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Research Article

Existence and Stability of a Caputo Variable-Order Boundary Value Problem

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In this study, we investigate the existence of a solution to the boundary value problem (BVP) of variable-order Caputo-type fractional differential equation by converting it into an equivalent standard Caputo (BVP) of the fractional constant order with the help of the generalized intervals and the piecewise constant functions. All our results in this study are proved by using Darbo's fixed-point theorem and the Ulam–Hyers (UH) stability definition. A numerical example is given at the end to support and validate the potentiality of our obtained results.

1. Introduction

Fractional differential equations of a constant order or fractional calculus, in general, have been studied by many researchers for more than three centuries compared to integer differential equations. In recent years, the notion of a variable-order operator is a much more recent improvement. Different authors have presented different definitions of variable-order differential; we refer to [1–6].

Several investigators have studied boundary value problems (BVPs) for different types of fractional differential equations (FDEs), for example, Adiguzel et al. [7] obtain a solution for a nonlinear (FDEs) of order $\alpha \in (2, 3]$, Benchora and Souid [8] obtain a solution for implicit fractional-order differential equations, and Zhang [9] discuss the existence of solutions for two point (BVPs) with singular (FDEs) of variable order.

Bai and Kong [10] studied the following problem:

$$\begin{cases} {}^{c}D_{0^+}^{\omega} y(t) = f\left(t, y(t), I_{0^+}^{\omega} y(t)\right), t \in \left[b_1, b_2\right], \omega \in]0, 1], 0 < b_1 < b_2 < \infty \\ y\left(b_1\right) = y_{b_1}, \end{cases} \tag{1}$$

where ${}^{c}D_{0^+}^{\omega}$ and $I_{0^+}^{\omega}$ are the Caputo–Hadamard derivative and Hadamard integral operators of constant order ω, respectively, f is a given continuous function, and $y_{b_1} \in \mathbb{R}$.

Some existence and Ulam stability properties for FDEs are studied by many authors (see [11, 12] and references cited therein).

Motivated by the above studies, we deal with the existence of solutions and the stability of the obtained solution in the sense of Ulam–Hyers (UH) to the following BVP of Caputo variable-order type:

$$\begin{cases} {}^{c}D_{0^+}^{\omega(t)}y(t) = f_1\left(t, y(t), I_{0^+}^{\omega(t)}y(t)\right), t \in J, \\ y(0) = 0, y(T) = 0, \end{cases} \tag{2}$$

where $J = [0, T]$, $0 < T < \infty$, $\omega(t)\colon J \longrightarrow (1, 2]$ is the variable order of the fractional derivatives, $f_1\colon J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, and the left Riemann–Liouville fractional integral (RLFI) of variable-order $\omega(t)$ for function $y(t)$ is (see, for example, [13–15])

$$I_{0^+}^{\omega(t)}y(t) = \int_0^t \frac{(t-s)^{\omega(t)-1}}{\Gamma(\omega(t))}y(s)ds, \quad t \in J, \tag{3}$$

and the left Caputo fractional derivative (CFD) of variable-order $\omega(t)$ for function $y(t)$ is (see, for example, [13–15])

$${}^{c}D_{0^+}^{\omega(t)}y(t) = \int_0^t \frac{(t-s)^{1-\omega(t)}}{\Gamma(2-\omega(t))}y^{(2)}(s)ds, \quad t \in J. \tag{4}$$

Remark 1. In (2), the variable order $\omega(t)\colon J \longrightarrow (1, 2]$, but the RLFI could be defined for any $\omega(t)\colon J \longrightarrow (0, \infty)$.

Remark 2. In the case of a constant order ω in equations (2) and (3), the RLFI and CFD coincide with the standard Riemann–Liouville fractional integral and Caputo fractional derivative, respectively (see [13, 14, 16]).

Recall the following properties of fractional derivatives and integrals [16].

2. Preliminaries

This section introduces some important fundamental definitions and results that will be needed in this paper.

Denote by $C(J, \mathbb{R})$ the Banach space of continuous functions $y\colon J \longrightarrow \mathbb{R}$, with the norm

$$\|y\| = \sup\{|y(t)|\colon t \in J\}. \tag{5}$$

Lemma 1. *Assume that $\beta_1 > 0$, $b_1, b_2 > 0$, $f_2 \in L^1(b_1, b_2)$, and ${}^{c}D_{b_1^+}^{\beta_1}f_2 \in L^1(b_1, b_2)$. Then, the differential equation,*

$${}^{c}D_{b_1^+}^{\beta_1}f_2 = 0, \tag{6}$$

has a unique solution:

$$f_2(t) = \lambda_0 + \lambda_1(t-b_1) + \lambda_2(t-b_1)^2 + \cdots + \lambda_{n-1}(t-b_1)^{n-1}, \tag{7}$$

where $n-1 < \beta_1 \le n$ and $\lambda_{\mathcal{J}} \in \mathbb{R}$, $j = 0, 1, \ldots, n-1$.

Lemma 2. *Let $\beta_1 > 0$, $b_1, b_2 > 0$, $f_2 \in L^1(b_1, b_2)$, and ${}^{c}D_{b_1^+}^{\beta_1}f_2 \in L^1(b_1, b_2)$. Then,*

$$I_{b_1^+}^{\beta_1}\,{}^{c}D_{b_1^+}^{\beta_1}f_2(t) = f_2(t) + \lambda_0 + \lambda_1(t-b_1) + \lambda_2(t-b_1)^2 + \cdots + \lambda_{n-1}(t-b_1)^{n-1}, \tag{8}$$

where $n-1 < \beta_1 \le n$ and $\lambda_j \in \mathbb{R}$, $j = 0, 1, \ldots, n-1$.

Lemma 3. *Let $\beta_1 > 0$, $b_1, b_2 > 0$, and $f_2 \in L^1(b_1, b_2)$. Then,*

$${}^{c}D_{b_1^+}^{\beta_1}I_{b_1^+}^{\beta_1}f_2(t) = f_2(t). \tag{9}$$

Lemma 4. *Let $\beta_1 > 0$, $b_1, b_2 > 0$, and $f_2 \in L^1(b_1, b_2)$. Then,*

$$I_{b_1^+}^{\beta_1}I_{b_1^+}^{\beta_2}f_2(t) = I_{b_1^+}^{\beta_2}I_{b_1^+}^{\beta_1}f_2(t) = I_{b_1^+}^{\beta_1+\beta_2}f_2(t). \tag{10}$$

Remark 3. It is to be hereby note that the semigroup property is not satisfied for general functions $\omega(t)$ and $\psi(t)$ (see [9, 17, 18]), that is,

$$I_{b_1^+}^{\omega(t)}I_{b_1^+}^{\psi(t)}f_2(t) \ne I_{b_1^+}^{\omega(t)+\psi(t)}f_2(t). \tag{11}$$

Example 1. Assume that

$$\omega(t) = t, \quad t \in [0,4], \psi(t) = \begin{cases} 2, & t \in [0,1] \\ 3, & t \in]1,4]. \end{cases} \quad f_2(t) = 2, \quad t \in [0,4],$$

$$\begin{aligned} I_{0^+}^{\omega(t)} I_{0^+}^{\psi(t)} f_2(t) &= \int_o^t \frac{(t-s)^{\omega(t)-1}}{\Gamma(\omega(t))} \int_0^s \frac{(s-\tau)^{\nu(s)-1}}{\Gamma(\nu(s))} f_2(\tau) d\tau\, ds, \\ &= \int_o^t \frac{(t-s)^{t-1}}{\Gamma(t)} \left[\int_0^1 \frac{(s-\tau)}{\Gamma(2)} 2\, d\tau + \int_1^s \frac{(s-\tau)^2}{\Gamma(3)} 2\, d\tau \right] ds, \\ &= \int_o^t \frac{(t-s)^{t-1}}{\Gamma(t)} \left[2s - 1 + \frac{(s-1)^3}{3} \right] ds, \end{aligned} \tag{12}$$

and

$$I_{0^+}^{\omega(t)+\psi(t)} f_2(t)| = \int_0^t \frac{(t-s)^{\omega(t)+\psi(t)-1}}{\Gamma(\omega(t)+\psi(t))} f_2(s) ds. \tag{13}$$

