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Research Article

A Study of Generalized Projective $\mathscr{P}$ – Curvature Tensor on Warped Product Manifolds

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The main aim of this study is to investigate the effects of the $\mathscr{P}-$curvature flatness, $\mathscr{P}-$divergence-free characteristic, and $\mathscr{P}-$symmetry of a warped product manifold on its base and fiber (factor) manifolds. It is proved that the base and the fiber manifolds of the $\mathscr{P}-$curvature flat warped manifold are Einstein manifold. Besides that, the forms of the $\mathscr{P}-$curvature tensor on the base and the fiber manifolds are obtained. The warped product manifold with $\mathscr{P}-$divergence-free characteristic is investigated, and amongst many results, it is proved that the factor manifolds are of constant scalar curvature. Finally, $\mathscr{P}-$symmetric warped product manifold is considered.

1. Introduction

Curvature tensors play a significant role in mathematics and physics. This is why many researchers have introduced and studied many curvature tensors in various ways, as well as they have shown the importance of these curvature tensors. For instance, the deviation of a space from constant curvature is measured by the concircular curvature tensor (for more details, see [1]). The Weyl curvature tensor describes the distorting but volume-preserving tidal effects of gravitation on a material body.

The $\mathscr{P}-$curvature tensor was first coined by De et al. in 2021 [2]. This curvature tensor is a good generalization of projective [3], conharmonic [4], $M-$projective [5], and the set of $\mathscr{W}_i-$curvature tensors which was introduced by Pokhariyal and Mishra [6–10]. This curvature tensor is given by

$$\mathscr{P}_{ijkl} = a_0 R_{ijkl} + a_1 g_{ij} R_{kl} + a_2 g_{ik} R_{jl} + a_3 g_{il} R_{jk} + a_4 g_{jk} R_{il} + a_5 g_{jl} R_{ik} + a_6 g_{kl} R_{ij}, \quad (1)$$

where a_i are constants, R_{ijkl} is the Riemann tensor, and R_{ij} is the Ricci tensor [2]. The authors studied this curvature tensor on pseudo-Riemannian manifolds and space times of general relativity. It is proved that pseudo-Riemannian manifolds M will be Einstein manifold if M admits a traceless $\mathscr{P}-$curvature tensor and will be of constant scalar curvature if M is of $\mathscr{P}-$curvature flat. Pseudo-Riemannian manifolds with $\mathscr{P}-$divergence-free characteristic were investigated in Gray's seven subspaces. As a final point, they studied perfect fluid space times when the $\mathscr{P}-$curvature tensor is flat, and in this case, many interesting results are obtained.

Geometers have considered all well-known curvature tensors on the warped product manifolds. For instance,

$\mathscr{W}_2$−curvature tensor on warped product manifolds is studied in [11]. Also, concircular curvature tensor on warped product manifold is considered in [12]. Motivated by these kinds of studies and others, this paper aims to investigate the $\mathscr{P}$−curvature tensor on the warped product manifolds.

This paper is organized as follows: In Section 2, the basic properties of the warped product manifold are presented. In Section 3, we consider the $\mathscr{P}$−curvature flat warped product manifold. We prove that the base and the fiber manifolds of the $\mathscr{P}$−curvature flat warped product manifold are Einstein manifold; also in this case, the forms of the $\mathscr{P}$−curvature tensor on the base and the fiber manifolds are obtained. Section 4 is devoted to study the $\mathscr{P}$−divergence-free warped product manifold. It is proven that the base and the fiber manifolds of the warped product manifold with $\mathscr{P}$−divergence-free characteristic are of constant scalar curvature. In addition, these factor manifolds of the $\mathscr{P}$−divergence-free warped product whose Ricci tensor is of Codazzi type are Ricci symmetric manifolds. Finally, we prove that the warped product manifold is $\mathscr{P}$−symmetric if and only if the base and the fiber manifolds are $\mathscr{P}$−symmetric manifolds.

2. On Singly Warped Product Manifold

Let $(\overline{M}, \overline{g})$ and $(\widetilde{M}, \widetilde{g})$ be two pseudo-Riemannian manifolds with dimensions $\dim \overline{M} = \overline{n}$ and $\dim \widetilde{M} = \widetilde{n} = n - \overline{n}$, where $n > \overline{n} > 1$. And, let $F: \overline{M} \longrightarrow (0, \infty)$ be a smooth positive function on $\overline{M}$. Consider the product manifold $\overline{M} \times \widetilde{M}$ with its natural projections $\pi: \overline{M} \times \widetilde{M} \longrightarrow \overline{M}$ and $\eta: \overline{M} \times \widetilde{M} \longrightarrow \widetilde{M}$. Then, the singly warped product manifold $M = \overline{M} \times_f \widetilde{M}$ is the product manifold $\overline{M} \times \widetilde{M}$ furnished with the metric tensor

$$g = \overline{g} \oplus F\widetilde{g}. \tag{2}$$

The manifold $\overline{M}$ is called the base manifold, whereas $\widetilde{M}$ is called the fiber manifold [13, 14]. A warped product manifold $M = \overline{M} \times_F \widetilde{M}$ is called trivial if the warping function F is constant. In this case, $M = \overline{M} \times_F \widetilde{M}$ is the Riemannian product $M = \overline{M} \times \widetilde{M}_F$, where $\widetilde{M}_F$ is the manifold $\widetilde{M}$ equipped with metric $F\widetilde{g}$, which is homothetic to $\widetilde{g}$.

Curvatures of the warped product manifold depend on the curvatures of its fiber and base manifolds. It is noted that the curvatures of the Riemannian product manifold split as a sum of the corresponding curvatures of the first and second factor manifolds since both of the metric and the Levi–Civita connection split as a sum. It is natural now to discuss the deviation in the relation between the different curvature formulas in warped product manifolds and their factor manifolds due to the existence of a nontrivial warping function.

Let $\partial/\partial x^a, \partial/\partial x^b, \ldots$ denote the basis vector fields on a neighborhood $\overline{U}$ of the base manifold $\overline{M}$, where $a, b, \ldots, \in \{1, \ldots, \overline{n}\}$, whereas $\partial/\partial x\ \alpha, \partial/\partial x^\beta, \ldots$ denote the basis vector fields on a neighborhood $\widetilde{U}$ of the fiber manifold $\widetilde{M}$, where $\alpha, \beta, \ldots . \in \{\overline{n} + 1, \ldots, n\}$. Likewise, $\partial/\partial x^i$, $\partial/\partial x^j, \ldots$ denote the basis vector fields on a neighborhood $\overline{U} \times \widetilde{U}$ of the warped product manifold $\overline{M} \times_F \widetilde{M}$, where $i, j, \ldots, \in \{1, \ldots, n\}$. The local components of the metric tensor $g = \overline{g} \times_F \widetilde{g}$ of the warped product manifold $\overline{M} \times_F \widetilde{M}$ are

$$g_{ij=} \begin{matrix} \overline{g}_{ab} & \text{for } i = a, j = b, \\ F\widetilde{g}_{\alpha\beta} & \text{for } i = \alpha, j = \beta, \\ 0 & \text{otherwise.} \end{matrix} \tag{3}$$

The local components Γ^h_{ij} of the Levi–Civita connection on the warped product $M = \overline{M} \times_F \widetilde{M}$ are as follows:

$$\begin{aligned} &\Gamma^a_{bc} = \overline{\Gamma}^a_{bc}, \Gamma^\alpha_{\beta\gamma} = \widetilde{\Gamma}^\alpha_{\beta\gamma}, \\ &\Gamma^a_{\alpha\beta} = \frac{1}{2}\overline{g}^{ab} F_b \widetilde{G}_{\alpha\beta}, \Gamma^\alpha_{a\beta} = \frac{1}{2F} F_a \delta^\alpha_\beta, \\ &\Gamma^\alpha_{ab} = \Gamma^a_{\alpha b} = 0, \end{aligned} \tag{4}$$

where $F_a = \partial_a F = \partial F/\partial x^a$ and $\widetilde{G}_{\alpha\beta\gamma\delta} = \widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\delta} - \widetilde{g}_{\alpha\delta}\widetilde{g}_{\beta\gamma}$.

