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Research Article

(k, l)-Anonymity in Wheel-Related Social Graphs Measured on the Base of k-Metric Antidimension

Jiang-Hua Tang,[1] Tahira Noreen,[2] Muhammad Salman ,[2] Masood Ur Rehman ,[3] and Jia-Bao Liu [4]

[1]*Department of General Education, Anhui Xinhua University, Hefei 230088, China*
[2]*Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan*
[3]*Department of Basic Sciences, Balochistan University of Engineering and Technology Khuzdar, Khuzdar 89100, Pakistan*
[4]*School of Mathematics and Physics, Anhui Jianzhu University, Hefei, Anhui 230601, China*

Correspondence should be addressed to Masood Ur Rehman; masoodqau27@gmail.com

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For the study and valuation of social graphs, which affect an extensive range of applications such as community decision-making support and recommender systems, it is highly recommended to sustain the resistance of a social graph G to active attacks. In this regard, a novel privacy measure, called the (k, l)-anonymity, is used since the last few years on the base of k-metric antidimension of G in which l is the maximum number of attacker nodes defining the k-metric antidimension of G for the smallest positive integer k. The k-metric antidimension of G is the smallest number of attacker nodes less than or equal to l such that other k nodes in G cannot be uniquely identified by the attacker nodes. In this paper, we consider four families of wheel-related social graphs, namely, Jahangir graphs, helm graphs, flower graphs, and sunflower graphs. By determining their k-metric antidimension, we prove that each social graph of these families is the maximum degree metric antidimensional, where the degree of a vertex is the number of vertices linked with that vertex.

1. Introduction

Since 2016, a novel privacy measure, "the (k, l) anonymity," is defined and used, for the sake of a social graph confrontation from various active attacks, in connection with the concept of k-metric antidimension. Trujillo-Rasua and Yero defined, studied in detail, and promoted the idea of k-metric antidimension, which provides a basis for the privacy measure (k, l)–anonymity [7]. They defined this privacy measure as follows:

> *"The (k, l)-anonymity for a social graph G will be preserved according to active attacks if the k-metric antidimension of G is bounded above by l for the least positive integer k, where l is an upper bound on the expected number of attacker nodes."*

Accordingly, it can be seen that having a k-antimetric generator (a set defining the k-metric antidimension) as the set of attacker nodes, the probability of unique identification of other nodes by an adversary in a social graph is less than or equal to $(1/k)$.

Besides, providing many significant theoretical properties of the k-metric antidimension of graphs, Trujillo-Rasua and Yero also supplied the k-metric antidimension of complete graphs, paths, cycles, complete bipartite graphs, and trees [7]. This significant work of Trujillo-Rasua and Yero attracted many researchers to work on this idea, and therefore, the literature has been updated with the following remarkable contributions up till now:

(i) Trujillo-Rasua and Yero further contributed by characterizing 1-metric antidimensional trees and unicyclic graphs [8]

(ii) Mauw et al. contributed by providing a privacy-preserving graph transformation, which improves privacy in social network graphs by contracting active attacks [6]

(iii) Čangalović et al. contributed by considering wheels and grid graphs in the context of the k–metric antidimension [1]

(iv) DasGupta et al. contributed by analyzing and evaluating privacy-violation properties of eight social network graphs [4]

(v) Kratica et al. contributed by investigating the k–metric antidimension of two families of generalized Petersen graphs $\mathrm{GP}(n,1)$ (also called prism graphs) and $\mathrm{GP}(n,2)$ [5]

(vi) Zhang and Gao and, later on, Chatterjee et al. contributed by proving that the problem of finding the k-metric antidimension of a graph is, generally, an NP-complete problem [3, 9]

Inspired by all these contributions and, particularly, motivated by the work done by Čangalović et al. on wheel graphs, we place our contribution by extending the study of (k,l)-anonymity privacy measure based on k-metric antidimension towards four families of wheel-related social graphs.

2. Basic Works

Let $G = (V(G), E(G))$ be a simple and connected graph. We denote two adjacent vertices x and y by $x \sim y$ and nonadjacent by $x \nsim y$ in G. Two vertices of G are said to be neighbors of each other if there is an edge between them. The (open) neighborhood of a vertex x in G is $N(x) = \{y \in V(G)\colon\ y \sim x \in E(G)\}$. The neighborhood $N(x)$ is closed if it includes x and is denoted by $N[x]$. The number of vertices adjacent with a vertex x is called its degree and is denoted by $\mathrm{d}(x)$. The maximum degree in G is $\Delta = \max_{x\in V(G)}\mathrm{d}(x)$. The metric on G is a mapping $d\colon V(G)\times V(G) \longrightarrow \mathbb{Z}^+ \cup \{0\}$ defined by $\mathrm{d}(x,y) = l$, where l is the length of the number of edges in the shortest path between vertices x and y in G. A vertex u of G identifies a pair (x,y) of vertices in G if $\mathrm{d}(x,u) \neq \mathrm{d}(y,u)$. The sum $G + H$ of two graphs G and H is obtained by joining each vertex of G with every vertex of H. We refer the book in [2] for nonmentioned graphical notations and terminologies used in this paper.

Let $S = \{s_1, s_2, \ldots, s_t\} \subseteq V(G)$ be an ordered set. Then, the metric code, or simply code, of a vertex $u \in V(G)$ with respect to S is the t-vector $c_S(u) = (\mathrm{d}(u,s_1), \mathrm{d}(u,s_2), \ldots, \mathrm{d}(u,s_t))$. A chosen set S of vertices of G unique identifies each pair (x,y) of vertices in G if $c_S(x) \neq c_S(y)$. The following concepts are defined by Trujillo-Rasua and Yero in [7]:

(i) A set S of vertices of G is called a k–antimetric generator (k-antiresolving set) for G if k is the largest positive integer such that k vertices of G, other than the vertices in S, are not uniquely identified by S; *i.e.*, for every vertex $w \in V(G) - S$, there exist at least $k-1$ different vertices $u_1, \ldots, u_{k-1} \in V(G) - S$ such that $c_S(w) = c_S(u_1) = \ldots = c_S(u_{k-1})$

(ii) The cardinality of the smallest k-antimetric generator for G is called the k-metric antidimension of G, denoted by $a\dim_k(G)$, and such a smallest generator is known as k-antimetric basis of G

(iii) If k is the largest positive integer such that G has a k-antimetric generator, then G is said to be k-metric antidimensional graph

If S is a set of vertices of a graph G, then it has been defined as a relation on $V(G) - S$ according to the vertices having equal metric codes with respect to S as follows.

2.1. Equivalence Relation and Classes [5, 7]. Let $S \subset V(G)$ be a set of vertices of a connected graph G and let ρ_S be a relation on $V(G) - S$ defined by

$$\begin{aligned}&\text{for all } x, y \in V(G) - S,\\ &x\,\rho_S\,y \Leftrightarrow c_S(a) = c_S(b).\end{aligned} \tag{1}$$

This relation is an equivalence relation and partitioned $V(G) - S$ into classes, say $S_1, \ldots, S_m$, called the equivalence classes corresponds to the relation ρ_S.

Accordingly, we get the following useful property from [5].

Remark 1 (see [5]). For a fixed integer $k \geq 1$, a set S is a k-antimetric generator for G if and only if $\min_{i=1}^{m}|S_i| = k$, where each S_i, $1 \leq i \leq m$, is an equivalence class defined by the relation ρ_S.

3. Wheel-Related Social Graphs

In this section, we consider five wheel-related social graphs. The (k,l)-anonymity of one of them, called a wheel graph, has been measured previously in [1], by investigating its k-metric antidimension. Here, we focus to investigate the k-metric antidimension of other four graphs. For $n \geq 3$, a wheel graph is $W_{1,n} = K_1 + C_n$, where K_1 is the trivial graph having only one vertex v, and C_n is a cycle graph with vertices in $V = V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Accordingly, the vertex set of this graph is $V(W_{1,n}) = \{v\} \cup V$ and edge set is $E(W_{1,n}) = \{v \sim v_i, v_i \sim v_{i+1}\,;\ 1 \leq i \leq n\}$, where the indices greater than n or less than 1 will be taken modulo n. Each edge $v \sim v_i$ is called a spoke in a wheel graph. One such graph is depicted in Figure 1.

In 2018, Čangalović et al. supplied the following investigations.

Observation 1 (see [1])

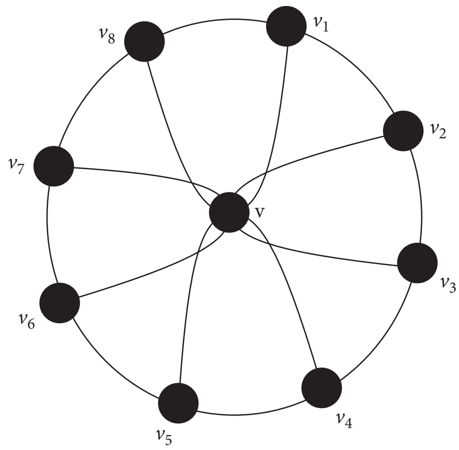

FIGURE 1: One wheel graph $W_{1,8}$.

