

Research Article

Consensus Conditions for a Class of Fractional-Order Nonlinear Multiagent Systems with Constant and Time-Varying Time Delays

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The consensus problem for a class of fractional-order nonlinear multiagent systems with a distributed protocol containing input time delay is investigated in this paper. Consider both cases of constant time delay and time-varying delay, the delay-independent consensus conditions are obtained to achieve the consensus of the systems, respectively, by adopting the linear matrix inequality (LMI) methods and stability theory of fractional-order systems. As illustrated by the numerical examples, the proposed theoretical results work well and accurately.

1. Introduction

The consensus problem based on distributed coordinated control of multiagent systems is widely applied in various engineering fields, such as the collaboration of mobile robotics, formation flight control for multi-unmanned aerial vehicles, the scheduling of intelligent transportation, etc. [1–6]. The distributed coordinated control strategy has advantages of high reliability, fast response speed, and flexible operation. The literature review illustrates that the consensus topic on the coordination of multiagent systems has been studied from a variety of perspectives, including the design of consensus protocols, exploration of consensus criteria, and actual application prospects. The consensus has achieved remarkably in theoretical research such as time-delay dependent consensus [7, 8], optimal consensus, finite time consensus, and consensus of higher-order systems [9–13]. The recently fruitful results on this topic refer to the following literature [14–16].

However, the results mentioned above are focused on integer-order systems. Actually, fractional calculus is more suitable to describe complex dynamics naturally. Compared with the integer-order case, the fractional-order model emphasizes time memory and nonlocal properties of the real

systems. For example, fractional calculus fits to model the dynamics of intelligent vehicles moving on the road surface with viscoelastic materials [17–19]. From the perspective of control theory, fractional calculus techniques could improve the performance index of control systems considering nonlinear factors, uncertainties, perturbations, etc. [20, 21]. Hence, it is significant to investigate the consensus problem in the frame work of fractional-order models. As we know, the consensus investigation of fractional-order systems was firstly shown in [22]. The convergence analysis of consensus of such kind of model was further studied in [23]. The consensus of fractional-order systems with input or communication delays was discussed in [24, 25]. The fractional-order leader-following consensus is also considered by constructing appropriate Lyapunov function in [26].

In the actual network environment, the network-induced delay is unavoidable due to the limited network bandwidth, irregular date change, and so on. These disadvantages will affect the system performance or even deteriorate the system's stability. Recently, the consensus issue of fractional-order multiagent systems with time-delay has attracted more attention. In [27], the fractional-order model with diverse communication delays is considered and sufficient consensus criteria were proposed by using the

frequency-domain analysis. The consensus of the fractional-order model with nonuniform input and communication delays is investigated [28]. The Lyapunov stability theory is also an efficient approach to estimate the consensus of complex dynamical systems in time-domain analysis. For example, the global asymptotical stability of fractional-order nonautonomous systems is obtained by constructing a feasible Lyapunov function [29]. In [30], exponential consensus of fractional-order delayed systems with a heterogeneous impulsive control strategy is investigated by using the comparison principle. The key issue of this technique is to propose a suitable Lyapunov function for the considered system [31, 32].

The main contribution of this work is estimating time-delay effect quantitatively to the consensus achievement of fractional-order multiagent systems. Firstly, considering the distributed coordination control with constant time delay, we obtain sufficient delay-independent criteria to achieve the consensus of fractional-order systems by constructing appropriate Lyapunov matrix inequalities. Secondly, the theoretical results are extended to time-varying delay case and the corresponding delay-independent consensus criterion is also obtained. In addition, the Lyapunov–Krasovskii function candidates for the fractional-order system with Caputo derivative are constructed properly.

The rest of this paper is organized as follows: In Section 2, preliminary knowledge about graph theory and fractional calculus are given and then the consensus of fractional-order multiagent systems is described briefly in Section 3. In Section 4, we propose sufficient consensus criteria to ensure that the consensus of fractional-order systems is achieved in the case of containing constant or time-varying delay. Numerical examples are given to illustrate the effectiveness of the theoretical results in Section 5. Finally, in Section 6, some concluding remarks are drawn.

