(h^{(0)})} \inf_{h\in V} J(h) = J(h^{(0)})$, for which $\mathscr{V}(h^{(0)})$ denotes $h^{(0)}$'s neighborhood system.

Definition 14. The function $J\colon \mathbb{H}((b_r), (p_r))_\delta \longrightarrow (-\infty, \infty]$ is said to be proper, when

$$\mathscr{D}(J) = \{f \in \mathbb{H}((b_r), (p_r))_\delta\colon J(f) < \infty\} \neq \varnothing. \quad (82)$$

Theorem 38. *If $\Xi \neq \varnothing$ and Ξ is a δ-closed subset of $\mathbb{H}((b_x), (p_x))_\delta$, with $\delta(h) = [\sum_{x=0}^{\infty} |b_x \widehat{h}_x|^{p_x}]^{(1/\varpi_p)}$, for all $h \in \mathbb{H}((b_x), (p_x))_\delta$, and $J\colon \Xi \longrightarrow (-\infty, \infty]$ is a proper, δ-lower semicontinuous function with $\inf_{h\in\Xi} J(h) > -\infty$, assume that $\lambda > 0$, $\{\eta_x\} \subset (0, \infty)$, and $h^{(0)} \in \Xi$ with $J(h^{(0)}) \leq \inf_{h\in\Xi} J(h) + \lambda$. So, we have $\{h^{(x)}\} \in \Xi$ which δ-converges to few $h^{(\lambda)}$, under the following conditions:*

(i) $\delta(h^{(\lambda)} - h^{(x)}) \leq (\lambda/2^x \eta_0)$, for every $x \in \mathbb{N}$.

(ii) $J(h^{(\lambda)}) + \sum_{x=0}^{\infty} \eta_x \delta(h^{(\lambda)} - h^{(x)}) \leq J(h^{(0)})$.

(iii) When $h \neq h^{(\lambda)}$, then $J(h^{(\lambda)}) + \sum_{x=0}^{\infty} \eta_x \delta(h^{(\lambda)} - h^{(x)}) < J(h) + \sum_{x=0}^{\infty} \eta_x \delta(h - h^{(x)})$.

Proof. Set $S(h^{(0)}) = \{h \in \Xi\colon J(h) + \eta_0 \delta(h - h^{(0)}) \leq J(h^{(0)})\}$. Since $h^{(0)} \in S(h^{(0)})$, then $S(h^{(0)}) \neq \varnothing$. As J is δ-lower semicontinuous, δ satisfies the Fatou property, and Ξ is δ-closed, we have that $S(h^{(0)})$ is δ-closed. Select $h^{(1)} \in S(h^{(0)})$ with

$$J\left(h^{(1)}\right) + \eta_0 \delta\left(h^{(1)} - h^{(0)}\right) \le \inf_{h \in S\left(h^{(0)}\right)} \left\{J(h) + \eta_0 \delta\left(h - h^{(0)}\right)\right\} + \frac{\lambda \eta_1}{2\eta_0}. \tag{83}$$

Next set

$$S\left(h^{(1)}\right) = \left\{h \in S\left(h^{(0)}\right) : J(h) + \sum_{i=0}^{1} \eta_i \delta\left(h - h^{(i)}\right) \le J\left(h^{(1)}\right) + \eta_0 \delta\left(h^{(1)} - h^{(0)}\right)\right\}. \tag{84}$$

Similar to $S(h^{(0)})$, one has $S(h^{(1)}) \ne \varnothing$ and δ-closed. Suppose that we have built $\{h^{(0)}, h^{(1)}, h^{(2)}, \ldots, h^{(x)}\}$ and $\{S(h^{(0)}), S(h^{(1)}), S(h^{(2)}), \ldots, S(h^{(x)})\}$. After that, select $h^{(x+1)} \in S(h^{(x)})$ with

$$\begin{aligned} &J\left(h^{(x+1)}\right) + \sum_{i=0}^{x} \eta_i \delta\left(h^{(x+1)} - h^{(i)}\right) \\ &\le \inf_{h \in S\left(h^{(x)}\right)} \left\{J(h) + \sum_{i=0}^{x} \eta_i \delta\left(h - h^{(i)}\right)\right\} + \frac{\lambda \eta_x}{2^x \eta_0}. \end{aligned} \tag{85}$$

Suppose

$$S\left(h^{(x+1)}\right) := \left\{h \in S\left(h^{(x)}\right) : J(h) + \sum_{i=0}^{x+1} \eta_i \delta\left(h - h^{(i)}\right) \le J\left(h^{(x+1)}\right) + \sum_{i=0}^{x} \eta_i \delta\left(h^{(x+1)} - h^{(i)}\right)\right\}. \tag{86}$$

