Review Article

On State-Dependent Delay Intuitionistic Fuzzy Partial Functional Differential Equations with Integral Boundary Conditions

Bouchra Ben Amma , Said Melliani , and Lalla Saadia Chadli

*Laboratory of Applied Mathematics and Scientific Computing, Faculty of Sciences and Technologies,
Sultan Moulay Slimane University, BP 523, 23000 Beni Mellal, Morocco*

Correspondence should be addressed to Bouchra Ben Amma; bouchrabenamma@gmail.com

Received 10 May 2021; Accepted 14 June 2021; Published 5 July 2021

Academic Editor: Niansheng Tang

Copyright © 2021 Bouchra Ben Amma et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper addresses the issue of the existence and uniqueness of intuitionistic fuzzy solutions for some classes of partial functional differential equations with state-dependent delay in a new weighted complete metric space. Theorems on the existence and uniqueness of intuitionistic fuzzy solutions for these problems with integral boundary conditions are established under some sufficient assumptions. Some numerical examples of applications of the main result of this work are presented.

1. Introduction

In 1965, Zadeh [1] proposed the concept of fuzzy sets with the purpose to model ambiguity, uncertainty, and vagueness in complicated systems. It can be considered as an extension of the usual (crisp) set theory. It has greater pliability to capture different aspects of incompleteness, imperfection, and uncertainty in data about many situations. The membership μ of an element of a fuzzy set is a single value between 0 and 1. Therefore, in reality, it may not always be true that the degree of nonmembership ν of an element in a fuzzy set is equal to 1 minus the membership degree ($1 - \mu$) because there may be some uncertainty degree. Thus, since the fuzzy set has no means to characterize the neutral state, neither support nor oppose, Atanassov [2] included the nonmembership function and defined the degree of uncertainty as $1 - \mu - \nu$. He introduced the topic of intuitionistic fuzzy sets (IFSs) as an extension of the standard fuzzy sets [3]. Several applications of IFS theory in diverse fields have been carried out; they are a very necessary and powerful tool in modeling imprecision; IFSs have been flourished in many different fields, an intuitionistic fuzzy approach to artificial intelligence [4, 5], medical diagnosis [6], drug selection [7], along with pattern recognition [8], microelectronic fault analysis [9], and decision-making problems [10, 11].

The concept of intuitionistic fuzzy differential equations has a rich history and is one of the actively developing topics of intuitionistic fuzzy set theory. The first paper on intuitionistic fuzzy differential equations was by Melliani and Chadli [12], and recently, the authors established some intuitionistic fuzzy solutions for such equations. They proved the existence and uniqueness of intuitionistic fuzzy solutions under some assumptions for these intuitionistic fuzzy differential equations using different procedures, see [13, 14]. The topic of intuitionistic fuzzy functional differential equations with delay is very rare: in [15], they first elaborated conditions that can be satisfied by intuitionistic fuzzy functional differential equations with delay and that guarantee the existence and uniqueness of local and global intuitionistic fuzzy solutions for such equations. Numerical algorithms for treatment of IFDEs are still evolving, see [16–20] for applications.

The notion of intuitionistic fuzzy partial differential equations was first studied in 2000 [21], and intuitionistic fuzzy partial differential equations with integral boundary conditions were first introduced in 2019 [22], where they proved existence and uniqueness results and defined a procedure to calculate the intuitionistic fuzzy solutions of such problems. However, according to our knowledge, research on the properties of solutions of intuitionistic fuzzy

partial functional differential equations is very limited. The study of intuitionistic fuzzy partial differential equations with state-dependent delay has been initiated recently in 2019 [23], in which they considered the existence and uniqueness of intuitionistic fuzzy solutions for these types of equations with local, nonlocal, and integral boundary conditions, based on some complicated assumptions on the

domain. But those hypotheses can be reduced to be milder. In the present paper, our models can be considered as an extension and complementation of the previous results into new weighted metric spaces of intuitionistic fuzzy valued functions. We study state-dependent delay intuitionistic fuzzy partial functional differential equations with integral boundary conditions of the following forms:

$$\left\{ \begin{array}{l} \frac{\partial^2 \langle \phi_1, \phi_2 \rangle (\lambda, \theta)}{\partial \lambda \partial \theta} = G_1(\lambda, \theta, \langle \phi_1, \phi_2 \rangle_{(\lambda, \theta)}), \quad (\lambda, \theta) \in I_c \times I_d, \\ \langle \phi_1, \phi_2 \rangle (\lambda, 0) + \int_0^\theta L_1(\lambda) \langle \phi_1, \phi_2 \rangle (\lambda, \sigma) d\sigma = f_1(\lambda, \theta), \quad (\lambda, \theta) \in I_c \times I_d, \\ \langle \phi_1, \phi_2 \rangle (0, \theta) + \int_0^\lambda L_2(\theta) \langle \phi_1, \phi_2 \rangle (\zeta, \theta) d\zeta = f_2(\lambda, \theta), \quad (\lambda, \theta) \in I_c \times I_d, \end{array} \right. \quad (1)$$

$$\left\{ \begin{array}{l} \frac{\partial^2 \langle \phi_1, \phi_2 \rangle (\lambda, \theta)}{\partial \lambda \partial \theta} = \frac{\partial (q(\lambda, \theta) \langle \phi_1, \phi_2 \rangle (\lambda, \theta))}{\partial \theta} + G_2(\lambda, \theta, \langle \phi_1, \phi_2 \rangle_{(\lambda, \theta)}), \quad (\lambda, \theta) \in I_c \times I_d, \\ \langle \phi_1, \phi_2 \rangle (\lambda, 0) + \int_0^\theta L_1(\lambda) \langle \phi_1, \phi_2 \rangle (\lambda, \sigma) d\sigma = f_1(\lambda, \theta), \quad (\lambda, \theta) \in I_c \times I_d, \\ \langle \phi_1, \phi_2 \rangle (0, \theta) + \int_0^\lambda L_2(\theta) \langle \phi_1, \phi_2 \rangle (\zeta, \theta) d\zeta = f_2(\lambda, \theta), \quad (\lambda, \theta) \in I_c \times I_d, \end{array} \right. \quad (2)$$

where $G_1, G_2: I_c \times I_d \times IF_n \longrightarrow IF_n$ is continuous and $q \in C(I_c \times I_d, \mathbb{R})$, $L_1 \in C(I_c, \mathbb{R})$, $L_2 \in C(I_d, \mathbb{R})$, $f_1 \in C(I_c \times I_d, IF_n)$, and $f_2 \in C(I_c \times I_d, IF_n)$ are given functions.

Motivated by this consideration, first, we shall discuss the existence and uniqueness of the intuitionistic fuzzy solutions to those problems by using some appropriate Banach fixed point theorems under suitable conditions. The main contributions of this work are as follows: introducing the integral boundary value problem for intuitionistic fuzzy partial differential equations with the state-dependent delays and defining their solution; further developing theoretical results on the existence and uniqueness of the intuitionistic fuzzy solution; defining the intuitionistic fuzzy solution to these problems with integral boundary conditions through a corresponding parametric problem.

Delay differential equations have been used in many fields for a long time. Researchers have considered state-dependent delay differential equations (SDDEs) for the last 50 years. However, more precisely integral initial boundary value problems with state-dependent delays constitute a very interesting and important class of applied problems; they have been used in modeling scientific phenomena in various applications in different fields such as biology, neural networks, control theory, physics, and medicine, see [24–33], for details. Linking the two aspects introduced, intuitionistic fuzzy mathematics and partial functional differential equations with state-dependent delays, we obtain intuitionistic fuzzy partial functional differential

equations (IFPFDEs) with state-dependent delays, which will attract great interest both in mathematics and in applications. No result has been reported on the existing models of state-dependent delay IFPFDEs with integral boundary initial conditions, and among the purposes of this work is to fill this gap. This paper initiates the study of such equations: solving state-dependent delay IFPFDEs with integral boundary initial conditions, providing some definitions for intuitionistic fuzzy solutions of such problems, progressing the main results on the existence and uniqueness of intuitionistic fuzzy solutions under suitable hypotheses in new weighted metric spaces, without any restraints in the data, displaying rigorous well-posedness and regularity analysis of the state-dependent delay intuitionistic fuzzy PFDEs, and establishing numerical examples that can be considered as a contribution to the subject of our present work and will be introduced elsewhere.