2. Preliminaries

In this section, some preliminary knowledge about the concepts of algebraic graph theory and fractional calculus are introduced. Meanwhile, the relevant important assumption is presented.

2.1. Graph Theory. A digraph represents $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, the network topology, among agents, in which $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and $\mathcal{A} = (a_{ij})_{N \times N}$ stand for the node set of agents, edge set of joined agents, and weighted adjacency matrix of \mathcal{G} , respectively. If there is a directed edge from node j to i , then $(j, i) \in \mathcal{E}$, and we note agent j is a input neighbor of agent i with $a_{ij} > 0$ and output neighbor of agent i with $a_{ji} > 0$. In other words, the information can be transformed from agent j to agent i ; otherwise, $a_{ij} = 0$. The input degree matrix is $\mathcal{D} = \text{diag}\{d_1, d_2, \dots, d_N\}$ and $d_i = \sum_{j=1, j \neq i}^N a_{ij}$; the Laplacian matrix L of the weighted digraph \mathcal{G} is defined as $L = (l_{ij}) \in \mathbb{R}^{N \times N} = \mathcal{D} - \mathcal{A}$. Let $D = \text{diag}\{\widehat{d}_1, \widehat{d}_2, \dots, \widehat{d}_N\}$ be the leader adjacency matrix of the union graph $\mathcal{G} = \mathcal{G} \cup 0$; then, we can denote $\bar{L} = L + D = (\bar{l}_{ij})_{N \times N}$.

Assumption 1. A directed graph contains a directed spanning tree if there exists a leader node 0, such that it has directed paths to all other following nodes in \mathcal{G} .

Assumption 2. The matrix pair (A, B) is stabilizable.

2.2. Fractional Calculus. The Riemann–Liouville and the Caputo fractional-order derivatives are two commonly used definitions. The autonomous fractional-order systems modeled with Caputo derivative could be converted to the similar initial value problem (IVP) and could also have definite physical meaning. Hence, we will use the fractional-order derivative with Caputo definition in this paper.

The Caputo fractional-order derivative [33] is defined as

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t - \tau)^{\alpha - n + 1}} d\tau. \quad (1)$$

where n is the integer satisfying $n - 1 < \alpha \leq n$ and $\Gamma(z)$ is the Gamma function satisfying $\Gamma(z + 1) = z\Gamma(z)$ for $z > 0$. In this paper, we consider the case of $0 < \alpha \leq 1$. Similarly, the definition of fractional-order integral [33] is

$${}_0 I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t - \tau)^{1 - \alpha}} d\tau, \quad (2)$$

and the following formula ${}_0^C D_t^\alpha ({}_0 I_t^\alpha f(t)) = f(t)$ holds. As the Caputo fractional-order derivative β is close to 1, the property $\lim_{\beta \rightarrow 1^-} [{}_0^C D_t^\beta f(t)] = \dot{f}(t)$ holds if $f(t)$ is differentiable [32].

Lemma 1 (see [31]). *Let $x(t) \in \mathbb{R}^n$ be a vector of differentiable function. Then, there exists some $t_0 > 0$ such that for any $t \geq t_0$, the following inequality holds:*

$$\frac{1}{2} \cdot {}_0^C D_t^\alpha (x^T(t) P x(t)) \leq x^T(t) P {}_0^C D_t^\alpha x(t), \quad (3)$$

where $P \in \mathbb{R}^{n \times n}$ is a symmetric and positive definite or semidefinite matrix.