So, we obtain

$$\begin{aligned} I_{0^+}^{\omega(t)} I_{0^+}^{\psi(t)} f_2(t)|_{t=3} &= \int_0^3 \frac{(3-s)^2}{\Gamma(3)} \left[2s - 1 + \frac{(s-1)^3}{3} \right] ds \\ &= \frac{21}{10}, \\ I_{0^+}^{\omega(t)+\psi(t)} f_2(t)|_{t=3} &= \int_0^3 \frac{(3-s)^{\omega(t)+\psi(t)-1}}{\Gamma(\omega(t)+\psi(t))} f_2(s) ds, \\ &= \int_0^1 \frac{(3-s)^4}{\Gamma(5)} 2\, ds + \int_1^3 \frac{(3-s)^5}{\Gamma(6)} 2\, ds, \\ &= \frac{1}{2} \int_0^1 \left(s^4 - 12s^3 + 54s^2 - 108s + 81 \right) ds, \\ &\quad + \frac{1}{60} \int_1^3 \left(-s^5 + 15s^4 - 90s^3 + 270s^2 \right. \\ &\quad \left. - 405s + 243 \right) ds, \\ &= \frac{665}{180}. \end{aligned} \tag{14}$$

Therefore, we obtain

$$I_{0^+}^{\omega(t)} I_{0^+}^{\psi(t)} f_2(t)|_{t=3} \neq I_{0^+}^{\omega(t)+\psi(t)} f_2(t)|_{t=3}. \tag{15}$$

Definition 1 (see [19–21]). Let $A \subset \mathbb{R}$, where A is named a generalized interval if it is either an interval, or $\{b_1\}$ or $\varnothing$.

A finite set $\mathcal{P}$ is named a partition of A if each x in A lies in just one among the generalized intervals E in $\mathcal{P}$.

A function $g: A \longrightarrow \mathbb{R}$ is defined to be piecewise constant with respect to partition $\mathcal{P}$ of A if g admits constant values on E, for any $E \in \mathcal{P}$.

Zhang et al. [22] gave very interesting result.

Lemma 5. *If $u \in C(J,(1,2])$, then both of the following holds:*

(a) For $f_2 \in C(J,\mathbb{R})$, $I_{0^+}^{\omega(t)} f_2(t) \in C(J,\mathbb{R})$

(b) For $f_2 \in C_\kappa(J,\mathbb{R}) = \{f_2(t) \in C(J,\mathbb{R}), t^\kappa f_2(t) \in C(J,\mathbb{R}), 0 \le \kappa \le 1\}$, the variable-order fractional integral $I_{0^+}^{\omega(t)} f_2(t)$ exists for any points on J

Definition 2 (see [23]). Let Ω be a bounded subset of the Banach space X. The Kuratowski measure of noncompactness (KMNC) is a mapping $\xi: \Omega \longrightarrow [0,\infty]$ which is defined as follows:

$$\xi(D) = \inf\left\{\varepsilon > 0: \exists \left(D_{\mathcal{J}}\right)_{\mathcal{J}=1,2,\ldots,n} \subset X, D \subseteq \cup_{\mathcal{J}=1}^n D_{\mathcal{J}}, \operatorname{diam}\left(D_{\mathcal{J}}\right) \le \varepsilon\right\}, \tag{16}$$

where

$$\operatorname{diam}\left(D_{\mathcal{J}}\right) = \sup\left\{\|x-y\|: x, y \in D_{\mathcal{J}}\right\}. \tag{17}$$

The KMNC satisfies the following properties.

Proposition 1 (see [23, 24]). *Let X be a Banach space and D, D_1, and D_2 be bounded subsets of X. Then,*

(1) $\xi(D) = 0$ if and only if $\overline{D}$ is compact

(2) $\xi(\phi) = 0$

(3) $\xi(D) = \xi(\overline{D}) = \xi(conv\ D)$

(4) $D_1 \subset D_2$ implies $\xi(D_1) \le \xi(D_2)$

(5) $\xi(D_1 + D_2) \le \xi(D_1) + \xi(D_2)$

(6) $\xi(\alpha D) = |\alpha|\xi(D), \alpha \in IR$

(7) $\xi(D_1 \cup D_2) = \max\{\xi(D_1), \xi(D_2)\}$

(8) $\xi(D_1 \cap D_2) = \min\{\xi(D_1), \xi(D_2)\}$

(9) $\xi(D + x_0) = \xi(D)$ for any $x_0 \in X$

Lemma 6 (see [25]). *Let $B \subset C(J,X)$ be a bounded and equicontinuous set; then,*

(i) The function $\xi(B(t))$ is continuous for $t \in J$, and

$$\widehat{\xi}(B) = \sup_{t \in J} \xi(B(t)). \tag{18}$$

(ii) $\xi\left(\int_0^T x(\rho)d\rho : x \in B\right) \leq \int_0^T \xi(B(\rho))d\rho$, *where*

$$B(\rho) = \{x(\rho) \colon x \in B\}, \quad \rho \in J. \tag{19}$$

Theorem 1 (DFPT, see [23]). *Let X be a Banach space and $\mathfrak{F}$ be a nonempty, bounded, closed, and convex subset of X and $\Upsilon\colon \mathfrak{F} \longrightarrow \mathfrak{F}$ is a continuous operator satisfying*

$$\xi(\Upsilon(G)) \leq k\xi(G), k \in [0,1), \quad \text{for any } G(\neq \varnothing) \subset \mathfrak{F}, \tag{20}$$

i.e., Υ is $k -$ set contractions.

Then, Υ has at least one fixed point in $\mathfrak{F}$.

Definition 3 (see [11]). The BVP (2) is (UH) stable if $\exists \lambda_{f_1} > 0, \forall \varepsilon > 0, \forall z \in C(J, \mathbb{R})$ satisfies the following inequality:

$$\left|{}^c D_{0^+}^{\omega(t)} z(t) - f_1\left(t, z(t), I_{0^+}^{\omega(t)} z(t)\right)\right| \leq \varepsilon, t \in J, \tag{21}$$

where $\exists\, y \in C(J, \mathbb{R})$ solution of BVP (2) with

$$|z(t) - y(t)| \leq \lambda_{f_1} \epsilon, t \in J. \tag{22}$$

3. Existence of Solution

In this section, we investigate the existence of solution for a BVP of a Caputo-type fractional differential equation using DFPT and KMNC.

Let us introduce the following assumptions.

(H1) Let $n \in \mathbb{N}$ be an integer,

$$\mathscr{P} = \{J_1 \coloneqq [0, T_1], J_2 \coloneqq (T_1, T_2], J_3 \coloneqq (T_2, T_3] \ldots J_n \coloneqq (T_{n-1}, T]\}, \tag{23}$$

a partition of the interval J, and let $\omega(t)\colon J \longrightarrow (1,2]$ be a piecewise constant function with respect to $\mathscr{P}$, i.e.,

$$\omega(t) = \sum_{j=1}^{n} \omega_j I_j(t) = \begin{cases} \omega_1, & \text{if } t \in J_1, \\ \omega_2, & \text{if } t \in J_2, \\ . & \\ . & \\ . & \\ \omega_n, & \text{if } t \in J_n, \end{cases} \tag{24}$$

where $1 < \omega_j \leq 2$ are constants, and I_j is the indicator of the interval $J_j \coloneqq (T_{j-1}, T_j]$, $j = 1, 2, \ldots, n$ (with $T_0 = 0$ and $T_n = T$), such that

$$I_j(t) = \begin{cases} 1, & \text{for } t \in J_j, \\ 0, & \text{for elsewhere.} \end{cases} \tag{25}$$

Remark 4. According to remark of Benchohra (p.20 in [26]), it is not difficult to show that condition (H2) and the following inequality,

$$\xi\left(t^\kappa \left|f_1(t, B_1, B_2)\right|\right) \leq K\xi(B_1) + L\xi(B_2), \tag{26}$$

are equivalent for any bounded sets $B_1, B_2 \subset X$ and for each $t \in J$.