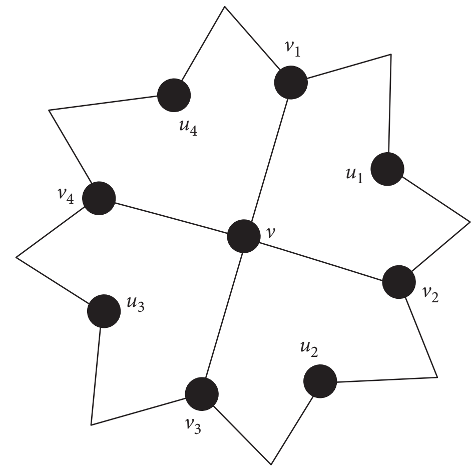

FIGURE 2: One Jahangir graph J_8.

$$adim_k(W_{1,3}) = \begin{cases} 1, & \text{for } k = 3, \\ 2, & \text{for } k = 2, \\ 3, & \text{for } k = 1, \end{cases} \quad (2)$$

$$adim_k(W_{1,4}) = \begin{cases} 1, & \text{for } k = 1, 4, \\ 2, & \text{for } k = 3, \\ 3, & \text{for } k = 2, \end{cases} \quad (3)$$

$$adim_k(W_{1,5}) = \begin{cases} 1, & \text{for } k = 2, 5, \\ 2, & \text{for } k = 1. \end{cases} \quad (4)$$

Theorem 1 (see [1]). *For all $n \geq 6$,*

$$adim_k(W_{1,n}) = \begin{cases} 1, & \text{for } k = 3, n, \\ 2, & \text{for } k = 1, 2. \end{cases} \quad (5)$$

The rest of the section is aimed to investigate the k-metric antidimensions of Jahangir graphs, helm graphs, flower graphs, and sunflower graphs.

3.1. Jahangir Graphs. For $n \geq 2$, a Jahangir (Gear) graph, J_{2n}, is obtained from a wheel graph $W_{1,2n} = K_1 + C_{2n}$ by deleting alternating spokes from the wheel. Let $V(C_{2n}) = V \cup U$, where $V = \{v_1, v_2, \ldots, v_n\}$ and $U = \{u_1, u_2, \ldots, u_n\}$. Then, the vertex set of a Jahangir graph is $V(J_{2n}) = \{v\} \cup V \cup U$ and its edge set is $E(J_{2n}) = \{v \sim v_i, v_i \sim u_i, v_i \sim u_{i-1}\ ;\ 1 \leq i \leq n\}$, where the indices greater than n or less than 1 will be taken modulo n. Figure 2 depicts graphical view of one Jahangir graph.

The following observation is easy to verify for $n = 2, 3$, and 4.

Observation 2

$$adim_k(J_4) = \begin{cases} 1, & \text{for } k = 1, 2, \\ 2, & \text{for } k = 3, \end{cases} \quad (6)$$

$$adim_k(J_6) = \begin{cases} 1, & \text{for } k = 1, 3, \\ 3, & \text{for } k = 2, \end{cases} \quad (7)$$

and $adim_k(J_8) = 1$ for each $k \in \{1, 2, 4\}$.

For all values of $n \geq 5$, the following result provides the k-metric antidimension of Jahangir graphs.

Theorem 2. *For $n \geq 5$, let J_{2n} be a Jahangir graph. Then,*

$$adim_k(J_{2n}) = \begin{cases} 1, & \text{for } k = 2, 3, n, \\ 2, & \text{for } k = 1. \end{cases} \quad (8)$$

Proof. First of all, it is worthy to note that $N(v) = V(C_{2n})$ and $N(v_i) = \{v, u_i, u_{i-1}\}$, for any $v_i \in V$, and $N(u_i) = \{v_i, v_{i+1}\}$, for any $u_i \in U$. Now, we need to discuss the following seven claims.

Claim 1: the set $S = \{v\}$ is an n-antimetric generator for J_{2n}.

Note that $c_S(x) = (1)$, for all $x \in V$, and $c_S(y) = (2)$, for all $y \in U$. According to the relation ρ_S, there are only two equivalence classes each has cardinality n. So, the result followed by Remark 1.

Claim 2: every singleton subset of V is a 3-antimetric generator for J_{2n}.

Let $S = \{v_i\} \subset V$ for any fixed $1 \leq i \leq n$. Then, $c_S(x) = (1)$, for all $x \in N(v_i)$, $c_S(y) = (2)$, for all $y \in V - \{v_i\}$, and $c_S(z) = (3)$, for all $z \in U - \{u_i, u_{i-1}\}$. Hence, the relation ρ_S supplies three equivalence classes $S_1 = N(v_i)$, $S_2 = V - \{v_i\}$, and $S_3 = U - \{u_i, u_{i-1}\}$. Thus,

$\min_{i=1}^{3}|S_i| = 3$, and hence, S is 3-antimetric generator, by Remark 1.

Claim 3: every singleton subset of U is a 2-antimetric generator for J_{2n}.

Let $S = \{u_i\} \subset U$; then, metric codes of the vertices are

$$c_S(x) = (1)\,\forall\, x \in N(u_i);\; c_S(u_{i+1}) = (2) = c_S(u_{i-1}) = c_S(v), \tag{9}$$

$$c_S(y) = (3)\,\forall\, y \in V - N(u_i);\; c_S(z) = (4)\,\forall\, z \in U - \{u_{i-1}, u_i, u_{i+1}\}. \tag{10}$$

Clearly, we receive four equivalence classes according to the relation ρ_S, $S_1 = N(u_i)$, $S_2 = N(v_i)$, $S_3 = V - N(u_i)$, and $S_4 = U - \{u_{i-1}, u_i, u_{i+1}\}$. Hence, $\min_{i=1}^{4}|S_i| = 2$, and S is a 2−antimetric generator, by Remark 1.

Claim 4: every 2 element subset of $V(J_{2n})$ is either 1-antimetric generator or 2-antimetric generator for J_{2n}.

Let S be a 2-element subset of $V(J_{2n})$. Then, we have the following two cases to discuss.

Case 1 (S contains v): here, we have two subcases.

Subcase 1.1: let $S = \{v, x\}$ with $x \in V$; then, $c_S(p) = (2,1) = c_S(q)$, for distinct $p, q \in N(x) - \{v\}$:

$$c_S(y) = (1,2)\forall\, y \in V - \{x\};\; c_S(z) = (2,3)\,\forall\, z \in U - N(x). \tag{11}$$

So, the equivalence classes corresponds to the relation ρ_S are $S_1 = N(x) - \{v\}$, $S_2 = V - \{x\}$, and $S_3 = U - N(x)$. Here, $\min_{i=1}^{3}|S_i| = 2$, which implies that S is a 2-antimetric generator, by Remark 1.

Subcase 1.2: let $S = \{v, x\}$ with $x \in U$; then, $c_S(t) = (1,1)$, for $t \in N(x)$:

$$c_S(p) = (2,2) = c_S(p') \text{ for } p, p' \in U \text{ such that } d(p,x) = d(p',x) = 2, \tag{12}$$

$$c_S(y) = (1,3)\,\forall\, y \in V - N(x);\; c_S(z) = (2,4)\forall z \in U - \{p, x, p'\}. \tag{13}$$

Due to the above metric coding, it is clear that we find four equivalence classes according to the relation ρ_S, which are $S_1 = N(x)$, $S_2 = \{p, p'\}$, $S_3 = V - N(x)$, and $S_4 = U - \{p, x, p'\}$. Thus, $\min_{i=1}^{4}|S_i| = 2$, which implies that S is a 2−antimetric generator, by Remark 1.

Case 2 (S does not contain v): again, we have three subcases to discuss.

Subcase 2.1: let $S \subset V$ and $S = \{v, v'\}$. Then, $d(v, v') = 2$. If $N(v) \cap N(v') = \{x\} \subset U$, then a vertex $y \in U$, such that $d(y, x) = 2$, has the unique metric code from the set $\{(1,3), (3,1)\}$ with respect to S. If no vertex from U is a common neighbor of v and v', then the vertex v has the unique metric code $(1,1)$ with respect to S.

Subcase 2.2: let $S \subset U$ and $S = \{u, u'\}$. Then, either $d(u, u') = 2$ or $d(u, u') = 4$. In the former case, a vertex $v \in V$, such that $v \in N(u) \cap N(u')$, has the unique metric code $(1,1)$. In the later case, we have two possibilities. If there is a vertex $x \in U$ such that $d(x, u) = 2 = d(x, u'')$, then a vertex $y \in U$, with $d(y, u) = 2$ and $d(y, u') = 4$, has the unique metric code $(2,4)$ with respect to S. If there is no such x in U, then the vertex v has the unique metric code $(2,2)$ with respect to S.

Subcase 2.3: let $S = \{u, v\}$ for $u \in U$ and $v \in V$. Then, either $d(u, v) = 1$ or $d(u, v) = 3$. For the later case, let $N(v) = \{v, x, y\}$, where $x, y \in U$ are distinct vertices. Here, we have two possibilities. If one of the neighbors x and y of v, say x, has the property that $d(x, u) = 2$, then the neighbor y of v has the unique metric code $(4,1)$ with respect to S. If $d(x, u) = 4 = d(y, u)$, then the vertex v has the unique metric code $(2,1)$ with respect to S. In the former case, v is a one neighbor of u from V, and the other neighbor of u from V has the unique metric code $(1,2)$ with respect to S.

In each possibility of these subcases, the relation ρ_S proposes at least one singleton equivalence class, which follows that $\min_i|S_i| = 1$. Hence, S is an 1−antimetric generator, by Remark 1.