Lemma 2 (Schur complement [34]). *If T_1 , T_2 , and T_3 are matrices and $T_3 > 0$, then*

$$\begin{pmatrix} T_1 & T_2^T \\ T_2 & -T_3 \end{pmatrix} < 0, \quad \iff T_1 + T_2^T T_3^{-1} T_2 < 0. \quad (4)$$

3. Problem Statement

In this section, we will consider the general fractional-order nonlinear multiagent systems containing N following agents over the directed network topology; the dynamics of the i th agent is

$${}_0^C D_t^\alpha x_i(t) = A x_i(t) + f(t, x_i(t)) + B u_i(t), \quad (5)$$

$i = 1, 2, \dots, N$. Here, $x_i(t) \in \mathbb{R}^n$, $f(t, x_i(t)) \in \mathbb{R}^n$, and $u_i(t) \in \mathbb{R}^m$ denote the state, nonlinear factor, and control input of agent i , respectively. $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are

constant matrices. The leader agent indexed by 0 has dynamics

$${}^C_0D_t^\alpha x_0(t) = Ax_0(t) + f(t, x_0(t)). \quad (6)$$

For simplicity, we suppose the nonlinear function $f(t, x_i(t))$ is continuous and satisfies the Lipschitz

condition; i.e., for a given constant $\rho > 0$, $\|f(t, x_i(t)) - f(t, x_0(t))\| \leq \rho \|x_i(t) - x_0(t)\|$.

Considering the complex communication environment, time-delay-induced blocking or delay is unavoidable. In order to realize the leader-following consensus of systems (5)-(6), we propose the following distributed control protocol with time delay:

$$u_i(t) = -K \sum_{j=1}^N a_{ij} (x_i(t-\tau) - x_j(t-\tau)) - K\hat{d}_i (x_i(t-\tau) - x_0(t-\tau)), \quad (7)$$

where K is the control gain matrix to be designed later. If agent i is connected to the leader, $\hat{d}_i = 1$; otherwise, $\hat{d}_i = 0$.

Definition 1. The leader-following consensus of systems (5)-(6) is said to be achieved, if the states of agents satisfy

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_0(t)\| \rightarrow 0, \quad i = 1, 2, \dots, N, \quad (8)$$

for any initial condition.

The aim of this paper is to discuss the feasible consensus conditions to the system. The measurement error between agent i and the leader is defined as $\varepsilon_i(t) = x_i(t) - x_0(t)$; multiagent systems (5)-(6) controlled by (7) can be rewritten as

$$\begin{aligned} {}^C_0D_t^\alpha \varepsilon_i(t) &= A\varepsilon_i(t) - BK \sum_{j=1}^N a_{ij} (\varepsilon_i(t-\tau) - \varepsilon_j(t-\tau)) \\ &\quad - BK\hat{d}_i \varepsilon_i(t-\tau) + f(t, x_i(t)) - f(t, x_0(t)). \end{aligned} \quad (9)$$

By using the Kronecker product, the vector form of (9) is expressed as

$${}^C_0D_t^\alpha \varepsilon(t) = (I_N \otimes A)\varepsilon(t) - (\bar{L} \otimes BK)\varepsilon(t-\tau) + F(t), \quad (10)$$

where $\varepsilon(t) = [\varepsilon_1^T(t), \varepsilon_2^T(t), \dots, \varepsilon_N^T(t)]^T$ and $F(t) = [f(t, x_1(t)) - f(t, x_0(t)), \dots, f(t, x_N(t)) - f(t, x_0(t))]^T$. The leader-following consensus of systems (5)-(6) is equivalent to the corresponding Lyapunov stability problem of system (10).

4. Main Results

Now, we will analyze the consensus problem of systems (5)-(6) by adopting the distributed control containing time

delay. Considering both case of constant and time-varying delay, we will design the suitable distributed controllers and give sufficient conditions to ensure the consensus of the system, respectively.

4.1. Constant Time Delay

Theorem 1. Suppose that Assumptions 1 and 2 hold and there exist symmetric, positive definite matrices P and Q and positive constants ε_1 and ε_2 such that the following inequalities hold:

$$A^T P + PA + Q + \varepsilon_1 \lambda_{\max}(\bar{L})I + \frac{\rho^2}{\varepsilon_2} I + \varepsilon_2 P^2 < 0, \quad (11)$$

$$\frac{\lambda_{\max}(PBB^T P)}{\varepsilon_1} I - Q < 0. \quad (12)$$

Then, the distributed controller designed in (7) with the control gain $K = B^T P$ ensures that the leader-follower consensus of systems (5)-(6) can be reached asymptotically.