Furthermore, for a given set B of functions $v\colon J \longrightarrow X$, we denote flushleft:

$$B(t) = \{v(t), v \in B\}, \ t \in J, \tag{27}$$

and

$$B(J) = \{v(t)\colon \ v \in B, \ t \in J\}. \tag{28}$$

The symbol $E_j = C(J_j, \mathbb{R})$, which indicated the Banach space of continuous functions $y\colon J_j \longrightarrow \mathbb{R}$ equipped with the norm

$$\|y\|_{E_j} = \sup_{t \in J_j} |y(t)|, \tag{29}$$

where $j \in \{1, 2, \ldots, n\}$,

Then, for $t \in J_j$, $j = 1, 2, \ldots, n$, the left Caputo fractional derivative (CFD), defined by (4), could be presented as a sum of left Caputo fractional derivatives of constant orders $\omega_l, l = 1, 2, \ldots, j$,

$$\begin{aligned} {}^c D_{0^+}^{\omega(t)} y(t) &= \int_0^t \frac{(t-s)^{1-\omega(t)}}{\Gamma(2-\omega(t))} y^{(2)}(s) ds, \\ &= \int_0^{T_1} \frac{(t-s)^{1-\omega_1}}{\Gamma(2-\omega_1)} y^{(2)}(s) ds + \int_{T_1}^{T_2} \frac{(t-s)^{1-\omega_2}}{\Gamma(2-\omega_2)} y^{(2)}(s) ds + \cdots + \int_{T_{j-1}}^{t} \frac{(t-s)^{1-\omega_j}}{\Gamma\left(2-\omega_j\right)} y^{(2)}(s) ds. \end{aligned} \tag{30}$$

Thus, by (30), BVP (2) can be written, for any $t \in J_j, \quad j = 1, 2, \ldots, n$ as the following form:

$$\int_0^{T_1} \frac{(t-s)^{1-\omega_1}}{\Gamma(2-\omega_1)} y^{(2)}(s)ds + \int_{T_1}^{T_2} \frac{(t-s)^{1-\omega_2}}{\Gamma(2-\omega_2)} y^{(2)}(s)ds + \cdots + \int_{T_{j-1}}^{t} \frac{(t-s)^{1-\omega_j}}{\Gamma(2-\omega_{\mathcal{J}})} y^{(2)}(s)ds = f_1\left(t, y(t), I_{0^+}^{\omega_j} y(t)\right). \tag{31}$$

Now, we will present the definition of the solution to BVP (2).

Definition 4. BVP (2) has a solution if there are functions $y_j, j = 1, 2, \ldots, n,$ so that $y_j \in C([0, T_j], \mathbb{R})$ fulfilling equation (31) and $y_j(0) = 0 = y_j(T_j)$.

Let the function $y \in C(J, \mathbb{R})$ be a solution of integral (31), such that $y(t) \equiv 0$ on $t \in [0, T_{j-1}]$. Then, (31) is reduced to

$$^{c}D_{T_{j-1}^+}^{\omega_j} y(t) = f_1\left(t, y(t), I_{T_{j-1}^+}^{\omega_j} y(t)\right), \quad t \in J_j. \tag{32}$$

We consider the following auxiliary BVP:

$$\begin{cases} ^{c}D_{T_{j-1}^+}^{\omega_j} y(t) = f_1\left(t, y(t), I_{T_{j-1}^+}^{\omega_j} y(t)\right), t \in J_j \\ y\left(T_{j-1}\right) = 0, y\left(T_j\right) = 0. \end{cases} \tag{33}$$

The following lemma is necessary in our next analysis of BVP (33).

Lemma 7. *Let $f_1 \in C(J_j \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Suppose that there exists a number $\kappa \in (0, 1)$ such that $t^\kappa f_1 \in C(J_{\mathcal{J}} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, for any $j \in \{1, 2, \ldots, n\}$.*

Then, the solution of BVP (33) can be expressed by the integral equation:

$$y(t) = -\left(T_j - T_{j-1}\right)^{-1}\left(t - T_{j-1}\right) I_{T_{j-1}^+}^{\omega_j} f_1\left(T_j, y\left(T_j\right), I_{T_{j-1}^+}^{\omega_j} y\left(T_j\right)\right) + I_{T_{j-1}^+}^{\omega_j} f_1\left(t, y(t), I_{T_{j-1}^+}^{\omega_j} y(t)\right). \tag{34}$$

Proof. Let $y \in E_j$ is a solution of BVP (33). Taking (RLFI) $I_{T_{j-1}^+}^{\omega_j}$ to both sides of (33) and using Lemma 2, we find

$$y(t) = \lambda_1 + \lambda_2\left(t - T_{j-1}\right) + \frac{1}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{t} (t-s)^{\omega_j - 1} f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) ds, \quad t \in J_j. \tag{35}$$

By $y(T_{j-1}) = 0$, we get $\lambda_1 = 0$.
By $y(T_j) = 0$, we observe that

$$\lambda_2 = -\left(T_j - T_{j-1}\right)^{-1} I_{T_{j-1}^+}^{\omega_j} f_1\left(T_j, y\left(T_j\right), I_{T_{j-1}^+}^{\omega_j} y\left(T_j\right)\right). \tag{36}$$

Then,

$$\begin{aligned} y(t) = &-\left(T_j - T_{j-1}\right)^{-1}\left(t - T_{j-1}\right) I_{T_{j-1}^+}^{\omega_j} f_1\left(T_j, y\left(T_j\right), I_{T_{j-1}^+}^{\omega_j} y\left(T_j\right)\right) \\ &+ I_{T_{j-1}^+}^{\omega_j} f_1\left(t, y(t), I_{T_{j-1}^+}^{\omega_j} y(t)\right), \quad t \in J_j. \end{aligned} \tag{37}$$

Conversely, let $y \in E_j$ be solution of integral equation (34). Employing the operator (CFD) $^{c}D_{T_{j-1}^+}^{\omega_j}$ to both sides of (34) and Lemma 3, we deduce that y is the solution of BVP (33).

Based on concept of MNCK and DFPT, we have the following theorem for the existence of a solution for BVP (33). ☐

Theorem 2. *In addition to the conditions of Lemma 7, suppose that there exist constants $K, L > 0$, such that, for any $x_l, z_l \in \mathbb{R}$, $l = 1, 2$, $t \in J_j$, and*

$$t^\kappa \left| f_1\left(t, x_1, z_1\right) - f_1\left(t, x_2, z_2\right) \right| \leq K\left|x_1 - x_2\right| + L\left|z_1 - z_2\right|, \tag{38}$$

the following inequality holds:

$$\frac{2\left(T_j - T_{j-1}\right)^{\omega_j - 1}\left(T_j^{1-\kappa} - T_{j-1}^{1-\kappa}\right)}{(1-\kappa)\Gamma\left(\omega_j\right)} \left(K + L\frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)}\right) < 1. \tag{39}$$

Then, BVP (33) has at least one solution E_j.

Proof. Consider the operator $W\colon E_j \longrightarrow E_j$ defined by

$$
\begin{aligned}
Wy(t) = &-\left(T_j - T_{j-1}\right)^{-1}\left(t - T_{j-1}\right) I_{T_{j-1}^{+}}^{\omega_j} f_1\left(T_j, y\left(T_j\right), I_{T_{j-1}^{+}}^{\omega_j} y\left(T_j\right)\right) \\
&+ \frac{1}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{t} (t-s)^{\omega_{j-1}} f_1\left(s, y(s), I_{T_{j-1}^{+}}^{\omega_j} y(s)\right) ds, \quad t \in J_j.
\end{aligned}
\tag{40}
$$

From the properties of fractional integrals and the continuity of function $t^{\kappa} f_1$, then the operator $W\colon E_j \longrightarrow E_j$ defined in (40) is well defined.

