Research Article

The Maximal Difference of Different Powers of an Element Modulo n

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In this paper, we investigate the maximal difference of integer powers of an element modulo n . Let $(a)_n$ denote the integer b with $1 \leq b \leq n$ such that $a \equiv b \pmod{n}$ for any integer a . Using the bounds for exponential sums, we obtain a lower bound of the function $H_{m_1, m_2}(n) = \max\{|(a^{m_1})_n - (a^{m_2})_n| : 1 \leq a \leq n, (a, n) = 1\}$, which gives $n - H_{m_1, m_2}(n) = O(n^{(3/4)+o(1)})$.

1. Introduction

Let $n \geq 3$ be an integer and $1 \leq a \leq n$ be an integer with $(a, n) = 1$. We know that there exists a unique integer c such that $ac \equiv 1 \pmod{n}$, where c is called the inverse of a modulo n with $1 \leq c < n$. We denote c by $\bar{a}$. The difference between an element and its inverse modulo n has been studied by several mathematicians.

Zhang [1] is the first person to explicitly study the distribution between an integer and its inverse modulo n , proving that

$$\sum_{a=1}^n (a - \bar{a})^{2k} = \frac{\phi(n)n^{2k}}{(2k+1)(k+1)} + O\left(4^k n^{((4k+1)/2)} d^2(n) \ln^2(n)\right), \quad (1)$$

where $\phi(n)$ is the Euler function and $d(n)$ is the divisor function.

Zhang [2] gives some further results for the distribution of $|a - \bar{a}|$. For any $0 < \delta \leq 1$, he defines that

$$S(n, \delta) = \#\{a : 0 \leq a \leq 1, (a, n) = 1, |a - \bar{a}| < \delta n\}, \quad (2)$$

where $\#\{\Delta\}$ denotes the number of the elements in $\{\Delta\}$ and shows that

$$S(n, \delta) = \delta(2 - \delta)\phi(n) + O\left(n^{(1/2)} d^2(n) \ln^3(n)\right). \quad (3)$$

Meanwhile, Zheng [3] obtains similar results in a more general context. More generalizations of [1, 2] are in recent papers [4–7].

Our object in this paper is to study the maximal difference of different powers of an element modulo n . The investigation for the maximal difference of different powers of a modulo n is motivated by the problem of the maximal difference of $|a - \bar{a}|$, where a and $\bar{a}$ run through the set $\{0, 1, \dots, n-1\}$ such that $a\bar{a} \equiv 1 \pmod{n}$. Khan [8] defines

$$M(n) = \max\{|a - \bar{a}| : 1 \leq a \leq n, (a, n) = 1\}, \quad (4)$$

proving that

$$M(n) \leq [n - 2\sqrt{n-1}]. \quad (5)$$

Later, Khan and Shaparlinski [9] show that

$$n - M(n) = o\left(n^{(3/4)+o(1)}\right). \quad (6)$$

Let $n \geq 3$ be an integer and $\mathbb{Z}_n$ denote residues' modulo n . In this paper, we assume these residues to consist of the elements $\{0, 1, 2, \dots, n-1\}$. We write $\mathbb{Z}_n^*$ to be the set of elements relatively prime to $\mathbb{Z}_n$. Let $(a)_n$ be the integer b with $1 \leq b \leq n$ such that $a \equiv b \pmod{n}$ for any integer a . Define

$$H_{m_1, m_2}(n) := \max\{|(a^{m_1})_n - (a^{m_2})_n| : 1 \leq a \leq n, a \in \mathbb{Z}_n^*\}. \quad (7)$$

We prove the following theorems.

Theorem 1. *Let $n \geq 3$ be an integer. Then, for arbitrary given positive integers m_1, m_2 , we have*

$$n - H_{m_1, m_2}(n) = O(n^{(3/4)+o(1)}). \quad (8)$$

In fact, we obtain a more general result which gives a lower bound for H_{m_1, m_2} :

Theorem 2. *Let $n \geq 3$ be an integer. Then, for arbitrary given positive integers m_1, m_2 , we have*

$$H_{m_1, m_2}(n) \geq n - \frac{4 \left(t^{\omega(n)} n^{(3/2)} d(n) + \left(t^{\omega(n)} n^{(3/2)} d(n) \right)^2 + \zeta(3/2) n^{(1/2)+o(1)} \sigma^2(n) \phi(n) \right)^{(1/2)}}{\phi(n)} + 2, \quad (9)$$

where $t = \max\{m_1, m_2\}$.