Proof. Consider a Lyapunov–Krasovskii function candidate defined as

$$V(t) = {}_0I_t^{1-\alpha} \varepsilon^T(t) P \varepsilon(t) + \int_{t-\tau}^t \varepsilon^T(s) (I_N \otimes Q) \varepsilon(s) ds, \quad (13)$$

based on the property of Caputo fractional-order derivative; the derivative of $V(t)$ can be rewritten as

$$\begin{aligned} \dot{V}(t) &= \lim_{\beta \rightarrow 1^-} [{}^C_0D_t^\beta] {}_0I_t^{1-\alpha} \varepsilon^T(t) P \varepsilon(t) + \lim_{\beta \rightarrow 1^-} [{}^C_0D_t^\beta] \int_{t-\tau}^t \varepsilon^T(s) (I_N \otimes Q) \varepsilon(s) ds \\ &= {}^C_0D_t^\alpha \varepsilon^T(t) P \varepsilon(t) + \lim_{\beta \rightarrow 1^-} [{}^C_0D_t^\beta] \int_{t-\tau}^t \varepsilon^T(s) (I_N \otimes Q) \varepsilon(s) ds. \end{aligned} \quad (14)$$

Lemma 1 can be used to calculate the upper bound of the derivative of $V(t)$ along the trajectory of system (10); then, we obtain

$$\begin{aligned}
\dot{V}(t) &\leq 2\varepsilon^T(t)P_0^C D_t^\alpha \varepsilon(t) + \varepsilon^T(t)(I_N \otimes Q)\varepsilon(t) - \varepsilon^T(t-\tau)(I_N \otimes Q)\varepsilon(t-\tau) \\
&\leq 2\varepsilon^T(t)(I_N \otimes P)[I_N \otimes A\varepsilon(t) - \bar{L} \otimes BK\varepsilon(t-\tau) + F(t)] + \varepsilon^T(t)(I_N \otimes Q)\varepsilon(t) \\
&\quad - \varepsilon^T(t-\tau)(I_N \otimes Q)\varepsilon(t-\tau) \\
&\leq \varepsilon^T(t)(I_N \otimes (A^T P + PA))\varepsilon(t) + 2\varepsilon^T(t)(I_N \otimes P)F(t) + \varepsilon^T(t)(I_N \otimes Q)\varepsilon(t) \\
&\quad - 2\varepsilon^T(t)(\bar{L} \otimes PBB^T P)\varepsilon(t-\tau) - \varepsilon^T(t-\tau)(I_N \otimes Q)\varepsilon(t-\tau).
\end{aligned} \tag{15}$$

By using the inequality $2x^T y \leq ax^T x + (1/a)y^T y$ for any $a > 0$, it has

$$-2\varepsilon^T(t)(\bar{L} \otimes PBB^T P)\varepsilon(t-\tau) \leq \epsilon_1 \lambda_{\max^2}(\bar{L})\varepsilon^T(t)\varepsilon(t) + \frac{\lambda_{\max^2}(PBB^T P)}{\epsilon_1} \varepsilon^T(t-\tau)\varepsilon(t-\tau). \tag{16}$$

Similarly, with the Lipschitz condition,

$$2\varepsilon^T(t)(I_N \otimes P)F(t) \leq \epsilon_2 \varepsilon^T(t)\varepsilon(t) + \frac{\rho^2}{\epsilon_2} \varepsilon^T(t)P^2\varepsilon(t). \tag{17}$$

Hence, we deduce that

$$\begin{aligned}
\dot{V}(t) &\leq \varepsilon^T(t)(I_N \otimes (A^T P + PA))\varepsilon(t) + \epsilon_1 \lambda_{\max^2}(\bar{L})\varepsilon^T(t)\varepsilon(t) + \epsilon_2 \varepsilon^T(t)\varepsilon(t) + \frac{\rho^2}{\epsilon_2} \varepsilon^T(t)P^2\varepsilon(t) \\
&\quad + \varepsilon^T(t)(I_N \otimes Q)\varepsilon(t) + \frac{\lambda_{\max^2}(PBB^T P)}{\epsilon_1} \varepsilon^T(t-\tau)\varepsilon(t-\tau) - \varepsilon^T(t-\tau)(I_N \otimes Q)\varepsilon(t-\tau) \\
&\leq \varepsilon^T(t)\left(A^T P + PA + Q + \epsilon_1 \lambda_{\max^2}(\bar{L})I + \epsilon_2 I + \frac{\rho^2}{\epsilon_2} P^2\right)\varepsilon(t) \\
&\quad + \varepsilon^T(t-\tau)\left(\frac{\lambda_{\max^2}(PBB^T P)}{\epsilon_1} I - Q\right)\varepsilon(t-\tau).
\end{aligned} \tag{18}$$