Let

$$
R_j \geq \frac{2 f^{*}\left(T_j - T_{j-1}\right)^{\omega_j} / \Gamma\left(\omega_j\right)}{1 - 2\left(T_j - T_{j-1}\right)^{\omega_j - 1}\left(T_j^{1-\kappa} - T_{j-1}^{1-\kappa}\right) / (1-\kappa)\Gamma\left(\omega_j\right)\left(K + L\left(T_j - T_{j-1}\right)^{\omega_j} / \Gamma\left(\omega_j + 1\right)\right)},
\tag{41}
$$

with

$$
f^{*} = \sup_{t \in J_j}\left|f_1(t, 0, 0)\right|.
\tag{42}
$$

We consider the set

$$
B_{R_j} = \left\{y \in E_j, \|y\|_{E_j} \leq R_j\right\}.
\tag{43}
$$

Clearly, B_{R_j} is nonempty, bounded, closed, and convex.

Now, we prove that W satisfies the assumption of Theorem 1.

Step 1 : $W\left(B_{R_j}\right) \subseteq \left(B_{R_j}\right)$.

For $y \in B_{R_j}$ and by (H2), we obtain

$$
\begin{aligned}
|Wy(t)| \leq &\frac{\left(T_j - T_{j-1}\right)^{-1}\left(t - T_{j-1}\right)}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j} \left(T_j - s\right)^{\omega_j - 1}\left|f_1\left(s, y(s), I_{T_{j-1}^{+}}^{\omega_j} y(s)\right)\right| ds, \\
&+ \frac{1}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{t} (t-s)^{\omega_j - 1}\left|f_1\left(s, y(s), I_{T_{j-1}^{+}}^{\omega_j} y(s)\right)\right| ds, \\
\leq &\frac{2}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j} \left(T_j - s\right)^{\omega_j - 1}\left|f_1\left(s, y(s), I_{T_{j-1}^{+}}^{\omega_j} y(s)\right)\right| ds, \\
\leq &\frac{2}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j} \left(T_j - s\right)^{\omega_j - 1}\left|f_1\left(s, y(s), I_{T_{j-1}^{+}}^{\omega_j} y(s)\right) - f_1(s, 0, 0)\right| ds, \\
&+ \frac{2}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j} \left(T_j - s\right)^{\omega_j - 1}\left|f_1(s, 0, 0)\right| ds, \\
\leq &\frac{2}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j} \left(T_j - s\right)^{\omega_j - 1} s^{-\kappa}\left(K|y(s)| + L\left|I_{T_{j-1}^{+}}^{\omega_j} y(s)\right|\right) ds + \frac{2 f^{*}\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j\right)},
\end{aligned}
$$

$$\leq \frac{2\left(T_j - T_{j-1}\right)^{\omega_j - 1}}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j} s^{-\kappa}\left(K + L\frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)}\right)|y(s)|ds + \frac{2f^*\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j\right)},$$
$$\leq \frac{2\left(T_j - T_{j-1}\right)^{\omega_j - 1}\left(T_j^{1-\kappa} - T_{j-1}^{1-\kappa}\right)}{(1-\kappa)\Gamma\left(\omega_j\right)}\left(K + L\frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)}\right)R_j + \frac{2f^*\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j\right)}, \tag{44}$$
$$\leq R_j,$$

which means that $W(B_{R_j}) \subseteq B_{R_j}$.

Step 2: W is continuous.

Let (y_n) be a sequence such that $(y_n) \longrightarrow y$ in E_j and $t \in J_j$. Then,

$$\left|(Wy_n)(t) - (Wy)(t)\right| \leq \frac{\left(T_j - T_{j-1}\right)^{-1}\left(t - T_{j-1}\right)}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j} \left(T_j - s\right)^{\omega_j - 1}\left|f_1\left(s, y_n(s), I_{T_{j-1}^+}^{\omega_j} y_n(s)\right) - f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right)\right|ds$$
$$+ \frac{1}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{t} (t-s)^{\omega_j - 1}\left|f_1\left(s, y_n(s), I_{T_{j-1}^+}^{\omega_j} y_n(s)\right) - f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right)\right|ds$$
$$\leq \frac{2}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j} \left(T_j - s\right)^{\omega_j - 1}\left|f_1\left(s, y_n(s), I_{T_{j-1}^+}^{\omega_j} y_n(s)\right) - f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right)\right|ds$$
$$\leq \frac{2}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j} s^{-\kappa}\left(T_j - s\right)^{\omega_j - 1}\left(K\left|y_n(s) - y(s)\right| + LI_{T_{j-1}^+}^{\omega_j}\left|y_n(s) - y(s)\right|\right)ds$$
$$\leq \frac{2K}{\Gamma\left(\omega_j\right)}\left\|y_n - y\right\|_{E_j} \int_{T_{j-1}}^{T_j} s^{-\kappa}\left(T_j - s\right)^{\omega_j - 1}ds$$
$$+ \frac{2L}{\Gamma\left(\omega_j\right)}\left\|I_{T_{j-1}^+}^{\omega_j}\left(y_n - y\right)\right\|_{E_j} \int_{T_{j-1}}^{T_j} s^{-\kappa}\left(T_j - s\right)^{\omega_j - 1}ds$$
$$\leq \frac{2K}{\Gamma\left(\omega_j\right)}\left\|y_n - y\right\|_{E_j} \int_{T_{j-1}}^{T_j} s^{-\kappa}\left(T_j - s\right)^{\omega_j - 1}ds$$
$$+ \frac{2L\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j\right)\Gamma\left(\omega_j + 1\right)}\left\|y_n - y\right\|_{E_j} \int_{T_{j-1}}^{T_j} s^{-\kappa}\left(T_j - s\right)^{\omega_j - 1}ds$$
$$\leq \left(\frac{2K}{\Gamma\left(\omega_j\right)} + \frac{2L\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j\right)\Gamma\left(\omega_j + 1\right)}\right)\left\|y_n - y\right\|_{E_j} \int_{T_{j-1}}^{T_j} s^{-\kappa}\left(T_j - s\right)^{\omega_j - 1}ds$$
$$\leq \frac{2\left(T_j - T_{j-1}\right)^{\omega_j - 1}\left(T_j^{1-\kappa} - T_{j-1}^{1-\kappa}\right)}{(1-\kappa)\Gamma\left(\omega_j\right)}\left(K + L\frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)}\right)\left\|y_n - y\right\|_{E_j}. \tag{45}$$

That is, we obtain

$$\left\|(Wy_n) - (Wy)\right\|_{E_j} \longrightarrow 0 \,\text{as}\, n \longrightarrow \infty. \tag{46}$$

Thus, the operator W is a continuous on E_j.

Step 3: W is bounded and equicontinuous.

By Step 1, we have $\|W(y)\|_{E_j} \leq R_j$ which means that $W(B_{R_j})$ is bounded. It remains to verify that $W(B_{R_j})$ is equicontinuous.

