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Research Article

On the $(\alpha-\psi)$-Contractive Mappings in C^*-Algebra Valued b-Metric Spaces and Fixed Point Theorems

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In this paper, for a unital C^*-algebra A, we introduce a version of α-ψ-contractive mappings in C^*-algebra valued b-metric spaces, and we prove some Banach fixed point theorems and give some examples to illustrate our results.

1. Introduction

Ma et al. [1] introduced the notion of C^*-algebra valued metric spaces, where the set of real number was replaced by the positive cone of a unital C^*-algebra. Later in [2], the class of C^*-algebra valued b-metric spaces is considered. Many results are introduced in this direction (see [3–10]). The notion of α-ψ-contractive mappings in metric spaces was introduced by Samet et al. [11]. Later in [12], Samet developed the notion of α-ψ-contractive mappings in b-metric spaces. Several results have been introduced in some related studies of α-admissible and $\alpha-\psi$-contractive mappings and related fixed point theorems [13–22]. In this present work, we introduced a version of $\alpha-\psi$-contractive mapping in a unital C^*-algebra valued b-metric spaces and proved some basic Banach fixed point theorems.

Some nontrivial examples are given to support our results. Suppose that A is a unital C^*-algebra with a unit I_A. Set $A_h=\{x\in A\colon x=x^*\}$. An element $x\in A$ is a positive element, if $x=x^*$ and $\sigma(x)\subset\mathbb{R}^+$ is the spectrum of x. We define a partial ordering $\preceq$ on A as $x\preceq y$ if $0_A\preceq y-x$, where 0_A means the zero element in A, and we let A^+ denote the $\{x\in A\colon x\succeq 0_A\}$ and $|x|=(x^*x)^{1/2}$.

Lemma 1. *Suppose that A is a unital C^*-algebra with unit I_A. The following holds:*

(1) If $a\in A$, with $\|a\|<1/2$, then $1-a$ is invertible and $\|a(1-a)^{-1}\|<1$

(2) For any $x\in A$ and $a,b\in A^+$, such that $a\preceq b$, we have x^*ax and x^*bx which are positive element and $x^*ax\preceq x^*bx$

(3) If $0_A\preceq a\preceq b$, then $\|a\|\leq\|b\|$

(4) If $a,b\in A^+$ and $ab=ba$, then $a.b\succeq 0_A$

(5) Let A' denote the set $\{a\in A\colon ab=ba\forall b\in A\}$ and let $a\in A'$, if $b,c\in A$ with $b\succeq c\succeq 0_A$ and $1-a\in(A')^+$ is an invertible element, then $(I_A-a)^{-1}b\preceq(I_A-a)^{-1}c$

We refer [23] for more C^* algebra details.

Definition 1. Let X be a nonempty set and $b\succeq I_A$, $b\in A'$, suppose the mapping $d_A\colon X\times X\longrightarrow A$ satisfies the following:

(1) $\mathrm{d}_A(x,y)\succeq 0_A$ for all $x,y\in X$ and $\mathrm{d}_A(x,y)=0_A\Leftrightarrow x=y$.

(2) $\mathrm{d}_A(x,y)=\mathrm{d}_A(y,x)$ for all $x,y\in X$.

(3) $\mathrm{d}_A(x,z)\preceq b[\mathrm{d}_A(x,y)+\mathrm{d}_A(y,z)]$ for all $x,y,z\in X$, where 0_A is zero element in A and I_A is the unit element in A. Then, d_A is called a C^*-algebra valued b-metric on X and (X,A,d_A) is called C^*-algebra valued b-metric space.

Definition 2. Let (X, A, d_A) be a C^*-algebra valued b-metric space, $x \in X$, and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X, then

(i) $\{x_n\}_{n=1}^{\infty}$ is convergent to x whenever, for every $c \in A$ with $c \succ 0_A$, there is a natural number $N \in \mathbb{N}$ such that

$$d_A(x_n, x) \preceq c, \tag{1}$$

for all $n > N$. We denote this by $\lim_{n \longrightarrow \infty} x_n = x$ or $x_n \longrightarrow x$ as $n \longrightarrow +\infty$.

