Research Article

Some Bohr-Type Inequalities for Bounded Analytic Functions

Hong Wang

School of Big Data and Artificial Intelligence, Chizhou University, Chizhou 247000, China

Correspondence should be addressed to Hong Wang; hongw@czu.edu.cn

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In this paper, some new versions of Bohr-type inequalities with one parameter or involving convex combination for bounded analytic functions of Schwarz function are established. Some previous inequalities are generalized. All the results are sharp.

1. Introduction and Preliminaries

Let $\mathbb{H}$ denote the class of analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ defined in the unit disk $\mathbb{D} := \{z \in \mathbb{C}: |z| < 1\}$ such that $|f(z)| < 1$ in $\mathbb{D}$. The Bohr's theorem states that if $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{H}$, then

$$\sum_{k=0}^{\infty} |a_k| |z|^k \leq 1, \quad \text{for } |z| \leq \frac{1}{3}, \quad (1)$$

the value $1/3$ is the best possible, and the inequality (1) is called classical Bohr inequality. In 1914, Bohr originally established the inequality (1) for $|z| < 1/6$ [1]. But subsequently later, Wiener, Riesz, and Schur independently proved the inequality (1) for $|z| < 1/3$, and the value is proved to be sharp [2–4].

Bohr phenomenon and Bohr radius problem are very interesting. Many generalized forms are studied and a lot of Bohr-type inequalities are obtained. Kayumov considered the Bohr-type inequalities for odd analytic functions and some other classes of analytic function in [5]. Bhowmik and Das [6] studied Bohr's phenomenon for subordinating families of certain univalent functions. Huang et al. [7] refined the Bohr inequality by allowing Schwarz function in place of the initial variable of the functions. Hu et al. [8, 9] established some Bohr inequalities with one parameter or involving convex combination. Abu-Muhanna [10] investigated the Bohr phenomenon for the class of analytic functions from the unit disk into the punctured unit disk.

Some authors considered the Bohr phenomenon for functions defined on other domains, such as concave-wedge domain [11, 12], convex domain [13], and the exterior of a compact domain [14].

The Bohr-type inequalities were also considered in other branches of mathematics. The analogous Bohr's radius was also studied for K-quasiconformal harmonic mappings by Liu and Ponnusamy [15]. Djakov and Ramanujan [16], Aizenberg [17], and Bayart et al. [18] generalized the Bohr inequality to the case of higher dimensions. Abu-Muhanna and Gunatillake [19] considered Bohr phenomenon in weighted Hardy-Hilbert spaces. Blasco [20] obtained the Bohr's radius of a Banach space. In [21], Paulsen explored Bohr's radius problem in Banach algebras. For more discussion on the Bohr-type inequalities, we can refer to [22–25].

We recall some results as follows.

Theorem 1 (see [26]). Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{H}$, $N \in \mathbb{N} = \{1, 2, 3, \dots\}$, then

$$|f(z)| + \sum_{k=N}^{\infty} |a_k| |z|^k \leq 1, \quad \text{for } |z| \leq \alpha, \quad (2)$$

where α is the positive root of the equation

$$2(1+r)r^N - (1-r)^2 = 0. \quad (3)$$

The radius α is the best possible. Moreover,

$$|f(z)|^2 + \sum_{k=N}^{\infty} |a_k| r^k \leq 1, \quad \text{for } |z| \leq \beta, \quad (4)$$

where β is the positive root of the equation

$$(1+r)r^N - (1-r)^2 = 0. \quad (5)$$

The radius β is the best possible.

Theorem 2 (see [27]). Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{H}$, and $s \in \mathbb{N}$, then

$$|f(z)| + \sum_{k=1}^{\infty} |a_{sk}| |z|^{sk} \leq 1, \quad \text{for } |z| \leq \gamma, \quad (6)$$

where γ is the positive root of the equation

$$r^{s+1} + 3r^s + r - 1. \quad (7)$$

The radius γ is the best possible.

Theorem 3 (see [27]). Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{H}$, then

$$|f(z)| + |f'(z)| |z| + \sum_{k=2}^{\infty} |a_k| |z|^k \leq 1, \quad \text{for } |z| \leq \frac{\sqrt{17}-3}{4}, \quad (8)$$

where $(\sqrt{17}-3)/4$ is the best possible.

Theorem 4 (see [28]). Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{H}$, $a := |a_0|$, then

$$\sum_{k=0}^{\infty} |a_k| |z|^k + \left(\frac{1}{1+a} + \frac{|z|}{1-|z|} \right) \sum_{k=1}^{\infty} |a_k|^2 |z|^{2k} \leq 1, \quad \text{for } |z| \leq \frac{1}{2+a}, \quad (9)$$

and the numbers $1/(2+a)$ and $1/(1+a)$ cannot be improved. Moreover,

$$a^2 + \sum_{k=1}^{\infty} |a_k| |z|^k + \left(\frac{1}{1+a} + \frac{|z|}{1-|z|} \right) \sum_{k=1}^{\infty} |a_k|^2 |z|^{2k} \leq 1, \quad \text{for } |z| \leq \frac{1}{2}, \quad (10)$$

and the numbers $1/2 + a$ and $1/(1+a)$ cannot be improved.