For $y \in B_{R_{\mathcal{J}}}$ and $t_1, t_2 \in J_j, t_1 < t_2$, we have

$$
\begin{aligned}
\left|(Wy)(t_2) - (Wy)(t_1)\right| = & \left| -\frac{(T_j - T_{j-1})^{-1}(t_2 - T_{j-1})}{\Gamma(\omega_{\mathcal{J}})} \int_{T_{j-1}}^{T_j} (T_j - s)^{\omega_j - 1} f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) ds \right. \\
& + \frac{1}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t_2} (t_2 - s)^{\omega_j - 1} f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) ds \\
& + \frac{(T_j - T_{j-1})^{-1}(t_1 - T_{j-1})}{\Gamma(\omega_j)} \int_{T_{j-1}}^{T_j} (T_j - s)^{\omega_j - 1} f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) ds \\
& \left. - \frac{1}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t_1} (t_1 - s)^{\omega_j - 1} f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) ds \right| \\
\le & \frac{(T_j - T_{j-1})^{-1}}{\Gamma(\omega_j)} \left((t_2 - T_{j-1}) - (t_1 - T_{j-1})\right) \int_{T_{j-1}}^{T_j} (T_j - s)^{\omega_j - 1} \left| f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) \right| ds \\
& + \frac{1}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t_1} \left((t_2 - s)^{\omega_j - 1} - (t_1 - s)^{\omega_j - 1}\right) \left| f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) \right| ds \\
& + \frac{1}{\Gamma(\omega_j)} \int_{t_1}^{t_2} (t_2 - s)^{\omega_j - 1} \left| f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) \right| ds \\
\le & \frac{(T_j - T_{j-1})^{-1}}{\Gamma(\omega_j)} \left((t_2 - T_{j-1}) - (t_1 - T_{j-1})\right) \int_{T_{j-1}}^{T_j} (T_j - s)^{\omega_j - 1} \left| f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) - f_1(s, 0, 0) \right| ds \\
& + \frac{(T_j - T_{j-1})^{-1}}{\Gamma(\omega_j)} \left((t_2 - T_{j-1}) - (t_1 - T_{j-1})\right) \int_{T_{j-1}}^{T_j} (T_j - s)^{\omega_j - 1} \left| f_1(s, 0, 0) \right| ds \\
& + \frac{1}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t_1} \left((t_2 - s)^{\omega_j - 1} - (t_1 - s)^{\omega_j - 1}\right) \left| f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) - f_1(s, 0, 0) \right| ds \\
& + \frac{1}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t_1} \left((t_2 - s)^{\omega_j - 1} - (t_1 - s)^{\omega_j - 1}\right) \left| f_1(s, 0, 0) \right| ds \\
& + \frac{1}{\Gamma(\omega_j)} \int_{t_1}^{t_2} (t_2 - s)^{\omega_j - 1} \left| f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) - f_1(s, 0, 0) \right| ds + \frac{1}{\Gamma(\omega_j)} \int_{t_1}^{t_2} (t_2 - s)^{\omega_j - 1} \left| f_1(s, 0, 0) \right| ds \\
\le & \frac{(T_j - T_{j-1})^{-1}}{\Gamma(\omega_j)} \left((t_2 - T_{j-1}) - (t_1 - T_{j-1})\right) \int_{T_{j-1}}^{T_j} (T_j - s)^{\omega_j - 1} s^{-\kappa} \left(K|y(s)| + L\left|I_{T_{j-1}^+}^{\omega_j} y(s)\right|\right) ds \\
& + \frac{f^*(T_j - T_{j-1})^{-1}}{\Gamma(\omega_j)} \left((t_2 - T_{j-1}) - (t_1 - T_{j-1})\right) \int_{T_{j-1}}^{T_j} (T_j - s)^{\omega_j - 1} ds \\
& + \frac{1}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t_1} s^{-\kappa} \left((t_2 - s)^{\omega_j - 1} - (t_1 - s)^{\omega_j - 1}\right) \left(K|y(s)| + L\left|I_{T_{j-1}^+}^{\omega_j} y(s)\right|\right) ds \\
& + \frac{f^*}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t_1} \left((t_2 - s)^{\omega_j - 1} - (t_1 - s)^{\omega_j - 1}\right) ds \\
& + \frac{1}{\Gamma(\omega_j)} \int_{t_1}^{t_2} s^{-\kappa} (t_2 - s)^{\omega_j - 1} \left(K|y(s)| + L\left|I_{T_{j-1}^+}^{\omega_j} y(s)\right|\right) ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{f^*}{\Gamma(\omega_j)}\int_{t_1}^{t_2}(t_2-s)^{\omega_j-1}ds\\
&\leq\frac{(T_j-T_{j-1})^{\omega_j-2}}{\Gamma(\omega_{\mathcal{J}})}\left((t_2-T_{j-1})-(t_1-T_{j-1})\right)\left(K\|y\|_{E_j}+L\left\|I_{T_{j-1}^+}^{\omega_j}y\right\|_{E_j}\right)\int_{T_{j-1}}^{T_j}s^{-\kappa}ds\\
&+\frac{f^*(T_j-T_{j-1})^{\omega_j-1}}{\Gamma(\omega_j+1)}\left((t_2-T_{j-1})-(t_1-T_{j-1})\right)\\
&+\frac{1}{\Gamma(\omega_j)}\left(K\|y\|_{E_j}+L\left\|I_{T_{j-1}^+}^{\omega_j}y\right\|_{E_j}\right)\int_{T_{j-1}}^{t_1}s^{-\kappa}\left((t_2-t_1)^{\omega_j-1}\right)ds\\
&+\frac{f^*}{\Gamma(\omega_j)}\left(\frac{(t_2-T_{j-1})^{\omega_j}}{\omega_j}-\frac{(t_2-t_1)^{\omega_j}}{\omega_j}-\frac{(t_1-T_{j-1})^{\omega_j}}{\omega_j}\right)\\
&+\frac{(t_2-t_1)^{\omega_j-1}}{\Gamma(\omega_j)}\left(K\|y\|_{E_j}+L\left\|I_{T_{j-1}^+}^{\omega_j}y\right\|_{E_j}\right)\int_{t_1}^{t_2}s^{-\kappa}ds+\frac{f^*}{\Gamma(\omega_j)}\frac{(t_2-t_1)^{\omega_j}}{\omega_j}\\
&\leq\frac{(T_j-T_{j-1})^{\omega_j-2}\left(T_j^{1-\kappa}-T_{j-1}^{1-\kappa}\right)}{(1-\kappa)\Gamma(\omega_j)}\left((t_2-T_{j-1})-(t_1-T_{j-1})\right)\\
&\left(K\|y\|_{E_j}+L\frac{(T_j-T_{j-1})^{\omega_j}}{\Gamma(\omega_j+1)}\|y\|_{E_j}\right)+\frac{f^*(T_j-T_{j-1})^{\omega_j-1}}{\Gamma(\omega_j+1)}\left((t_2-T_{j-1})-(t_1-T_{j-1})\right)\\
&+\left(\frac{\left(t_1^{1-\kappa}-T_{j-1}^{1-\kappa}\right)(t_2-t_1)^{\omega_j-1}}{(1-\kappa)\Gamma(\omega_j)}\right)\left(K\|y\|_{E_j}+L\frac{(T_j-T_{j-1})^{\omega_j}}{\Gamma(\omega_j+1)}\|y\|_{E_j}\right)\\
&+\frac{f^*}{\Gamma(\omega_j+1)}\left((t_2-T_{j-1})^{\omega_j}-(t_2-t_1)^{\omega_j}-(t_1-T_{j-1})^{\omega_j}\right)+\frac{\left(t_2^{1-\kappa}-t_1^{1-\kappa}\right)(t_2-t_1)^{\omega_j-1}}{(1-\kappa)\Gamma(\omega_j)}\\
&\left(K\|y\|_{E_j}+L\frac{(T_j-T_{j-1})^{\omega_j}}{\Gamma(\omega_j+1)}\|y\|_{E_j}\right)+\frac{f^*(t_2-t_1)^{\omega_j}}{\Gamma(\omega_j+1)}\\
&\leq\left(\frac{(T_j-T_{j-1})^{\omega_j-2}\left(T_j^{1-\kappa}-T_{j-1}^{1-\kappa}\right)}{(1-\kappa)\Gamma(\omega_j)}\left(K+L\frac{(T_j-T_{j-1})^{\omega_j}}{\Gamma(\omega_j+1)}\right)\|y\|_{E_j}+\frac{f^*(T_j-T_{j-1})^{\omega_j-1}}{\Gamma(\omega_j+1)}\right)\left((t_2-T_{j-1})-(t_1-T_{j-1})\right)\\
&+\left(\frac{t_2^{1-\kappa}-T_{j-1}^{1-\kappa}}{(1-\kappa)\Gamma(\omega_j)}\left(K+L\frac{(T_j-T_{j-1})^{\omega_j}}{\Gamma(\omega_j+1)}\right)\|y\|_{E_j}\right)(t_2-t_1)^{\omega_j-1}\\
&+\frac{f^*}{\Gamma(\omega_j+1)}\left((t_2-T_{j-1})^{\omega_j}-(t_1-T_{j-1})^{\omega_j}\right).
\end{aligned}
\tag{47}
$$

Hence, $\|(Wy)(t_2)-(Wy)(t_1)\|_{E_j}\longrightarrow 0$ as $|t_2-t_1|\longrightarrow 0$. It implies that $T(B_{R_j})$ is equicontinuous.