Claim 5: for $n \geq 7$, the set $A' = \{v_i, v, x\} \subseteq V(J_{2n})$ is a 2−antimetric generator whenever $x \in (V \cup U) - \{u_{i-2}, v_{i-1}, u_{i-1}, v_i, u_i, v_{i+1}, u_{i+1}\}$. Otherwise, A' is 1−antimetric generator for J_{2n}.

If $x \in (V \cup U) - \{u_{i-2}, v_{i-1}, u_{i-1}, v_i, u_i, v_{i+1}, u_{i+1}\}$, the following two cases are need to be discussed for x.

Case 1: whenever $x \in U - \{u_{i-2}, u_{i-1}, u_i, u_{i+1}\}$, metric codes with respect to A' are

$$c_{A'}(t) = (1,2,4)\forall t \in N(v_i) - \{v\},\quad c_{A'}(p) = (2,1,1)\,\forall\, p \in N(x), \tag{14}$$

$$c_{A'}(q) = (2,1,3)\,\forall\, q \in V - N(x),\quad c_{A'}(u_{i+1}) = (3,2,2) = c_{A'}(u_{i+3}) \text{ for } 1 \leq i \leq n, \tag{15}$$

$$c_{A'}(y) = (3,2,4)\forall\, y \in U - \{u_{i-1}, u_i, x, u_{i+2}, u_{i+3}\}. \tag{16}$$

So, the equivalence classes corresponds to the relation $\rho_{A'}$ are $S_1 = N(v_i) - \{v\}$, $S_2 = N(x)$, $S_3 = V - N(x)$, $S_4 = \{u_{i+1}, u_{i+3}\}$, and $S_5 = U - \{u_{i-1}, u_i, x, u_{i+1}, u_{i+2}, u_{i+3}\}$. Note that $\min_{i=1}^{5}|S_i| = 2$, so Remark 1 yields the required result.

Case 2: whenever $x \in V - \{v_{i-1}, v_i, v_{i+1}\}$, metric codes with respect to A' are

$$\begin{aligned} c_{A'}(f) &= (1,2,3)\ \forall\, f \in N(v_i) - \{v\};\\ c_{A'}(g) &= (3,2,1) \forall\, g \in N(x) - \{v\}, \end{aligned} \tag{17}$$

$$\begin{aligned} c_{A'}(h) &= (2,1,2) \forall\, h \in V - \{v_i, x\};\\ c_{A'}(m) &= (3,2,3) \forall\, m \in U - (N(v_i) \cup N(x)). \end{aligned} \tag{18}$$

Hence, the equivalence classes corresponds to relation $\rho_{A'}$ are $S_1 = N(v_i) - \{v\}$, $S_2 = N(x) - \{v\}$, $S_3 = V - \{v_i, x\}$, and $S_4 = U - (N(v_i) \cup N(x))$. It follows that $\min_{i=1}^{4}|S_i| = 2$ because $n \geq 7$. Thus, Remark 1 proves that A' is a 2-antimetric generator.

Now, if $x \in \{u_{i-2}, v_{i-1}, u_{i-1}, u_i, v_{i+1}, u_{i+1}\}$, then again we have two cases.

Case 1: whenever $x \in \{v_{i-1}, v_{i+1}\}$, we have a vertex $t \in N(x) - \{v\}$ such that $c_{A'}(t) \neq c_{A'}(t')$, for all $t' \in V(C_{2n}) - \{t\}$

Case 2: whenever $x \in \{u_{i-2}, u_{i-1}, u_i, u_{i+1}\}$, we have a vertex $t \in U$ with $\mathrm{d}(t, x) = 2$ and $c_{A'}(t) \neq c_{A'}(t')$, for all $t' \in V(C_{2n}) - \{t\}$

In both the cases, we get at least one singleton equivalence class $\{t\}$ with respect to the relation $\rho_{A'}$, which implies that $\min_i|S_i| = 1$. Hence, A' is a 1−antimetric generator, by Remark 1.

Claim 6: except $n = 5, 7$, the set $B = \{u_i, v, u\} \subseteq V(J_{2n})$ is a 2-antimetric generator whenever $u \in U - \{u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}\}$. Otherwise, B is a 1−antimetric generator for J_{2n}.

Whenever $u \in U - \{u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}\}$, then the metric coding with respect to B is

$$\begin{aligned} c_B(p) &= (1,1,3) \forall\, p \in N(u_i),\ c_B(u_{i-1}) = (2,2,4)\\ &= c_S(u_{i+1}),\ c_B(q) = (3,1,1)\ \forall\, q \in N(u), \end{aligned} \tag{19}$$

$$\begin{aligned} c_B(x') &= (4,2,2) = c_S(y'), \text{where } x', y'\\ &\in U \text{ such that } \mathrm{d}(x', u) = \mathrm{d}(y', u) = 2, \end{aligned} \tag{20}$$

$$\begin{aligned} c_B(x) &= (3,1,3) \forall x \in V - (N(u_i) \cup N(u)),\ c_B(z)\\ &= (4,2,4) \forall\, z \in U - \{u_{i-1}, u_i, u_{i+1}, x', u, y'\}. \end{aligned} \tag{21}$$

Hence, we have the equivalence classes $S_1 = N(u_i)$, $S_2 = \{u_{i-1}, u_{i+1}\}$, $S_3 = N(u)$, $S_4 = \{x', y'\}$, $S_5 = V-(N(u_i) \cup N(u))$, and $S_6 = U - \{u_{i-1}, u_i, u_{i+1}, x', u, y'\}$ in accordance with the relation ρ_B. Thus, $\min_{i=1}^{6}|S_i| = 2$, o B is a 2-antimetric generator, by Remark 1.

Next, whenever $u \in \{u_{i-2}, u_{i-1}, u_{i+1}, u_{i+2}\}$, then

$$\begin{aligned} c_B(v_i) &= (1,1,1) \text{ when } u = u_{i-1},\\ c_B(v_{i+1}) &= (1,1,1) \text{ when } u = u_{i+1}, \end{aligned} \tag{22}$$

$$\begin{aligned} c_B(v_{i-1}) &= (2,2,2) \text{ when } u = u_{i-2},\\ c_B(v_{i+1}) &= (2,2,2) \text{ when } u = u_{i+2}. \end{aligned} \tag{23}$$

In each possibility, the given metric code is unique, which provides a singleton equivalence class according to the relation ρ_B. Hence, $\min_i|S_i| = 1$, and B is a 1−antimetric generator, by Remark 1.

Claim 7: every set $S \subseteq V(J_{2n})$ of cardinality $t \geq 3$ is a 1−antimetric generator for J_{2n}, except the sets A' and B discussed in Claims 5 and 6, respectively.

If S contains the vertex v, then there exists a vertex $g \in U$ (or $g \in V$) such that g is a neighbor of some element in S, and $c_S(g) \neq c_S(g')$, for any $g' \in V(J_{2n}) - \{g\}$. If S does not contain the vertex v, then v has the unique metric code with respect to S. In both the cases, we get a singleton equivalence class according to the relation ρ_S. Hence, $\min_i|S_i| = 1$, and Remark 1 implies that S is a 1-antimetric generator.

All these claims conclude the proof with the following points:

(i) For $k \in \{4, 5 \ldots, n-1\}$, there does not exist a k-antimetric generator for J_{2n}.

(ii) Claims 1, 2, and 3 provide that $a\dim_1(J_{2n}) \geq 2$. Furthermore, there exist 1-antimetric generators for J_{2n} of cardinality at least 2, by Claims 4 to 7. It follows that $a\dim_1(J_{2n}) = 2$.

(iii) Claim 3 provides the existence of a 2-antimetric generator for J_{2n} of cardinality 1, which yields that $a\dim_2(J_{2n}) = 1$.

(iv) Claim 2 provides the existence of a 3-antimetric generator for J_{2n} of cardinality 1, which implies that $a\dim_3(J_{2n}) = 1$.

(v) An n-antimetric generator for J_{2n} of cardinality 1 exists due to Claim 1, and hence, $a\dim_n(J_{2n}) = 1$. □

3.2. Helm Graphs. For $n \geq 3$, a helm graph, H_n, is obtained from a wheel graph $W_{1,n} = K_1 + C_n$ by attaching one leaf (a vertex of degree one) with each vertex of the cycle C_n. Let $U = \{u_1, u_2, \ldots, u_n\}$ be the set of leaves; then, the vertex set of a helm graph is $V(H_n) = V(W_{1,n}) \cup U$, and its edge set is $E(H_n) = E(W_{1,n}) \cup \{v_i \sim u_i;\ 1 \leq i \leq n\}$, where the indices greater than n or less than 1 will be taken modulo n. One helm graph is shown in Figure 3.

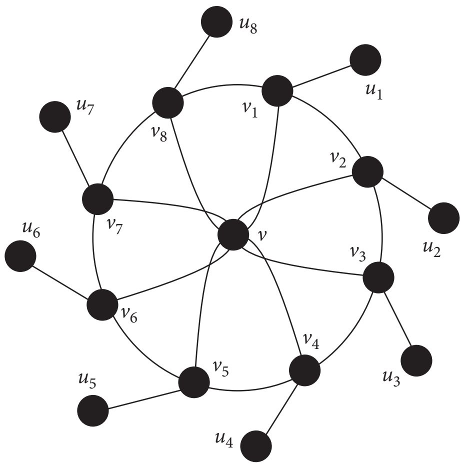

FIGURE 3: One helm graph H_8.