In order to establish our main results, we need the following lemmas, which will play the key role in proving the main results of this paper.

Lemma 1 (see [27]) (Schwarz-Pick lemma). Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{H}$, then

$$\left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)} f(z_2)} \right| \leq \left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right|, \quad \text{for } z_1, z_2 \in \mathbb{D}, \quad (11)$$

and the equality holds for distinct $z_1, z_2 \in \mathbb{D}$ if and only if $f(z)$ is a Möbius transformation.

In particular,

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad \text{for } z \in \mathbb{D}, \quad (12)$$

and the equality holds for some $z \in \mathbb{D}$, if and only if $f(z)$ is a Möbius transformation.

Lemma 2 (see [29]). Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{H}$, then $|a_k| \leq 1 - |a_0|^2$, for $k \in \mathbb{N}$.

Lemma 3. For $0 \leq x \leq x_0 \leq 1$, it holds that

$$f(x) := x + A(1 - x^2) \leq f(x_0), \quad \text{whenever } 0 \leq A \leq \frac{1}{2}. \quad (13)$$

The proof is simple. We omit it.

Lemma 4 (see [28]). Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{H}$, and $a := |a_0|$, then

$$\begin{aligned} & \sum_{k=0}^{\infty} |a_k| |z|^k + \left(\frac{1}{1+a} + \frac{|z|}{1-|z|} \right) \sum_{k=1}^{\infty} |a_k|^2 |z|^{2k} \\ & \leq a + (1 - a^2) \frac{|z|}{1-|z|}, \quad \text{for } |z| \in [0, 1). \end{aligned} \quad (14)$$

Let $\mathbb{S}_m := \{\omega \in \mathbb{H} : \omega(0) = \dots = \omega^{(m-1)}(0) = 0, \text{ and } \omega^{(m)}(0) \neq 0\}$, $m \in \mathbb{N} = \{1, 2, 3, \dots\}$, be the classes of Schwarz function [30, 31]. The aim of this paper is to establish some new versions of Bohr-type inequalities with one parameter or involving convex combination for bounded analytic functions of Schwarz function. Theorems 1–4 are generalized. All of our results are sharp.

2. Main Results

Theorem 5. Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{H}$, $a := |a_0|$ and $\omega_m \in \mathbb{S}_m$, $\psi_n \in \mathbb{S}_n$ for $m, n \in \mathbb{N}$. Then, for $\lambda \in (0, +\infty)$ and $N, s \in \mathbb{N}$, we have

$$|f(\omega_m(z))| + \lambda \sum_{k=N}^{\infty} |a_{sk}| |\psi_n(z)|^{sk} \leq 1, \quad (15)$$

for $|z| = r \leq R_{\lambda, N, m, n, s}$, where $R_{\lambda, N, m, n, s}$ is the unique root in $(0, 1)$ of the equation

$$2\lambda r^{Nns+m} + 2\lambda r^{Nns} + r^m + r^{ns} - r^{m+ns} - 1 = 0, \quad (16)$$

and the radius $R_{\lambda, N, m, n, s}$ is the best possible.

Proof. By the Schwarz lemma and the Schwarz-Pick lemma, respectively, we obtain

$$\begin{aligned} |\omega_m(z)| & \leq |z|^m, \\ |\psi_n(z)| & \leq |z|^n, \end{aligned} \quad (17)$$

$$|f(\omega_m(z))| \leq \frac{a + r^m}{1 + ar^m}, \quad \text{for } z \in D.$$

Then, by Lemma 2, we obtain

$$\begin{aligned} |f(\omega_m(z))| + \lambda \sum_{k=N}^{\infty} |a_{sk}| |\psi_n(z)|^{sk} & \leq \frac{a + r^m}{1 + ar^m} + \lambda (1 - a^2) \frac{r^{Nns}}{1 - r^{ns}} \\ & := A_{N, m, n, s}(a, r, \lambda). \end{aligned} \quad (18)$$

We just need to show that $A_{N,m,n,s}(a, r, \lambda) \leq 1$ holds for $r \leq R_{\lambda, N, m, n, s}$. That is to prove $A(a, r, \lambda) \leq 0$ for $r \leq R_{\lambda, N, m, n, s}$, where

$$\begin{aligned} A(a, r, \lambda) &= (a + r^m)(1 - r^{ns}) + \lambda(1 - a^2)r^{Nns}(1 + ar^m) - (1 + ar^m)(1 - r^{ns}) \\ &= (a + r^m - 1 - ar^m)(1 - r^{ns}) + \lambda(1 - a^2)r^{Nns}(1 + ar^m) \\ &= (1 - a)[(r^m - 1)(1 - r^{ns}) + \lambda(1 + a)r^{Nns}(1 + ar^m)] \\ &\leq (1 - a)[(r^m - 1)(1 - r^{ns}) + 2\lambda r^{Nns}(1 + r^m)]. \end{aligned} \quad (19)$$