(ii) $\{x_n\}_{n=1}^{\infty}$ is said to be a Cauchy sequence whenever, for every $c \in A$ with $c \succ 0_A$, there is a natural number $N \in \mathbb{N}$ such that

$$d_A(x_n, x_m) \preceq c, \tag{2}$$

for all $n, m > N$.

Lemma 2 *(i) $\{x_n\}_{n=1}^{\infty}$ is a convergence sequence in X if for any element $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n > N$, $\|d(x_n, x)\| \leq \varepsilon$.*

(ii) $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X, if for any $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $\|d_A(x_n, x_m)\| \leq \varepsilon$, for all $n, m > N$. We say that (X, A, d_A) is a complete C^*-algebra valued b-metric space if every Cauchy sequence is convergent with respect to A.

Example 1. Let $X = \mathbb{R}$ and $A = M_2(\mathbb{C})$ be the set of all 2×2 matrices with entries in $\mathbb{C}$, and $M_2(\mathbb{C})$ is a C^*-algebra with the matrix norm. Define

$$d_A(a, b) = \begin{pmatrix} \lambda_{1|a_{11} - b_{11}|^p} & 0 \\ 0 & \lambda_{1|a_{11} - b_{11}|^p} \end{pmatrix}, \tag{3}$$

where $a = (a_{ij})_{i,j=1}^2$ and $b = (b_{ij})_{i,j=1}^2$ are two 2×2-matrices, $a_{ij}, b_{ij} \in \mathbb{C}$, for all $i, j = 1, 2$, $\lambda_1, \lambda_2 > 0$.

One can define a partial ordering $(\preceq)$ on $M_2(\mathbb{C})$ as following $a \preceq b$ if and only if $|a_{ij}| \leq |b_{ij}| \forall i, j = 1, 2$.

And an element $a \succeq 0$ is positive in $M_2(\mathbb{C})$ if and only if $|a_{ij}| \geq 0$ for all $i, j = 1, 2$, we denote $M_2(\mathbb{C})^+$ the set of all positive element in $M_2(\mathbb{C})$. Then, $(X, M_2(\mathbb{C}), d_{M_2(\mathbb{C})})$ is C^*-algebra valued b-metric space.

Definition 4. If ψ: $A \longrightarrow B$ is a linear mapping in C^*-algebra, it is said to be positive if $\psi(A^+) \subseteq B^+$. In this case, $\psi(A_h) \subseteq B_h$, and the restriction map: ψ: $A_h \longrightarrow B_h$ is increasing.

Definition 5. Suppose that A and B are C^*-algebras. A mapping ψ: $A \longrightarrow B$ is said to be C^*-homomorphism if

(a) $\psi(ax + by) = a\psi(x) + b\psi(y)$ for all $a, b \in \mathbb{C}$ and $x, y \in A$

(b) $\psi(xy) = \psi(x)\psi(y) \forall x, y \in A$

(c) $\psi(x^*) = \psi(x)^* \forall x \in A$

(d) ψ maps the unit in A to the unit in B

Definition 6. Let Ψ_A be the set of positive functions ψ_A: $A^+ \longrightarrow A^+$ satisfying the following conditions:

(a) $\psi_A(a)$ is continuous and nondecreasing

(b) $\psi_A(a) = 0$ iff $a = 0$

(c) $\sum_{n=1}^{\infty} \psi_A^n(a) < \infty$, $\lim_{n \longrightarrow \infty} \psi_A^n(a) = 0$ for each $a \succ 0$, where ψ_A^n is nth iterate of ψ_A

(d) The series $\sum_{k=0}^{\infty} b^k \psi_A^k(a) < \infty$ for $a \succ 0$ is increasing and continuous at 0

Corollary 1. *Every C^*-homomorphism is contractive and hence bounded.*

Lemma 3. *Every $*$-homomorphism is positive.*

2. Main Results

In [11] Samet et al. and in [12] Samet introduced the concept of α-ψ-contractive mappings in metric space and α-ψ-contractive mappings in b-metric space, respectively. Here, we will develop the definitions in case of unital C^*-algebra and study some Banach fixed point theorems.