Step 4: W is *k-set contractions.*

Let $t\in J_j$ and $B\in B_{R_j}$; then,

$$\begin{aligned}
\xi(W(B)(t)) &= \xi((Wy)(t), y \in B)\\
&\leq \left\{ \frac{\left(T_j - T_{j-1}\right)^{-1}\left(t - T_{j-1}\right)}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j} \left(T_j - s\right)^{\omega_j - 1} \xi f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) ds \right.\\
&\quad \left. + \frac{1}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{t} (t-s)^{\omega_j - 1} \xi f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) ds, y \in B \right\}.
\end{aligned} \tag{48}$$

By Remark 4, we have, for each $s \in J_j$,

$$\begin{aligned}
\xi(W(B)(t)) &\leq \left\{ \frac{\left(T_j - T_{j-1}\right)^{-1}\left(t - T_{j-1}\right)}{\Gamma\left(\omega_j\right)} \right.\\
&\quad \int_{T_{j-1}}^{T_j} \left(T_j - s\right)^{\omega_j - 1} \left[K\widehat{\xi}(B) \int_{T_{j-1}}^{T_j} s^{-\kappa} ds + L \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)} \widehat{\xi}(B) \int_{T_{j-1}}^{T_j} s^{-\kappa} ds \right]\\
&\quad \left. + \frac{1}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{t} (t-s)^{\omega_j - 1} \left[K\widehat{\xi}(B) \int_{T_{j-1}}^{t} s^{-\kappa} ds + L \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)} \widehat{\xi}(B) \int_{T_{j-1}}^{t} s^{-\kappa} ds \right], y \in B \right\}\\
&\leq \left\{ \frac{\left(T_j - T_{j-1}\right)^{\omega_j - 2}\left(t - T_{j-1}\right)}{\Gamma\left(\omega_j\right)} \right.\\
&\quad \int_{T_{j-1}}^{T_j} \left[K\widehat{\xi}(B) \int_{T_{j-1}}^{T_j} s^{-\kappa} ds + L \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)} \widehat{\xi}(B) \int_{T_{j-1}}^{T_j} s^{-\kappa} ds \right]\\
&\quad \left. + \frac{\left(t - T_{j-1}\right)^{\omega_j - 1}}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{t} \left[K\widehat{\xi}(B) \int_{T_{j-1}}^{t} s^{-\kappa} ds + L \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)} \widehat{\xi}(B) \int_{T_{j-1}}^{t} s^{-\kappa} ds \right], y \in B \right\}\\
&\leq \frac{\left(T_j^{1-\kappa} - T_{j-1}^{1-\kappa}\right)\left(T_j - T_{j-1}\right)^{\omega_j - 2}\left(t - T_{j-1}\right)}{(1-\kappa)\Gamma\left(\omega_j\right)} \left(K + L \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)} \right) \widehat{\xi}(B)\\
&\quad + \frac{\left(t^{1-\kappa} - T_{j-1}^{1-\kappa}\right)\left(t - T_{j-1}\right)^{\omega_j - 1}}{(1-\kappa)\Gamma\left(\omega_j\right)} \left(K + L \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)} \right) \widehat{\xi}(B)\\
&\leq \frac{2\left(T_j^{1-\kappa} - T_{j-1}^{1-\kappa}\right)\left(T_j - T_{j-1}\right)^{\omega_j - 1}}{(1-\kappa)\Gamma\left(\omega_j\right)} \left(K + L \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)} \right) \widehat{\xi}(B).
\end{aligned} \tag{49}$$

Thus,

$$\widehat{\xi}(WB) \leq \frac{2\left(T_j^{1-\kappa} - T_{j-1}^{1-\kappa}\right)\left(T_j - T_{j-1}\right)^{\omega_j - 1}}{(1-\kappa)\Gamma\left(\omega_j\right)} \left(K + L \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)} \right) \widehat{\xi}(B). \tag{50}$$

So, BVP (33) has at least a solution $\widetilde{y}_j \in B_{R_j}$. Since $B_{R_j} \subset E_j$, we have completed the proof of Theorem 2.

Now, we will be interested in proving the existence of solution for BVP (2). We begin by presenting the following assumption.

(H2) Let $f_1: J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, and there exists $\kappa \in (0,1)$ such that $t^{\kappa} f_1 \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and there exist a constants $K, L > 0$, such that$t^{\kappa}|f_1(t,x_1,z_1) - f_1(t,x_2,z_2)| \le K|x_1 - x_2| + L|z_1 - z_2|$, for any $x_1, x_2, z_1, z_2 \in \mathbb{R}$ and $t \in J$. □

Theorem 3. *Let (H1) and (H2) hold and inequality (39) be satisfied for any $j \in \{1,2,\ldots,n\}$. Then, BVP (2) has at least one solution in $C(J,\mathbb{R})$.*

Proof. By Theorem 2, BVP (33) possesses a solution $\widetilde{y}_j \in E_j$, $j \in \{1,2,\ldots,n\}$.

We define the function

$$y_j = \begin{cases} 0, t \in \left[0, T_{j-1}\right], \\ \widetilde{y}_j, t \in J_j, \end{cases} \quad j \in \{1,2,\ldots,n\}. \tag{51}$$

Thus, for $t \in J_{\mathcal{J}}$, the integral equation (31) has the solution $y_{\mathcal{J}} \in C([0,T_{\mathcal{J}}],\mathbb{R})$ with $y_{\mathcal{J}}(0) = 0$ and $y_{\mathcal{J}}(T_{\mathcal{J}}) = \widetilde{y}_{\mathcal{J}}(T_{\mathcal{J}}) = 0$.

Then, the function,

$$y(t) = \begin{cases} y_1(t), \quad t \in J_1, \\ y_2(t) = \begin{cases} 0, \quad t \in J_1, \\ \widetilde{y}_2, \quad t \in J_2 \end{cases} \\ . \\ . \\ . \\ y_n(t) = \begin{cases} 0, \quad t \in \left[0, T_{j-1}\right], \\ \widetilde{y}_j, \quad t \in J_j \end{cases} \end{cases} \tag{52}$$

is a solution of BVP (2) in $C(J,\mathbb{R})$. □

4. The Stability

In this section, we show that BVP (2) is UH stable.

Theorem 4. *Let all the conditions of Theorem 3 be satisfied. Then, BVP (2) is UH stable.*

Proof. Let the function $z(t)$ from $z \in C(J_j,\mathbb{R})$ satisfy inequality (21).