The rest of the paper is organised as follows: in Section 2, we will recall some basic definitions and preliminary facts which will be used throughout the following sections. In Section 3, we provide the existence and uniqueness of intuitionistic fuzzy solutions for partial functional differential equations with integral boundary conditions under suitable conditions in a new weighted metric space. In Section 4, we illustrate our abstract results by numerical applications. Finally, Section 5 gives the main conclusions and possible future works in this area.

2. Preliminaries

We define IF_n as follows:

$$\begin{aligned} IF_n &= IF(\mathbb{R}^n) \\ &= \{ \langle \phi_1, \phi_2 \rangle : \mathbb{R}^n \longrightarrow [0, 1] \times [0, 1], \\ &\quad |\forall \lambda \in \mathbb{R}^n, 0 \leq \phi_1(\lambda) + \phi_2(\lambda) \leq 1 \}. \end{aligned} \quad (3)$$

We called $\langle \phi_1, \phi_2 \rangle \in IF_n$ as an intuitionistic fuzzy number if it verifies the following assumptions:

- (a) $\langle \phi_1, \phi_2 \rangle$ is normal, i.e., there exists λ_0 and $\lambda_1 \in \mathbb{R}^n$ such that $\phi_1(\lambda_0) = 1$ and $\phi_1 = 2(\lambda_1) = 1$
- (b) ϕ_1 is fuzzy convex, and ϕ_2 is fuzzy concave
- (c) ϕ_1 is upper semicontinuous, and ϕ_2 is lower semicontinuous

- (d) $\text{supp} \langle \phi_1, \phi_2 \rangle = cl\{\lambda \in \mathbb{R}^n : |\phi_2(\lambda) < 1\}$ is bounded

We define the upper and lower α -cuts of $\langle \phi_1, \phi_2 \rangle \in IF_n$ with $\alpha \in [0, 1]$ by the following forms:

$$\begin{aligned} [\langle \phi_1, \phi_2 \rangle]^\alpha &= \{ \lambda \in \mathbb{R}^n : \phi_2(\lambda) \leq 1 - \alpha \}, \\ [\langle \phi_1, \phi_2 \rangle]_\alpha &= \{ \lambda \in \mathbb{R}^n : \phi_1(\lambda) \geq \alpha \}. \end{aligned} \quad (4)$$

We define $0_{(1,0)} \in IF_n$ as

$$0_{(1,0)}(s) = \begin{cases} (1, 0), & s = 0, \\ (0, 1), & s \neq 0. \end{cases} \quad (5)$$

Let $\langle \phi_1, \phi_2 \rangle$ and $\langle \phi'_1, \phi'_2 \rangle \in IF_n$ and $\zeta \in \mathbb{R}$; then,

$$\begin{aligned} (\langle \phi_1, \phi_2 \rangle + \langle \phi'_1, \phi'_2 \rangle)(r) &= \left(\sup_{r=m+n} \min(\phi_1(m), \phi'_1(n)), \inf_{r=m+n} \max(\phi_2(m), \phi'_2(n)) \right), \\ \zeta \langle \phi_1, \phi_2 \rangle &= \begin{cases} \langle \zeta v, \zeta w \rangle, & \text{if } \zeta \neq 0, \\ 0_{(1,0)}, & \text{if } \zeta = 0. \end{cases} \end{aligned} \quad (6)$$

For $\langle \phi_1, \phi_2 \rangle$ and $\langle \phi'_1, \phi'_2 \rangle \in IF_n$ and $\zeta \in \mathbb{R}$, therefore

$$\begin{aligned} [\langle \phi_1, \phi_2 \rangle + \langle \phi'_1, \phi'_2 \rangle]^\alpha &= [\langle \phi_1, \phi_2 \rangle]^\alpha + [\langle \phi'_1, \phi'_2 \rangle]^\alpha, \\ [\zeta \langle \phi_1, \phi_2 \rangle]^\alpha &= \zeta [\langle \phi_1, \phi_2 \rangle]^\alpha, \\ [\langle \phi_1, \phi_2 \rangle + \langle \phi'_1, \phi'_2 \rangle]_\alpha &= [\langle \phi_1, \phi_2 \rangle]_\alpha + [\langle \phi'_1, \phi'_2 \rangle]_\alpha, \\ [\zeta \langle \phi_1, \phi_2 \rangle]_\alpha &= \zeta [\langle \phi_1, \phi_2 \rangle]_\alpha. \end{aligned} \quad (7)$$