Definition 7 (see [11]). Let T: $X \longrightarrow X$ be self map and α: $X \times X \longrightarrow [0, +\infty)$. Then, T is called α-admissible if for all $x, y \in X$ with $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

Definition 8. Let X be a nonempty set and α_A: $X \times X \longrightarrow (A')^+$ be a function, we say that the self map T is α_A-admissible if $(x, y) \in X \times X$, $\alpha_A(x, y) \succeq I_A \Rightarrow \alpha_A(Tx, Ty) \succeq I_A$, where I_A the unit of A.

Definition 9. Let (X, A, d_A) be a C^*-algebra valued b-metric space and T: $X \longrightarrow X$ is mapping, we say that T is an α_A-ψ_A-contractive mapping if there exist two functions α_A: $X \times X \longrightarrow A^+$ and $\psi_A \in \Psi_A$ such that

$$\alpha_A(x, y) d_A(Tx, Ty) \preceq \psi_A(d_A(x, y)), \tag{4}$$

for all $x, y \in X$.

Theorem 1 (Banach version fixed point). *Let (X, A, d_A) be a complete C^*-algebra valued b-metric space and T: $X \longrightarrow X$ be an α_A-ψ_A-contractive mapping satisfying the following conditions:*

(i) T is α_A-admissible

(ii) There exists $x_0 \in X$ such that $\alpha_A(x_0, Tx_0) \succeq I_A$

(iii) T is continuous

Then, T has a fixed point in X.

Proof. Let $x_0 \in X$ such that $\alpha_A(x_0, Tx_0) \succeq I_A$ and define a sequence $\{x_n\}_{n=0}^{\infty}$ in X such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point for T.

Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, since T is α_A-admissible, we get

$$\begin{aligned}\alpha_A(x_0, x_1) = \alpha_A(x_0, Tx_0) \succeq I_A \Rightarrow \\ \alpha_A(Tx_0, T^2x_0) = \alpha_A(x_1, x_2) \succeq I_A.\end{aligned} \tag{5}$$

By induction, we have

$$\alpha_A(x_n, x_{n+1}) \succeq I_A, \quad \text{for all } n \in \mathbb{N}. \tag{6}$$

By inequalities (4) and (6), we get

$$\begin{aligned} d_A(x_n, x_{n+1}) &= d_A(Tx_{n-1}, Tx_n), \\ &\preceq \alpha_A(x_{n-1}, x_n) d_A(Tx_{n-1}, Tx_n) \\ &\preceq \psi_A(d_A(x_{n-1}, x_n)). \end{aligned} \tag{7}$$

By induction, we obtain

$$d_A(x_n, x_{n+1}) \preceq \psi_A^n(d_A(x_0, x_1)), \quad \text{for all } n \in \mathbb{N}. \tag{8}$$

For $m \geq 1$ and $p \geq 1$, it follows that

$$\begin{aligned} d_A(x_m, x_{m+p}) &\preceq b[d_A(x_m, x_{m+1}) + d_A(x_{m+1}, x_{m+p})] \\ &\preceq bd_A(x_m, x_{m+1}) + b^2 d_A(x_{m+1}, x_{m+2}) + \cdots + b^{p-1} d_A(x_{m+p-2}, x_{m+p-1}) + b^p d_A(x_{m+p-1}, x_{m+p}) \\ &\preceq b\psi_A^m(d_A(x_0, x_1)) + b^2 \psi_A^{m+1}(d_A(x_0, x_1)) + b^{p-1}\psi_A^{m+p-2}(d_A(x_0, x_1)) + b^p \psi_A^{m+p-1}(d_A(x_0, x_1)) \\ &= \sum_{k=1}^{p-1} b^k \psi_A^{m+k-1}(d_A(x_0, x_1)) + b^p \psi_A^{m+p-1}(d_A(x_0, x_1)). \end{aligned} \tag{9}$$