Therefore, $\dot{V}(t)$ is negative definite, which implies that the consensus of systems (5)-(6) is reached.

To check whether the algebraic Riccati inequality (11) can be solved, it suffices to determine a positive definite solution of an associated Lyapunov matrix inequality. \square

Lemma 3. Assume that P is a positive definite matrix satisfying the following Lyapunov inequality:

$$A^T P + PA + 2Q + \epsilon_1 \lambda_{\max^2}(\bar{L})I + \epsilon_2 I < 0, \tag{19}$$

where ϵ_1 and ϵ_2 be positive scalars. Then, P is also a solution of the algebraic Riccati inequality (11) provided that

$$\frac{\rho^2}{\epsilon_2} P^2 - Q < 0. \tag{20}$$

Proof. From (19) and (20), for all $\varepsilon \in \mathbb{R}^n$, we conclude that

$$\begin{aligned}
&\varepsilon^T\left(A^T P + PA + Q + \epsilon_1 \lambda_{\max^2}(\bar{L})I + \epsilon_2 I + \frac{\rho^2}{\epsilon_2} P^2\right)\varepsilon \\
&\leq \left(\frac{\rho^2}{\epsilon_2} P^2 - Q\right)\|\varepsilon\|^2 < 0,
\end{aligned} \tag{21}$$

is negative due to condition (20). Hence, inequality (11) holds.

According to the Schur complement theorem, inequalities (11) and (12) can be rewritten as the following form. \square

$$\begin{pmatrix} A^T P + PA + Q + (\epsilon_1 \lambda_{\max}^2(\bar{L}) + \epsilon_2) I & \frac{\rho}{\sqrt{\epsilon_2}} P & 0 \\ \frac{\rho}{\sqrt{\epsilon_2}} P & -I & 0 \\ 0 & 0 & Q - \frac{\lambda_{\max}^2(PBB^T P)}{\epsilon_1} I \end{pmatrix} < 0. \quad (22)$$

Then, the leader-follower consensus of systems (5)-(6) can be reached asymptotically.

Proof. From the proof of Theorem 1 and Lemma 2 (Schur complement), it implies that the consensus of systems (5)-(6) is realized. \square

4.2. *Effects of Time-Varying Delay.* Furthermore, we will extend the corresponding distributed control protocol (7) with time-varying delay to investigate the leader-following consensus problem of systems (5)-(6):

$$\begin{aligned} u_i(t) = & -K \sum_{j=1}^N a_{ij} (x_i(t - \tau(t)) - tx_j(t - \tau(t))) \\ & - K \hat{d}_i (x_i(t - \tau(t)) - x_0(t - \tau(t))). \end{aligned} \quad (23)$$

Theorem 2. Suppose that Assumptions 1 and 2 hold and there exist symmetric, positive definite matrices P and Q and positive constants ϵ_1 and ϵ_2 such that the following LMI holds:

Assumption 3. The time-varying delay satisfies the following conditions: (i) There exists $\tau_1 > 0$ such that $0 \leq \tau(t) \leq \tau_1$. (ii) There exists $\tau_2 > 0$ such that $\dot{\tau}(t) \leq 1 - \tau_2$.