Obviously, it is enough to show that $(r^m - 1)(1 - r^{ns}) + 2\lambda r^{Nns}(1 + r^m) \leq 0$ holds for $r \leq R_{\lambda, N, m, n, s}$. Let

$$\begin{aligned} g(r) &= (r^m - 1)(1 - r^{ns}) + 2\lambda r^{Nns}(1 + r^m) \\ &= 2\lambda r^{Nns+m} + 2\lambda r^{Nns} + r^m + r^{ns} - r^{m+ns} - 1. \end{aligned} \quad (20)$$

Then, we have

$$\begin{aligned} g'(r) &= 2\lambda(Nns + m)r^{Nns+m-1} + 2\lambda Nnsr^{Nns-1} \\ &\quad + mr^{m-1} + nsr^{ns-1} - (m + ns)r^{m+ns-1} \\ &= 2\lambda(Nns + m)r^{Nns+m-1} + 2\lambda Nnsr^{Nns-1} \\ &\quad + mr^{m-1}(1 - r^{ns}) + nsr^{ns-1}(1 - r^m) > 0. \end{aligned} \quad (21)$$

We claim that $g(r)$ is an increasing function of $r \in (0, 1)$ for fixing $\lambda \in (0, +\infty)$, $N \in \mathbb{N}$. Meanwhile, we observe that $g(0) = -1 < 0$ and $g(1) = 4\lambda > 0$. Then, there is a unique root $r^* \in (0, 1)$ such that $g(r^*) = 0$. Hence, $g(r) \leq 0$ holds for $r \leq r^* := R_{\lambda, N, m, n, s}$.

Now, we show that the radius $R_{\lambda, N, m, n, s}$ is the best possible. Let $a \in [0, 1)$,

$$\begin{aligned} \omega_m(z) &= z^m, \quad \psi_n(z) = z^n, \quad f(z) = \frac{a + z}{1 + az} \\ &= a + (1 - a^2) \sum_{k=1}^{\infty} (-a)^{k-1} z^k, \quad z \in D. \end{aligned} \quad (22)$$

Taking $z = r \in (0, 1)$, then the left side of inequality (15) reduces to

$$|f(r^m)| + \lambda \sum_{k=N}^{\infty} |a_{sk}| r^{nsk} = \frac{a + r^m}{1 + ar^m} + \lambda(1 - a^2) \frac{a^{Ns-1} r^{Nns}}{1 - a^s r^{ns}}. \quad (23)$$

Now, we just need to show that if $r > R_{\lambda, N, m, n, s}$, then there exists an $a \in [0, 1)$ such that the right side of (23) is greater than 1. That is to prove $l(a, r, \lambda) > 0$ for $r > R_{\lambda, N, m, n, s}$, where

$$\begin{aligned} l(a, r, \lambda) &= (a + r^m)(1 - a^s r^{ns}) + \lambda(1 - a^2)a^{Ns-1} r^{Nns}(1 + ar^m) - (1 + ar^m)(1 - a^s r^{ns}) \\ &= (1 - a^s r^{ns})(a + r^m - 1 - ar^m) + \lambda(1 - a^2)a^{Ns-1} r^{Nns}(1 + ar^m) \\ &= (1 - a)[(r^m - 1)(1 - a^s r^{ns}) + \lambda(1 + a)a^{Ns-1} r^{Nns}(1 + ar^m)]. \end{aligned} \quad (24)$$

Let

$$\begin{aligned} j(a, r, \lambda) &= (r^m - 1)(1 - a^s r^{ns}) \\ &\quad + \lambda(1 + a)a^{Ns-1} r^{Nns}(1 + ar^m). \end{aligned} \quad (25)$$

Observe that

$$\begin{aligned} \lim_{a \rightarrow 1^-} j(a, r, \lambda) &= 2\lambda r^{Nns+m} + 2\lambda r^{Nns} \\ &\quad + r^{ns} + r^m - r^{ns+m} - 1 = g(r). \end{aligned} \quad (26)$$

Thus, the monotonicity of $g(r)$ implies that if $r > R_{\lambda, N, m, n, s}$, then $g(r) > 0$. Therefore, by the continuity of $j(a, r, \lambda)$ for variable a , we have that if $r > R_{\lambda, N, m, n, s}$, there exists an $a \in [0, 1)$ such that $j(a, r, \lambda) > 0$, so does $l(a, r, \lambda)$.