Since $b \succeq I_A$, using Definition 6, we obtain

$$d_A(x_m, x_{m+p}) \preceq \sum_{k=1}^{p-1} b^k \psi_A^{m+k-1}(d_A(x_0, x_1)) + b^p \psi_A^{m+p-1}(d_A(x_0, x_1)) \longrightarrow 0_A, \quad \text{as } n \longrightarrow +\infty. \tag{10}$$

Thus, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Since (X, A, d_A) is complete, there exists $x \in X$ such that $x_n \longrightarrow x$ as $n \longrightarrow +\infty$, from continuity of T if it follows that $x_{n+1} = Tx_n \longrightarrow Tx$ as $n \longrightarrow +\infty$. And by uniqueness of the limit, we get $Tx = x$, that is, x is a fixed point of T. To prove the uniqueness of the fixed point, we will consider the following condition. (H_A): for all $x, y \in X$, there exists $z \in X$ such that $\alpha_A(x, z) \succeq I_A$ and $\alpha_A(y, z) \succeq I_A$.

Theorem 2. *Adding condition (H_A) to the hypothesis of Theorem 1, we obtain the uniqueness of the fixed point of T.*

Proof. Suppose that x and y are two fixed points of T. From (H_A), there exists $z \in X$ such that

$$\begin{aligned} \alpha_A(x, z) \succeq I_A, \\ \alpha_A(y, z) \succeq I_A. \end{aligned} \tag{11}$$

Since T is α_A-admissible, we get

$$\begin{aligned} \alpha_A(x, T^n z) \succeq I_A, \\ \alpha_A(y, T^n z) \succeq I_A, \\ \text{for all } n \in \mathbb{N}. \end{aligned} \tag{12}$$

Using (4) and (12), we obtain

$$\begin{aligned} d_A(x, T^n z) &= d_A(Tx, T(T^{n-1}z)), \\ &\preceq \alpha_A(x, T^{n-1}z) d_A(Tx, T(T^{n-1}z)) \\ &\preceq \psi_A^n(d_A(x, z)), \quad \text{for all } n \in \mathbb{N} \longrightarrow 0_A \text{ as } n \longrightarrow +\infty. \end{aligned} \tag{13}$$

Thus, $T^n z = x$. Similarly $T^n z = y$ as $n \longrightarrow +\infty$. So, the uniqueness of the limit gives $x = y$. This completes the proof.

Theorem 3 (Kannan version fixed point). *.Let (X, A, d_A) be a complete C^*-algebra valued b-metric space and $T: X \longrightarrow X$ be a mapping satisfying*

$$\alpha_A(x, y) d_A(Tx, Ty) \preceq \psi_A(d_A(Tx, x) + d_A(Ty, y)), \tag{14}$$

for $x, y \in X$, where

$$\alpha_A: X \times X \longrightarrow A^+, \quad \text{and } \psi_A \in \Psi_A, \tag{15}$$

and the following conditions holds:

(i) T is α_A-admissible

(ii) There exists $x_0 \in X$ such that $\alpha_A(x_0, Tx_0) \succeq I_A$

(iii) T is continuous

Then, T has a fixed point in X.

Proof. Following the proof of Theorem 1, we get

$$\alpha_A(x_n, x_{n+1}) \succeq I_A, \quad (16)$$

for all $n \in \mathbb{N}$.

By inequalities (14) and (16), we obtain

$$\begin{aligned} d_A(x_n, x_{n+1}) &= d_A(Tx_{n-1}, Tx_n), \\ &\preceq \alpha_A(x_{n-1}, x_n) d_A(Tx_{n-1}, Tx_n) \\ &\preceq \psi_A(d_A(Tx_{n-1}, x_{n-1}) + d_A(Tx_n, x_n)) \\ &= \psi_A(d_A(x_n, x_{n-1}) + d_A(x_{n+1}, x_n)) \\ &= \psi_A(d_A(x_n, x_{n-1}) + \psi_A d_A(x_n, x_{n+1})) \\ (1-\psi_A) d_A(x_n, x_{n+1}) &\preceq \psi_A(d_A(x_n, x_{n-1})) \\ d_A(x_n, x_{n+1}) &\preceq \psi_A(1-\psi_A)^{-1}(d_A(x_n, x_{n-1})), \end{aligned} \quad (17)$$

from Lemma 1 and Definition 6, and let $\varphi = \psi_A (1-\psi_A)^{-1} = \varphi_A \sum_{n=0}^{\infty} \psi_A^n = \sum_{n=0}^{\infty} \psi_A^n < \infty$.