Theorem 3. Suppose that Assumptions 1-3 are fulfilled and that there exist positive constants ϵ_1 and ϵ_2 and symmetric, positive definite matrices P and Q such that the following LMI holds:

$$\begin{pmatrix} A^T P + PA + Q + (\epsilon_1 \lambda_{\max}^2(\bar{L}) + \epsilon_2) I & \frac{\rho}{\sqrt{\epsilon_2}} P & 0 \\ \frac{\rho}{\sqrt{\epsilon_2}} P & -I & 0 \\ 0 & 0 & \tau_2 Q - \frac{\lambda_{\max}^2(PBB^T P)}{\epsilon_1} I \end{pmatrix} < 0. \quad (24)$$

Then, the distributed controller designed in (23) with the control gain $K = B^T P$ ensures that the leader-follower consensus of systems (5)-(6) can be reached asymptotically.

Proof. We consider the Lyapunov–Krasovskii function candidate

$$V(t) = {}_0 I_t^{1-\alpha} \varepsilon^T(t) P \varepsilon(t) + \int_{t-\tau(t)}^t \varepsilon^T(s) (I_N \otimes Q) \varepsilon(s) ds. \tag{25}$$

Taking the derivative of (25) yields the estimate

$$\begin{aligned} \dot{V}(t) &= \lim_{\beta \rightarrow 1^-} [{}^C D_t^\beta] {}_0 I_t^{1-\alpha} \varepsilon^T(t) P \varepsilon(t) + \lim_{\beta \rightarrow 1^-} [{}^C D_t^\beta] \int_{t-\tau(t)}^t \varepsilon^T(s) (I_N \otimes Q) \varepsilon(s) ds \\ &= {}^C D_t^\alpha \varepsilon^T(t) P \varepsilon(t) + \lim_{\beta \rightarrow 1^-} [{}^C D_t^\beta] \int_{t-\tau(t)}^t \varepsilon^T(s) (I_N \otimes Q) \varepsilon(s) ds \\ &\leq 2\varepsilon^T(t) P {}^C D_t^\alpha \varepsilon(t) + \varepsilon^T(t) (I_N \otimes Q) \varepsilon(t) - (1 - \dot{\tau}(t)) \varepsilon^T(t - \tau(t)) (I_N \otimes Q) \varepsilon(t - \tau(t)). \end{aligned} \tag{26}$$

By Theorem 3, the derivative of $V(t)$ along the trajectory of system (10) is

$$\begin{aligned} \dot{V}(t) &\leq \varepsilon^T(t) (I_N \otimes (A^T P + PA)) \varepsilon(t) + 2\varepsilon^T(t) (I_N \otimes P) F(t) + \varepsilon^T(t) (I_N \otimes Q) \varepsilon(t) \\ &\quad - 2\varepsilon^T(t) (\bar{L} \otimes P B B^T P) \varepsilon(t - \tau(t)) - \tau_2 \varepsilon^T(t - \tau(t)) (I_N \otimes Q) \varepsilon(t - \tau(t)). \end{aligned} \tag{27}$$

Similar to the proof of Theorem 1, one has

$$\begin{aligned} \dot{V}(t) &\leq \varepsilon^T(t) \left(A^T P + PA + Q + \epsilon_1 \lambda_{\max^2}(\bar{L}) I + \epsilon_2 I + \frac{\rho^2}{\epsilon_2} P^2 \right) \varepsilon(t) \\ &\quad + \varepsilon^T(t - \tau(t)) \left(\frac{\lambda_{\max^2}(P B B^T P)}{\epsilon_1} I - \tau_2 Q \right) \varepsilon(t - \tau(t)). \end{aligned} \tag{28}$$

From (28), $\dot{V}(t)$ is negative definite; hence, it implies that the consensus of systems (5)-(6) is reached. \square

5. Numerical Examples

In the simulation, we will consider the networked multiagent system composed of a leader and four followers, as shown in Figure 1.

The system matrices are given as

$$\begin{aligned} A &= \begin{pmatrix} 9 & -15 \\ 18 & -20 \end{pmatrix}, \\ B &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{29}$$

A simple checking shows that (A, B) is stable. According to the graph theory, the Laplacian L and the matrix D are written as

$$\begin{aligned} L &= \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \\ D &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{30}$$

The maximum nonzero eigenvalue of $\bar{L} = L + D$, noted as $\lambda_{\max}(\bar{L}) = 3.7321$. The nonlinear function $f(x_i(t)) = k_2 x_{i1}(t) + 0.5 * (k_1 - k_2) (|x_{i1}(t) + 1| - |x_{i1}(t) - 1|)$, where $k_1 = -1.31$, $k_2 = 0.75$, and chosen $\rho = 1.31$ satisfies the Lipschitz condition [35]. The fractional order of systems (5)-(6) is $\alpha = 0.9$.