In Theorem 5, setting $s = 1$, then we have Corollary 1. $\square$

Corollary 1. Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{H}$, $a := |a_0|$ and $\omega_m \in \mathbb{S}_m$, $\psi_n \in \mathbb{S}_n$ for $m, n \in \mathbb{N}$. Then, for $\lambda \in (0, +\infty)$ and $N \in \mathbb{N}$, it holds that

$$|f(\omega_m(z))| + \lambda \sum_{k=N}^{\infty} |a_k| |\psi_n(z)|^k \leq 1, \quad (27)$$

for $|z| = r \leq R_{\lambda, N, m, n}$, where $R_{\lambda, N, m, n}$ is the unique root in $(0, 1)$ of the equation

$$2\lambda r^{Nn+m} + 2\lambda r^{Nn} + r^m + r^n - r^{m+n} - 1 = 0, \quad (28)$$

and the radius $R_{\lambda, N, m, n}$ is the best possible.

Remark 1

- (1) If $\omega_m(z) = \psi_n(z) = z$, $N=1$, and $\lambda = 1$ in Corollary 1, then it reduces to the first part of Theorem 1.
- (2) If $\omega_m(z) = \psi_n(z) = z$, $N=1$, and $\lambda = 1$ in Theorem 5, then it reduces to Theorem 2.

Theorem 6. Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{H}$, $a := |a_0|$ and $\omega_m \in \mathbb{S}_m$, $\psi_n \in \mathbb{S}_n$ for $m, n \in \mathbb{N}$. Then, for $\lambda \in (0, +\infty)$ and $N \in \mathbb{N}$, we have

$$|f(\omega_m(z))|^2 + \lambda \sum_{k=N}^{\infty} |a_{sk} \psi_n(z)|^{sk} \leq 1, \quad (29)$$

for $|z| = r \leq R_{\lambda, N, m, n, s}^*$, where $R_{\lambda, N, m, n, s}^*$ is the unique root in $(0, 1)$ of the equation

$$\begin{aligned} B(a, r, \lambda) &= (a + r^m)^2 (1 - r^{ns}) + \lambda (1 - a^2) r^{Nns} (1 + r^m)^2 - (1 + ar^m)^2 (1 - r^{ns}) \\ &= (1 - r^{ns}) (a + r^m + 1 + ar^m) (a + r^m - 1 - ar^m) + \lambda (1 - a^2) r^{Nns} (1 + ar^m)^2 \\ &= (1 - a^2) [(1 - r^{ns}) (1 + r^m) (r^m - 1) + \lambda r^{Nns} (1 + ar^m)^2] \\ &\leq (1 - a^2) (1 + r^m) [(1 - r^{ns}) (r^m - 1) + \lambda r^{Nns} (1 + r^m)]. \end{aligned} \quad (32)$$

It is sufficient for us to prove $(1 - r^{ns}) (r^m - 1) + \lambda r^{Nns} (1 + r^m) \leq 0$ holds for $r \leq R_{\lambda, N, m, n, s}^*$. Let

$$\begin{aligned} g(r) &= (1 - r^{ns}) (r^m - 1) + \lambda r^{Nns} (1 + r^m) \\ &= \lambda r^{Nns+m} + \lambda r^{Nns} + r^m + r^{ns} - r^{ns+m} - 1, \end{aligned} \quad (33)$$

then we have

$$\begin{aligned} g'(r) &= \lambda (Nns + m) r^{Nns+m-1} + \lambda Nns r^{Nns-1} \\ &\quad + m r^{m-1} + n s r^{ns-1} - (m + ns) r^{m+ns-1} \\ &= \lambda (Nns + m) r^{Nns+m-1} + \lambda Nns r^{Nns-1} \\ &\quad + n s r^{ns-1} (1 - r^m) + m r^{m-1} (1 - r^{ns}) > 0. \end{aligned} \quad (34)$$

Let

$$\begin{aligned} j(a, r, \lambda) &= (r^m - 1) (1 - ar^{ns}) + \lambda a^{Ns-1} r^{Nns} (1 + ar^m) \\ &= \lambda r^{Nn+m} a^{Ns} + \lambda r^{Nns} a^{Ns-1} + r^m + ar^{ns} - ar^{ns+m} - 1. \end{aligned} \quad (37)$$

$$\lambda r^{Nns+m} + \lambda r^{Nns} + r^{ns} + r^m - r^{ns+m} - 1 = 0, \quad (30)$$

and the radius $R_{\lambda, N, m, n, s}^*$ is the best possible.