So, we get $d_A(x_n, x_{n+1}) \preceq \varphi_A(d_A(x_n, x_{n+1}))$.

By induction, we obtain

$$d_A(x_n, x_{n+1}) \preceq \varphi_A^n(d_A(x_0, x_1)), \quad \text{for all } n \in \mathbb{N}. \quad (18)$$

For $m \geq 1$ and $p \geq 1$, it follows by similar calculation in Theorem 1 that

$$d_A(x_m, x_{m+p}) = \sum_{k=1}^{p-1} b^k \varphi_A^{m+k-1}(d_A(x_0, x_1)) + b^p \varphi_A^{m+p-1}(d_A(x_0, x_1)). \quad (19)$$

Since $b \succeq I_A$, using Definition 6, we obtain

$$d_A(x_m, x_{m+p}) \preceq \sum_{k=1}^{p-1} b^k \varphi_A^{m+k-1}(d_A(x_0, x_1)) + b^p \varphi_A^{m+p-1}(d_A(x_0, x_1)) \longrightarrow 0_A, \quad \text{as } n \longrightarrow +\infty. \quad (20)$$

Thus, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Since (X, A, d_A) is complete, there exists $x \in X$ such that $x_n \longrightarrow x$ as $n \longrightarrow +\infty$, and from continuity of T, it follows that $x_{n+1} = Tx_n \longrightarrow Tx$ as $n \longrightarrow +\infty$.

And by uniqueness of the limit, we get $Tx = x$; that is, x is a fixed point of T.

Now, if $y(\neq x)$ is another fixed point of T, then

$$\begin{aligned} 0_A \preceq d_A(x, y) &= d_A(Tx, Ty), \\ &\preceq \alpha_A(x, y) d_A(Tx, Ty) \\ &\preceq psi_A(d_A(Tx, x) + d_A(Ty, y)) \\ &= \psi_A(d_A(x, x) + d_A(y, y)) \\ &= psi_A(0) = 0_A. \end{aligned} \quad (21)$$

This implies that $d_A(x, y) = 0_A$. That is, $x = y$, and this complete the proof.

Theorem 4 (Banach–Kannan version fixed point). *Let (X, A, d_A) be a complete C^*-algebra valued b-metric space and $T\colon X \longrightarrow X$ be a mapping satisfying*

$$\alpha_A(x, y) d_A(Tx, Ty) \preceq \psi_A(d_A(x, y) + d_A(Tx, x) + d_A(Ty, y)), \quad (22)$$

for $x, y \in X$, where

$$\alpha_A\colon X \times X \longrightarrow A^+, \quad \text{and } \psi_A \in \Psi_A, \quad (23)$$

such that $\psi_A(1-\psi_A)^{-1} \preceq 1/2 I_A$, and the following conditions hold:

(i) T is α_A-admissible

(ii) There exists $x_0 \in X$ such that $\alpha_A(x_0, Tx_0) \succeq I_A$

(iii) T is continuous

Then, T has a fixed point in X.

Proof. Following the proof of Theorem 1, we get

$$\alpha_A(x_n, x_{n+1}) \succeq I_A, \quad \text{for all } n \in \mathbb{N}. \quad (24)$$

By using inequalities (22) and (24), we have

$$\begin{aligned} d_A(x_n, x_{n+1}) &= d_A(Tx_{n-1}, Tx_n), \\ &\preceq \alpha_A(x_{n-1}, x_n) d_A(Tx_{n-1}, Tx_n) \\ &\preceq \psi_A(d_A(x_{n-1}, x_n) + d_A(Tx_{n-1}, x_{n-1}) \\ &\quad + d_A(Tx_n, x_n)) \\ &= \psi_A(d_A(x_n, x_{n-1}) 2I_A + d_A(x_n, x_{n+1})), \end{aligned} \quad (25)$$

