

Research Article

The Optimal Graph Whose Least Eigenvalue is Minimal among All Graphs via 1-2 Adjacency Matrix

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All graphs under consideration are finite, simple, connected, and undirected. Adjacency matrix of a graph G is 0,1 matrix

$A = [a_{ij}] = \begin{cases} 0, & \text{if } v_i = v_j \text{ or } d(v_i, v_j) \geq 2 \\ 1, & \text{if } d(v_i, v_j) = 1. \end{cases}$ Here in this paper, we discussed new type of adjacency matrix known by 1-2 adjacency

matrix defined as $A_{1,2}(G) = [a_{ij}] = \begin{cases} 0, & \text{if } v_i = v_j \text{ or } d(v_i, v_j) \geq 3 \\ 1, & \text{if } d(v_i, v_j) = 2 \end{cases}$, from eigenvalues of the graph, we mean eigenvalues of the 1-2

adjacency matrix. Let T_n^c be the set of the complement of trees of order n . In this paper, we characterized a unique graph whose least eigenvalue is minimal among all the graphs in T_n^c .

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Its order is $|V(G)|$, denoted by n , and its size is $|E(G)|$, denoted by m . The distance between two vertices u and v of a graph G is the length of a shortest path between them. The 1-2 adjacency matrix of G is defined to be the matrix $A_{1,2}(G) = [a_{ij}]$ of order n with $a_{ij} = 1$ if $1 \leq d(v_i, v_j) \leq 2$ and a_{ij} is zero for the rest cases. The solution of $\det(xI - A_{1,2}(G))$ is the eigenvalues of G . Since $A_{1,2}(G)$ is always symmetric and real, all the eigenvalues can be arranged as $\lambda_1(G) \leq \lambda_2(G) \leq \dots \leq \lambda_n(G)$. We write $\lambda_{\min}(G)$ for the least eigenvalue of G . One can find that $\lambda_n(G)$ is exactly the spectral radius of G .

The least eigenvalue of any graph is nonpositive. For only disconnected graph, it is equal to zero. Otherwise, for graphs with at least one edge, it is less than or equal to -1 (by the interlacing theorem, see [1], p-19); it is equal to -1 if and only if G is a disjoint union of complete graphs or equivalently G^c is a null graph or a complete multipartite graph. If G contains $K_{1,2}$ as an induced subgraph, then $\lambda_{\min}(G) \leq \lambda_{\min}(K_{1,2}) = -\sqrt{2}$ (again by the interlacing theorem). For 1-2 adjacency matrix, $\lambda_{\min}(K_{1,2}) = -1$, and if a

graph G containing P_5 is an induced subgraph, then $\lambda_{\min}(G) \leq 1$.

There are many results in the literature concerning the largest eigenvalue (spectral radius) of simple graph; see, e.g., [2] or [1]. Javid examined different families of graphs to pick optimal graph with respect to least eigenvalues via usual adjacency matrix in their respective graph classes in [3–5]. Lubna et al. examined square power graph of G for their least eigenvalue [6]. Recently, the problem of minimizing the least eigenvalue of graphs subject to one or more parameters has received more attention. Bell et al. [7, 8] discussed the least eigenvalue of connected graphs with prescribed order and size. Fan et al. [9] discussed the least eigenvalue of complement of trees. Liu et al. [10] discussed the least eigenvalue of unicyclic graphs with pendant vertices. Fan et al. [9] discussed the least eigenvalue of unicyclic graphs with given girth. Petrovic et al. [11] discussed the least eigenvalue of bicyclic graphs.

In a family of graphs, a graph is called minimizing if the least eigenvalue of its adjacency matrix is minimum in the set of the least eigenvalues of all the graphs. Denote by T_n, T_n^c the set of trees of order n and the set of the complements of trees order n , respectively. In this paper, we determine the

unique minimizing graph in T_n^c for $n \geq 6$, which is not the complement of the star graph, and we are limiting our discussion to the set $T_n^c - K_n^c$ where K_n^c denotes all those graphs whose λ_{\min} is -1 .