On the warped product $M = \overline{M} \times_F \widetilde{M}$, the local components of the Riemannian curvature tensor R_{ijkl} are given by [15–18]

$$R_{abc\ d} = \overline{R}_{abc\ d}, \tag{5}$$

$$R_{\alpha\beta\gamma\delta} = F\widetilde{R}_{\alpha\beta\gamma\delta} - \frac{1}{4}\overline{\Delta}F\widetilde{G}_{\alpha\beta\gamma\delta}, \tag{6}$$

$$R_{\alpha ab\beta} = \frac{-1}{2} T_{ab}\widetilde{g}_{\alpha\beta}, \tag{7}$$

where $\overline{\Delta}F = \overline{g}^{ab} F_a F_b$ and T_{ab} is a tensor of type $(0, 2)$ with local components $T_{ab} = \overline{\nabla}_b F_a - 1/2F F_a F_b$ and $T_{\alpha\beta} = T_{a\alpha} = 0$.

The local components of the Ricci curvature R_{ij} of the warped product $M = \overline{M} \times_F \widetilde{M}$ are the following [16, 17]:

$$R_{ab} = \overline{R}_{ab} - \frac{\widetilde{n}}{2F} T_{ab}, \tag{8}$$

$$R_{a\alpha} = 0, \tag{9}$$

$$R_{\alpha\beta} = \widetilde{R}_{\alpha\beta} - \frac{1}{2}\left[tr(T) + \frac{\widetilde{n} - 1}{2F}\overline{\Delta}F\right]\widetilde{g}_{\alpha\beta}, \tag{10}$$

where $tr(T) = \overline{g}^{ab} T_{ab}$.

It is well known that [16, 17]

$$\begin{aligned} &R_{abc\ d;e} = \overline{R}_{abc\ d;e}, \\ &R_{\alpha\beta\gamma\delta;\varepsilon} = F\widetilde{R}_{\alpha\beta\gamma\delta;\varepsilon}, \\ &R_{\alpha ab\beta;\varepsilon} = 0, \end{aligned} \tag{11}$$

where "semicolon" refers to the covariant derivative with respect to the metric.

Also,

$$R_{ab;c} = \overline{R}_{ab;c} - \frac{\widetilde{n}}{2}\nabla_c\left(\frac{T_{ab}}{F}\right),$$
$$R_{\alpha\beta;\gamma} = \widetilde{R}_{\alpha\beta;\gamma}, \tag{12}$$
$$R_{ab;\varepsilon} = 0.$$

3. P – Curvature Flat Singly Warped Product Manifolds

In this section, we consider that the warped product manifold $M = \overline{M}\times_F\widetilde{M}$ is a $\mathscr{P}$–curvature flat manifold. The local components of the considered $\mathscr{P}$–curvature tensor of the warped product manifold $\overline{M}\times_F\widetilde{M}$, which in general do not vanish identically, are the following $\mathscr{P}_{abcd}$, $\mathscr{P}_{\alpha ab\beta}$, $\mathscr{P}_{\alpha\beta\gamma\delta}$, and $\mathscr{P}_{ab\alpha\beta}$, whereas the local components $\mathscr{P}_{abc\alpha}$ and $\mathscr{P}_{a\alpha\beta\gamma}$ vanish.

Let us calculate the first component of the $\mathscr{P}$–curvature tensor of the warped product manifold M which is

$$\mathscr{P}_{\alpha ab\beta} = a_0R_{\alpha ab\beta} + a_1g_{\alpha a}R_{b\beta} + a_2g_{\alpha b}R_{a\beta} + a_3g_{\alpha\beta}R_{ab} + a_4g_{ab}R_{\alpha\beta} + a_5g_{a\beta}R_{\alpha b} + a_6g_{b\beta}R_{\alpha a}. \tag{13}$$

In virtue of (3) and (9), we have

$$\mathscr{P}_{\alpha ab\beta} = a_0R_{\alpha ab\beta} + a_3g_{\alpha\beta}R_{ab} + a_4g_{ab}R_{\alpha\beta}. \tag{14}$$

Utilizing equations (3), (7), (8), and (10) in equation (14), we infer

$$\mathscr{P}_{\alpha ab\beta} = \frac{-a_0}{2}T_{ab}\widetilde{g}_{\alpha\beta} + a_3F\widetilde{g}_{\alpha\beta}\left(\overline{R}_{ab} - \frac{\widetilde{n}}{2F}T_{ab}\right) + a_4\overline{g}_{ab}\left\{\widetilde{R}_{\alpha\beta} - \frac{1}{2}\left[tr(T) + \frac{\widetilde{n}-1}{2F}\overline{\Delta}F\right]\widetilde{g}_{\alpha\beta}\right\}. \tag{15}$$

Suppose that M is $\mathscr{P}$–curvature flat, that is, $\mathscr{P}_{\alpha ab\beta} = 0$. Thus,

$$\frac{-a_0}{2}T_{ab}\widetilde{g}_{\alpha\beta} + a_3F\widetilde{g}_{\alpha\beta}\left(\overline{R}_{ab} - \frac{\widetilde{n}}{2F}T_{ab}\right) + a_4\overline{g}_{ab}\left\{\widetilde{R}_{\alpha\beta} - \frac{1}{2}\left[tr(T) + \frac{\widetilde{n}-1}{2F}\overline{\Delta}F\right]\widetilde{g}_{\alpha\beta}\right\} = 0. \tag{16}$$

A contraction with $\widetilde{g}^{\alpha\beta}$ implies

$$0 = \frac{-a_0}{2}T_{ab}\widetilde{n} + a_3F\widetilde{n}\left(\overline{R}_{ab} - \frac{\widetilde{n}}{2F}T_{ab}\right) + a_4\overline{g}_{ab}\left\{\widetilde{R} - \frac{1}{2}\left[tr(T) + \frac{\widetilde{n}-1}{2F}\overline{\Delta}F\right]\widetilde{n}\right\},$$
$$\overline{R}_{ab} = \frac{a_0}{2a_3F}T_{ab} + \frac{\widetilde{n}}{2F}T_{ab} - \frac{a_4\overline{g}_{ab}}{a_3F}\left(\frac{\widetilde{R}}{\widetilde{n}} - \frac{1}{2}\left[tr(T) + \frac{\widetilde{n}-1}{2F}\overline{\Delta}F\right]\right). \tag{17}$$