By applying Theorem 2, one can obtain the following.

Corollary 1. *Let $n \geq 3$ be an integer. Then, for arbitrary given positive integers m_1, m_2 , we have*

$$n - H_{m_1, m_2}(n) \leq \left(4 \zeta^{(1/2)} \left(\frac{3}{2} \right) + o(1) \right) n^{(9/4)+o(1)} \phi^{-(3/2)}(n). \quad (10)$$

2. Preliminaries

2.1. Notations. In this paper, $[t]$ is the integer part of t . We denote the group of units of $\mathbb{Z}_n$ by $\mathbb{Z}_n^*$; thus, the cardinality of $\mathbb{Z}_n^*$ is given by the Euler function $\phi(n)$. Let $\omega(n)$ be the counting function of the number of distinct primes. The number and the sum of the divisor function are defined by

$$d(n) = \sum_{k|n} 1, \quad (11)$$

$$\sigma(n) = \sum_{k|n, k>0} k = n \sum_{k|n, k>0} \frac{1}{k},$$

and the Riemann zeta function is written by

$$\zeta(s) = \sum_{z=1}^{\infty} \frac{1}{z^s}. \quad (12)$$

In what follows, we use the Landau symbol O with the understanding that any implied constants are absolute. For given functions F and G , the notation $F = O(G)$ is equivalent to the statement that the inequality $|F| \leq c|G|$ holds with some constant $c \geq 0$.

We recall the identity which follows from the formula of the sum of a geometric progression:

$$\sum_{r=0}^{n-1} e\left(\frac{rw}{n}\right) = \begin{cases} n, & w \equiv 0 \pmod{n} \\ 0, & w \not\equiv 0 \pmod{n} \end{cases}, \quad (13)$$

where $e(z) = e^{2\pi iz}$.

2.2. Several Lemmas. We need several lemmas to prove our theorems.

Lemma 1. *For any $d | n$, with $d \leq n$, there holds*

$$\sum_{\substack{r \in \mathbb{Z}_n^* \\ (r, n) = d}} \left| \sum_{x=0}^{h-1} e\left(\frac{rx}{n}\right) \right| \leq \frac{nh}{d}. \quad (14)$$

Proof. (see Lemma 17.3 in [10]). $\square$

Lemma 2. *Let p be a prime and u be an integer with $(u, p) = 1$. Then, for any integers $k \geq 2$ and $\alpha \geq 2$, there holds*

$$\left| \sum_{\substack{a=1 \\ p \nmid a}}^{p^\alpha} e\left(\frac{ua^k}{p^\alpha}\right) \right| \leq kp^{(\alpha/2)}. \quad (15)$$

Proof. This can be easily obtained from formula (0.5) in [11]. $\square$

Lemma 3. *Let $n > 1$ be an integer and k be a fixed positive integer. For any integer u with $(u, n) = 1$, we have the following estimation:*

$$\left| \sum_{a \in \mathbb{Z}_n^*} e\left(\frac{ua^k}{n}\right) \right| \leq k^{\omega(n)} n^{(1/2)}. \quad (16)$$

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ be the prime factor decomposition of n . Note that

$$\sum_{a \in \mathbb{Z}_n^*} e\left(\frac{ua^k}{n}\right) = \prod_{i=1}^r \left(\sum_{a \in \mathbb{Z}_{p_i^{\alpha_i}}^*} e\left(\frac{u(n/p_i^{\alpha_i})^{k-1} a^k}{p_i^{\alpha_i}}\right) \right). \quad (17)$$