Case 1. Constant time delay.

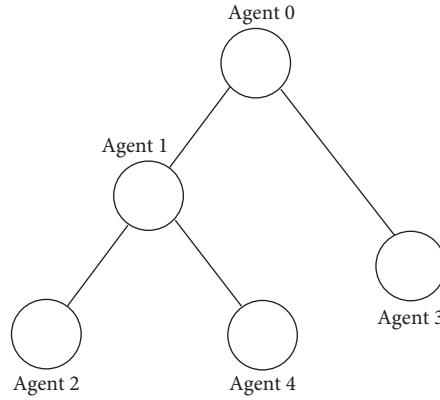


FIGURE 1: Topology structure of the multiagent systems.

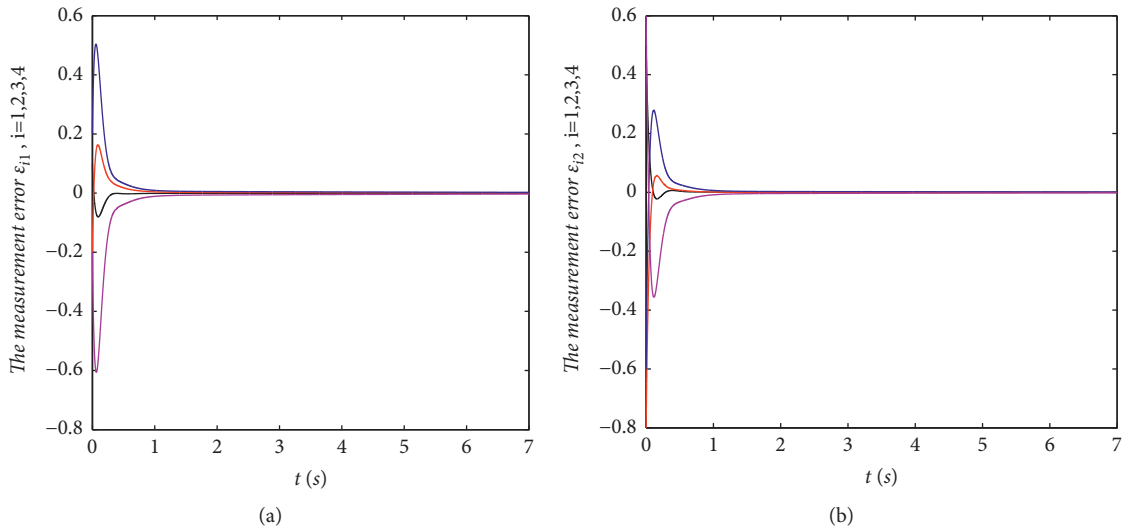


FIGURE 2: (a) The trajectory of measurement error ϵ_{i1} ; (b) the trajectory of measurement error ϵ_{i2} .

Applying Theorem 1 and setting $\tau = 0.8$, $\epsilon_1 = 0.1$, and $\epsilon_2 = 1$, the corresponding feasible solutions of (11)-(12) are found to be

$$\begin{aligned}
 P &= \begin{pmatrix} 3.2835 & -2.7690 \\ -2.7690 & 3.0576 \end{pmatrix}, \\
 Q &= \begin{pmatrix} 4.1417 & -3.5605 \\ -3.5605 & 4.4048 \end{pmatrix}.
 \end{aligned} \tag{31}$$

Obviously, P and Q are symmetric, positive definite matrices. Also, they are satisfied to the LMI of Theorem 2. Substituting $K = B^T P$ to system (10), the measurement error $\epsilon_i(t)$ between agent i and the leader will converge to 0 as shown in Figures 2(a) and 2(b). It implies the consensus of systems (5)-(6) will be achieved.