One more contraction with g^{ab} gives

$$0 = \frac{-a_0\overline{n}}{2}tr(T) + a_3\widetilde{n}F\left(\overline{R} - \frac{\widetilde{n}}{2F}tr(T)\right) + a_4\overline{n}\left\{\widetilde{R} - \frac{1}{2}\left[tr(T) + \frac{\widetilde{n}-1}{2F}\overline{\Delta}F\right]\widetilde{n}\right\},$$
$$\widetilde{R} = \frac{a_0\widetilde{n}}{2a_4\overline{n}}tr(T) - \frac{a_3\widetilde{n}F}{a_4\overline{n}}\left(\overline{R} - \frac{\widetilde{n}}{2F}tr(T)\right) + \frac{\widetilde{n}}{2}\left[tr(T) + \frac{\widetilde{n}-1}{2F}\overline{\Delta}F\right]. \tag{18}$$

Equation (17) and (18) together imply

$$\overline{R}_{ab} = \frac{\overline{g}_{ab}}{\overline{n}}\overline{R}, \tag{19}$$

which means that the base manifold $\overline{M}$ is Einstein manifold.

Contracting equation (16) with $\overline{g}^{ab}$ gives

$$0=\frac{-a_0}{2}tr(T)\tilde{g}_{\alpha\beta}+a_3F\tilde{g}_{\alpha\beta}\left(\overline{R}-\frac{\tilde{n}}{2F}tr(T)\right)+a_4\overline{n}\left\{\tilde{R}_{\alpha\beta}-\frac{1}{2}\left[tr(T)+\frac{\tilde{n}-1}{2F}\overline{\Delta}F\right]\tilde{g}_{\alpha\beta}\right\},$$
$$\tilde{R}_{\alpha\beta}=\frac{a_0}{2a_4\overline{n}}tr(T)\tilde{g}_{\alpha\beta}-\frac{a_3}{a_4\overline{n}}F\tilde{g}_{\alpha\beta}\left(\overline{R}-\frac{\tilde{n}}{2F}tr(T)\right)+\frac{1}{2}\left[tr(T)+\frac{\tilde{n}-1}{2F}\overline{\Delta}F\right]\tilde{g}_{\alpha\beta}. \tag{20}$$

Multiplying equation (20) with $\tilde{g}^{\alpha\beta}$ implies

$$0=\frac{-a_0}{2}tr(T)\tilde{n}+a_3\tilde{n}F\left(\overline{R}-\frac{\tilde{n}}{2F}tr(T)\right)+a_4\overline{n}\left\{\tilde{R}-\frac{1}{2}\left[tr(T)+\frac{\tilde{n}-1}{2F}\overline{\Delta}F\right]\tilde{n}\right\}=0,$$
$$\overline{R}=\frac{-a_0}{2a_3F}tr(T)+\frac{\tilde{n}}{2F}tr(T)-\frac{a_4\overline{n}}{a_3\tilde{n}F}\left\{\tilde{R}-\frac{1}{2}\left[tr(T)+\frac{\tilde{n}-1}{2F}\overline{\Delta}F\right]\tilde{n}\right\}. \tag{21}$$

Substituting equation (21) into (20), we have

$$\tilde{R}_{\alpha\beta}=\frac{\tilde{g}_{\alpha\beta}}{\tilde{n}}\tilde{R}, \tag{22}$$

which means that the fiber manifold $\tilde{M}$ is Einstein manifold.

The second component of the $\mathscr{P}$–curvature tensor is

$$\mathscr{P}_{ab\alpha\beta}=a_0R_{ab\alpha\beta}+a_1g_{ab}R_{\alpha\beta}+a_2g_{a\alpha}R_{b\beta}+a_3g_{\alpha\beta}R_{b\alpha}+a_4g_{b\alpha}R_{a\beta}+a_5g_{b\beta}R_{a\alpha}+a_6g_{\alpha\beta}R_{ab}. \tag{23}$$

In virtue of (3) and (9), one gets

$$\mathscr{P}_{ab\alpha\beta}=a_1g_{ab}R_{\alpha\beta}+a_6g_{\alpha\beta}R_{ab}. \tag{24}$$

The use of equations (3), (7), (8), and (10) implies

$$\mathscr{P}_{ab\alpha\beta}=a_1\overline{g}_{ab}\left(\tilde{R}_{\alpha\beta}-\frac{1}{2}\left[tr(T)+\frac{\tilde{n}-1}{2F}\overline{\Delta}F\right]\tilde{g}_{\alpha\beta}\right)+a_6F\tilde{g}_{\alpha\beta}\left(\overline{R}_{ab}-\frac{\tilde{n}}{2F}T_{ab}\right). \tag{25}$$

Now, consider that $\mathscr{P}$–curvature tensor is flat; that is, $\mathscr{P}_{ab\alpha\beta}=0$, and hence,

$$a_1\overline{g}_{ab}\left(\tilde{R}_{\alpha\beta}-\frac{1}{2}\left[tr(T)+\frac{\tilde{n}-1}{2F}\overline{\Delta}F\right]\tilde{g}_{\alpha\beta}\right)+a_6F\tilde{g}_{\alpha\beta}\left(\overline{R}_{ab}-\frac{\tilde{n}}{2F}T_{ab}\right)=0. \tag{26}$$

A contraction with $\tilde{g}^{\alpha\beta}$ implies

$$0=a_1\overline{g}_{ab}\left(\tilde{R}-\frac{1}{2}\left[tr(T)+\frac{\tilde{n}-1}{2F}\overline{\Delta}F\right]\tilde{n}\right)+a_6F\tilde{n}\left(\overline{R}_{ab}-\frac{\tilde{n}}{2F}T_{ab}\right),$$
$$\overline{R}_{ab}=\frac{\tilde{n}}{2F}T_{ab}-\frac{a_1\overline{g}_{ab}}{a_6F\tilde{n}}\left(\tilde{R}-\frac{1}{2}\left[tr(T)+\frac{\tilde{n}-1}{2F}\overline{\Delta}F\right]\tilde{n}\right). \tag{27}$$

Again, contracting equation (27) with $\overline{g}^{ab}$ gives

$$0=a_1\overline{n}\left(\tilde{R}-\frac{1}{2}\left[tr(T)+\frac{\tilde{n}-1}{2F}\overline{\Delta}F\right]\tilde{n}\right)+a_6F\tilde{n}\left(\overline{R}-\frac{\tilde{n}}{2F}tr(T)\right),$$
$$\tilde{R}=\frac{1}{2}\left[tr(T)+\frac{\tilde{n}-1}{2F}\overline{\Delta}F\right]\tilde{n}-\frac{a_6F\tilde{n}}{a_1\overline{n}}\left(\overline{R}-\frac{\tilde{n}}{2F}tr(T)\right). \tag{28}$$

Combining the previous two equations, we reveal that

$$\overline{R}_{ab} = \frac{\overline{g}_{ab}}{\overline{n}}\overline{R}. \tag{29}$$

Similarly, we can obtain

$$\widetilde{R}_{\alpha\beta} = \frac{\widetilde{g}_{\alpha\beta}}{\widetilde{n}}\widetilde{R}. \tag{30}$$

From the above discussion, we are in a position to state the following.