In fact, if m and l pass through a reduced residue system modulo n_1 and n_2 , respectively, $mn_2 + ln_1$ passes through a reduced system modulo $n_1 n_2$. One can write

$$\begin{aligned}
\sum_{a \in \mathbb{Z}_n^*} e\left(\frac{ua^k}{n}\right) &= \sum_{m \in \mathbb{Z}_{n_1}^*} \sum_{l \in \mathbb{Z}_{n_2}^*} e\left(\frac{u(mn_2 + \ln_1)^k}{n_1 n_2}\right) \\
&= \sum_{m \in \mathbb{Z}_{n_1}^*} e\left(\frac{u(mn_2)^k}{n_1 n_2}\right) \sum_{l \in \mathbb{Z}_{n_2}^*} e\left(\frac{u(\ln_1)^k}{n_1 n_2}\right) \\
&= \sum_{m \in \mathbb{Z}_{n_1}^*} e\left(\frac{un_2^{k-1} m^k}{n_1}\right) \sum_{l \in \mathbb{Z}_{n_2}^*} e\left(\frac{un_1^{k-1} l^k}{n_2}\right).
\end{aligned} \quad (18)$$

This yields identity (17). By Lemma 2, we get Lemma 3. $\square$

Lemma 4. Let m_1, m_2 be fixed integers. Then, the bound,

$$\max_{(u_1, u_2, n) = g} \left| \sum_{a \in \mathbb{Z}_n^*} e\left(\frac{u_1 a^{m_1} + u_2 a^{m_2}}{n}\right) \right| \leq g^{(1/2)} n^{(1/2)+o(1)}, \quad (19)$$

holds.

Proof. It is easily obtained from Lemma 1 in [12]. $\square$

To introduce the following lemma better, we define

$$S(d, f) = \sum_{\substack{r, s \in \mathbb{Z}_n^* \\ r+s > 0 \\ (r, n) = d \\ (s, n) = f}} \left| \sum_{x \in \mathbb{Z}_n^*} e\left(\frac{rx^{m_1} + sx^{m_2}}{n}\right) \right| \left| \sum_{x=0}^{h-1} e\left(\frac{rx}{n}\right) \right|^2 \left| \sum_{x=0}^{h-1} e\left(\frac{sx}{n}\right) \right|^2. \quad (20)$$

Lemma 5. For any $d|n$ and $f|n$, we have

$$S(d, f) \leq \begin{cases} m_1^{\omega(n)} n^{(3/2)} h^3 d^{-(1/2)}, & f = n; \\ m_2^{\omega(n)} n^{(3/2)} h^3 f^{-(1/2)}, & d = n; \\ n^{(5/2)} h^2 \frac{(d, f)^{(1/2)}}{df}, & d < n, f < n. \end{cases} \quad (21)$$

Proof. We begin with the case when $f = n$. Apparently, this indicates that $s = 0$ and $r \neq 0$. From Lemma 1 and 3, we derive

$$\begin{aligned}
S(d, n) &= h^2 \sum_{\substack{r \in \mathbb{Z}_n^* \\ (r, n) = d}} \left| \sum_{x \in \mathbb{Z}_n^*} e\left(\frac{rx^{m_1}}{n}\right) \right| \left| \sum_{x=0}^{h-1} e\left(\frac{rx}{n}\right) \right|^2 \\
&= h^2 \sum_{\substack{r \in \mathbb{Z}_n^* \\ (r, n) = d}} d \left| \sum_{x \in \mathbb{Z}_{n/d}^*} e\left(\frac{r/d x^{m_1}}{n/d}\right) \right| \left| \sum_{x=0}^{h-1} e\left(\frac{rx}{n}\right) \right|^2 \\
&\leq n^{(3/2)} h^3 m_1^{\omega(n)} d^{-(1/2)}.
\end{aligned} \quad (22)$$

Using a similar method, we have the estimation,

$$S(n, f) \leq n^{(3/2)} h^3 m_2^{\omega(n)} f^{-(1/2)}. \quad (23)$$

We now consider the case $d < n$ and $f < n$. By Lemma 1 and 4, one obtains

$$\begin{aligned}
S(d, f) &\leq n^{(1/2)+o(1)} \sum_{\substack{r \in \mathbb{Z}_n^* \\ (r, n) = d}} \sum_{\substack{s \in \mathbb{Z}_n^* \\ (s, n) = f}} (r, s, n)^{(1/2)} \\
&\quad \left| \sum_{x=0}^{h-1} e\left(\frac{rx}{n}\right) \right|^2 \left| \sum_{x=0}^{h-1} e\left(\frac{sx}{n}\right) \right|^2 \leq n^{(1/2)+o(1)} (d, f, n)^{(1/2)} \\
&\quad \frac{n^2 h^2}{df} \leq n^{(5/2)+o(1)} h^2 \frac{(d, f)^{(1/2)}}{df}.
\end{aligned} \quad (24)$$