Definition 1. Let $\alpha \in [0, 1]$ and $\langle \phi'_1, \phi'_2 \rangle \in IF_n$; the following sets are defined by

$$\begin{aligned} [\langle \phi_1, \phi_2 \rangle]_l^+(\alpha) &= \inf\{\lambda \in \mathbb{R}^n | \phi_1(\lambda) \geq \alpha\}, \\ [\langle \phi_1, \phi_2 \rangle]_r^+(\alpha) &= \sup\{\lambda \in \mathbb{R}^n | \phi_1(\lambda) \geq \alpha\}, \\ [\langle \phi_1, \phi_2 \rangle]_l^-(\alpha) &= \inf\{\lambda \in \mathbb{R}^n | \phi_2(\lambda) \leq 1 - \alpha\}, \\ [\langle \phi_1, \phi_2 \rangle]_r^-(\alpha) &= \sup\{\lambda \in \mathbb{R}^n | \phi_2(\lambda) \leq 1 - \alpha\}. \end{aligned} \quad (8)$$

Remark 1

$$\begin{aligned} [\langle \phi_1, \phi_2 \rangle]_\alpha &= [[\langle \phi_1, \phi_2 \rangle]_l^+(\alpha), [\langle \phi_1, \phi_2 \rangle]_r^+(\alpha)], \\ [\langle \phi_1, \phi_2 \rangle]^\alpha &= [[\langle \phi_1, \phi_2 \rangle]_l^-(\alpha), [\langle \phi_1, \phi_2 \rangle]_r^-(\alpha)]. \end{aligned} \quad (9)$$

Proposition 1 (see [34]). For all α_1 and $\alpha_2 \in [0, 1]$ and $\langle \phi_1, \phi_2 \rangle \in IF_n$,

$$(a) [\langle \phi_1, \phi_2 \rangle]_{\alpha_1} \subset [\langle \phi_1, \phi_2 \rangle]^{\alpha_1}$$

(b) $[\langle \phi_1, \phi_2 \rangle]_{\alpha_1}$ and $[\langle \phi_1, \phi_2 \rangle]^{\alpha_1}$ are nonempty compact convex sets in $\mathbb{R}^n$

(c) if $\alpha_1 \leq \alpha_2$, therefore $[\langle \phi_1, \phi_2 \rangle]_{\alpha_2} \subset [\langle \phi_1, \phi_2 \rangle]_{\alpha_1}$ and $[\langle \phi_1, \phi_2 \rangle]^{\alpha_2} \subset [\langle \phi_1, \phi_2 \rangle]^{\alpha_1}$

(d) If $\alpha_n \nearrow \alpha_1$, therefore $[\langle \phi_1, \phi_2 \rangle]_{\alpha_1} = \bigcap_n [\langle \phi_1, \phi_2 \rangle]_{\alpha_n}$ and $[\langle \phi_1, \phi_2 \rangle]^{\alpha_1} = \bigcap_n [\langle \phi_1, \phi_2 \rangle]^{\alpha_n}$

Let $\alpha \in [0, 1]$; we denote N (any set) as follows:

$$\begin{aligned} N_\alpha &= \{ \lambda \in \mathbb{R}^n : \phi_1(\lambda) \geq \alpha \}, \\ N^\alpha &= \{ \lambda \in \mathbb{R}^n : \phi_2(\lambda) \leq 1 - \alpha \}. \end{aligned} \quad (10)$$