It is an easy task to verify the following observation.

Observation 3. When $n \in \{3,5\}$, $adim_k(H_n) = 1$, for $k = 1,2,\ldots,n$. While

$$adim_k(H_4) = \begin{cases} 1, & \text{for } k = 1,4, \\ 5, & \text{for } k = 2, \end{cases} \tag{24}$$

$$adim_k(H_6) = \begin{cases} 1, & \text{for } k = 1,3,6, \\ 3, & \text{for } k = 2. \end{cases} \tag{25}$$

Theorem 3. *For $n \geq 7$, let H_n be a helm graph. Then,*

$$adim_k(H_n) = \begin{cases} 1, & \text{for } k = 1,4,n, \\ 2, & \text{for } k = 3, \\ 3, & \text{for } k = 2. \end{cases} \tag{26}$$

Proof. It is significant to keep in hand the neighborhoods $N(v) = V$ and $N(v_i) = \{v, v_{i+1}, v_{i-1}, u_i\}$, for any $v_i \in V$, and $N(u_i) = \{v_i\}$, for any leaf $u_i \in U$. Now, we have to discuss the following eight claims.

Claim 1: the set $S = \{v\}$ is an n–antimetric generator for H_n.

Note that $c_S(x) = (1)$, for all $x \in V$, and $c_S(y) = (2)$, for all $y \in U$. There are only two equivalence classes $S_1 = V$ and $S_2 = U$, both of cardinality n, according to the relation ρ_S. Hence, $\min_{i=1}^{2}|S_i| = n$, which implies that S is an n-antimetric generator, by Remark 1.

Claim 2: every single leaf form a 1-antimetric generator for H_n.

Let $S = \{u_i\}$ be a set of one leaf $u_i \in U$. Then, $c_S(v_i) = (1)$, where $v_i \in N(u_i)$, and no other vertex of H_n has the code similar to v_i. It follows that there exists a singleton equivalence class due to the relation ρ_S. Accordingly, Remark 1 refers that S is an 1–antimetric generator.

Claim 3: every singleton subset of V is a 4-antimetric generator for H_n.

Let $S = \{v_i\} \subset V$, for any fixed $1 \leq i \leq n$; then, $c_S(x) = (1)$, for all $x \in N(v_i)$, $c_S(y) = (2)$, for all $y \in \{u_{i-1}, u_{i+1}\} \cup (V - \{v_{i-1}, v_i, v_{i+1}\})$, and $c_S(z) = (3)$, for all $z \in U - \{u_{i-1}, u_i, u_{i+1}\}$. Thus, the relation ρ_S produces three equivalence classes $S_1 = N(v_i)$, $S_2 = \{u_{i-1}, u_{i+1}\} \cup (V - \{v_{i-1}, v_i, v_{i+1}\})$, and $S_3 = U - \{u_{i-1}, u_i, u_{i+1}\}$. Hence, $\min_{i=1}^{3}|S_i| = 4$, and S is a 4-antimetric generator for H_n, by Remark 1.

Claim 4: the set $S' = \{u_i, w\}$ is a 3-antimetric generator for H_n whenever $w \in N(u_i)$. Otherwise, S' is a 1-antimetric generator.

Whenever $w \in N(u_i)$, we have $w = v_i$, and the metric codes with respect to S' are

$$\begin{aligned} &c_{S'}(x) = (2,1)\,\forall\, x \in N(v_i) - \{u_i\}; \\ &c_{S'}(y) = (4,3) \forall y \in U - \{u_{i-1}, u_i, u_{i+1}\}, \end{aligned} \tag{27}$$

$$c_{S'}(z) = (3,2) \forall z \in \{u_{i-1}, u_{i+1}\} \cup \left(V - \{v_{i-1}, v_i, v_{i+1}\}\right). \tag{28}$$

The equivalence classes corresponds to the relation $\rho_{S'}$ are $S_1 = N(v_i) - \{u_i\}$, $S_2 = U - \{u_{i-1}, u_i, u_{i+1}\}$, and $S_3 = \{u_{i-1}, u_{i+1}\} \cup (V - \{v_{i-1}, v_i, v_{i+1}\})$. It follows, by Remark 1, that S' is a 3-antimetric generator since $\min_{i=1}^{3}|S_i| = 3$.

When $w \in N(u_i)$, then the vertex $v_i \in N(u_i)$ has the metric code which is not similar to the metric code of any other vertex of H_n. This creates at least one singleton equivalence class $\{v_i\}$ in accordance with the relation $\rho_{S'}$, which implies that S' is a 1-antimetric generator for H_n, by Remark 1.

Claim 5: every 2–element subset of $V(H_n)$, except the set S' of Claim 4, is a 1-antimetric generator for H_n.

Let S be a 2–element subset of $V(H_n)$ and $S \neq S'$. Then, we discuss the following two cases:

Case 1 (S does not contain v): let $S \subset U$. When $S = \{u_i, u_{i+2}\}$, the vertex u_{i+1} has the unique metric code $(3,3)$ with respect to S. When $S = \{u_i, u_{i-2}\}$, the vertex u_{i-1} has the unique metric code $(3,3)$ with respect to S. Otherwise, the vertex v has the unique metric code $(2,2)$ with respect to S. Next, we let $S \subset V$. When $S = \{v_i, v_{i+2}\}$, the vertex v_{i-1} has the unique metric code $(1,2)$ with respect to S. Otherwise, the vertex v has the unique metric code $(1,1)$ with respect to S.

Case 2 (S contains v): let $S = \{v_i, v\}$; then, the leaf $u_i \in N(v_i)$ has the unique metric code $(1,2)$ with respect to S.

Each possibility in both the cases yields at least one singleton equivalence class according to the relation ρ_S, which implies that S is a 1-antimetric generator, by Remark 1.

Claim 6: the set $E = \{u_i, v_i, v\} \subset V(H_n)$ is a 2-antimetric generator for H_n.

Note that $c_E(u_{i-1}) = (3,2,2) = c_S(u_{i+1})$, $c_E(v_{i-1}) = (2,1,1) = c_S(v_{i+1})$,

$$\begin{aligned} c_E(x) &= (3,2,1), \forall x \in V - \{v_{i-1}, v_i, v_{i+1}\}, \\ c_E(y) &= (4,3,2) \forall y \in U - \{u_{i-1}, u_i, u_{i+1}\}. \end{aligned} \tag{29}$$

So, there are four equivalence classes $S_1 = \{u_{i-1}, u_{i+1}\}$, $S_2 = \{v_{i-1}, v_{i+1}\}$, $S_3 = V - \{v_{i-1}, v_i, v_{i+1}\}$, and $S_4 = U - \{u_{i-1}, u_i, u_{i+1}\}$ with respect to the relation ρ_E. That is, $\min_{i=1}^{4}|S_i| = 2$, and Remark 1 assists that E is a 2-antimetric generator.

Claim 7: for even values of $n \geq 8$, the set $S = \{v, v_i, u_i, v_{i+(n/2)}, u_{i+(n/2)}\} \subset V(H_n)$ is a 2-antimetric generator for H_n.

Note the metric coding of the vertices with respect to S is as follows:

$$\begin{aligned} c_S(v_{i-1}) &= (2,1,1,2,3) = c_S(v_{i+1}), \\ c_S(u_{i-1}) &= (3,2,2,3,4) = c_S(u_{i+1}), \end{aligned} \tag{30}$$

$$c_S(x) = (3,2,1,2,3) \forall x \in V - \{v_{i-1}, v_i, v_{i+1}, v_{i+(n/2)-1}, v_{i+(n/2)+1}\}, \tag{31}$$

$$c_S(y) = (4,3,2,3,4) \forall y \in U - \{u_{i-1}, u_i, u_{i+1}, u_{i+(n/2)-1}, u_{i+(n/2)+1}\}, \tag{32}$$

$$\begin{aligned} c_S(v_{i+(n/2)-1}) &= (3,2,1,1,2) = c_S(v_{i+(n/2)+1}), \\ c_S(u_{i+(n/2)-1}) &= (4,3,2,2,3) = c_S(u_{i+(n/2)+1}). \end{aligned} \tag{33}$$

Hence, the classes according to the relation ρ_S are $S_1 = \{v_{i-1}, v_{i+1}\}$, $S_2 = \{u_{i-1}, u_{i+1}\}$, $S_3 = \{v_{i+(n/2)-1}, v_{i+(n//)2+1}\}$, $S_4 = \{u_{i+(n/2)-1}, u_{i+(n/2)+1}\}$, $S_5 = V - \{v_{i-1}, v_i, v_{i+1}, v_{i+(n/2)-1}, v_{i+(n/2)+1}\}$, and $S_6 = U - \{u_{i-1}, u_i, u_{i+1}, u_{i+(n/2)-1}, u_{i+(n/2)+1}\}$. Here, $\min_{i=1}^{6}|S_i| = 2$, which implies that S is a 2-antimetric generator, by Remark 1.

Claim 8: any subset of $V(H_n)$ of cardinality $t \geq 3$ is a 1-antimetric generator for H_n, except the sets E and S considered in Claims 6 and 7, respectively.