This completes the proof. $\square$

2.3. Proofs of the Theorems. Now, we complete the proofs of our main results.

Proof. of Theorem 2. We denote by V the number of the solution of congruences:

$$\begin{aligned}
x^{m_1} &\equiv h - y + z \pmod{n}, \\
x^{m_2} &\equiv -h + u - v \pmod{n}.
\end{aligned} \quad (25)$$

In the following, we show the range of h in the case that congruences (25) are solvable for $x \in \mathbb{Z}_n^*$, and integers $0 \leq y, z, u, v \leq h-1$. Then, it can be easily deduced that

$$H_{m_1, m_2}(n) \geq n - 4h + 2. \quad (26)$$

Applying the exponential sums identity (13), we have

$$\begin{aligned}
V &= \sum_{x \in \mathbb{Z}_n^*} \sum_{y, z, u, v=0}^{h-1} \frac{1}{n^2} \\
&\quad \sum_{r, s \in \mathbb{Z}_n} e\left(\frac{r(x^{m_1} - h + y - z) + s(x^{m_2} + h - u + v)}{n}\right) \\
&= \frac{1}{n^2} \sum_{r, s \in \mathbb{Z}_n} e\left(\frac{-rh + sh}{n}\right) \sum_{x \in \mathbb{Z}_n^*} e\left(\frac{rx^{m_1} + sx^{m_2}}{n}\right) \\
&\quad \left| \sum_{x=0}^{h-1} e\left(\frac{rx}{n}\right) \right|^2 \left| \sum_{x=0}^{h-1} e\left(\frac{sx}{n}\right) \right|^2.
\end{aligned} \quad (27)$$

Provided that $r = 0$ and $s = 0$, the right-hand side of equation (27) is $(\phi(n)h^4/n^2)$. Then, we have

$$\begin{aligned}
\left| V - \frac{\phi(n)h^4}{n^2} \right| &\leq \frac{1}{n^2} \sum_{\substack{r, s \in \mathbb{Z}_n^* \\ r+s > 0}} \left| \sum_{x \in \mathbb{Z}_n^*} e\left(\frac{rx^{m_1} + sx^{m_2}}{n}\right) \right| \\
&\quad \left| \sum_{x=0}^{h-1} e\left(\frac{rx}{n}\right) \right|^2 \left| \sum_{x=0}^{h-1} e\left(\frac{sx}{n}\right) \right|^2.
\end{aligned} \quad (28)$$

From Lemma 5, one can write

$$\begin{aligned}
 \left| V - \frac{\phi(n)h^4}{n^2} \right| &\leq \frac{1}{n^2} \sum_{r+s>0} \left| \sum_{x \in \mathbb{Z}_n^*} e\left(\frac{rx^{m_1} + sx^{m_2}}{n}\right) \right| \left| \sum_{x=0}^{h-1} e\left(\frac{rx}{n}\right) \right|^2 \left| \sum_{x=0}^{h-1} e\left(\frac{sx}{n}\right) \right|^2 \\
 &= \frac{1}{n^2} \left(\sum_{d|n, d < n} S(d, n) + \sum_{f|n, f < n} S(n, f) + \sum_{\substack{d|n, f|n \\ d, f < n}} S(d, f) \right) \\
 &\leq \frac{1}{n^2} \left((m_1^{\omega(n)} + m_2^{\omega(n)}) n^{(3/2)} h^3 d(n) + n^{(5/2)+o(1)} h^2 \sum_{d|n, f|n} \frac{(d, f)^{(1/2)}}{df} \right).
 \end{aligned} \tag{29}$$

We know

$$\begin{aligned}
 \sum_{d|n, f|n} \frac{(d, f)^{(1/2)}}{df} &\leq \sum_{k|n} k^{(1/2)} \sum_{d|(n/k)} \sum_{f|(n/k)} \frac{1}{k^2 df} \\
 &= \sum_{k|n} k^{-(3/2)} \sum_{d|(n/k)} \sum_{f|(n/k)} \frac{1}{df} \leq \zeta\left(\frac{3}{2}\right) \left(\frac{\sigma(n)}{n}\right)^2.
 \end{aligned} \tag{30}$$