Theorem 1. *Let $M = \overline{M}\times_F\widetilde{M}$ be a $\mathscr{P}-$curvature flat singly warped product manifold furnished with the metric tensor $g = \overline{g}\times_F\widetilde{g}$. Then, the base and the fiber manifolds of the warped product manifold $\overline{M}\times_F\widetilde{M}$ are Einstein manifold.*

Moving on to the next component, we have

$$\begin{aligned}\mathscr{P}_{abc\ d} &= a_0R_{abc\ d} + a_1g_{ab}R_{c\ d} + a_2g_{ac}R_{b\ d} + a_3g_{a\ d}R_{bc}\\ &\quad + a_4g_{bc}R_{a\ d} + a_5g_{b\ d}R_{ac} + a_6g_{c\ d}R_{ab}.\end{aligned} \tag{31}$$

Using (3), (8), and (5), we obtain

$$\begin{aligned}\mathscr{P}_{abc\ d} &= a_0\overline{R}_{abc\ d} + a_1\overline{g}_{ab}\left(\overline{R}_{c\ d} - \frac{\widetilde{n}}{2F}T_{c\ d}\right) + a_2\overline{g}_{ac}\left(\overline{R}_{b\ d} - \frac{\widetilde{n}}{2F}T_{b\ d}\right) + a_3\overline{g}_{a\ d}\left(\overline{R}_{bc} - \frac{\widetilde{n}}{2F}T_{bc}\right)\\ &\quad + a_4\overline{g}_{bc}\left(\overline{R}_{a\ d} - \frac{\widetilde{n}}{2F}T_{a\ d}\right) + a_5\overline{g}_{b\ d}\left(\overline{R}_{ac} - \frac{\widetilde{n}}{2F}T_{ac}\right) + a_6\overline{g}_{c\ d}\left(\overline{R}_{ab} - \frac{\widetilde{n}}{2F}T_{ab}\right).\end{aligned} \tag{32}$$

The previous equation can be rewritten in the following form:

$$\begin{aligned}\mathscr{P}_{abc\ d} &= a_0\overline{R}_{abc\ d} + a_1\overline{g}_{ab}\overline{R}_{c\ d} + a_2\overline{g}_{ac}\overline{R}_{b\ d} + a_3\overline{g}_{a\ d}\overline{R}_{bc} + a_4\overline{g}_{bc}T_{a\ d} + a_5\overline{g}_{b\ d}\overline{R}_{ac} + a_6\overline{g}_{c\ d}\overline{R}_{ab}\\ &\quad - \frac{\widetilde{n}}{2F}\left[a_1\overline{g}_{ab}T_{c\ d} + a_2\overline{g}_{ac}T_{b\ d} + a_3\overline{g}_{a\ d}T_{bc} + a_4\overline{g}_{bc}T_{a\ d} + a_5\overline{g}_{b\ d}T_{ac} + a_6\overline{g}_{c\ d}T_{ab}\right].\end{aligned} \tag{33}$$

Remember that the $\mathscr{P}-$curvature tensor on the base manifold is of the form

$$\begin{aligned}\overline{\mathscr{P}}_{abcd} &= a_0\overline{R}_{abc\ d} + a_1\overline{g}_{ab}\overline{R}_{c\ d} + a_2\overline{g}_{ac}\overline{R}_{b\ d} + a_3\overline{g}_{a\ d}\overline{R}_{bc}\\ &\quad + a_4\overline{g}_{bc}T_{a\ d} + a_5\overline{g}_{b\ d}\overline{R}_{ac} + a_6\overline{g}_{c\ d}\overline{R}_{ab}.\end{aligned} \tag{34}$$

Consequently, we can obtain the following:

$$\begin{aligned}\mathscr{P}_{abcd} &= \overline{\mathscr{P}}_{abc\ d} - \frac{\widetilde{n}}{2F}\left[a_1\overline{g}_{ab}T_{c\ d} + a_2\overline{g}_{ac}T_{b\ d} + a_3\overline{g}_{a\ d}T_{bc}\right.\\ &\quad \left. + a_4\overline{g}_{bc}T_{a\ d} + a_5\overline{g}_{b\ d}T_{ac} + a_6\overline{g}_{c\ d}T_{ab}\right].\end{aligned} \tag{35}$$

Suppose that $M = \overline{M}\times_F\widetilde{M}$ is a $\mathscr{P}-$curvature flat; that is, $\mathscr{P}_{abcd} = 0$. This leads to

$$\begin{aligned}\overline{\mathscr{P}}_{abc\ d} &= \frac{\widetilde{n}}{2F}\left[a_1\overline{g}_{ab}T_{c\ d} + a_2\overline{g}_{ac}T_{b\ d} + a_3\overline{g}_{a\ d}T_{bc} + a_4\overline{g}_{bc}T_{a\ d}\right.\\ &\quad \left. + a_5\overline{g}_{b\ d}T_{ac} + a_6\overline{g}_{c\ d}T_{ab}\right],\end{aligned} \tag{36}$$

which is the form of the $\mathscr{P}-$curvature tensor of the base manifold $\overline{M}$. Thus, we can state the following theorem.

Theorem 2. *Let $M = \overline{M}\times_F\widetilde{M}$ be a $\mathscr{P}-$curvature flat singly warped product manifold equipped with the metric tensor $g = \overline{g}\times_F\widetilde{g}$. Then, the $\mathscr{P}-$curvature tensor on the base manifold $\overline{M}$ is given by*

$$\begin{aligned}\overline{\mathscr{P}}_{abc\ d} &= \frac{\widetilde{n}}{2F}\left[a_1\overline{g}_{ab}T_{c\ d} + a_2\overline{g}_{ac}T_{b\ d} + a_3\overline{g}_{a\ d}T_{bc} + a_4\overline{g}_{bc}T_{a\ d}\right.\\ &\quad \left. + a_5\overline{g}_{b\ d}T_{ac} + a_6\overline{g}_{c\ d}T_{ab}\right].\end{aligned} \tag{37}$$

Assume that $T_{ab} = 0$; then, equation (36) implies

$$\overline{\mathscr{P}}_{abc\ d} = 0, \tag{38}$$

which means that the base manifold is $\mathscr{P}-$curvature flat.