Theorem 1 (see [7]). *If G is a graph whose least eigenvalue is minimal among all the connected graphs, then*

- (i) G is either a bipartite graph
- (ii) G is a join of two nested split graphs

Theorem 2 (see [12]). *Let $K(n, m)$ denote the family of all connected bipartite graphs and let $G \in K(n, m)$ be the graph such that its least eigenvalue is minimal among all the connected bipartite graphs, then G must be a double nested graph.*

2. Main Results

We begin with some definitions. Given a graph G of order n , a vector $X \in \mathbb{R}^n$ is called defined on G if there exists an injective function f from $V(G)$ to the entries of X ; simply written $X_v = f(v)$ for each $v \in V(G)$, if X is an eigenvector of $A_{1,2}(G)$, then it is naturally defined on $V(G)$ and every entry of X corresponds to the vertex $v \in V(G)$. One can easily prove that

$$X^T A_{1,2}(G)X = 2 \sum_{d(u,v) \in \{1,2\}} X_u X_v, \quad (1)$$

and λ is an eigenvalue of G corresponding to the eigenvector X iff $X \neq 0$ and

$$\lambda X_u = \sum_{v \in N_G(u)} X_v, \quad \text{for each vertex } u \in V(G), \quad (2)$$

where $N_G(u)$ denotes the neighbourhood of u in the graph G . Equation (2) is called the eigen equation for the graph G . Now, for any arbitrary unit vector $X \in \mathbb{R}^n$,

$$\lambda_{\min}(G) \leq X^T A_{1,2}(G)X, \quad (3)$$

and it will be equal if and only if X is a least eigenvector of G . Now, let G^c denote the complement of a graph G . One can easily proof that $A_{1,2}(G^c) = J - I - A_{1,2}(G)$, where I and J denote the identity matrix and ones matrix of the same size as $A_{1,2}(G)$. Now, for any vector X ,

$$X^T A_{1,2}(G^c)X = X^T (J - I)X - X^T A_{1,2}(G)X. \quad (4)$$

A graph G is said to be a tree if and only if there is a path between any pair of vertices and has no cycle. A tree is said to be a star if and only if there exists a vertex of degree $n - 1$ and all other vertices have degree 1. We introduce a special tree, denoted by $T(p, q)$, which is obtained from two disjoint stars $K_{1,p}$ ($p \geq 1$) and $K_{1,q}$ ($q \geq 1$) by joining one pendant vertex of $K_{1,p}$ and one pendant vertex of $K_{1,q}$. The complement of $T(p, q)$ is $G^c = (T(p, q))^c$, as shown in Figure 1.

Lemma 1. *Given $n \geq 6$, for any positive integer p, q such that $p \geq q \geq 1$ and $p + q + 2 = n$, $\lambda_{\min}(T(p, q))^c \{1, 2\} \geq$*

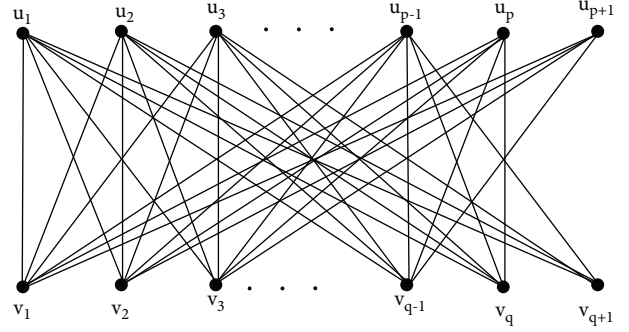


FIGURE 1: $G^c = (T(p, q))^c$ graph with $m = pq + p + q - 2$.

$\lambda_{\min}(T(\lfloor (n-2)/2 \rfloor, \lfloor (n-2)/2 \rfloor)^c \{1, 2\})$, with equality if and only if $p = \lfloor (n-2)/2 \rfloor$ and $q = \lfloor (n-2)/2 \rfloor$.

Proof. Suppose $n \geq 6$. Let λ_1 be the least eigenvalue of $(T(p, q))^c$ and X be the corresponding least eigenvector. Let X_i denote the entries of X corresponding to the vertex v_i for $i = 1, 2, 3, 4, 5, 6$. Now, by eigen equation (2), all the vertices attached at v_2 and v_5 have the same values given by X , say X_1 and X_6 . Now, as $\lambda_{\min}(T(p, q))^c < -1$, so by an eigen equation,

$$\begin{aligned} \lambda_1 X_1 &= X_4 + X_5 + (q-1)X_6, \\ \lambda_1 X_2 &= X_5 + (q-1)X_6, \\ \lambda_1 X_3 &= (q-1)X_6, \\ \lambda_1 X_4 &= (p-1)X_1, \\ \lambda_1 X_5 &= (p-1)X_1 + X_2, \\ \lambda_1 X_6 &= (p-1)X_1 + X_2 + X_3. \end{aligned} \quad (5)$$