Remark that $t = \max\{m_1, m_2\}$, so we can get the inequality

$$\begin{aligned}
 |n^2 V - \phi(n)h^4| &\leq (m_1^{\omega(n)} + m_2^{\omega(n)}) n^{(3/2)} h^3 d(n) \\
 &\quad + \zeta\left(\frac{3}{2}\right) \sigma^2(n) n^{(1/2)+o(1)} h^2 \leq 2t^{\omega(n)} n^{(3/2)} h^3 d(n) \\
 &\quad + \zeta\left(\frac{3}{2}\right) \sigma^2(n) n^{(1/2)+o(1)} h^2.
 \end{aligned} \tag{31}$$

After a simple calculation, one can write

$$h \geq \left(t^{\omega(n)} n^{(3/2)} d(n) + \left((t^{\omega(n)} n^{(3/2)} d(n))^2 + \zeta\left(\frac{3}{2}\right) n^{(1/2)+o(1)} \sigma^2(n) \phi(n) \right)^{(1/2)} \right) \phi^{-1}(n). \tag{35}$$

This finishes the proof of Theorem 2. $\square$

Proof. of Corollary 1. Applying Theorem 2, one can obtain

$$\begin{aligned}
 n - H_{m_1, m_2}(n) &\leq 4 \left(t^{\omega(n)} n^{(3/2)} d(n) + \left((t^{\omega(n)} n^{(3/2)} d(n))^2 + \zeta\left(\frac{3}{2}\right) n^{(1/2)+o(1)} \sigma^2(n) \phi(n) \right)^{(1/2)} \right) \phi^{-1}(n) \\
 &\leq 4 \left(2t^{\omega(n)} n^{(3/2)} d(n) + \zeta^{(1/2)}\left(\frac{3}{2}\right) n^{(1/4)+o(1)} \sigma(n) \phi^{(1/2)}(n) \right) \phi^{-1}(n)
 \end{aligned}$$

$$\begin{aligned}
 n^2 V &\geq h^2 (\phi(n)h^2 - 2t^{\omega(n)} n^{(3/2)} h d(n) \\
 &\quad - \zeta\left(\frac{3}{2}\right) \sigma^2(n) n^{(1/2)+o(1)}).
 \end{aligned} \tag{32}$$

The right-hand side of (32) has to be positive. It suffices to show that

$$\phi(n)h^2 - 2t^{\omega(n)} n^{(3/2)} h d(n) - \zeta\left(\frac{3}{2}\right) \sigma^2(n) n^{(1/2)+o(1)} \geq 0. \tag{33}$$

Multiplying both side of inequality (33) by $\phi(n)$ and calculating briefly, we have

$$\begin{aligned}
 (\phi(n)h - n^{(3/2)} d(n) t^{\omega(n)})^2 &\geq (n^{(3/2)} d(n) t^{\omega(n)})^2 \\
 &\quad + \zeta\left(\frac{3}{2}\right) \sigma^2(n) n^{(1/2)+o(1)}.
 \end{aligned} \tag{34}$$

Hence,

$$\begin{aligned}
&\leq 8t^{\omega(n)} n^{(3/2)} d(n) \phi^{-1}(n) + 4\zeta^{(1/2)} \left(\frac{3}{2}\right) n^{(9/4)+o(1)} \phi^{-(3/2)}(n) \\
&\leq \left(4\zeta^{(1/2)} \left(\frac{3}{2}\right) + o(1)\right) n^{(9/4)+o(1)} \phi^{-(3/2)}(n),
\end{aligned} \tag{36}$$

where we use the bounds

$$\begin{aligned}
\sigma(n)\phi(n) &\leq n^2, \\
\omega(n) &= O\left(\frac{\log n}{\log \log n}\right), \\
\log d(n) &= O\left(\frac{\log n}{\log \log n}\right).
\end{aligned} \tag{37}$$

The proof of Corollary 1 is complete. $\square$

Proof. of Theorem 1. Combining Corollary 1 with the bounds,

$$\frac{n}{\phi(n)} = O(\log \log(n)). \tag{38}$$

See Theorem 328 in [10]; Theorem 1 follows immediately. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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