Lemma 1 (see [34]). Let $\{N_\alpha, \alpha \in [0, 1]\}$ and $\{N^\alpha, \alpha \in [0, 1]\}$ be two families of subsets of $\mathbb{R}^n$; verify the assumptions of Proposition 1 if ϕ_1 and ϕ_2 are defined as follows:

$$\begin{aligned} \phi_1(\lambda) &= \begin{cases} 0, & \text{if } \lambda \notin N_0, \\ \sup\{\alpha \in [0, 1] : \lambda \in N_\alpha\}, & \text{if } \lambda \in N_0, \end{cases} \\ \phi_2(\lambda) &= \begin{cases} 1, & \text{if } \lambda \notin N^0, \\ 1 - \sup\{\alpha \in [0, 1] : \lambda \in N^\alpha\}, & \text{if } \lambda \in N^0. \end{cases} \end{aligned} \quad (11)$$

Then, $\langle \phi_1, \phi_2 \rangle \in IF_n$.

We define the following metric by

$$\begin{aligned}
d_{\infty}^n(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle) &= \frac{1}{4} \sup_{0 < \alpha \leq 1} \left\| [\langle \phi_1, \phi_2 \rangle]_r^+(\alpha) - [\langle \phi'_1, \phi'_2 \rangle]_r^+(\alpha) \right\| \\
&+ \frac{1}{4} \sup_{0 < \alpha \leq 1} \left\| [\langle \phi_1, \phi_2 \rangle]_l^+(\alpha) - [\langle \phi'_1, \phi'_2 \rangle]_l^+(\alpha) \right\| \\
&+ \frac{1}{4} \sup_{0 < \alpha \leq 1} \left\| [\langle \phi_1, \phi_2 \rangle]_r^-(\alpha) - [\langle \phi'_1, \phi'_2 \rangle]_r^-(\alpha) \right\| \\
&+ \frac{1}{4} \sup_{0 < \alpha \leq 1} \left\| [\langle \phi_1, \phi_2 \rangle]_l^-(\alpha) - [\langle \phi'_1, \phi'_2 \rangle]_l^-(\alpha) \right\|.
\end{aligned} \tag{12}$$

where $\|\cdot\|$ is the usual Euclidean norm in $\mathbb{R}^n$.

Theorem 1 (see [34]). d_{∞}^n is a metric on IF_n .

Theorem 2 (see [34]). (IF_n, d_{∞}^n) is a complete metric space.

We denote

- (i) $I_c = [0, c]$, $I_d = [0, d]$, $I_0 = [-r, 0] \times [-r, 0]$,
 $I_r = [-r, c] \times [-r, d]$, and $\widehat{I}_r = I_r \setminus (0, c] \times (0, d]$
- (ii) $C(I, IF_n)$: the space of all continuous mappings defined over I_i into IF_n ($i = c, d, 0$, and r)

For $\beta > 0$ arbitrary, it can be shown that $(C(I_r, IF_n), D_{\beta})$ is a complete metric space, where the supremum weighted metric D_{β} on $C(I_r, IF_n)$ is defined by

$$\begin{aligned}
D_{\beta}(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle) \\
= \sup_{(r_1, r_2) \in I_r} \left\{ d_{\infty}^n(\langle \phi_1, \phi_2 \rangle(r_1, r_2), \langle \phi'_1, \phi'_2 \rangle(r_1, r_2)) e^{-\beta(r_1 + r_2)} \right\}.
\end{aligned} \tag{13}$$

Definition 2. A mapping $G_1: I_r \longrightarrow IF_n$ is continuous at point $(\lambda_0, \theta_0) \in I_r$ provided for any arbitrary $\varepsilon > 0$; there exists an $\sigma(\varepsilon)$ such that

$$d_{\infty}^n(G_1(\lambda, \theta), G_1(\lambda_0, \theta_0)) < \varepsilon, \tag{14}$$

whenever $\max\{|\lambda - \lambda_0|, |\theta - \theta_0|\} < \sigma(\varepsilon) \forall (\lambda, \theta) \in I_r$.