Since ψ_A is additive, we get

$$
\begin{aligned}
(1-\psi_A)d_A(x_n,x_{n+1}) &\preceq 2I_A\psi_A(d_A(x_n,x_{n-1}))\\
d_A(x_n,x_{n+1}) &\preceq 2I_A(1-\psi_A)^{-1}\psi_A(d_A(x_n,x_{n-1})),
\end{aligned}
\tag{26}
$$

and putting $(1-\psi_A)^{-1}\psi_A=\varphi_A(I_A/2)$, we get

$$
d_A(x_n,x_{n+1})\preceq\varphi_A^n d_A(x_0,x_1), \tag{27}
$$

for all $n\in\mathbb{N}$; for $m\geq 1$ and $p\geq 1$ and by similar calculation as in the proof of Theorem 1, we get

$$
\begin{aligned}
d_A\left(x_m,x_{m+p}\right) \preceq &\sum_{k=1}^{p-1} b^n\varphi_A^{m+k+1}(d_A(x_0,x_1))\\
&+b^p\varphi_A^{m+p-1}(d_A(x_0,x_1))\longrightarrow 0_A,
\end{aligned}
\tag{28}
$$

as $n\longrightarrow+\infty$. This x is a fixed point of T.

To prove the uniqueness part of the fixed point of T, if $y(\neq x)$ is another fixed point of T, we have

$$
\begin{aligned}
0_A &\preceq d_A(x,y)=d_A(Tx,Ty),\\
&\preceq\alpha_A(x,y)d_A(Tx,Ty)\\
&\preceq psi_A(d_A(x,y)+d_A(Tx,x)+d_A(Ty,y))\\
&=\psi_A(d_A(x,y)+d_A(x,x)+d_A(y,y))\\
&\preceq\psi_A(d_A(x,y))\\
&\preceq d_A(x,y).
\end{aligned}
\tag{29}
$$

This is a contraction, so $d_A(x,y)=0_A$, and this gives $x=y$. This completes the proof.

Example 2. Let $X=\mathbb{R}$ and $A=M_2(\mathbb{C})$ as given in Example 1, define $T\colon X\longrightarrow X$, by $Tx=x/2$, and $\alpha_{M_2(\mathbb{C})}\colon X\times X\longrightarrow M_2(\mathbb{C})^+$ and $\alpha_{M_2(\mathbb{C})}(x,y)=I_{M_2(\mathbb{C})}$, so $\alpha_{M_2(\mathbb{C})}(Tx,Ty)=\alpha_{M_2(\mathbb{C})}(x/2,y/2)=I_{M_2(\mathbb{C})}$; thus, T is $\alpha_{M_2(\mathbb{C})}$–admissible, where $M_2(\mathbb{C})^+$ is the set of all positive elements in $M_2(\mathbb{C})$. Define $\psi_{M_2(\mathbb{C})}\colon M_2(\mathbb{C})^+\longrightarrow M_2(\mathbb{C})^+$, $\psi_{M_2(\mathbb{C})}(a)=a/2$. This is clear that $\alpha_{M_2(\mathbb{C})}-\psi_{M_2(\mathbb{C})}$-contractive mapping and satisfies $\alpha_{M_2(\mathbb{C})}(x,y).(d_{M_2(\mathbb{C})}(Tx,Ty))\preceq\psi_{M_2(\mathbb{C})}(d_{M_2(\mathbb{C})}(x,y))$ for all $x,y\in X$.

3. Applications

In this section, we shall apply Theorem 1 to prove the existence and uniqueness of solution an integral equation in C^*-algebra.