Therefore, we construct the sequences $\{h^{(x)}\}$ and $\{S(h^{(x)})\}$ by induction. For constant $x \in \mathbb{N}$, assume $y \in S(h^{(x)})$. One can see that

$$J(y) + \sum_{i=0}^{x} \eta_i \delta\left(y - h^{(i)}\right) \le J\left(h^{(x)}\right) + \sum_{i=0}^{x-1} \eta_i \delta\left(h^{(x)} - h^{(i)}\right), \tag{87}$$

which gives

$$\begin{aligned} \eta_x \delta\left(y - h^{(x)}\right) &\le J\left(h^{(x)}\right) + \sum_{i=0}^{x-1} \eta_i \delta\left(h^{(x)} - h^{(i)}\right) - \left[J(y) + \sum_{i=0}^{x-1} \eta \delta\left(y - h^{(i)}\right)_i\right] \\ &\le J\left(h^{(x)}\right) + \sum_{i=0}^{x-1} \eta_i \delta\left(h^{(x)} - h^{(i)}\right) - \inf_{h \in S\left(h^{(x-1)}\right)} \left[J(h) + \sum_{i=0}^{x-1} \eta_i \delta\left(h - h^{(i)}\right)\right] \le \frac{\lambda \eta_x}{2^x \eta_0}. \end{aligned} \tag{88}$$

Since $\{S(h^{(x)})\}$ is decreasing with $h^{(x)} \in S(h^{(x)})$, for each $x \in \mathbb{N}$, one has

$$\delta\left(h^{(x+q)} - h^{(x)}\right) \leq \frac{\lambda}{2^x \eta_0}, \tag{89}$$

for each $x, q \in \mathbb{N}$, which gives that $\{h^{(x)}\}$ is $\delta-$ Cauchy. Since $\mathbb{H}((b_x), (p_x))_\delta$ is $\delta-$ Banach space, $\{h^{(x)}\}$ has $\delta-$ limits $h^{(\lambda)}$ and $\bigcap_{x \in \mathbb{N}} S(h^{(x)}) = \{h^{(\lambda)}\}$ satisfies. As $h^{(x+1)} \in S(h^{(x)})$, one has

$$J\left(h^{(x+1)}\right) + \sum_{i=0}^{x} \eta_i \delta\left(h^{(x+1)} - h^{(i)}\right) \leq J\left(h^{(x)}\right) + \sum_{i=0}^{x-1} \eta_i \delta\left(h^{(x)} - h^{(i)}\right), \tag{90}$$

which implies that $\left\{J(h^{(x)}) + \sum_{i=0}^{x-1} \eta_i \delta(h^{(x)} - h^{(i)})\right\}$ is decreasing. After that, assume $h \neq h^{(\lambda)}$. So, we get $r \in \mathbb{N}$ for which $h \notin S(h^{(x)})$, for each $x \geq r$, i.e.,

$$J\left(h^{(x)}\right) + \sum_{i=0}^{x-1} \eta_i \delta\left(h^{(x)} - h^{(i)}\right) < J(h) + \sum_{i=0}^{x} \eta_i \delta\left(h - h^{(i)}\right). \tag{91}$$

As $h^{(\lambda)} \in S(h^{(x)})$, with $x \geq r$, one can see that

$$\begin{aligned} J\left(h^{(\lambda)}\right) + \sum_{i=0}^{x} \eta_i \delta\left(h^{(\lambda)} - h^{(i)}\right) &\leq J\left(h^{(x)}\right) + \sum_{i=0}^{x-1} \eta_i \delta\left(h^{(x)} - h^{(i)}\right) \\ &\leq J\left(h^{(r)}\right) + \sum_{i=0}^{r-1} \eta_i \delta\left(h^{(r)} - h^{(i)}\right). \end{aligned} \tag{92}$$

As $x \longrightarrow \infty$ in the previous inequality, one gets

$$J\left(h^{(\lambda)}\right) + \sum_{i=0}^{\infty} \eta_i \delta\left(h^{(\lambda)} - h^{(i)}\right) \leq J\left(h^{(r)}\right) + \sum_{i=0}^{r-1} \eta_i \delta\left(h^{(r)} - h^{(i)}\right) < J(h) + \sum_{i=0}^{r} \eta_i \delta\left(h - h^{(i)}\right) \leq J(h) + \sum_{i=0}^{\infty} \eta_i \delta\left(h - h^{(i)}\right). \tag{93}$$

This implies that

$$J\left(h^{(\lambda)}\right) + \sum_{x=0}^{\infty} \eta_x \delta\left(h^{(\lambda)} - h^{(x)}\right) < J(h) + \sum_{x=0}^{\infty} \eta_x \delta\left(h - h^{(x)}\right). \tag{94}$$

This finishes the proof. □

We discuss the concept of Caristi's fixed point theorem in $\mathbb{H}((b_x), (p_x))_\delta$ using Theorem 38.