Corollary 1. *The base manifold $\overline{M}$ of the warped product manifold M is $\mathscr{P}-$curvature flat if the warped product manifold M is $\mathscr{P}-$curvature flat and $T_{ab} = 0$.*

The last component of the $\mathscr{P}-$curvature tensor is

$$\begin{aligned}\mathscr{P}_{\alpha\beta\gamma\delta} &= a_0R_{\alpha\beta\gamma\delta} + a_1g_{\alpha\beta}R_{\gamma\delta} + a_2g_{\alpha\gamma}R_{\beta\delta} + a_3g_{\alpha\delta}R_{\beta\gamma}\\ &\quad + a_4g_{\beta\gamma}R_{\alpha\delta} + a_5g_{\beta\delta}R_{\alpha\gamma} + a_6g_{\gamma\delta}R_{\alpha\beta}.\end{aligned} \tag{39}$$

Using (6) and (10), we obtain

$$\begin{aligned}\mathscr{P}_{\alpha\beta\gamma\delta} &= a_0\left(F\widetilde{R}_{\alpha\beta\gamma\delta} - \frac{1}{4}\overline{\Delta}F\widetilde{G}_{\alpha\beta\gamma\delta}\right) + a_1F\widetilde{g}_{\alpha\beta}\left\{\widetilde{R}_{\gamma\delta} - \frac{1}{2}\left[tr(T) + \frac{\widetilde{n}-1}{2F}\overline{\Delta}F\right]\widetilde{g}_{\gamma\delta}\right\} + a_2F\widetilde{g}_{\alpha\gamma}\left\{\widetilde{R}_{\beta\delta} - \frac{1}{2}\left[tr(T) + \frac{\widetilde{n}-1}{2F}\overline{\Delta}F\right]\widetilde{g}_{\beta\delta}\right\}\\ &+ a_3F\widetilde{g}_{\alpha\delta}\left\{\widetilde{R}_{\beta\gamma} - \frac{1}{2}\left[tr(T) + \frac{\widetilde{n}-1}{2F}\overline{\Delta}F\right]\widetilde{g}_{\beta\gamma}\right\}\\ &+ a_4F\widetilde{g}_{\beta\gamma}\left\{\widetilde{R}_{\alpha\delta} - \frac{1}{2}\left[tr(T) + \frac{\widetilde{n}-1}{2F}\overline{\Delta}F\right]\widetilde{g}_{\alpha\delta}\right\} + a_5F\widetilde{g}_{\beta\delta}\left\{\widetilde{R}_{\alpha\gamma} - \frac{1}{2}\left[tr(T) + \frac{\widetilde{n}-1}{2F}\overline{\Delta}F\right]\widetilde{g}_{\alpha\gamma}\right\}\\ &+ a_6F\widetilde{g}_{\gamma\delta}\left\{\widetilde{R}_{\alpha\beta} - \frac{1}{2}\left[tr(T) + \frac{\widetilde{n}-1}{2F}\overline{\Delta}F\right]\widetilde{g}_{\alpha\beta}\right\}.\end{aligned} \tag{40}$$

The previous equation can be rewritten in the following form:

$$\begin{aligned}\mathscr{P}_{\alpha\beta\gamma\delta} &= Fa_0\widetilde{R}_{\alpha\beta\gamma\delta} + a_1\widetilde{g}_{\alpha\beta}\widetilde{R}_{\gamma\delta} + a_2\widetilde{g}_{\alpha\gamma}\widetilde{R}_{\beta\delta} + a_3\widetilde{g}_{\alpha\delta}\widetilde{R}_{\beta\gamma} + a_4\widetilde{g}_{\beta\gamma}\widetilde{R}_{\alpha\delta} + a_5\widetilde{g}_{\beta\delta}\widetilde{R}_{\alpha\gamma} + a_6\widetilde{g}_{\gamma\delta}\widetilde{R}_{\alpha\beta}\\ &- \frac{1}{2}\left(tr(T) + \frac{\widetilde{n}-1}{2F}\overline{\Delta}F\right)\frac{1}{4}a_0\overline{\Delta}F\widetilde{G}_{\alpha\beta\gamma\delta} + (a_1+a_6)F\widetilde{g}_{\alpha\beta}\widetilde{g}_{\gamma\delta} + (a_2+a_5)F\widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\delta} + (a_3+a_4)F\widetilde{g}_{\alpha\delta}\widetilde{g}_{\beta\gamma}.\end{aligned} \tag{41}$$

The $\mathscr{P}$−curvature tensor on the fiber manifold is given by

$$\begin{aligned}\widetilde{\mathscr{P}}_{\alpha\beta\gamma\delta} &= a_0\widetilde{R}_{\alpha\beta\gamma\delta} + a_1\widetilde{g}_{\alpha\beta}\widetilde{R}_{\gamma\delta} + a_2\widetilde{g}_{\alpha\gamma}\widetilde{R}_{\beta\delta} + a_3\widetilde{g}_{\alpha\delta}\widetilde{R}_{\beta\gamma}\\ &+ a_4\widetilde{g}_{\beta\gamma}\widetilde{R}_{\alpha\delta} + a_5\widetilde{g}_{\beta\delta}\widetilde{R}_{\alpha\gamma} + a_6\widetilde{g}_{\gamma\delta}\widetilde{R}_{\alpha\beta}.\end{aligned} \tag{42}$$

Thus,

$$\begin{aligned}\mathscr{P}_{\alpha\beta\gamma\delta} &= F\widetilde{\mathscr{P}}_{\alpha\beta\gamma\delta} - \left(tr(T) + \frac{\widetilde{n}-1}{2F}\overline{\Delta}F\right)\frac{1}{4}a_0P\widetilde{G}_{\alpha\beta\gamma\delta}\\ &+ (a_1+a_6)F\widetilde{g}_{\alpha\beta}\widetilde{g}_{\gamma\delta} + (a_2+a_5)F\widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\delta}\\ &+ (a_3+a_4)F\widetilde{g}_{\alpha\delta}\widetilde{g}_{\beta\gamma}.\end{aligned} \tag{43}$$

If M is $\mathscr{P}$−curvature flat, that is, $\mathscr{P}_{\alpha\beta\gamma\delta} = 0$, then

$$\begin{aligned}\widetilde{\mathscr{P}}_{\alpha\beta\gamma\delta} &= \frac{1}{2}\left(tr(T) + \frac{\widetilde{n}-1}{2F}\overline{\Delta}F\right)\frac{1}{4F}a_0\overline{\Delta}F\widetilde{G}_{\alpha\beta\gamma\delta}\\ &+ (a_1+a_6)\widetilde{g}_{\alpha\beta}\widetilde{g}_{\gamma\delta} + (a_2+a_5)\widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\delta} + (a_3+a_4)\widetilde{g}_{\alpha\delta}\widetilde{g}_{\beta\gamma}.\end{aligned} \tag{44}$$

Thus, we have the following.

Theorem 3. *Let $M = \overline{M}\times_F\widetilde{M}$ be a $\mathscr{P}$−curvature flat singly warped product manifold with the metric tensor $g = \overline{g}\times_F\widetilde{g}$. Then, the $\mathscr{P}$−curvature tensor on the fiber manifold is of the form*

$$\begin{aligned}\widetilde{\mathscr{P}}_{\alpha\beta\gamma\delta} &= \left(tr(T) + \frac{\widetilde{n}-1}{2F}\overline{\Delta}F\right)\left\{\frac{1}{4F}a_0P\widetilde{G}_{\alpha\beta\gamma\delta} + (a_1+a_6)\widetilde{g}_{\alpha\beta}\widetilde{g}_{\gamma\delta}\right.\\ &\left. + (a_2+a_5)\widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\delta} + (a_3+a_4)\widetilde{g}_{\alpha\delta}\widetilde{g}_{\beta\gamma}\right.\end{aligned} \tag{45}$$

4. P – Divergence-Free Warped Product Manifold

The divergence of the $\mathscr{P}$−curvature tensor is given by [2]

$$\begin{aligned}\nabla_h\mathscr{P}^h_{jkl} &= a_1\nabla_jR_{kl} + (a_2-a_0)\nabla_kR_{jl} + (a_3+a_0)\nabla_lR_{kj}\\ &+ \frac{1}{2}a_4g_{jk}\nabla_lR + \frac{1}{2}a_5g_{jl}\nabla_kR + \frac{1}{2}a_6g_{kl}\nabla_jR.\end{aligned} \tag{46}$$