Transform (5) into a matrix equation $(B - \lambda_1 I)X' = 0$, where $X' = (X_1, X_2, X_3, X_4, X_5, X_6)^T$ and the matrix B of order 6 is omitted. We get

$$\begin{aligned} f(\lambda; p, q) &= \det(B - \lambda I) = (p + q - pq - 1) \\ &\quad - \lambda^2(2p + 2q - 3pq - 1) - \lambda^4(p + q + pq - 2) + \lambda^6, \end{aligned} \quad (6)$$

and

$$f(\lambda; p, q) - f(\lambda; p-1, q+1) = (p-q-1)(\lambda^4 - 3\lambda^2 + 1). \quad (7)$$

Note that λ_1 is a least root of $f(\lambda; p, q)$. In addition, $f(-1, 6180; p, q) = 33.2670 + 0.00024593(pq) - 11.0894(p+q) < 0$ as $p+q = n-2 \geq 4$, which implies $\lambda_1 < -1.6180$.

If $p \geq q + 2$, by the above discussion, for $\lambda_1 < -1.6180$, we have

$$f(\lambda; p, q) - f(\lambda; p-1, q+1) < 0. \quad (8)$$

In particular, $f(\lambda_1; p-1, q+1) < 0$, which implies

$$\lambda_{\min}(T(p, q))^c \geq \lambda_{\min}(T(p-1, q+1))^c. \quad (9)$$

The result now follows. \square

Lemma 2. Let T^c be a graph in $T_n^c - K_n^c$ and let X be the least eigenvector of T , then X contains at least two positive and two negative entries.

Proof. Assume on contrary that X contains exactly one positive entry corresponding to the vertex w , i.e., $X_w > 0$. Since T is not a star, there exists a vertex $u \in N_T(w)$ such that $N_T(u) \setminus w \neq \emptyset$. Considering the eigen equation (2) on the vertex u for the graph T^c , we have

$$0 \leq \lambda_1(T^c)X_u = \sum_{d(v,u) \in \{1,2\}} X_v \leq 0. \tag{10}$$

This implies that $X_u = 0$ and $X_v = 0$ for each $v \in N_T(u)$, now as

$$\lambda_1(T^c)X_w = \sum_{d(v,w) \in \{1,2\}} X_v. \tag{11}$$

Now, let $v_0 \in N_T(u), w$. Then, by (2) and (11),

$$\begin{aligned} [1 + \lambda_1(T^c)]X_{v_0} &= X_{v_0} + \sum_{v \in d(v,v_0) \in \{1,2\} \setminus v_0} X_v, \\ &= \sum_{v \in d(v,v_0) \in \{1,2\}} X_v, \\ &= X_w + \sum_{d(v,v_0) \in \{1,2\} \setminus w} X_v, \\ &= X_w + \lambda_1(T^c)X_w, \\ &= [1 + \lambda_1(T^c)]X_w. \end{aligned} \tag{12}$$

This implies $X_{v_0} = X_w$. Since $\lambda_1(T^c) < 1$ as T is not a star, a contradiction, so our supposition is wrong and hence X contains at least two positive entries. Similarly, if we consider $-X$, we also get X which contains at least two negative entries. The result follows. \square

Before we state other results, we need some definitions.

Let G be a bipartite graph with colour classes U and V such that $V(G) = U \cup V$. G is said to be a double nested graph if $U = \{U_1 \cup U_2 \cup \dots \cup U_k\}$ and $V = \{V_1 \cup V_2 \cup \dots \cup V_k\}$ such that each $u_i \in U_1$ is adjacent to each and every vertex in V , and similarly, each $u_i \in U_2$ is adjacent to each and every vertex in $V = \{V_1 \cup V_2 \cup \dots \cup V_{k-1}\}$ for $i = \{1, 2, 3, \dots, k\}$.

Now, as we know from [12], if the two vertices u and w belong to the same colour class and $x_u \geq x_w$, then $\deg(u) \geq \deg(w)$.