Theorem 39. *If $\Xi \neq \varnothing$ and Ξ is a δ-closed subset of $\mathbb{H}((b_x), (p_x))_\delta$, under $\delta(h) = [\sum_{x=0}^{\infty} |b_x \widehat{h_x}|^{p_x}]^{1/\varpi_p}$, with $h \in \mathbb{H}((b_x), (p_x))_\delta$, let $\lambda > 0$ and $\{\eta_n\}$ with $0 < \nu = \sum_{x=0}^{\infty} \eta_x < \infty$. U: $\Xi \longrightarrow \Xi$ is a mapping and there is a function J: $\Xi \longrightarrow (-\infty, \infty]$ which is a proper and δ-lower semicontinuous under $\inf_{h \in \Xi} J(h) > -\infty$ and*

(1) $\delta(U(h) - g) - \delta(h - g) \leq \delta(U(h) - h)$, for any $h, g \in \Xi$.

(2) $\delta(U(h) - h) \leq J(h) - J(U(h))$, for any $h \in \Xi$.

Hence, there is a fixed point of U in Ξ.

Proof. As $0 < \nu = \sum_{x=0}^{\infty} \eta_x < \infty$, we have that $J_1 := \nu J$ is proper, bounded from below, and δ-lower semicontinuous. If $h \in \Xi$, one has

$$\nu \delta(U(h) - h) \leq J_1(h) - J_1(U(h)). \tag{95}$$

As $\inf_{h \in \Xi} J_1(h) > -\infty$, there is $h^{(0)} \in \Xi$ with $J_1(h^{(0)}) < \inf_{h \in \Xi} J_1(h) + \lambda$. From Theorem 38, there is $\{h^{(x)}\}$ which δ-converges to few $h^{(\lambda)} \in \Xi$, with

$$J_1\left(h^{(\lambda)}\right) + \sum_{x=0}^{\infty} \eta_x \delta\left(h^{(\lambda)} - h^{(x)}\right) < J_1(h) + \sum_{x=0}^{\infty} \eta_x \delta\left(h - h^{(x)}\right), \tag{96}$$

for all $h \neq h^{(\lambda)}$. Suppose that $U(h^{(\lambda)}) \neq h^{(\lambda)}$, and one has

$$J_1\left(h^{(\lambda)}\right) + \sum_{x=0}^{\infty} \eta_x \delta\left(h^{(\lambda)} - h^{(x)}\right) < J_1\left(U\left(h^{(\lambda)}\right)\right) + \sum_{x=0}^{\infty} \eta_x \delta\left(U\left(h^{(\lambda)}\right) - h^{(x)}\right), \tag{97}$$

which gives

$$J_1\left(h^{(\lambda)}\right) - J_1\left(U\left(h^{(\lambda)}\right)\right) < \sum_{x=0}^{\infty} \eta_x \delta\left(U\left(h^{(\lambda)}\right) - h^{(x)}\right) - \sum_{x=0}^{\infty} \eta_x \delta\left(h^{(\lambda)} - h^{(x)}\right) = \sum_{x=0}^{\infty} \eta_x \left(\delta\left(U\left(h^{(\lambda)}\right) - h^{(x)}\right) - \delta\left(h^{(\lambda)} - h^{(x)}\right)\right). \tag{98}$$

From condition (6), one can see that

$$J_1\left(h^{(\lambda)}\right) - J_1\left(U\left(h^{(\lambda)}\right)\right) < \sum_{x=0}^{\infty} \eta_x \delta\left(U\left(h^{(\lambda)}\right) - h^{(\lambda)}\right) = \nu\delta\left(U\left(h^{(\lambda)}\right) - h^{(\lambda)}\right). \tag{99}$$

Inequality (6) gives

$$\nu\delta\left(U\left(h^{(\lambda)}\right) - h^{(\lambda)}\right) \leq J_1\left(h^{(\lambda)}\right) - J_1\left(U\left(h^{(\lambda)}\right)\right) < \nu\delta\left(U\left(h^{(\lambda)}\right)\right) - h^{(\lambda)}. \tag{100}$$

We have a contradiction. Hence, $U(h^{(\lambda)}) = h^{(\lambda)}$. This completes the proof. □

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

This study was funded by the University of Jeddah, Saudi Arabia, under grant no. UJ-20-084-DR. The authors, therefore, acknowledge with thanks the University's technical and financial support.

References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[2] R. Kannan, "Some results on fixed points-II," *The American Mathematical Monthly*, vol. 76, no. 4, pp. 405–408, 1969.

[3] S. J. H. Ghoncheh, "Some Fixed point theorems for Kannan mapping in the modular spaces," *Ciencia e Natura*, vol. 37, pp. 462–466, 2015.