If $\mathscr{P}$−curvature tensor is divergence-free, that is, $\nabla_h\mathscr{P}^h_{jkl} = 0$, then

$$\begin{aligned}0 &= a_1\nabla_jR_{kl} + (a_2-a_0)\nabla_kR_{jl} + (a_3+a_0)\nabla_lR_{kj}\\ &+ \frac{1}{2}a_4g_{jk}\nabla_lR + \frac{1}{2}a_5g_{jl}\nabla_kR + \frac{1}{2}a_6g_{kl}\nabla_jR.\end{aligned} \tag{47}$$

Contracting with g^{kl} and using the relation $\nabla_kR^k_j = 1/2\nabla_jR$, we get

$$0 = (2a_1 + a_2 + a_3 + a_4 + a_5 + a_6n)\nabla_jR. \tag{48}$$

If $(2a_1 + a_2 + a_3 + a_4 + a_5 + a_6n) \neq 0$, then $\nabla_jR = 0$. And hence, equation (46) reduces to

$$a_1\nabla_jR_{kl} + (a_2-a_0)\nabla_kR_{jl} + (a_3+a_0)\nabla_lR_{kj} = 0. \tag{49}$$

We thus have the following.

Lemma 1. *A warped manifold $\overline{M}\times_F\widetilde{M}$ with divergence-free $\mathscr{P}$−curvature tensor is of constant scalar curvature and Ricci tensor satisfies equation (49), provided* $(2a_1 + a_2 + a_3 + a_4 + a_5 + a_6n) \neq 0$.

The divergence component $\nabla_h \mathscr{P}^h_{\alpha\beta\gamma}$ of the $\mathscr{P}$−curvature tensor on the warped product manifold $\overline{M}\times_F\widetilde{M}$ is

$$\nabla_h \mathscr{P}^h_{\alpha\beta\gamma} = a_1\nabla_\alpha R_{\beta\gamma} + (a_2 - a_0)\nabla_\beta R_{\alpha\gamma} + (a_3 + a_0)\nabla_\gamma R_{\beta\alpha} + \frac{1}{2}a_4 g_{\alpha\beta}\nabla_\gamma R + \frac{1}{2}a_5 g_{\alpha\gamma}\nabla_\beta R + \frac{1}{2}a_6 g_{\beta\gamma}\nabla_\alpha R. \quad (50)$$

If $\overline{M}\times_F\widetilde{M}$ is $\mathscr{P}$−divergence-free, that is, $\nabla_h \mathscr{P}^h_{\alpha\beta\gamma} = 0$, then

$$0 = a_1\nabla_\alpha R_{\beta\gamma} + (a_2 - a_0)\nabla_\beta R_{\alpha\gamma} + (a_3 + a_0)\nabla_\gamma R_{\beta\alpha} + \frac{1}{2}a_4 g_{\alpha\beta}\nabla_\gamma R + \frac{1}{2}a_5 g_{\alpha\gamma}\nabla_\beta R + \frac{1}{2}a_6 g_{\beta\gamma}\nabla_\alpha R. \quad (51)$$

Using the obtained result in the previous lemma, we can have

$$a_1\nabla_\alpha R_{\beta\gamma} + (a_2 - a_0)\nabla_\beta R_{\alpha\gamma} + (a_3 + a_0)\nabla_\gamma R_{\beta\alpha} = 0. \quad (52)$$

In view of equation (12), we infer

$$a_1\nabla_\alpha \widetilde{R}_{\beta\gamma} + (a_2 - a_0)\nabla_\beta \widetilde{R}_{\alpha\gamma} + (a_3 + a_0)\nabla_\gamma \widetilde{R}_{\beta\alpha} = 0. \quad (53)$$

Contracting with $\widetilde{g}^{\beta\gamma}$ and using $\nabla_\alpha \widetilde{R}^\alpha_\beta = 1/2\nabla_\beta \widetilde{R}$, we get

$$(2a_1 + a_2 + a_3)\nabla_\alpha \widetilde{R} = 0, \quad (54)$$

If $2a_1 + a_2 + a_3 \neq 0$, then

$$\nabla_\alpha \widetilde{R} = 0, \quad (55)$$

which means that the fiber manifold $\widetilde{M}$ of the warped product manifold is of constant scalar curvature. Thus, we can state the following theorem.

Theorem 4. *The fiber manifold $\widetilde{M}$ of $\mathscr{P}$−divergence-free warped product manifold M is of constant scalar curvature.*

The next divergence component is

$$\nabla_h \mathscr{P}^h_{abc} = a_1\nabla_a R_{bc} + \nabla_b R_{ac} + (a_3 + a_0)\nabla_c R_{ab} + \frac{1}{2}a_4 g_{ab}\nabla_c R + \frac{1}{2}a_5 g_{ac}\nabla_b R + \frac{1}{2}a_6 g_{bc}\nabla_a R. \quad (56)$$

Assuming that M is $\mathscr{P}$−divergence-free and utilizing the obtained result in the previous lemma, we have

$$a_1\nabla_a R_{bc} + (a_2 - a_0)\nabla_b R_{ac} + (a_3 + a_0)\nabla_c R_{ab} = 0. \quad (57)$$

In virtue of equation (12), we get

$$0 = a_1\nabla_a \overline{R}_{bc} + (a_2 - a_0)\nabla_b \overline{R}_{ac} + (a_3 + a_0)\nabla_c \overline{R}_{ab} - \frac{\widetilde{n}}{2}\left\{\nabla_a\left(\frac{T_{bc}}{F}\right) + \nabla_b\left(\frac{T_{ac}}{F}\right) + \nabla_c\left(\frac{T_{ab}}{F}\right)\right\}. \quad (58)$$

If $T_{ab} = 0$, then

$$a_1\nabla_a \overline{R}_{bc} + (a_2 - a_0)\nabla_b \overline{R}_{ac} + (a_3 + a_0)\nabla_c \overline{R}_{ab} = 0. \quad (59)$$

Multiplying this with g^{bc} and using $\nabla_a \overline{R}^a_b = 1/2\nabla_b \overline{R}$, we have

$$(2a_1 + a_2 + a_3)\nabla_a \overline{R} = 0. \quad (60)$$

If $2a_1 + a_2 + a_3 \neq 0$, then

$$\nabla_a \overline{R} = 0, \quad (61)$$

which means that the base manifold $\overline{M}$ of the warped product manifold M is of constant scalar curvature. Thus, we conclude the following.