Proof. Inequality (17) and Lemma 2 lead to that

$$\begin{aligned} |f(\omega_m(z))|^2 + \lambda \sum_{k=N}^{\infty} |a_{sk} \psi_n(z)|^{sk} \\ \leq \left(\frac{a + r^m}{1 + ar^m} \right)^2 + \lambda (1 - a^2) \frac{r^{Nns}}{1 - r^{ns}} := B_{N, m, n, s}(a, r, \lambda). \end{aligned} \quad (31)$$

We just need to show that $B_{N, m, n, s}(a, r, \lambda) \leq 1$ holds for $r \leq R_{\lambda, N, m, n, s}^*$. That is to prove $B(a, r, \lambda) \leq 0$ for $r \leq R_{\lambda, N, m, n, s}^*$, where

Obviously, $g(r)$ is an increasing function of $r \in (0, 1)$ for fix $\lambda \in (0, +\infty)$, $N, s \in \mathbb{N}$, and we also have $g(0) = -1 < 0$, $g(1) = 2\lambda > 0$. Then, there is a unique root $r^* \in (0, 1)$ such that $g(r^*) = 0$. Hence, $g(r) \leq 0$ holds for $r \leq r^* := R_{\lambda, N, m, n, s}^*$.

Now, we show that the radius $R_{\lambda, N, m, n, s}^*$ is the best possible. We still consider the functions $\omega_m(z)$, $\psi_n(z)$, $f(z)$ as in (22). Taking $z = r \in (0, 1)$, then the left side of inequality (29) reduces to

$$|f(r^m)|^2 + \lambda \sum_{k=N}^{\infty} |a_{sk}| r^{nsk} = \left(\frac{a + r^m}{1 + ar^m} \right)^2 + \lambda (1 - a^2) \frac{a^{Ns-1} r^{Nns}}{1 - ar^{ns}}. \quad (35)$$

Now, we just need to show that if $r > R_{\lambda, N, m, n, s}^*$, then there exists an $a \in [0, 1)$ such that the right side of (35) is greater than 1. That is to prove $l(a, r, \lambda) > 0$, where

$$\begin{aligned} l(a, r, \lambda) &= (a + r^m)^2 (1 - ar^{ns}) + \lambda (1 - a^2) a^{Ns-1} r^{Nns} (1 + ar^m)^2 - (1 + ar^m)^2 (1 - ar^{ns}) \\ &= (1 - ar^{ns}) (a + r^m + 1 + ar^m) (a + r^m - 1 - ar^m) + \lambda (1 - a^2) a^{Ns-1} r^{Nns} (1 + ar^m)^2 \\ &= (1 - a^2) (1 + r^m) [(r^m - 1) (1 - ar^{ns}) + \lambda a^{Ns-1} r^{Nns} (1 + ar^m)]. \end{aligned} \quad (36)$$

Observe that

$$\begin{aligned} \lim_{a \rightarrow 1^-} j(a, r, \lambda) &= \lambda r^{Nn+m} + \lambda r^{Nns} + r^m \\ &\quad + r^{ns} - r^{ns+m} - 1 = g(r). \end{aligned} \quad (38)$$

Thus, the monotonicity of $g(r)$ implies that if $r > R_{\lambda, N, m, n, s}^*$, then $g(r) > 0$. Therefore, by the continuity of $j(a, r, \lambda)$ for variable a , we have that if $r > R_{\lambda, N, m, n, s}^*$, there exists an $a \in [0, 1)$ such that inequality $j(a, r, \lambda) > 0$, so does $l(a, r, \lambda)$.

In Theorem 6, setting $s = 1$, then we have Corollary 2. $\square$

Corollary 2. Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{H}$, $a := |a_0|$ and $\omega_m \in \mathbb{S}_m$, $\psi_n \in \mathbb{S}_n$ for $m, n \in \mathbb{N}$. Then, for $\lambda \in (0, +\infty)$ and $N \in \mathbb{N}$, we have

$$|f(\omega_m(z))|^2 + \lambda \sum_{k=N}^{\infty} |a_k \psi_n(z)|^k \leq 1, \quad (39)$$

for $|z| = r \leq R_{\lambda, N, m, n}^*$, where $R_{\lambda, N, m, n}$ is the unique root in $(0, 1)$ of the equation

$$\lambda r^{Nn+m} + \lambda r^{Nn} + r^{ns} + r^m - r^{n+m} - 1 = 0, \quad (40)$$

and the radius $R_{\lambda, N, m, n}^*$ is the best possible.

Remark 2. If $\omega_m(z) = \psi_n(z) = z$ and $\lambda = 1$ in Corollary 2, then it reduces to the second part of Theorem 1.

Theorem 7. Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{H}$, $a := |a_0|$, and $\omega_m \in \mathbb{S}_m$ for $m \in \mathbb{N}$. Then, for $\lambda \in (0, +\infty)$ and $N \in \mathbb{N}$, we have

$$|f(\omega_m(z))| + |f'(\omega_m(z))| |\omega_m(z)| + \lambda \sum_{k=N}^{\infty} |a_k \omega_m(z)|^k \leq 1, \quad (41)$$

for $|z| = r \leq R_{\lambda, N, m}$, where $R_{\lambda, N, m}$ is the unique root in $(0, \sqrt[m]{\sqrt{2}-1})$ of the equation

$$2\lambda r^{Nm} (1+r^m)^2 + (1-r^m)(r^{2m} + 2r^m - 1) = 0, \quad (42)$$

and the radius $R_{\lambda, N, m}$ is the best possible.