Example 3. Let E be a compact Hausdorff space, we denote by $C(E)$ the algebra of all complex-valued continuous functions on E with pointwise addition and multiplication. The algebra $C(E)$ with the involution defined by $f^*(t)=\overline{f(t)}$ for each $f\in C(E)$, $t\in E$ and with the norm $\|f\|_\infty=\sup\{|f(t)|,t\in E\}$ is a commutative C^*-algebra, with unit $I_{C(E)}$ is the constant function. Let $C^+(E)=\{f\in C(E)\colon \overline{f(t)}=f(t), f(t)\geq 0\}$ denote the positive Cone of $C(E)$, with partial order relation $f\preceq g$ if and only if $f(t)\leq g(t)$. Put $d_{C(E)}\colon C(E)\times C(E)\longrightarrow C(E)$ as $d_{C(E)}(f,g)=\sup_{t\in E}\{|f(t)-g(t)|^p\}.I_{C(E)}$. It is clear that $(C(E),C(E),d_{C(E)})$ is a complete C^*-algebra valued b-metric space.

Theorem 5 (Application). *Consider the integral equation*

$$
x(t)=\int_E F(t,x(s))\mathrm{d}s+h(t), \tag{30}
$$

where E is the compact topological Hausdorff space. Suppose

(1) $F\colon E\times\mathbb{R}\longrightarrow\mathbb{R}$.

(2) There exists a continuous function $\phi\colon E\times E\longrightarrow\mathbb{R}$ *and* $k\in(0,1)$ *such that*

$$
|F(t,f(s))-F(t,g(s))|\leq k|\phi(t,s)||f(s)-g(s)|, \tag{31}
$$

for all $f,g\in C(E)$, $t,s\in E$.

(3) $\sup_{t\in E}\int_E|\phi(t,s)|ds\leq 1$, *then the integral equation (30) has a unique solution* $x^*=\overline{x}\in C(E)$.

Proof. Let $X=C(E)$ and $A=C(E),d_{C(E)}$ as in the Example3, $(C(E),C(E),d_{C(E)})$ is a complete C^*-algebra valued b-metric space, and let $T\colon C(E)(E)\longrightarrow C(E)$ given by $Tx(t)=\int_E F(t,x(s))\mathrm{d}s+h(t)$, $x,h\in C(E),t,s\in E$. $\alpha_{C(E)}\colon C(E)\times C(E)\longrightarrow C^+(E)$ defined by $\alpha_{C(E)}(f,g)=(f,g).I_{(E)}$. And $\psi_{C(E)}\colon C^+(E)\longrightarrow C^+(E)$ defined by $\psi_{C(E)}(f)=f$.

Now,

$$
\begin{aligned}
d_{C(E)}(Tf,Tg)&=\sup_{t\in E}\{|Tf(t)-Tg(t)|^p\}.I_{C(E)},\\
&=\sup_{t\in E}\int_E|F(t,f(s))\mathrm{d}s-F(t,g(s))\mathrm{d}s|^p.I_{C(E)}\mathrm{d}t\\
&\leq\sup_{t\in E}\int_E|\phi(t,s)\mathrm{d}s|^p|f(s)-g(s)|^p.I_{C(E)}\mathrm{d}t\\
&\leq k^p d_{C(E)}(f,g).
\end{aligned}
\tag{32}
$$

Put $A=k^p$ since $k\in(0,1)$, this gives $\|A\|\leq 1$, and we get

$$
\begin{aligned}
\alpha_{C(E)}\left(d_{C(E)}(Tf,Tg)\right)&\leq\|A\|\psi_{C(E)}\left(d_{C(E)}(f,g)\right)\\
&\leq\psi_{C(E)}\left(d_{C(E)}(f,g)\right),
\end{aligned}
\tag{33}
$$

for all $f,g\in C(E)$.

Thus, T is an $\alpha_{C(E)}-\psi_{C(E)}$-contractive mapping and satisfies Theorem 1. So, T has a unique fixed point, and the integral equation (30) has a unique solution $x^*=\overline{x}\in C(E)$.

4. Conclusions

In this paper, we define a new version of α_A-ψ_A-admissible in the case of self mappings $T\colon A\longrightarrow A$. We prove the principal Banach fixed point theorem, Kannan fixed point theorem, and Banach–Kannan fixed point theorem in the C^*-algebra valued b-metric space, which generalized the given results in [1, 2, 11, 12, 24].

Data Availability

No data were used to support the results.

Conflicts of Interest

The authors of this research declare that they have no conflicts of interest.

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