Let $U = \{u_1, u_2, \dots, u_k\}$ and $V = \{v_1, v_2, \dots, v_k\}$ be the colour classes such that $x_{u_1} \geq x_{u_2} \geq x_{u_3} \geq \dots \geq x_{u_k} \geq$ and $x_{v_1} \geq x_{v_2} \geq x_{v_3} \geq \dots \geq x_{v_k} \geq$, where x_{u_i} and x_{v_i} for $i = \{1, 2, 3, \dots, k\}$ are the entries of the corresponding eigenvector X of G .

Lemma 3 (see [12]). Let G be a graph with the above assumptions. Then,

- (1) The vertices u_1 and v_1 are adjacent
- (2) The degree of u_1 and v_1 is complete

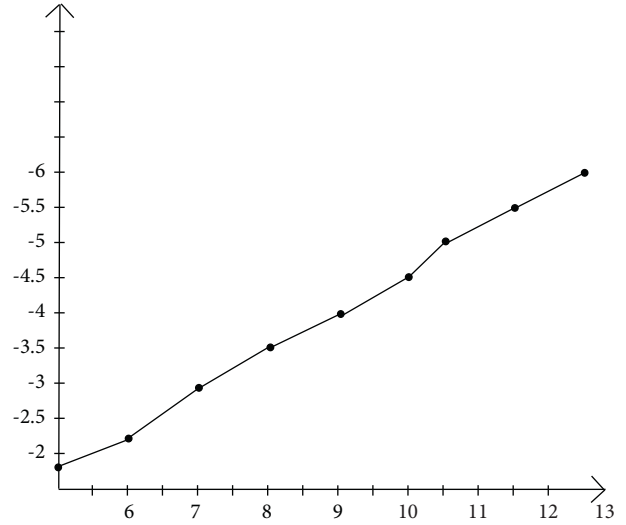


FIGURE 2: The behavior of eigenvalues with respect to number of vertices.

- (3) If the vertex u is adjacent to v_i , then u is adjacent to v_j for $j < k$, and if the vertex v is adjacent to u_i , then v is adjacent to u_j for $j < k$

Observation 1. If G is a bipartite of order n and size m , then $\lambda_{\min}G$ goes to its lower bound by increasing the size of G .

Bell [12] discussed the behavior of maximum eigenvalue by increasing the number of edges and kept $V(G)$ fixed. We here show the behavior of least eigenvalue of a $(T(p, q))^c$ graph by fixing the number of edges at $m = pq + p + q - 2$ and let the number of vertices be increased. In Figure 2, the horizontal axes show vertices and vertical axes represent least eigenvalues.

Theorem 3. Let T be a tree of order $n \geq 4$. Then,

$$\lambda_{\min}(T^c)_{\{1,2\}} \geq \lambda_{\min}\left(T\left(\frac{(n-2)}{2}, \frac{(n-2)}{2}\right)^c\right)_{\{1,2\}}. \tag{13}$$

Proof. As we know,

$$\lambda_{\min}(T(p, q))^c_{\{1,2\}} \geq \lambda_{\min}\left(T\left(\left\lceil \frac{(n-2)}{2} \right\rceil, \left\lfloor \frac{(n-2)}{2} \right\rfloor\right)^c\right)_{\{1,2\}}. \tag{14}$$

By Theorems 1 and 2, it is enough to show that $(T(p, q))^c$ is a largest double nested graph at $1 - 2$ distance. Let $(T(p, q))^c$ be the graph such that $\lambda_{\min}(T(p, q))^c$ is the least one among all the graphs and let X be the least eigenvector. Now, let G be the graph obtained from $(T(p, q))^c$ by rotating the edge $r_i s$ to $r_i t$ such that $s \neq t$ for $i = \{1, 2, \dots, k\}$, then $\lambda_{\min}G \leq \lambda_{\min}(T(p, q))^c$, and similarly, if s and t belong to the same colour class s.t $x_s \geq x_t$, then $\deg(s) \geq \deg(t)$, and so by Lemma 3, it will be a double nested graph. Next, we will show that it is the largest possible double nested graph at $1 - 2$ distance. As the size $(T(p, q))^c$ is $pq - 3$, so all we need to show is that the size cannot exceed

$pq - 3$. Suppose on contrary that $m > (pq - 3)$, then either $m = pq - 2$ or $m = pq - 1$. In both cases, the connected $T(p, q)$ at 1 - 2 distance is not possible, which completes the proof. \square

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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