Let W be a subset of $V(H_n)$ with $|W| = t \geq 3$ and $W \neq E, S$. Then, note the following two possibilities:

(1) W contains v: if $|W| = 3$ and $W = \{u, v, v\}$ with $u \in U$, $v \in V$ but $v \in N(u)$ (because this case is already discussed in Claim 6). Then, the vertex $x \in N(u)$ has the unique metric code from the set $\{(1,1,1), (1,1,2)\}$ with respect to W. If $|W| \geq 4$, then a neighbor of some $w \in W - \{v\}$ receives the unique metric code with respect to W.

(2) W does not contain v: the vertex v has the unique metric code with respect to W.

In both the possibilities, we get at least one singleton equivalence class according to the relation ρ_W. Thus, $\min_i |S_i| = 1$, and Remark 1 provides that S is a 1-antimetric generator.

The proof will reach to its end by discussing the following points on the base of formerly discussed claims:

(i) For $k \in \{5, 6, \ldots, n-1\}$, there does not exist a k-antimetric generator for H_n.

(ii) We find a 1-antimetric generator for H_n of (1) cardinality 1 due to Claim 2, (2) cardinality 2 due to Claims 4 and 5, and (3) cardinality $t \geq 3$ due to Claim 8. Since a 1-antimetric generator for H_n of cardinality 1 is the smallest one, so $a\dim_1(H_n) = 1$.

(iii) Claim 6 assures the existence of a 2-antimetric generator for H_n of cardinality 3 for all values of n, while Claim 7 assures the existence of a 2-antimetric generator for H_n of cardinality 5 just for even values of n. Moreover, no singleton set or 2-element set of vertices in H_n is a 2-antimetric generator for H_n due to Claims 1 to 5. It follows that $a\dim_2(H_n) = 3$.

(iv) We receive a 3–antimetric generator of cardinality 2 from Claim 4, and no singleton set is a 3-antimetric generator for H_n due to Claims 1 to 3, which implies that $a\dim_3(H_n) = 2$.

(v) Claim 3 assists that $a\dim_4(H_n) = 1$ because of the existence of a 4-antimetric generator for H_n of cardinality 1.

(vi) Finally, $a\dim_n(H_n) = 1$ due to an n-antimetric generator for H_n of cardinality 1 exists by Claim 1. □

3.3. Flower Graphs. For $n \geq 3$, a flower graph, F_n, is obtained from a helm graph H_n by joining its each leaf u_i to the vertex v of K_1. Accordingly, the vertex set of a flower graph is $V(F_n) = V(H_n)$, and its edge set is $E(F_n) = E(H_n) \cup \{v \sim u_i;\ 1 \leq i \leq n\}$, where the indices greater than n or less than 1 will be taken modulo n. Figure 4 provides graphical appearance of one flower graph.

The following observation is easy to understand for the flower graph F_3.

Observation 4

$$a\dim_k(F_3) = \begin{cases} 1, & \text{for } k = 2, 6, \\ 2, & \text{for } k = 1. \end{cases} \tag{34}$$

Theorem 4. *For all $n \geq 4$, let F_n be a flower graph. Then,*

$$a\dim_k(F_n) = \begin{cases} 1, & \text{for } k = 2, 4, 2n, \\ 2, & \text{for } k = 1, 3. \end{cases} \tag{35}$$

Proof. The following listed neighborhoods of the vertices of F_n will be used in the proof: $N(v) = V \cup U$ and $N(v_i) = \{v, v_{i+1}, v_{i-1}, u_i\}$, for any $v_i \in V$, and $N(u_i) = \{v, v_i\}$, for any $u_i \in U$. We have to discuss the following nine claims to prove the required result.

Claim 1: the set $S = \{v\}$ is a $2n$-antimetric generator for F_n.

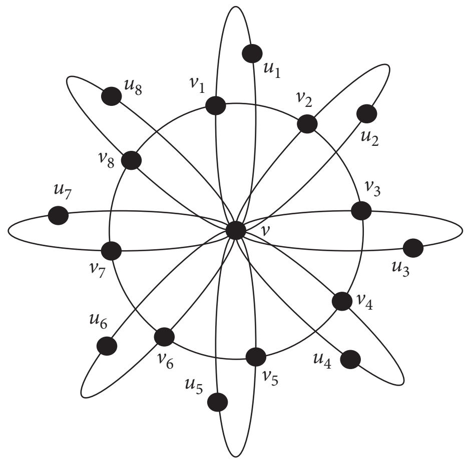

Figure 4: One flower graph F_8.

Note that $c_S(x) = (1)$, for all $x \in V(F_n)$. So, the only one equivalence class of cardinality $2n$ is produced by the relation ρ_S. Hence, S is a $2n$-antimetric generator.

Claim 2: every singleton subset of U is a 2–antimetric generator for F_n.

Let $S = \{u_i\} \subset U$ for any fixed $1 \le i \le n$; then, $c_S(x) = (1)$, for all $x \in N(u_i)$, and $c_S(y) = (2)$, for all $y \in V(F_n) - N[u_i]$. The relation ρ_S creates two equivalence classes $S_1 = N(u_i)$ and $S_2 = V(F_n) - N[u_i]$. It follows that $\min_{i=1}^{2}|S_i| = 2$, and Remark 1 proposes that S is a 2-antimetric generator.

Claim 3: every singleton subset of V is a 4-antimetric generator for F_n.

Let $S = \{v_i\} \subset V$ for any fixed $1 \le i \le n$; then, $c_S(x) = (1)$, for all $x \in N(v_i)$, and $c_S(y) = (2)$, for all $y \in V(F_n) - N[v_i]$. Thus, we get two equivalence classes $S_1 = N(v_i)$ and $S_2 = V(F_n) - N[v_i]$ by the relation ρ_S with $\min_{i=1}^{2}|S_i| = 4$. Hence, S is a 4-antimetric generator, by Remark 1.

Claim 4: the set $W = \{v_i, v\} \subset V(F_n)$ is a 3-antimetric generator for F_n.

Note the metric codes with respect to W is as follows: $c_W(x) = (1, 1)$, for all $x \in N(v_i) - \{v\}$, and $c_W(y) = (2, 1)$, for all $y \in V(F_n) - N[v_i]$. Here, the equivalence classes obtained through the relation ρ_W are $S_1 = N(v_i) - \{v\}$ and $S_2 = V(F_n) - N[v_i]$. Hence, $\min_{i=1}^{2}|S_i| = 3$, and Remark 1 implies that W is a 3-antimetric generator.

Claim 5: the set $W' = \{v_i, u\} \subset V(F_n)$ is a 2-antimetric generator for F_n, where $u \in \{u_{i-1}, u_{i+1}\}$.

The metric codes with respect W' are $c_{W'}(x) = (1, 1)$, for all $x \in N(u)$, $c_{W'}(y) = (1, 2)$, for all $y \in N(v_i) - N(u)$, and $c_{W'}(z) = (2, 2)$, for all $z \in V(F_n) - (N[v_i] \cup N[u])$. Accordingly, three equivalence classes $S_1 = N(u)$, $S_2 = N(v_i) - N(u)$, and $S_3 = V(F_n) - (N[v_i] \cup N[u])$ are generated by the relation $\rho_{W'}$ with $\min_{i=1}^{3}|S_i| = 2$. Therefore, Remark 1 provides that W' is a 2–antimetric generator.

Claim 6: any 2-element set, $S \subseteq V(F_n)$, is a 1-antimetric generator for F_n, except the sets W and W' discussed in Claims 4 and 5, respectively. We discuss the following two possibilities:

(1) Either $S \subset V$ or $S \subset U$ or $S = \{v_i, u_j\}$ with $S \ne W'$. Then, $c_S(v) = (1, 1)$ is the unique metric code in F_n.
(2) If $S = \{u_i, v\}$, then $c_S(v_i) = (1, 1)$ is the unique metric code in F_n.

In both the possibilities, we receive at least one singleton equivalence class, in accordance with the relation ρ_S, which implies that $\min_i |S_i| = 1$. Hence, Remark 1 yields that S is a 1-antimetric generator.

Claim 7: the set $M = \{u_i, v_i, v\} \subset V(F_n)$ is a 2-antimetric generator for F_n.

The metric coding with respect to M is

$$\begin{aligned} c_M(x) &= (2, 1, 1)\,\forall\, x \in N(v_i) - \{u_i, v\}; \\ c_M(y) &= (2, 2, 1)\forall\, y \in V(F_n) - \{v_{i-1}, v_{i+1}\}. \end{aligned} \tag{36}$$

It follows that $S_1 = N(v_i) - \{u_i, v\}$ and $S_2 = V(F_n) - N[v_i]$ are the equivalence classes produced by the relation ρ_M. Here, $\min_{i=1}^{2}|S_i| = 2$, which implies that M is a 2-antimetric generator, by Remark 1.

Claim 8: the set $M' = \{v_i, v, f\} \subset V(F_n)$ is a 3-antimetric generator for F_n whenever $f \in V - \{v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}\}$. Otherwise, M' is a 1–antimetric generator for F_n.