Case 2. Time-varying delay.

Given $\tau(t) = 0.8|\sin t|$, the parameters can be chosen as $\tau_1 = 0.8$ and $\tau_2 = 0.2$ according to Assumption 3. Applying

Theorem 3 and setting $\epsilon_1 = 0.1$ and $\epsilon_2 = 1$, the corresponding feasible solutions of (24) are found to be

$$\begin{aligned}
 P &= \begin{pmatrix} 2.9062 & -2.4268 \\ -2.4268 & 2.6575 \end{pmatrix}, \\
 Q &= \begin{pmatrix} 4.8981 & -4.7854 \\ -4.7854 & 5.8650 \end{pmatrix}.
 \end{aligned} \tag{32}$$

By straightforward checking, P and Q are symmetric, positive definite matrices. The measurement error $\epsilon_i(t)$ between agent i and the leader will converge to 0 as shown in Figures 3(a) and 3(b). It implies the consensus of systems (5)-(6) will be achieved. The numerical simulations illustrate that if the fixed or variable time-delay could be estimated, then we could choose suitable parameters of the controller to eliminate the error between any agent and the leader fastly and the leader-follower consensus of systems (5)-(6) can be reached asymptotically.

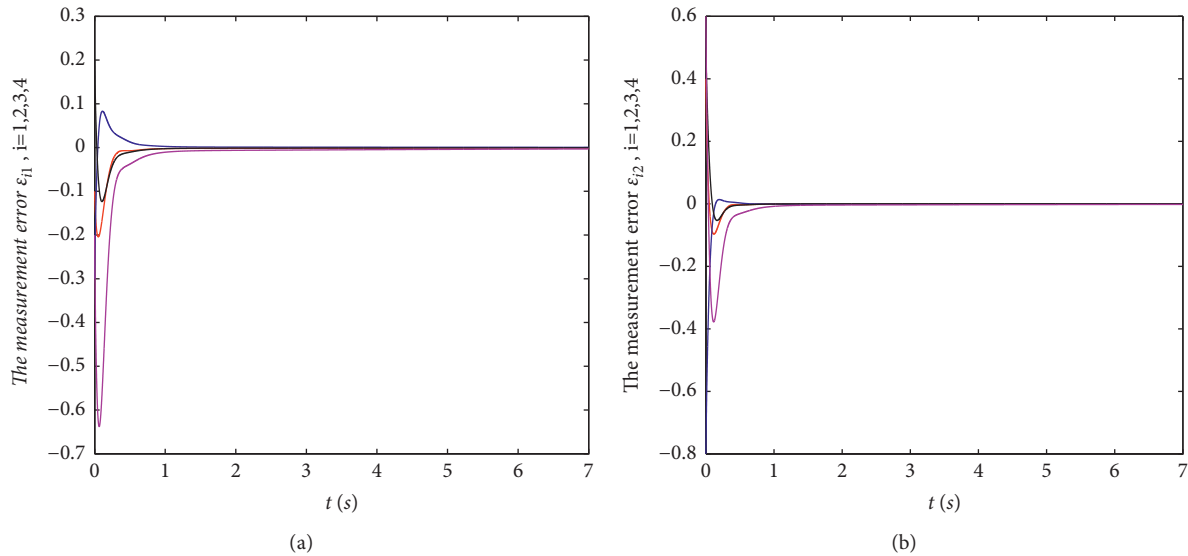


FIGURE 3: (a) The trajectory of measurement error ε_{i1} , (b) the trajectory of measurement error ε_{i2} .

6. Conclusions

This paper presents three main theorems to achieve the consensus of fractional-order multiagent systems containing input time delay. By using the graph theory, constructing the Lyapunov matrix inequality, and combining the stability of fractional-order time-delay systems, sufficient delay-independent consensus conditions are obtained. Numerical examples show that the proposed theorems and relevant calculation formula work efficiently and accurately. In the future work, we will consider extending the current work to the general vector systems with time-varying topologies.

Data Availability

Data are available on request to the corresponding author (15996301586@163.com).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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