Theorem 5. *The base manifold $\overline{M}$ of $\mathscr{P}$−divergence-free warped product manifold M is of constant scalar curvature, provided $T_{ab} = 0$.*

Now, consider the warped product M has a Codazzi Ricci tensor; that is, $\nabla_l R_{kj} = \nabla_k R_{jl}$. And consequently, M is of constant scalar curvature. Thus, equation (46) leads to

$$\nabla_h \mathscr{P}^h_{jkl} = [a_1 + a_2 + a_3]\nabla_j R_{kl}. \quad (62)$$

Proposition 1. *A warped product manifold M with Codazzi Ricci tensor is $\mathscr{P}$− divergence-free if and only if it has symmetric Ricc'i tensor, provided $a_1 + a_2 + a_3 \neq 0$.*

Now, the divergence component $\nabla_h \mathscr{P}^h_{\alpha\beta\gamma}$ of the $\mathscr{P}$−curvature tensor is

$$\nabla_h \mathscr{P}^h_{\alpha\beta\gamma} = [a_1 + a_2 + a_3]\nabla_\alpha R_{\beta\gamma}. \quad (63)$$

Assume that the warped product manifold M is $\mathscr{P}$−divergence-free, and hence, one gets

$$[a_1 + a_2 + a_3]\nabla_\alpha R_{\beta\gamma} = 0. \quad (64)$$

Using equation (12), we get

$$[a_1 + a_2 + a_3]\nabla_\alpha \widetilde{R}_{\beta\gamma} = 0. \quad (65)$$

If $a_1 + a_2 + a_3 \neq 0$, then

$$\nabla_\alpha \widetilde{R}_{\beta\gamma} = 0. \quad (66)$$

Thus, we have the folllowing.

Theorem 6. *Let M be a $\mathscr{P}$−divergence-free warped product whose Ricci tensor is of Codazzi type. Then, the Ricci tensor of the fiber manifold is symmetric.*

Also, the divergence component $\nabla_h \mathscr{P}^h_{abc}$ is

$$\nabla_h \mathscr{P}^h_{abc} = [a_1 + a_2 + a_3]\nabla_a R_{bc}. \quad (67)$$

If $\overline{M}\times_F\widetilde{M}$ is $\mathscr{P}$−divergence-free, then

$$[a_1 + a_2 + a_3]\nabla_a R_{bc} = 0. \quad (68)$$

In view of equation (12), we get

$$[a_1 + a_2 + a_3]\left[\overline{R}_{ab;c} - \frac{\widetilde{n}}{2}\nabla_c\left(\frac{T_{ab}}{F}\right)\right] = 0. \quad (69)$$

If $T_{ab} = 0$ and $a_1 + a_2 + a_3 \neq 0$, then

$$\overline{R}_{ab;c} = 0. \tag{70}$$

We thus can state the following.

Theorem 7. *Let M be a warped product with Codazzi Ricci tensor. Then the Ricci tensor of the base manifold $\overline{M}$ is symmetric.*

5. Semisymmetries of the $\mathscr{P}$ – Curvature Tensor

It is well known that a manifold M is said to be semi-symmetric if its Riemann tensor satisfies

$$(\nabla_m \nabla_n - \nabla_n \nabla_m) R_{ijkl} = 0. \tag{71}$$

A manifold M is said to be Ricci semisymmetric if its Ricci tensor satisfies

$$(\nabla_m \nabla_n - \nabla_n \nabla_m) R_{ij} = 0. \tag{72}$$

The $\mathscr{P}$–curvature tensor is called semisymmetric if

$$(\nabla_m \nabla_n - \nabla_n \nabla_m) \mathscr{P}_{ijkl} = 0. \tag{73}$$

Applying $(\nabla_m \nabla_n - \nabla_n \nabla_m)$ on both sides of equation (1), one can have

$$\begin{aligned} (\nabla_m \nabla_n - \nabla_n \nabla_m) \mathscr{P}_{ijkl} &= (\nabla_m \nabla_n - \nabla_n \nabla_m) \begin{bmatrix} a_0 R_{ijkl} + a_1 g_{ij} R_{kl} + a_2 g_{ik} R_{jl} + a_3 g_{il} R_{jk} \\ + a_4 g_{jk} R_{il} + a_5 g_{jl} R_{ik} + a_6 g_{kl} R_{ij} \end{bmatrix}, \\ (\nabla_m \nabla_n - \nabla_n \nabla_m) \mathscr{P}_{ijkl} &= a_0 (\nabla_m \nabla_n - \nabla_n \nabla_m) R_{ijkl} + a_1 g_{ij} (\nabla_m \nabla_n - \nabla_n \nabla_m) R_{kl} \\ &\quad + a_2 g_{ik} (\nabla_m \nabla_n - \nabla_n \nabla_m) R_{jl} + a_3 g_{il} (\nabla_m \nabla_n - \nabla_n \nabla_m) R_{jk} \\ &\quad + a_4 g_{jk} (\nabla_m \nabla_n - \nabla_n \nabla_m) R_{il} + a_5 g_{jl} (\nabla_m \nabla_n - \nabla_n \nabla_m) R_{ik} \\ &\quad + a_6 g_{kl} (\nabla_m \nabla_n - \nabla_n \nabla_m) R_{ij}. \end{aligned} \tag{74}$$

Thus, we have the following:

Proposition 2. *A pseudo-Riemannian manifold M admits a semisymmetric $\mathscr{P}$–curvature tensor if and only if M is semisymmetric.*

Now, assume that M has a semisymmetric $\mathscr{P}$–curvature tensor; that is,

$$(\nabla_m \nabla_n - \nabla_n \nabla_m) \mathscr{P}_{ijkl} = 0. \tag{75}$$

Thus, we have

$$\begin{aligned} 0 &= a_0 (\nabla_m \nabla_n - \nabla_n \nabla_m) R_{ijkl} + a_1 g_{ij} (\nabla_m \nabla_n - \nabla_n \nabla_m) R_{kl} + a_2 g_{ik} (\nabla_m \nabla_n - \nabla_n \nabla_m) R_{jl} \\ &\quad + a_3 g_{il} (\nabla_m \nabla_n - \nabla_n \nabla_m) R_{jk} + a_4 g_{jk} (\nabla_m \nabla_n - \nabla_n \nabla_m) R_{il} + a_5 g_{jl} (\nabla_m \nabla_n - \nabla_n \nabla_m) R_{ik} \\ &\quad + a_6 g_{kl} (\nabla_m \nabla_n - \nabla_n \nabla_m) R_{ij}. \end{aligned} \tag{76}$$

A contraction with g^{ik} implies

$$-a_5 g_{jl} (\nabla_m \nabla_n - \nabla_n \nabla_m) R = [a_0 + a_1 + a_2 n + a_3 + a_4 + a_6] (\nabla_m \nabla_n - \nabla_n \nabla_m) R_{jl}. \tag{77}$$

A multiplication with g^{jl} implies

$$[a_0 + a_1 + a_2 n + a_3 + a_4 + n a_5 + a_6] (\nabla_m \nabla_n - \nabla_n \nabla_m) R = 0. \tag{78}$$

If $a_0 + a_1 + a_2 n + a_3 + a_4 + n a_5 + a_6 \neq 0$, then

$$(\nabla_m \nabla_n - \nabla_n \nabla_m) R = 0. \tag{79}$$

And hence, equation (77) becomes

$$[a_0 + a_1 + a_2 n + a_3 + a_4 + a_6] (\nabla_m \nabla_n - \nabla_n \nabla_m) R_{jl} = 0. \tag{80}$$

If $a_0 + a_1 + a_2 n + a_3 + a_4 + a_6 \neq 0$, we have

$$(\nabla_m \nabla_n - \nabla_n \nabla_m) R_{jl} = 0, \tag{81}$$

which means that M is Ricci semisymmetric.