Proof. By the Schwarz lemma and the Schwarz-Pick lemma, respectively, we have

$$|\omega_m(z)| \leq r^m, \quad (43)$$

$$|f(\omega_m(z))| \leq \frac{r^m + a}{1 + ar^m}, \quad \text{for } z \in \mathbb{D}.$$

Then, by Lemmas 1–3, we obtain

$$\begin{aligned} & |f(\omega_m(z))| + |f'(\omega_m(z))| |\omega_m(z)| + \lambda \sum_{k=N}^{\infty} |a_k \omega_m(z)|^k \\ & \leq |f(\omega_m(z))| + \frac{1 - |f(\omega_m(z))|^2}{1 - |\omega_m(z)|^2} |\omega_m(z)| + \lambda \sum_{k=N}^{\infty} |a_k \omega_m(z)|^k \\ & \leq \frac{a + r^m}{1 + ar^m} + \left[1 - \left(\frac{a + r^m}{1 + ar^m} \right)^2 \right] \frac{r^m}{1 - r^{2m}} + \lambda (1 - a^2) \frac{r^{Nm}}{1 - r^m} \\ & = \frac{a + r^m}{1 + ar^m} + \frac{(1 - a^2)r^m}{(1 + ar^m)^2} + \lambda (1 - a^2) \frac{r^{Nm}}{1 - r^m} := C_{N, m}(a, r, \lambda), \end{aligned} \quad (44)$$

for $r \leq \sqrt[m]{\sqrt{2}-1}$, where $\sqrt[m]{\sqrt{2}-1}$ is the unique root of the equation $r^{2m} + 2r^m - 1 = 0$ in $(0, 1)$. We just need to show that $A_{N, m}(a, r, \lambda) \leq 1$ holds for $r \leq R_{\lambda, N, m}$. That is to prove $C(a, r, \lambda) \leq 0$ for $r \leq R_{\lambda, N, m}$, where

$$\begin{aligned} C(a, r, \lambda) &= [(a + r^m)(1 + ar^m) + (1 - a^2)r^m - (1 + ar^m)^2] (1 - r^m) + \lambda (1 - a^2) r^{Nm} (1 + ar^m)^2 \\ &= (1 - r^m)(1 + ar^m)(a + r^m - 1 - ar^m) + (1 - a^2)r^m (1 - r^m) + \lambda (1 - a^2) r^{Nm} (1 + ar^m)^2 \\ &= (1 - a)[(1 - r^m)(1 + ar^m)(r^m - 1) + (1 + a)r^m (1 - r^m) + \lambda (1 + a) r^{Nm} (1 + ar^m)^2]. \end{aligned} \quad (45)$$

Let

$$\begin{aligned} g(r) &= (1 - r^m)(1 + ar^m)(r^m - 1) + (1 + a)r^m (1 - r^m) \\ &\quad + \lambda (1 + a) r^{Nm} (1 + ar^m)^2, \end{aligned} \quad (46)$$

then

$$\begin{aligned} g(r) &= \lambda r^{Nm+2m} a^3 + \lambda r^{Nm} (r^{2m} + 2r^m) a^2 + [(1 - r^m)r^{2m} + \lambda r^{Nm} (2r^m + 1)] a + (1 - r^m)(2r^m - 1) + \lambda r^{Nm} \\ &\leq \lambda r^{Nm+2m} + \lambda r^{Nm} (r^{2m} + 2r^m) + (1 - r^m)r^{2m} + \lambda r^{Nm} (2r^m + 1) + (1 - r^m)(2r^m - 1) + \lambda r^{Nm} \\ &= 2\lambda r^{Nm} (r^m + 1)^2 + (1 - r^m)(r^{2m} + 2r^m - 1) := h(r). \end{aligned} \quad (47)$$

It is easy to verify that both $2\lambda r^{Nm}(r^m + 1)^2$ and $(1 - r^m)(r^{2m} + 2r^m - 1)$ are continuous and increasing functions for $r \in [0, \sqrt[m]{\sqrt{2} - 1}]$. Thus, $h(r)$ is a monotonically increasing function for $r \in [0, \sqrt[m]{\sqrt{2} - 1}]$. Furthermore, noting $h(0) = -1 < 0$ and $h(\sqrt[m]{\sqrt{2} - 1}) = 4\lambda(\sqrt{2} - 1)^N > 0$, thus there exists a unique real root $r^* \in (0, \sqrt[m]{\sqrt{2} - 1})$ such that $h(r^*) = 0$. Hence, it holds that $h(r) \leq 0$, for $r \leq r^* := R_{\lambda, N, m}$.