$f \in V - \{v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}\}$, and we have the metric codes of the vertices with respect to M' as follows:

$$\begin{aligned} c_{M'}(x) &= (1, 1, 2)\forall\, x \in N(v_i) - \{v\}; \\ c_{M'}(y) &= (2, 1, 1)\forall\, y \in N(f) - \{v\}, \end{aligned} \tag{37}$$

$$c_{M'}(z) = (2, 1, 2)\,\forall\, z \in V(F_n) - (N[v_i] \cup N[f]). \tag{38}$$

Thus, the relation $\rho_{M'}$ partitioned $V(F_n) - M'$ into three equivalence classes $S_1 = N(v_i) - \{v\}$, $S_2 = N(f) - \{v\}$, and $S_3 = V(F_n) - (N[v_i] \cup N[f])$, with $\min_{i=1}^{3}|S_i| = 3$. It follows, by Remark 1, that M' is a 3-antimetric generator.

Whenever $f \in \{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\}$, if $f = v_{i+1}$ or v_{i-1}, then a neighbor of f lying in U has the unique metric code $(2, 1, 1)$ with respect to M'. If $f = v_{i+2}$ or v_{i-2}, then v_{i+1} or v_{i-1}, respectively, has the unique metric code $(1, 1, 1)$ with respect to M'. In either cases, we obtain at least one singleton equivalence class according to the relation $\rho_{M'}$, which implies that M' is a 1-antimetric generator, by Remark 1.

Claim 9: any set $S \subseteq V(F_n)$ of cardinality $t \ge 3$ is a 1–antimetric generator for F_n, except the sets M and M' discussed in Claims 7 and 8, respectively.

We discuss the following two cases:

Case 1 (S does not contain v): in this case, the vertex v has the unique metric code with respect to S

Case 2 (S contains v): since $S \neq M, M'$, there exists a vertex $x \in N(s)$ for at least one $s \in S - \{v\}$ such that x has the unique metric code with respect to S

In both the cases, we get at least one singleton equivalence class according to the relation ρ_S, which implies that S is a 1−antimetric generator, by Remark 1.

We conclude the proof by discussing the following points using preceding claims:

(i) For $k \in \{5, 6, \ldots, 2n-1\}$, there does not exist a k-antimetric generator for F_n.

(ii) We get an 1-antimetric generator for F_n of (1) cardinality 2 by Claim 6 and (2) cardinality $t \geq 3$ by Claims 8 and 9. Furthermore, no singleton set possesses the property of 1−antimetric generator in F_n, by Claims 1 to 3. It follows that $adim_1(F_n) = 2$.

(iii) For F_n, Claim 2 promises the existence of a 2-antimetric generator of cardinality 1, Claim 5 promises the existence of a 2-antimetric generator of cardinality 2, and Claim 7 promises the existence of a 2-antimetric generator of cardinality 3. All these promises conclude that $adim_2(F_n) = 1$.

(iv) There exists a 3-antimetric generator for F_n of cardinality 2 due to Claim 4, and a 3-antimetric generator of cardinality 3 due to Claim 8. Thus, Claims 1 to 3 conclude that $adim_3(F_n) = 2$.

(v) Claim 3 declares the existence of a 4-antimetric generator for F_n of cardinality 1, which implies that $adim_4(F_n) = 1$.

(vi) A $2n$-antimetric generator for F_n exists due to Claim 1, so $adim_{2n}(F_n) = 1$. □

3.4. Sunflower Graphs. For $n \geq 3$, a sunflower graph, SF_n, is obtained from a wheel graph $W_{1,n} = K_1 + C_n$ by attaching one vertex u_i to every two consecutive vertices of the cycle C_n. Let $U = \{u_1, u_2, \ldots, u_n\}$; then, the vertex set of a sun flower graph is $V(SF_n) = V(W_{1,n}) \cup U$ and its edge set is $E(SF_n) = E(W_{1,n}) \cup \{u_i \sim v_i, u_i \sim v_{i+1}\ ;\ 1 \leq i \leq n\}$, where the indices greater than n or less than 1 will be taken modulo n. A graphical preview of this graph is displayed in Figure 5.

The following observation is an easy exercise to understand.

Observation 5. When $n \in \{3, 5, 6\}$, $adim_k(SF_n) = 1$, for $k = 1, 2, \ldots, n$. While

$$adim_k(SF_4) = \begin{cases} 1, & \text{for } k = 1, 3, 4, \\ 3, & \text{for } k = 2, \end{cases} \tag{39}$$

$$adim_k(SF_7) = \begin{cases} 1, & \text{for } k = 2, 3, 7, \\ 2, & \text{for } k = 1, \end{cases} \tag{40}$$

$$adim_k(SF_8) = \begin{cases} 1, & \text{for } k = 2, 4, 8, \\ 2, & \text{for } k = 1. \end{cases} \tag{41}$$

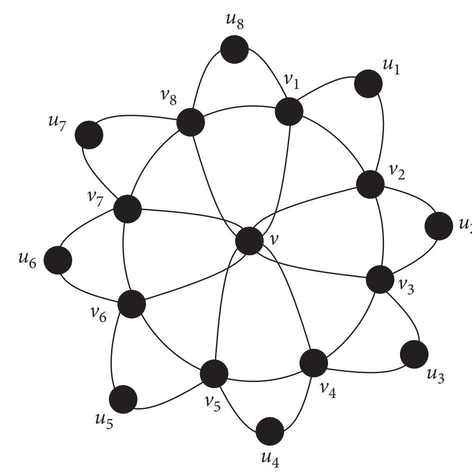

FIGURE 5: One sunflower graph SF_8.

Theorem 5. *For $n \geq 9$, let SF_n be a sunflower graph. Then,*

$$adim_k(SF_n) = \begin{cases} 1, & \text{for } k = 2, 5, n, \\ 2, & \text{for } k = 1. \end{cases} \tag{42}$$

Proof. The neighborhoods, $N(v) = V$ and $N(v_i) = \{v, v_{i+1}, v_{i-1}, u_i, u_{i-1}\}$ for any $v_i \in V$, and $N(u_i) = \{v_i, v_{i+1}\}$, for any $u_i \in U$, of the vertices in SF_n are useful to discuss the following nine claims.

Claim 1: the set $S = \{v\}$ is an n-antimetric generator for SF_n.

Note that $c_S(x) = (1)$, for all $x \in V$, and $c_S(y) = (2)$, for all $y \in U$. Thus, there are two equivalence classes V and U according to the relation ρ_S, and each class has n elements. Hence, Remark 1 yields that S is an n-antimetric generator.

Claim 2: every singleton subset S of V is a 5−antimetric generator for SF_n.

Let $S = \{v_i\} \subset V$ for any fixed $1 \leq i \leq n$. Then, $c_S(x) = (1)$, for all $x \in N(v_i)$:

$$\begin{aligned} &c_S(y) = (3) \forall\, y \in U - \{u_{i-2}, u_{i-1}, u_i, u_{i+1}\}; \\ &c_S(z) = (2) \forall\, z \in \{u_{i-2}, u_{i+1}\} \cup (V - \{v_{i-1}, v_i, v_{i+1}\}). \end{aligned} \tag{43}$$

Therefore, the equivalence classes, corresponding to the relation ρ_S, are $S_1 = N(v)$, $S_2 = \{u_{i-2}, u_{i+1}\} \cup (V - \{v_{i-1}, v_i, v_{i+1}\})$, and $S_3 = U - \{u_{i-2}, u_{i-1}, u_i, u_{i+1}\}$. It follows that $\min_{i=1}^{3} |S_i| = 5$, and S is a 5-antimetric generator, by Remark 1.

Claim 3: every singleton subset S of U is a 2−antimetric generator for SF_n.

Let $S = \{u_i\} \subset U$, for any fixed $1 \leq i \leq n$. Then,

$$c_S(q) = (1)\forall q \in N(u_i); \quad c_S(x) = (2)\forall x \in \{v\} \cup \{v_{i-1}, u_{i-1}, u_{i+1}, v_{i+2}\}, \tag{44}$$

$$c_S(y) = (3)\forall y \in \{u_{i-2}, u_{i+2}\} \cup (V - \{v_{i-1}, v_i, v_{i+1}, v_{i+2}\}), \tag{45}$$

$$c_S(z) = (4), \forall z \in U - \{u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}\}. \tag{46}$$

We have four equivalence classes $S_1 = N(u_i)$, $S_2 = \{v\} \cup \{v_{i-1}, u_{i-1}, u_{i+1}, v_{i+2}\}$, $S_3 = \{u_{i-2}, u_{i+2}\} \cup (V - \{v_{i-1}, v_i, v_{i+1}, v_{i+2}\})$, and $S_4 = U - \{u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}\}$, in accordance with the relation ρ_S. Thus, $\min_{i=1}^4 |S_i| = 2$, which implies that S is a 2-antimetric generator, by Remark 1.

Claim 4: the set $S = \{v_i, v\} \subset V(\mathrm{SF}_n)$ is a 2-antimetric generator for SF_n.

The metric coding with respect to S is listed as follows:

$$c_S(u_{i-1}) = (1,2) = c_S(u_i); \; c_S(v_{i-1}) = (1,1) = c_S(v_{i+1}); \; c_S(u_{i-2}) = (2,2) = c_S(u_{i+1}), \tag{47}$$

$$c_S(x) = (2,1)\forall x \in V - \{v_{i-1}, v_i, v_{i+1}\}; \quad c_S(y) = (3,2)\forall y \in U - \{u_{i-2}, u_{i-1}, u_i, u_{i+1}\}. \tag{48}$$

So, the relation ρ_S supplies five equivalence classes $S_1 = \{u_{i-1}, u_i\}$, $S_2 = \{v_{i-1}, v_{i+1}\}$, $S_3 = \{u_{i-2}, u_{i+1}\}$, $S_4 = V - \{v_{i-1}, v_i, v_{i+1}\}$, and $S_5 = U - \{u_{i-2}, u_{i-1}, u_i, u_{i+1}\}$. Hence, $\min_{i=1}^5 |S_i| = 2$, and S is a 2-antimetric generator, by Remark 1.