Proposition 3. *A pseudo-Riemannian manifold M with semisymmetric $\mathscr{P}$–curvature tensor is Ricci semisymmetric.*

Taking the covariant derivative of the first component of the $\mathscr{P}$–curvature tensor, which is given by equation (14), we infer

$$\mathscr{P}_{\alpha ab\beta;\varepsilon} = a_0 R_{\alpha ab\beta;\varepsilon} + a_3 g_{\alpha\beta} R_{ab;\varepsilon} + a_4 g_{ab} R_{\alpha\beta;\varepsilon}. \tag{82}$$

Using equations (11) and (12) in the previous equation, we have

$$\mathcal{P}_{\alpha ab\beta;\varepsilon} = a_4 \overline{g}_{ab} \widetilde{R}_{\alpha\beta;\varepsilon}. \tag{83}$$

Suppose that the $\mathcal{P}$−curvature tensor is symmetric; that is, $\mathcal{P}_{\alpha ab\beta;\varepsilon} = 0$. Thus,

$$a_4 \overline{g}_{ab} \widetilde{R}_{\alpha\beta;\varepsilon} = 0. \tag{84}$$

A contraction with $\overline{g}^{ab}$ gives

$$a_4 \overline{n} \widetilde{R}_{\alpha\beta;\varepsilon} = 0. \tag{85}$$

If $a_4 \neq 0$, one may have

$$\widetilde{R}_{\alpha\beta;\varepsilon} = 0, \tag{86}$$

which means that the fiber manifold $\widetilde{M}$ of the warped product manifold $\overline{M}\times_F\widetilde{M}$ is Ricci symmetric. We thus can state the following.

Theorem 8. *Let M be a warped product manifold with symmetric $\mathcal{P}$−curvature tensor. Then, the fiber manifold $\widetilde{M}$ of $\overline{M}\times_F\widetilde{M}$ is a Ricci symmetric manifold.*

The covariant derivative of the component $\mathcal{P}_{\alpha\beta\gamma\delta}$ is

$$\begin{aligned}\mathcal{P}_{\alpha\beta\gamma\delta;\varepsilon} = {} & a_0 R_{\alpha\beta\gamma\delta;\varepsilon} + a_1 g_{\alpha\beta} R_{\gamma\delta;\varepsilon} + a_2 g_{\alpha\gamma} R_{\beta\delta;\varepsilon} + a_3 g_{\alpha\delta} R_{\beta\gamma;\varepsilon} \\ & + a_4 g_{\beta\gamma} R_{\alpha\delta;\varepsilon} + a_5 g_{\beta\delta} R_{\alpha\gamma;\varepsilon} + a_6 g_{\gamma\delta} R_{\alpha\beta;\varepsilon}.\end{aligned} \tag{87}$$

The use of equations (3), (11), and (12) implies

$$\begin{aligned}\mathcal{P}_{\alpha\beta\gamma\delta;\varepsilon} &= F\big[a_0 \widetilde{R}_{\alpha\beta\gamma\delta;\varepsilon} + a_1 \widetilde{g}_{\alpha\beta} \widetilde{R}_{\gamma\delta;\varepsilon} + a_2 \widetilde{g}_{\alpha\gamma} \widetilde{R}_{\beta\delta;\varepsilon} + a_3 \widetilde{g}_{\alpha\delta} \widetilde{R}_{\beta\gamma;\varepsilon} \\ &\quad + a_4 \widetilde{g}_{\beta\gamma} \widetilde{R}_{\alpha\delta;\varepsilon} + a_5 \widetilde{g}_{\beta\delta} \widetilde{R}_{\alpha\gamma;\varepsilon} + a_6 \widetilde{g}_{\gamma\delta} \widetilde{R}_{\alpha\beta;\varepsilon}\big], \\ \mathcal{P}_{\alpha\beta\gamma\delta;\varepsilon} &= F \widetilde{\mathcal{P}}_{\alpha\beta\gamma\delta;\varepsilon}.\end{aligned} \tag{88}$$

Thus, we can state the following.

Theorem 9. *Let $\overline{M}\times_F\widetilde{M}$ be a warped product manifold with symmetric $\mathcal{P}$−curvature tensor. Then, the fiber manifold $\widetilde{M}$ has symmetric $\mathcal{P}$−curvature tensor.*

The covariant derivative of the component $\mathcal{P}_{abcd}$ is

$$\begin{aligned}\mathcal{P}_{abcd;e} = {} & a_0 R_{abcd;e} + a_1 g_{ab} R_{cd;e} + a_2 g_{ac} R_{bd;e} + a_3 g_{a\ d} R_{bc;e} \\ & + a_4 g_{bc} R_{ad;e} + a_5 g_{bd} R_{ac;e} + a_6 g_{cd} R_{ab;e}.\end{aligned} \tag{89}$$

Utilizing (3), (11), and (12) entails that

$$\begin{aligned}\mathcal{P}_{abcd;e} = {} & a_0 \overline{R}_{abcd;e} + a_1 \overline{g}_{ab} \overline{R}_{cd;e} + a_2 \overline{g}_{ac} \overline{R}_{bd;e} + a_3 \overline{g}_{ad} \overline{R}_{bc;e} + a_4 \overline{g}_{bc} \overline{R}_{ad;e} + a_5 \overline{g}_{bd} \overline{R}_{ac;e} + a_6 \overline{g}_{cd} \overline{R}_{ab;e} \\ & - \frac{\widetilde{n}}{2} \nabla_e \left[a_1 \overline{g}_{ab} \frac{T_{c\ d}}{F} + a_2 \overline{g}_{ac} \frac{T_{b\ d}}{F} + a_3 \overline{g}_{ad} \frac{T_{bc}}{F} + a_4 \overline{g}_{bc} \frac{T_{a\ d}}{F} + a_5 \overline{g}_{bd} \frac{T_{ac}}{F} + a_6 \overline{g}_{cd} \frac{T_{ab}}{F} \right], \\ \mathcal{P}_{abcd;e} = {} & \overline{\mathcal{P}}_{abcd;e} - \frac{\widetilde{n}}{2} \nabla_e \left[a_1 \overline{g}_{ab} \frac{T_{c\ d}}{F} + a_2 \overline{g}_{ac} \frac{T_{b\ d}}{F} + a_3 \overline{g}_{ad} \frac{T_{bc}}{F} \right. \\ & \left. + a_4 \overline{g}_{bc} \frac{T_{a\ d}}{F} + a_5 \overline{g}_{bd} \frac{T_{ac}}{F} + a_6 \overline{g}_{cd} \frac{T_{ab}}{F} \right].\end{aligned} \tag{90}$$

If $\nabla_e T_{ab} = 0$, we have

$$\mathcal{P}_{abcd;e} = \overline{\mathcal{P}}_{abcd;e}. \tag{91}$$

We thus can state the following theorem.

Theorem 10. *Let $\overline{M}\times_F\widetilde{M}$ be a warped product manifold M with symmetric $\mathcal{P}$−curvature tensor. Then, the base manifold $\overline{M}$ has a symmetric $\mathcal{P}$−curvature tensor, provided $\nabla_e T_{ab} = 0$.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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