Now, we show that the radius $R_{\lambda, N, m}$ is the best possible. Let $a \in [0, 1)$,

$$\omega_m(z) = z^m,$$

$$f(z) = \frac{a+z}{1+az} = a + (1-a^2) \sum_{k=1}^{\infty} (-a)^{k-1} z^k, \quad z \in \mathbb{D}. \quad (48)$$

Taking $z = r \in (0, 1)$, then the left side of inequality (41) reduces to

$$|f(r^m)| + |f'(r^m)| r^m + \lambda \sum_{k=N}^{\infty} |a_k| r^{mk} = \frac{a+r^m}{1+ar^m} + \frac{(1-a^2)r^m}{(1+ar^m)^2} + \lambda(1-a^2) \frac{a^{N-1}r^{Nm}}{1-ar^m}. \quad (49)$$

Now, we just need to show that if $r > R_{\lambda, N, m}$, then there exists an $a \in [0, 1)$ such that the right side of (49) is greater than 1. That is to prove $l(a, r, \lambda) > 0$ for $r > R_{\lambda, N, m}$, where

$$\begin{aligned} l(a, r, \lambda) &= (1-ar^m)(1+ar^m)(a+r^m-1-ar^m) + (1-a^2) \\ &\quad [r^m(1-ar^m) + \lambda a^{N-1}r^{Nm}(1+ar^m)^2] = (1-a) \\ &\quad [(r^m-1)(1-ar^m)(1+ar^m) + (1+a)r^m(1-ar^m) + \lambda(1+a)a^{N-1}r^{Nm}(1+ar^m)^2]. \end{aligned} \quad (50)$$

Let

$$\begin{aligned} j(a, r, \lambda) &= (r^m-1)(1-ar^m)(1+ar^m) + (1+a)r^m(1-ar^m) \\ &\quad + \lambda(1+a)a^{N-1}r^{Nm}(1+ar^m)^2. \end{aligned} \quad (51)$$

Observe that

$$\begin{aligned} \lim_{a \rightarrow 1^-} j(a, r, \lambda) &= 2\lambda r^{Nm}(1+r^m)^2 \\ &\quad + (1-r^m)(r^{2m} + 2r^m - 1) = h(r). \end{aligned} \quad (52)$$

Thus, the monotonicity of $h(r)$ implies that if $r \in (R_{\lambda, N, m}, \sqrt[m]{\sqrt{2} - 1})$, then $h(r) > 0$. Therefore, by the continuity of $j(a, r, \lambda)$ for variable a , we have that if $r \in (R_{\lambda, N, m}, \sqrt[m]{\sqrt{2} - 1})$, there exists an $a \in [0, 1)$ such that inequality $j(a, r, \lambda) > 0$, so does $l(a, r, \lambda)$. $\square$

Remark 3. If $\omega_m(z) = z$, $N = 2$, and $\lambda = 1$ in Theorem 7, then it reduces to Theorem 3.

Theorem 8. Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{H}$, $a := |a_0|$, and $\omega_m \in \mathbb{S}_m$, for $m \in \mathbb{N}$. Then, for $t \in [0, 1)$, we have

$$t|f(\omega_m(z))| + (1-t) \left[\sum_{k=0}^{\infty} |a_k| |\omega_m(z)|^k + \left(\frac{1}{1+a} + \frac{|\omega_m(z)|}{1-|\omega_m(z)|} \right) \sum_{k=1}^{\infty} |a_k|^2 |\omega_m(z)|^{2k} \right] \leq 1, \quad (53)$$

for $|z| = r \leq R_{t, m}$, where

$$R_{t, m} = \begin{cases} \sqrt[m]{\frac{1}{(1+a)\sqrt{1-t}+1}}, & \text{for } t \in \left[0, \frac{(1+a)^2-1}{(1+a)^2}\right) \cup \left(\frac{(1+a)^2-1}{(1+a)^2}, 1\right), \\ \sqrt[m]{\frac{1}{2}}, & \text{for } t = \frac{(1+a)^2-1}{(1+a)^2}, \end{cases} \quad (54)$$

and the radius $R_{t,m}$ is the best possible.

Proof. Inequalities (43) and Lemmas 1, 2 and 4 lead to that

$$\begin{aligned} & t|f(\omega_m(z))| + (1-t) \left[\sum_{k=0}^{\infty} |a_k| |\omega_m(z)|^k + \left(\frac{1}{1+a} + \frac{|\omega_m(z)|}{1-|\omega_m(z)|} \right) \sum_{k=1}^{\infty} |a_k|^2 |\omega_m(z)|^{2k} \right] \\ & \leq t \frac{a+r^m}{1+ar^m} + (1-t)a + (1-t)(1-a^2) \frac{r^m}{1-r^m} := D_m(a, r, t). \end{aligned} \quad (55)$$

We just need to show that $D_m(a, r, t) \leq 1$ holds for $r \leq R_{t,m}$. That is to prove $D(a, r, t) \leq 0$ for $r \leq R_{t,m}$, where

$$\begin{aligned} D(a, r, t) &= t(a+r^m)(1-r^m) + (1-t)(1+ar^m) \\ &\quad [a(1-r^m) + (1-a^2)r^m] - (1+ar^m)(1-r^m) \\ &= (1-a) \{ [-(1+a)^2t + (1+a)^2 - 1] r^{2m} + 2r^m - 1 \}. \end{aligned} \quad (56)$$

Let

$$g(r) = [-(1+a)^2t + (1+a)^2 - 1] r^{2m} + 2r^m - 1. \quad (57)$$

Then, we divide it into two cases to discuss.