Claim 5: the set $S = \{u_i, v\} \subset V(\mathrm{SF}_n)$ is a 2-antimetric generator for SF_n.

We have the following metric coding with respect to S: $c_S(t) = (1,1)$, for all $t \in N(u_i)$:

$$c_S(u_{i-1}) = (2,2) = c_S(u_{i+1}); \; c_S(v_{i-1}) = (2,1) = c_S(v_{i+2}); \; c_S(u_{i-2}) = (3,2) = c_S(u_{i+2}), \tag{49}$$

$c_S(x) = (3,1)$, for all $x \in V - (N(u_i) \cup \{v_{i-1}, v_{i+2}\})$, and $c_S(y) = (4,2)$, for all $y \in U - \{u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}\}$. So, the equivalence classes, in accordance with the relation ρ_S, are $S_1 = N(u_i)$, $S_2 = \{u_{i-1}, u_{i+1}\}$, $S_3 = \{v_{i-1}, v_{i+2}\}$, $s_4 = \{u_{i-2}, u_{i+2}\}$, $S_5 = V - (N(u_i) \cup \{v_{i-1}, v_{i+2}\})$, and $S_6 = U - \{u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}\}$. Here, $\min_{i=1}^6 |S_i| = 2$, which implies that S is a 2-antimetric generator, by Remark 1.

Claim 6: each 2-element set $S \subset V(\mathrm{SF}_n) - \{v\}$ is a 1−antimetric generator for SF_n.

We discuss the following three possibilities:

(1) Let $S = \{u, v\}$ for $u \in U$ and $v \in V$. If either $d(u,v) = 1$ or $d(u,v) = 2$, then there is a vertex p in V such that $p \in N(u) \cap N(v)$, and $c_S(p) = (1,1) \neq c_S(p')$, for any $p' \in V(\mathrm{SF}_n) - \{p\}$. If $d(u,v) = 3$, then there is a neighbor q of v from U such that $c_S(q) = (4,1) \neq c_S(q')$, for any $q' \in V(\mathrm{SF}_n) - \{q\}$. If $d(u,v) \geq 4$, then the vertex v has the unique metric code $(2,1)$ with respect to S.

(2) Let $S \subset V$ and $S = \{v, v'\}$. Then, either $d(v,v') = 1$ or $d(v,v') = 2$. In the former case, a vertex $u \in U$, for which $d(u,v) = 2$ and $d(u,v') = 3$, has the unique metric code $(2,3)$ with respect to S. In the later case, we have two discussions: If $N(v) \cap N(v') = \{v''\} \subset V$, then a vertex $u \in U$, such that $d(u,v'') = 2$, has the unique metric code from the set $\{(1,3),(3,1)\}$ with respect to S. If no vertex in V is a common neighbor of v and v', then the vertex v has the unique metric code $(1,1)$ with respect to S.

(3) Let $S \subset U$; then, $c_S(v) = (2,2)$ is the unique metric code in SF_n. In all these possibilities, we get at least one singleton equivalence class according to the relation ρ_S, which implies that $\min_i |S_i| = 1$. Hence, S is a 1-antimetric generator, by Remark 1.

Claim 7: the set $E = \{v_i, v, x\} \subset V(\mathrm{SF}_n)$ is a 2-antimetric generator of SF_n whenever $x \in V - \{v_{i-3}, v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$. Otherwise, E is a 1-antimetric generator. Whenever $x \in V - \{v_{i-3}, v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$, note that $c_E(u_{i-2}) = (2,2,3) = c_E(u_{i+1})$ and

$$c_E(u_{i-1}) = (1,2,3) = c_E(u_i), \text{ where } u_{i-1}, u_i \in N(v_i) - \{v_{i-1}, v, v_{i+1}\}, \tag{50}$$

$$c_E(v_{i-1}) = (1,1,2) = c_E(v_{i+1}), \text{ where } v_{i-1}, v_{i+1} \in N(v_i) - \{v, u_{i-1}, u_i\}, \tag{51}$$

$$c_E(h) = (3,2,2) = c_E(h'), \text{where } h, h' \in U \text{ with } d(h,x) = d(h',x) = 2, \tag{52}$$

$$c_E(g) = (2,1,1) = c_E(g'), \text{where } g, g' \in V \text{ with } d(g,x) = d(g',x) = 1, \tag{53}$$

$$c_E(l) = (3,2,1) = c_E(l'), \text{where } l, l' \in U \text{ with } d(l,x) = d(l',x) = 1, \tag{54}$$

$$c_E(y) = (2,1,2)\forall y \in V - \{v_{i-1}, v_i, v_{i+1}, g, x, g'\}, \tag{55}$$

$$c_E(z) = (3,2,3)\forall z \in U - \{u_{i-2}, u_{i-1}, u_i, u_{i+1}, h, h', l, l'\}. \tag{56}$$

Hence, we have eight equivalence classes $S_1 = \{u_{i-1}, u_i\}$, $S_2 = \{v_{i-1}, v_{i+1}\}$, $S_3 = \{u_{i-2}, u_{i+1}\}$, $S_4 = \{h, h'\}$, $S_5 = \{l, l'\}$, $S_6 = \{g, g'\}$, $s_7 = V - (S_2 \cup S_6)$, and $S_8 = U - (S_1 \cup S_3 \cup S_4 \cup S_5)$ in accordance with the relation ρ_E. It can be seen that $\min_{i=1}^8 |S_i| = 2$, which yields that E is a 2-antimetric generator, by Remark 1.

Whenever $x \in \{v_{i-3}, v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}, v_{i+3}\}$, we have a vertex $u \in U$ such that $c_E(u) \neq c_E(u')$ for any $u' \in V(\mathrm{SF}_n) - \{u\}$. Hence, we receive at least one singleton equivalence class due to the relation ρ_E, which implies that E is a 1-antimetric generator, by Remark 1.

Claim 8: the set $E' = \{u_i, v, a\} \subset V(\mathrm{SF}_n)$ is a 2-antimetric generator for SF_n whenever $a \in U - \{u_{i-4}, u_{i-3}, u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}\}$. Otherwise, E' is a 1-antimetric generator.

Whenever $a \in U - \{u_{i-4}, u_{i-3}, u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}\}$, we have the metric codes with respect to E' as follows:

$$c_{E'}(t) = (1,1,3)\forall t \in N(u_i);\quad c_{E'}(u_{i-2}) = (3,2,4) = c_{E'}(u_{i+2}), \tag{57}$$

$$c_{E'}(u_{i-1}) = (2,2,4) = c_{E'}(u_{i+1});\quad c_{E'}(v_{i-1}) = (2,1,3) = c_{E'}(v_{i+2}), \tag{58}$$

$$c_{E'}(p) = (3,1,1) = c_{E'}(p'), \text{where } p, p' \in N(a), \tag{59}$$

$$c_{E'}(q) = (4,2,2) = c_{E'}(q'), \text{ where } q, q' \in U \text{ such that } \mathrm{d}(q,a) = \mathrm{d}(q',a) = 2, \tag{60}$$

$$c_{E'}(r) = (3,1,2) = c_{E'}(r'), \text{where } r, r' \in V \text{ such that } \mathrm{d}(r,a) = \mathrm{d}(r',a) = 2, \tag{61}$$

$$c_{E'}(x) = (4,2,3) = c_{E'}(x'), \text{where } x, x' \in U \text{ such that } \mathrm{d}(x,a) = \mathrm{d}(x',a) = 3, \tag{62}$$

$$c_{E'}(y) = (3,1,3)\forall y \in V - (N(u_i) \cup N(a) \cup \{v_{i-1}, v_{i+2}, r, r'\}), \tag{63}$$

$$c_{E'}(z) = (4,2,4)\forall z \in U - \{u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}, x, q, l, q', x'\}. \tag{64}$$

Therefore, we get 10 equivalence classes $S_1 = N(u_i)$, $S_2 = \{u_{i-1}, u_{i+1}\}$, $S_3 = \{v_{i-1}, v_{i+2}\}$, $S_4 = \{u_{i-2}, u_{i+2}\}$, $S_5 = N(a)$, $S_6 = \{q, q'\}$, $S_7 = \{r, r'\}$, $S_8 = \{x, x'\}$, $S_9 = V - (S_1 \cup S_3 \cup S_5 \cup S_7)$, and $S_{10} = U - (S_2 \cup S_4 \cup S_6 \cup S_8)$ in accordance with the relation $\rho_{E'}$. It has been observed that $\min_{i=1}^{10}|S_i| = 2$, so E' is a 2–antimetric generator, by Remark 1.

Whenever $a \in \{u_{i-4}, u_{i-3}, u_{i-2}, u_{i-1}, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}\}$, we have a vertex $u \in U$ such that $c_{E'}(u) \neq c_{E'}(u')$, for any $u' \in V(\mathrm{SF}_n) - \{u\}$. Hence, we receive at least one singleton equivalence class by the relation $\rho_{E'}$, which implies that $\min_i|S_i| = 1$. Hence, E' is a 1–antimetric generator, by Remark 1.