[4] L. Diening, P. Harjulehto, P. Hästö, and M. Ružička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, Berlin, Germany, 2011.

[5] K. R. Rajagopal and M. Růžička, "On the modeling of electrorheological materials," *Mechanics Research Communications*, vol. 23, no. 4, pp. 401–407, 1996.

[6] M. Ružička, "Electrorheological fluids. Modeling and mathematical theory," *In Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2000.

[7] A. Pietsch, *Operator Ideals*, North-Holland Publishing Company, Amsterdam, Netherland, 1980.

[8] A. Pietsch, "Small ideals of operators," *Studia Mathematica*, vol. 51, no. 3, pp. 265–267, 1974.

[9] B. M. Makarov and N. Faried, "Some properties of operator ideals constructed by snumbers (In Russian)," in *Theory of Operators in Functional Spaces*Academy of Science, Siberian section, Novosibirsk, Russia, 1977.

[10] N. Faried and A. A. Bakery, "Small operator ideals formed by s numbers on generalized Cesáro and Orlicz sequence space-ssnumbers on generalized Cesàro and Orlicz sequence spaces," *Journal of Inequalities and Applications*, vol. 2018, no. 1, p. 357, 2018.

[11] T. Yaying, B. Hazarika, and M. Mursaleen, "On sequence space derived by the domain of q-Cesàro matrix in $\ell_p$space and the associated operator ideal," *Journal of Mathematical Analysis and Applications*, vol. 493, p. 2021, Article ID 124453, 2020.

[12] M. Mursaleen and A. K. Noman, "Compactness by the Hausdorff measure of noncompactness," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 8, pp. 2541–2557, 2010.

[13] M. Mursaleen and A. K. Noman, "Compactness of matrix operators on some new difference sequence spaces," *Linear Algebra and Its Applications*, vol. 436, no. 1, pp. 41–52, 2012.

[14] A. A. Bakery and O. S. K. Mohamed, "Kannan prequasi contraction maps on Nakano sequence spaces," *Journal of Function Spaces*, vol. 2020, Article ID 8871563, 10 pages, 2020.

[15] A. A. Bakery and O. S. K. Mohamed, "Kannan nonexpansive maps on generalized Cesàro backward difference sequence space of non-absolute type with applications to summable equations," *Journal of Inequalities and Applications*, vol. 103, 2021.

[16] A. A. Bakery and M. H. El Dewaik, "A generalization of Caristi's fixed point theorem in the variable exponent weighted formal power series space," *Journal of Function Spaces*, vol. 2021, Article ID 9919420, 18 pages, 2021.

[17] A. L. Shields, "Weighted shift operators and analytic function theory," *Math. Surveys Monographs*, vol. 13, 1974.

[18] K. Hedayatian, "On cyclicity in the space $H^p(\beta)$," *Taiwanese Journal of Mathematics*, vol. 8, no. 3, pp. 429–442, 2004.

[19] H. Emamirad and G. S. Heshmati, "Chaotic weighted shifts in Bargmann space," *Journal of Mathematical Analysis and Applications*, vol. 308, no. 1, pp. 36–46, 2005.

[20] N. Faried, A. Morsy, and Z. A. Hassanain, "ss-numbers of shift operators of formal entire functions," *Journal of Approximation Theory*, vol. 176, pp. 15–22, 2013.

[21] A. Pietsch, *Eigenvalues and s-numbers*, Cambridge University Press, New York, NY, USA, 1986.

[22] A. A. Bakery and A. R. Abou Elmatty, "A note on Nakano generalized difference sequence space," *Advances in Difference Equations*, vol. 620, p. 2020, 2020.

[23] N. Faried, Z. Abd El Kader, and A. A. Mehanna, "s-numbers of polynomials of shift operators on $\ell^p 1 \leq p \leq \infty$," *Journal of the Egyptian Mathematical Society*, vol. 1, pp. 31–37, 1993.

[24] B. Altay and F. Başar, "Generalization of the sequence space $\ell(p)$derived by weighted means," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 1, pp. 147–185, 2007.

[25] B. E. Rhoades, "Operators of $A-p$type," *Atti Accad, Naz, Lincei Rend, Cl, Sci, Fis, Mat, Natur*, vol. 59, no. 3-4, pp. 238–241, 1975.

[26] A. Pietsch, *Operator Ideals*, VEB Deutscher Verlag derWissenschaften, Berlin, Germany, 1978.

[27] A. Pietsch, "ss-Numbers of operators in Banach spaces," *Studia Mathematica*, vol. 51, no. 3, pp. 201–223, 1974.

[28] C. Farkas, "A generalized form of Ekeland's variational principle," *Analele Universitatii Ovidius Constanta - Seria Matematica*, vol. 20, no. 1, pp. 101–112, 2012.