Case 1. If $t \in [0, ((1+a)^2 - 1/(1+a)^2)) \cup ((1+a)^2 - 1/(1+a)^2, 1)$, then we have $g(r) \leq 0$ for $r \leq \sqrt[m]{1/((1+a)\sqrt{1-t+1})}$, where $\sqrt[m]{1/((1+a)\sqrt{1-t+1})}$ is the unique root in $(0, 1)$ of equation $[-(1+a)^2t + (1+a)^2 - 1]r^{2m} + 2r^m - 1 = 0$. Thus, $B(a, r, t) \leq 0$ for $r \leq \sqrt[m]{1/((1+a)\sqrt{1-t+1})}$.

Case 2. If $t = (1+a)^2 - 1/(1+a)^2$, then we have $g(r) \leq 0$ for $r \leq \sqrt[m]{1/2}$. Thus, $B(a, r, t) \leq 0$ for $r \leq \sqrt[m]{1/2}$.

Now, we show that the radius $R_{t,m}$ is the best possible. Let $a \in [0, 1]$; we still consider the functions $\omega_m(z)$, $f(z)$ as in (48). Taking $z = r \in (0, 1)$, then the left side of inequality (53) reduces to

$$\begin{aligned} & t|f(r^m)| + (1-t) \left[\sum_{k=0}^{\infty} |a_k| r^{mk} + \left(\frac{1}{1+a} + \frac{r^m}{1-r^m} \right) \sum_{k=1}^{\infty} |a_k|^2 r^{2mk} \right] \\ &= t \frac{a+r^m}{1+ar^m} + (1-t) \left[a + \sum_{k=1}^{\infty} (1-a^2) a^{k-1} r^{mk} + \left(\frac{1}{1+a} + \frac{r^m}{1-r^m} \right) \sum_{k=1}^{\infty} (1-a^2)^2 a^{2k-2} r^{2mk} \right] \\ &= t \frac{a+r^m}{1+ar^m} + (1-t) \left[a + (1-a^2) \frac{r^m}{1-r^m} \right] := D_m^*(a, r, t). \end{aligned} \quad (58)$$

Now, we just need to show that if $r > R_{t,m}$, then there exists an $a \in [0, 1]$ such that $D_m^*(a, r, t)$ is greater than 1. That is to prove $l(a, r, t) > 0$ for $r > R_{t,m}$, where

$$\begin{aligned} l(a, r, t) &= t(a+r^m)(1-r^m) + (1-t) \\ &\quad [a(1+ar^m)(1-r^m) + r^m(1+ar^m)] \\ &\quad - (1+ar^m)(1-r^m) \\ &= (1-a) \{ [-(1+a)^2t + (1+a)^2 - 1] r^{2m} + 2r^m - 1 \}. \end{aligned} \quad (59)$$

Let

$$h(r) = [-(1+a)^2t + (1+a)^2 - 1] r^{2m} + 2r^m - 1. \quad (60)$$

Next, we divide it into two cases to discuss.

Case 1. If $t \in [0, ((1+a)^2 - 1/(1+a)^2)) \cup ((1+a)^2 - 1/(1+a)^2, 1)$, then we have $h(r) \geq 0$ for $r \geq \sqrt[m]{1/((1+a)\sqrt{1-t+1})}$. Thus, $l(a, r, t) \geq 0$ for $r \geq \sqrt[m]{1/((1+a)\sqrt{1-t+1})}$.

Case 2. If $t = ((1+a)^2 - 1/(1+a)^2)$, then we have $h(r) \geq 0$ for $r \geq \sqrt[m]{1/2}$. Thus, $l(a, r, t) \geq 0$ for $r \geq \sqrt[m]{1/2}$. $\square$

Remark 4. If $\omega_m(z) = z$ and $t = 0$ in Theorem 8, then it reduces to Theorem 4.

3. Conclusion

We establish some new versions of Bohr-type inequalities with one parameter or involving convex combination for bounded analytic functions of Schwarz function, and we conclude that all of the corresponding Bohr radii in the paper are exact. These inequalities generalize some earlier results on the Bohr radius problem.

Data Availability

The author declares that this research is purely theoretical and does not associate with any data.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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