We define the functions:

$$z_j(t) = \begin{cases} 0, t \in \left[0, T_{j-1}\right], \\ z(t), \quad t \in J_j, \end{cases} \quad j \in \{1,2,\ldots,n\}. \tag{53}$$

By equality (30), for $j \in \{1,2,\ldots,n\}$ and $t \in J_j$, we obtain

$$^{c}D^{\omega(t)}_{T^{+}_{j-1}} z_j(t) = \int_{T_{j-1}}^{t} \frac{(t-s)^{1-\omega_j}}{\Gamma\left(2-\omega_j\right)} z^{(2)}(s)ds. \tag{54}$$

Taking the RLFI $I^{\omega_j}_{T^{+}_{j-1}}$ of both sides of inequality (21), we obtain

$$\begin{aligned} &\left| z_j(t) + \frac{\left(T_j - T_{j-1}\right)^{-1}\left(t - T_{j-1}\right)}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j} \left(T_j - s\right)^{\omega_{j-1}} f_1\left(s, z_j(s), I^{\omega_j}_{T^{+}_{j-1}} z_j(s)\right) ds, \right. \\ &\left. -\frac{1}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{t} (t-s)^{\omega_{j-1}} f_1\left(s, z_j(s), I^{\omega_j}_{T^{+}_{j-1}} z_j(s)\right) ds \right|, \\ &\le \varepsilon \int_{T_{j-1}}^{t} \frac{(t-s)^{\omega_j - 1}}{\Gamma\left(\omega_j\right)} ds \; \varepsilon \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)}, \\ &\le \varepsilon \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)}. \end{aligned} \tag{55}$$

According to Theorem 3, BVP (2) has a solution $y \in C(J,\mathbb{R})$ that is given for any $t \in J_j$, $j = 1,2,\ldots,n$, as $y(t) = y_j(t)$, where

$$y_j = \begin{cases} 0, \quad t \in \left[0, T_{j-1}\right], \\ \widetilde{y}_j, \quad t \in J_j, \end{cases} \tag{56}$$

and $\widetilde{y}_j$ is a solution of (33), which is given according to Lemma 7 by

$$\begin{aligned}\widetilde{y}_j(t) = &-\frac{\left(T_j - T_{j-1}\right)^{-1}\left(t - T_{j-1}\right)}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j} \left(T_j - s\right)^{\omega_{j-1}} f_1\left(s, \widetilde{y}_j(s), I_{T_{j-1}^+}^{\omega_j} \widetilde{y}_j(s)\right) ds \\ &+ \frac{1}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{t} (t-s)^{\omega_{j-1}} f_1\left(s, \widetilde{y}_j(s), I_{T_{j-1}^+}^{\omega_j} \widetilde{y}_j(s)\right) ds.\end{aligned} \tag{57}$$

Then, by equations (56) and (57), for $t \in J_j, j = 1, 2, \ldots, n$, we obtain

$$\begin{aligned}
|z(t) - y(t)| &= \left|z(t) - y_j(t)\right| = \left|z_j(t) - \widetilde{y}_j(t)\right|, \\
&= |z_j(t) + \frac{\left(T_j - T_{j-1}\right)^{-1}\left(t - T_{j-1}\right)}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j} \left(T_j - s\right)^{\omega_{j-1}} f_1\left(s, \widetilde{y}_j(s), I_{T_{j-1}^+}^{\omega_j} \widetilde{y}_j(s)\right) ds, \\
&\quad - \frac{1}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{t} (t-s)^{\omega_{j-1}} f_1\left(s, \widetilde{y}_j(s), I_{T_{j-1}^+}^{\omega_j} \widetilde{y}_j(s)\right) ds|, \\
&\leq |z_j(t) + \frac{\left(T_j - T_{j-1}\right)^{-1}\left(t - T_{j-1}\right)}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j} \left(T_j - s\right)^{\omega_{j-1}} f_1\left(s, z_j(s), I_{T_{j-1}^+}^{\omega_j} z_j(s)\right) ds, \\
&\quad - \frac{1}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{t} (t-s)^{\omega_{j-1}} f_1\left(s, z_j(s), I_{T_{j-1}^+}^{\omega_j} z_j(s)\right) ds| + \frac{\left(T_j - T_{j-1}\right)^{-1}\left(t - T_{j-1}\right)}{\Gamma\left(\omega_j\right)}, \\
&\int_{T_{j-1}}^{T_j} \left(T_j - s\right)^{\omega_j - 1} \left|f_1\left(s, z_j(s), I_{T_{j-1}^+}^{\omega_j} z_j\right) - f_1\left(s, \widetilde{y}_j(s), I_{T_{j-1}^+}^{\omega_j} \widetilde{y}_j\right)\right| ds, \\
&\quad + \frac{1}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{t} (t-s)^{\omega_j - 1} \left|f_1\left(s, z_j(s), I_{T_{j-1}^+}^{\omega_j} z_j\right) - f_1\left(s, \widetilde{y}_j(s), I_{T_{j-1}^+}^{\omega_j} \widetilde{y}_j\right)\right| ds, \\
&\leq \epsilon \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)} + \frac{\left(T_j - T_{j-1}\right)^{-1}\left(t - T_{j-1}\right)}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j} \left(T_j - s\right)^{\omega_j - 1} s^{-\kappa} \left(K\left|z_j(s) - \widetilde{y}_j(s)\right| + L I_{T_{j-1}^+}^{\omega_j} \left|z_j(s) - \widetilde{y}_j(s)\right|\right) ds, \\
&\quad + \frac{1}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{t} (t-s)^{\omega_j - 1} s^{-\kappa} \left(K\left|z_j(s) - \widetilde{y}_j(s)\right| + L I_{T_{j-1}^+}^{\omega_j} \left|z_j(s) - \widetilde{y}_j(s)\right|\right) ds, \\
&\leq \epsilon \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)} + \frac{\left(T_j - T_{j-1}\right)^{\omega_j - 1}}{\Gamma\left(\omega_j\right)} \left(K\left\|z_j - \widetilde{y}_j\right\|_{E_j} + L\left\|I_{T_{j-1}^+}^{\omega_j}\left(z_j - \widetilde{y}_j\right)\right\|_{E_j}\right) \int_{T_{j-1}}^{T_j} s^{-\kappa} ds, \\
&\quad + \frac{\left(T_{\mathcal{J}} - T_{\mathcal{J}-1}\right)^{\omega_{\mathcal{J}} - 1}}{\Gamma\left(\omega_{\mathcal{J}}\right)} \left(K\|z_{\mathcal{J}} - \widetilde{y}_{\mathcal{J}}\|_{E_{\mathcal{J}}} + L\|I_{T_{\mathcal{J}-1}^+}^{\omega_{\mathcal{J}}}\left(z_{\mathcal{J}} - \widetilde{y}_{\mathcal{J}}\right)\|_{E_{\mathcal{J}}}\right) \int_{T_{\mathcal{J}-1}}^{t} s^{-\kappa} ds
\end{aligned}$$

$$\begin{aligned}
&\leq \epsilon\frac{\left(T_{\mathcal{J}}-T_{\mathcal{J}-1}\right)^{\omega_{\mathcal{J}}}}{\Gamma\left(\omega_{\mathcal{J}}+1\right)}+\frac{\left(T_{\mathcal{J}}-T_{\mathcal{J}-1}\right)^{\omega_{\mathcal{J}}-1}\left(T_{\mathcal{J}-1}^{1-\kappa}-T_{\mathcal{J}-1}^{1-\kappa}\right)}{(1-\kappa)\Gamma\left(\omega_{\mathcal{J}}\right)}\left(K\|z_{\mathcal{J}}-\tilde{y}_{\mathcal{J}}\|_{E_{\mathcal{J}}}+L\frac{\left(T_{\mathcal{J}}-T_{\mathcal{J}-1}\right)^{\omega_{\mathcal{J}}}}{\Gamma\left(\omega_{\mathcal{J}}+1\right)}\|z_{\mathcal{J}}-\tilde{y}_{\mathcal{J}}\|_{E_{\mathcal{J}}}\right)\\
&\quad+\frac{\left(T_{\mathcal{J}}-T_{\mathcal{J}-1}\right)^{\omega_{\mathcal{J}}-1}\left(t^{1-\kappa}-T_{\mathcal{J}-1}^{1-\kappa}\right)}{(1-\kappa)\Gamma\left(\omega_{\mathcal{J}}\right)}\left(K\|z_{\mathcal{J}}-\tilde{y}_{\mathcal{J}}\|_{E_{\mathcal{J}}}+L\frac{\left(T_{\mathcal{J}}-T_{\mathcal{J}-1}\right)^{\omega_{\mathcal{J}}}}{\Gamma\left(\omega_{\mathcal{J}}+1\right)}\|z_{\mathcal{J}}-\tilde{y}_{\mathcal{J}}\|_{E_{\mathcal{J}}}\right)\\
&\leq \varepsilon\frac{\left(T_{\mathcal{J}}-T_{\mathcal{J}-1}\right)^{\omega_{\mathcal{J}}}}{\Gamma\left(\omega_{\mathcal{J}}+1\right)}+\frac{2\left(T_{\mathcal{J}}-T_{\mathcal{J}-1}\right)^{\omega_{\mathcal{J}}-1}\left(T_{\mathcal{J}}^{1-\kappa}-T_{\mathcal{J}-1}^{1-\kappa}\right)}{(1-\kappa)\Gamma\left(\omega_{\mathcal{J}}\right)}\left(K+L\frac{\left(T_{\mathcal{J}}-T_{\mathcal{J}-1}\right)^{\omega_{\mathcal{J}}}}{\Gamma\left(\omega_{\mathcal{J}}+1\right)}\right)\|z_{\mathcal{J}}-\tilde{y}_{\mathcal{J}}\|_{E_{\mathcal{J}}}\\
&\leq \varepsilon\frac{\left(T_{\mathcal{J}}-T_{\mathcal{J}-1}\right)^{\omega_{\mathcal{J}}}}{\Gamma\left(\omega_{\mathcal{J}}+1\right)}+\mu\|z-y\|,
\end{aligned}\tag{58}$$