Definition 3. A mapping $G_1: I_r \times IF_n \longrightarrow IF_n$ is called continuous at point $(\lambda_0, \theta_0, \langle \phi_1, \phi_2 \rangle_0) \in I_r \times IF_n$ provided for any arbitrary $\varepsilon > 0$; there exists an $\sigma(\varepsilon)$ such that

$$d_{\infty}^n(G_1(\lambda, \theta, \langle \phi_1, \phi_2 \rangle), G_1(\lambda_0, \theta_0, \langle \phi_1, \phi_2 \rangle_0)) < \varepsilon, \tag{15}$$

whenever $\max\{|\lambda - \lambda_0|, |\theta - \theta_0|\} < \sigma(\varepsilon)$ and $d_{\infty}^n(\langle \phi_1, \phi_2 \rangle, \langle \phi_1, \phi_2 \rangle_0) < \sigma(\varepsilon) \forall (\lambda, \theta) \in I_r, \langle \phi_1, \phi_2 \rangle \in IF_n$.

Definition 4. A mapping $G_1: I_c \times I_d \longrightarrow IF_n$ is called strongly measurable if for $\forall \alpha \in [0, 1]$, the set-valued mappings $g_{\alpha}: I_c \times I_d \longrightarrow P_k(\mathbb{R}^n)$ defined by $g_{\alpha}(\lambda, \theta) = [G_1(\lambda, \theta)]_{\alpha}$ and $g^{\alpha}: I_c \times I_d \longrightarrow P_k(\mathbb{R}^n)$ defined by $g^{\alpha}(\lambda, \theta) = [G_1(\lambda, \theta)]^{\alpha}$ are (Lebesgue) measurable.

Definition 5. $G_1: I_c \times I_d \longrightarrow IF_n$ is integrably bounded if there exists an integrable function $h: I_c \times I_d \longrightarrow \mathbb{R}^n$ such that $\|(\zeta, \sigma)\| \leq h(\lambda, \theta)$ holds for any $(\zeta, \sigma) \in \text{supp}(G_1(\lambda, \theta))$, $(\lambda, \theta) \in I_c \times I_d$.

Definition 6. Suppose $G_1: I_c \times I_d \longrightarrow IF_n$ is integrably bounded and strongly measurable for each $\alpha \in (0, 1]$; then,

$$\begin{aligned}
\left[\int_0^c \int_0^d G_1(\lambda, \theta) d\theta d\lambda \right]_{\alpha} &= \int_0^c \int_0^d [G_1(\lambda, \theta)]_{\alpha} d\theta d\lambda \\
&= \left\{ \int_0^c \int_0^d G(\lambda, \theta) d\theta d\lambda \mid G: I_c \times I_d \longrightarrow \mathbb{R}^n \text{ is a measurable selection for } G_{1\alpha} \right\}, \\
\left[\int_0^c \int_0^d G_1(\lambda, \theta) d\theta d\lambda \right]^{\alpha} &= \int_0^c \int_0^d [G_1(\lambda, \theta)]^{\alpha} d\theta d\lambda \\
&= \left\{ \int_0^c \int_0^d G(\lambda, \theta) d\theta d\lambda \mid G: I_c \times I_d \longrightarrow \mathbb{R}^n \text{ is a measurable selection for } G_1^{\alpha} \right\}.
\end{aligned} \tag{16}$$

Let $G_1, G_2: I_c \times I_d \longrightarrow IF_n$ be integrable and $k \in \mathbb{R}$. The integral has the elementary properties as follows:

$$\begin{aligned}
(i) \int_0^c \int_0^d [G_1(\lambda, \theta) + G_2(\lambda, \theta)] d\theta d\lambda &= \int_0^c \int_0^d G_1(\lambda, \theta) d\theta d\lambda \\
&+ \int_0^c \int_0^d G_2(\lambda, \theta) d\theta d\lambda
\end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad & \int_0^c \int_0^d k G_1(\lambda, \theta) d\theta d\lambda = k \int_0^c \int_0^d G_1 G_1(\lambda, \theta) d\theta d\lambda \\ \text{(iii)} \quad & d_\infty^n \left(\int_0^c \int_0^d G_1(\lambda, \theta) d\theta d\lambda, \int_0^c \int_0^d G_2(\lambda, \theta) d\theta d\lambda \right) \leq \\ & \int_0^c \int_0^d d_\infty^n(G_1(\lambda, \theta), G_2(\lambda, \theta)) d\theta d\lambda \end{aligned}$$