Claim 9: except the sets $E, E' \subset V(\mathrm{SF}_n)$ discussed in Claims 7 and 8, respectively, each set $S \subseteq V(\mathrm{SF}_n)$ of cardinality $k \geq 3$ is a 1–antimetric generator for SF_n.

We have to discuss the following two cases:

Case 1 (S contains v): let $|S| \geq 4$ (because the case, when $|S| = 3$, has been discussed in Claims 7 and 8). Then, there a vertex $x \in V$ such that x is a neighbor of some $s \in S$ whenever $S \cap V = \varnothing$, or there a vertex $x \in U$ such that x is a neighbor of some $s \in S$ whenever either $S \cap U = \varnothing$ or $S \cap V \neq \varnothing \neq S \cap U$, and we get the unique metric code of x with respect to S.

Case 2 (S does not contain v): whenever $S \subseteq V$ or $S \subseteq U$, the vertex v has the unique metric code of x with respect to S. Whenever $S \cap U \neq \varnothing \neq S \cap V$, there is a vertex $x \in U$ (or $x \in V$) such that $\mathrm{d}(x,s) = 1$ for some element $s \in S$, and $c_S(x) \neq c_S(x')$, for any $x' \in V(\mathrm{SF}_n) - \{x\}$.

In both the cases, the relation ρ_S supplies at least one singleton equivalence class, which yields that $\min_i |S_i| = 1$. Hence, S is a 1-antimetric generator, by Remark 1.

These claims complete the proof with the following deductions:

(i) There does not exist a k–antimetric generator for SF_n when $k \in \{3, 4, 6, 7, \ldots, n-1\}$.

(ii) Claim 6 supplies a 1–antimetric generator for SF_n of cardinality 2, and Claims 7 to 9 supply a 1–antimetric generator for SF_n of cardinality $t \geq 3$. Claims 1 to 3 provide the guaranty of nonexistence of singleton 1–antimetric generator for SF_n. It follows that $a\dim_1(\mathrm{SF}_n) = 2$.

(iii) The existence of a 2–antimetric generator for SF_n of cardinalities 1, 2, and 3 is assured by Claim 3, by Claims 4 and 5, and by Claims 7 and 8, respectively. Accordingly, $a\dim_2(\mathrm{SF}_n) = 1$.

(iv) $a\dim_5(\mathrm{SF}_n) = 1$ because of the existence of a 5–antimetric generator for SF_n of cardinality 1 in Claim 2.

(v) Claim 1 provides an n–antimetric generator for SF_n of cardinality 1, which yields that $a\dim_n(\mathrm{SF}_n) = 1$. □

4. Concluding Remarks

For a connected graph G, the number $\mathrm{rad}(G) = \min\{\mathrm{ecc}(x) = \max_{y \in V(G)} \mathrm{d}(x,y);\ x \in V(G)\}$ is called the radius of G, where $\mathrm{ecc}(x)$ is the eccentricity of x. The center of G is a subgraph $G[X]$ induced by the set $X = \{x \in V(G):\ \mathrm{ecc}(x) = \mathrm{rad}(G)\}$. It has been observed the following useful properties about a k–antimetric dimensional graph in [7].

Remarks 2 (see [7])

(a) If a connected graph is k–metric antidimensional, then $1 \leq k \leq \Delta$

(b) If the center of a connected graph is trivial, then it is k–metric antidimensional for some $k \geq 2$

It can be easily seen that each wheel-related social graph, considered in this paper, has the trivial center. Remark 2 (a) insures that each of these graphs has k−metric antidimension, for some $k \geq 1$, and it must be k−metric antidimensional for some $k \geq 2$. So, naturally, it raises the following two questions:

Q1. For how many and for which values of $k \geq 1$ a wheel-related social graph admits k−metric antidimension?

Q2. For which maximum value of k, $2 \leq k \leq \Delta$, a wheel-related social graph is k−metric antidimensional?

The results of Čangalović et al., proved in [1] and listed in Observation 1 and Theorem 1, were the pioneers to address the answers of questions Q1 and Q2. These results revealed that (1) when $n = 3, 5$, a wheel graph $W_{1,n}$ admits k−metric antidimension for three values of $k \in \{1, 2, n\}$ and (2) when $n = 4$ and for all $n \geq 6$, $W_{1,n}$ admits k−metric antidimension for four values of $k \in \{1, 2, 3, n\}$. It follows that $W_{1,n}$ is Δ−metric antidimensional.

To extend the study of (k, l)−anonymity based on the k−metric antidimension, we considered four graphs related to wheel graphs in this article. By investigating their k−metric antidimension, we addressed the answers of questions Q1 and Q2 as follows:

(i) For a Jahangir graph J_{2n}, Observation 2 and Theorem 2 revealed that (1) J_4 admits k−metric antidimension for three values of $k \in \{1, 2, 3\}$, (2) when $n = 3, 4$, J_{2n} admits k−metric antidimension for three values of $k \in \{1, 2, n\}$, and (3) for all $n \geq 5$, J_{2n} admits k−metric antidimension for four values of $k \in \{1, 2, 3, n\}$

(ii) For a helm graph H_n, Observation 3 and Theorem 3 revealed that (1) when $n = 3, 4, 5$, H_n admits k−metric antidimension for three values of $k \in \{1, 2, n\}$, (2) H_6 admits k−metric antidimension for four values of $k \in \{1, 2, 3, 6\}$, and (3) for all $n \geq 7$, H_n admits k−metric antidimension for five values of $k \in \{1, 2, 3, 4, n\}$

(iii) For a flower graph F_n, Observation 4 and Theorem 4 revealed that (1) F_3 admits k−metric antidimension for three values of $k \in \{1, 2, 6\}$ and (2) for all $n \geq 4$, F_n admits k−metric antidimension for five values of $k \in \{1, 2, 3, 4, 2n\}$.

(iv) For a sunflower graph SF_n, Observation 5 and Theorem 5 revealed that (1) when $n = 3, 5, 6$, SF_n admits k−metric antidimension for three values of $k \in \{1, 2, n\}$, (2) when $n = 4, 7$, SF_n admits k−metric antidimension for four values of $k \in \{1, 2, 3, n\}$, (3) SF_8 admits k−metric antidimension for four values of $k \in \{1, 2, 4, 8\}$, and (4) for all $n \geq 9$, SF_n admits k−metric antidimension for four values of $k \in \{1, 2, 5, n\}$.

From all these results, it can be concluded that each considered wheel-related social graph is Δ−metric antidimensional.

Furthermore, according to the computed k−metric antidimension of wheel-related social graphs, we investigated that each of them meets the (k, l)−anonymity in the following ways (skipping particular cases which can be observed straightforwardly).

Wheel graph: for $n \geq 6$ and $\Delta = n$ and by Theorem 1, we have

$$\begin{array}{lccccc} k & : & 1 & 2 & 3 & \Delta \\ k-\text{metric antidimension} & : & 2 & 2 & 1 & 1 \\ (k,l)-\text{anonymity} & : & (1,2) & (2,2) & (3,1) & (\Delta,1) \end{array}. \tag{65}$$

Jahangir graph: for $n \geq 5$ and $\Delta = n$ and by Theorem 2, we have

$$\begin{array}{lccccc} k & : & 1 & 2 & 3 & \Delta \\ k-\text{metric antidimension} & : & 2 & 1 & 1 & 1 \\ (k,l)-\text{anonymity} & : & (1,2) & (2,1) & (3,1) & (\Delta,1) \end{array}. \tag{66}$$

Helm graph: for $n \geq 7$ and $\Delta = n$ and by Theorem 3, we have

$$\begin{array}{lcccccc} k & : & 1 & 2 & 3 & 4 & \Delta \\ k-\text{metric antidimension} & : & 1 & 3 & 2 & 1 & 1 \\ (k,l)-\text{anonymity} & : & (1,1) & (2,3) & (3,2) & (4,1) & (\Delta,1) \end{array}. \tag{67}$$

Flower graph: for $n \geq 4$ and $\Delta = 2n$ and by Theorem 4, we have

$$\begin{array}{lcccccc} k & : & 1 & 2 & 3 & 4 & \Delta \\ k-\text{metric antidimension} & : & 2 & 1 & 2 & 1 & 1 \\ (k,l)-\text{anonymity} & : & (1,2) & (2,1) & (3,2) & (4,1) & (\Delta,1) \end{array}. \tag{68}$$

Sunflower graph: for $n \geq 9$ and $\Delta = n$ and by Theorem 5, we have

$$\begin{array}{llllll} k & : & 1 & 2 & 5 & \Delta \\ k-\text{metric antidimension} & : & 2 & 1 & 1 & 1 \\ (k,l)-\text{anonymity} & : & (1,2) & (2,1) & (3,1) & (\Delta,1) \end{array}. \quad (69)$$

The (k,l)−anonymity, measured on the base of k−metric antidimension for the maximum value of $k = \Delta$, assures that a user can be reidentified with the probability less than or equal $(1/\Delta)$ by a rival controlling only single attacker node v in every considered wheel-related social graph. It is remarkably interesting to leave the following conjecture for the readers.

Conjecture 1. *Each wheel-related social graph and generalizations of wheels are* Δ−*metric antidimensional and meet* $(\Delta, 1)$−*anonymity.*

Data Availability

The figures, tables, and other data used to support this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interests regarding the publication of this paper.

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