where

$$\mu=\max_{j=1,2,\ldots,n}\frac{2\left(T_j-T_{j-1}\right)^{\omega_j-1}\left(T_j^{1-\kappa}-T_{j-1}^{1-\kappa}\right)}{(1-\kappa)\Gamma\left(\omega_j\right)}\left(K+L\frac{\left(T_j-T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j+1\right)}\right).\tag{59}$$

Then,

$$\|z-y\|(1-\mu)\leq\frac{\left(T_j-T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j+1\right)}\varepsilon.\tag{60}$$

Thus, for $t\in J_{\mathcal{J}}$, we have

$$|z(t)-y(t)|\leq\|z-y\|\leq\frac{\left(T_j-T_{j-1}\right)^{\omega_j}}{(1-\mu)\Gamma\left(\omega_j 1\right)}\varepsilon:=\lambda_{f_1}\varepsilon.\tag{61}$$

Therefore, BVP (2) is UH stable. □

5. Illustration

In this section, we construct an illustrative example to express the validity of the obtained results.

Example 2. Consider the following BVP:

$$\begin{cases}{}^{c}D_{0^+}^{\omega(t)}y(t)=\dfrac{t^{-1/3}e^{-t}}{\left(e^{et^2/1+t}+4e^{2t}+1\right)\left(1+|y(t)|+\left|I_0^{\omega(t)}y(t)\right|\right)}, & t\in J:=[0,2],\\ y(0)=0,\quad y(2)=0.\end{cases}\tag{62}$$

Suppose that

$$\begin{aligned}
&f_1(t,x,z)=\frac{t^{-1/3}e^{-t}}{\left(e^{e^{t^2/1+t}}+4e^{2t}+1\right)(1+y+z)},\ (t,x,z)\in[0,2]\times[0,+\infty)\times[0,+\infty).\\
&\omega(t)=\begin{cases}\dfrac{3}{2}, & t\in J_1:=[0,1],\\[2ex] \dfrac{9}{5}, & t\in J_2:=]1,2].\end{cases}
\end{aligned}\tag{63}$$

Then, we have

$$t^{1/3}\left|f_1(t,x_1,z_1)-f_1(t,x_2,z_2)\right| = \left|\frac{e^{-t}}{\left(e^{et^2/1+t}+4e^{2t}+1\right)}\left(\frac{1}{1+x_1+z_1}-\frac{1}{1+x_2+z_2}\right)\right|$$
$$=\frac{e^{-t}\left(\left|x_1-x_2\right|+\left|z_1-z_2\right|\right)}{\left(e^{et^2/1+t}+4e^{2t}+1\right)\left(1+x_1+z_1\right)\left(1+x_2+z_2\right)}$$
$$\leq\frac{e^{-t}}{\left(e^{et^2/1+t}+4e^{2t}+1\right)\left(\left|x_1-x_2\right|+\left|z_1-z_2\right|\right)}$$
$$\leq\frac{1}{(e+5)}\left|x_1-x_2\right|+\frac{1}{(e+5)}\left|z_1-z_2\right|. \tag{64}$$

Hence, condition (H2) holds with $\kappa=1/3$ and $K=L=1/e+5$.

According to (33) and by (15), we have the following two auxiliary BVP:

$$\begin{cases} {}^cD_{0^+}^{3/2}y(t)=\dfrac{t^{-1/3}e^{-t}}{\left(e^{e^{t^2/1+t}}+4e^{2t}+1\right)\left(1+|y(t)|+\left|I_0^{3/2}y(t)\right|\right)}, & t\in J_1,\\ y(0)=0, \quad y(1)=0, \end{cases} \tag{65}$$

and

$$\begin{cases} {}^cD_{1^+}^{9/5}y(t)=\dfrac{t^{-1/3}e^{-t}}{\left(e^{et^2/1+t}+4e^{2t}+1\right)\left(1+|y(t)|+\left|I_0^{9/5}y(t)\right|\right)}, & t\in J_2,\\ y(1)=0, \quad y(2)=0. \end{cases} \tag{66}$$

Next, we shall check that condition (39) is satisfied for $j=1$. Indeed,

$$\frac{\left(T_1^{1-\kappa}-T_0^{1-\kappa}\right)\left(T_1-T_0\right)^{\omega_1-1}}{(1-\kappa)\Gamma\left(\omega_1\right)}\left(2K+\frac{2L\left(T_1-T_0\right)^{\omega_1}}{\Gamma\left(\omega_1+1\right)}\right)=\frac{2}{2/3(e+5)\Gamma(3/2)}\left(1+\frac{1}{\Gamma(5/2)}\right)\simeq 0.7685<1. \tag{67}$$

Accordingly, condition (39) is achieved. By Theorem 2, BVP (65) has a solution $\widetilde{y}_1\in E_1$.

We shall check that condition (39) is satisfied for $j=2$. Indeed,

$$\frac{\left(T_2^{1-\kappa}-T_1^{1-\kappa}\right)\left(T_2-T_1\right)^{\omega_2-1}}{(1-\kappa)\Gamma\left(\omega_2\right)}\left(2K+\frac{2L\left(T_2-T_1\right)^{\omega_2}}{\Gamma\left(\omega_2+1\right)}\right)=\frac{2^{2/3}-1}{2/3\Gamma(9/5)}\,\frac{2}{e+5}\left(1+\frac{1}{\Gamma(14/5)}\right)\simeq 0.3913<1. \tag{68}$$

Thus, condition (39) is satisfied.

By Theorem 2, BVP (66) has a solution $\widetilde{y}_2\in E_2$.

Thus, by Theorem 3, BVP (62) possesses a solution:

$$y(t)=\begin{cases} \widetilde{y}_1(t), & t\in J_1,\\ y_2(t), & t\in J_2, \end{cases} \tag{69}$$

where

$$y_2(t) = \begin{cases} 0, & t \in J_1, \\ \tilde{y}_2(t), & t \in J_2. \end{cases} \quad (70)$$

According to Theorem 4, BVP (62) is UH stable.

6. Conclusion

In this work, we presented results about the existence and uniqueness of solutions to the BVP of Caputo fractional differential equations of variable-order $\omega(t)$, where $\omega(t)$: $[0, T] \longrightarrow (1, 2]$ is a piecewise constant function. All our results are based on Darbo's fixed-point theorem, and we studied Ulam–Hyers (UH) stability of solutions to our problem. Finally, we illustrated the theoretical findings by a numerical example.

The variable-order BVPs are important and interesting to all researchers. Therefore, all results in this paper show a great potential to be applied in various applications of multidisciplinary sciences.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

All authors declare that they have no conflicts of interest.

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