Definition 7. Let $\langle \phi_1, \phi_2 \rangle$ and $\langle \phi'_1, \phi'_2 \rangle \in \text{IF}_n$; we call the Hukuhara difference as an intuitionistic fuzzy number $\langle \phi_1, \phi_2 \rangle \in \text{IF}_n$; if it exists, such that

$$\langle \phi_1, \phi_2 \rangle - \langle \phi'_1, \phi'_2 \rangle = \langle \phi_1, \phi_2 \rangle \iff \langle \phi_1, \phi_2 \rangle = \langle \phi'_1, \phi'_2 \rangle + \langle \phi_1, \phi_2 \rangle. \quad (17)$$

Definition 8. Let $G_1: [c, d] \longrightarrow \text{IF}_n$ be differentiable, and suppose that the derivative g' is integrable over $[c, d]$; if the Hukuhara difference $G_1(d) - G_1(c)$ exists in IF_n , then we have $\int_c^d G_1'(t) dt = G_1(d) - G_1(c)$.

Definition 9. Let $G_1: I_c \times I_d \longrightarrow \text{IF}_n$. The intuitionistic fuzzy partial derivative of g with respect to x at the point $(\lambda_0, \theta_0) \in I_c \times I_d$ is the intuitionistic fuzzy quantity

$(\partial G_1(\lambda_0, \theta_0)/\partial \lambda) \in \text{IF}_n$ such that $\forall \Delta t > 0$ sufficiently small, the H-differences $G_1(\lambda_0 + \Delta t, \theta_0) - G_1(\lambda_0, \theta_0)$ and $G_1(\lambda_0, \theta_0) - G_1(\lambda_0 - \Delta t, \theta_0)$ exist in IF_n and the limits

$$\begin{aligned} \frac{\partial G_1(\lambda_0, \theta_0)}{\partial \lambda} &= \lim_{\Delta t \rightarrow 0^+} \frac{G_1(\lambda_0 + \Delta t, \theta_0) - G_1(\lambda_0, \theta_0)}{\Delta t}, \\ \frac{\partial G_1(\lambda_0, \theta_0)}{\partial \lambda} &= \lim_{\Delta t \rightarrow 0^+} \frac{G_1(\lambda_0, \theta_0) - G_1(\lambda_0 - \Delta t, \theta_0)}{\Delta t}. \end{aligned} \quad (18)$$

The intuitionistic fuzzy partial derivative of g with respect to θ at the point $(\lambda_0, \theta_0) \in I_c \times I_d$ and higher order of intuitionistic fuzzy partial derivative of g are defined similarly.

3. The Main Results

We prove the existence-uniqueness of the intuitionistic fuzzy solution for the following intuitionistic fuzzy functional partial differential equation with integral boundary conditions:

$$\left\{ \begin{aligned} & \frac{\partial^2 \langle \phi_1, \phi_2 \rangle(\lambda, \theta)}{\partial \lambda \partial \theta} = G_1(\lambda, \theta, \langle \phi_1, \phi_2 \rangle_{(\lambda, \theta)}), & (\lambda, \theta) \in I_c \times I_d := [0, c] \times [0, d], \\ & \langle \phi_1, \phi_2 \rangle(\lambda, \theta) = \eta(\lambda, \theta), & (\lambda, \theta) \in \widehat{I}_r, \\ & \langle \phi_1, \phi_2 \rangle(\lambda, 0) + \int_0^\theta L_1(\lambda) \langle \phi_1, \phi_2 \rangle(\lambda, \sigma) d\sigma = f_1(\lambda, \theta), & (\lambda, \theta) \in I_c \times I_d, \\ & \langle \phi_1, \phi_2 \rangle(0, \theta) + \int_0^\lambda L_2(\theta) \langle \phi_1, \phi_2 \rangle(\zeta, \theta) d\zeta = f_2(\lambda, \theta), & (\lambda, \theta) \in I_c \times I_d. \end{aligned} \right. \quad (19)$$

Assume that $G_1: I_r \times C(I_0, \text{IF}_n) \longrightarrow \text{IF}_n$ and $\eta \in C(I_0, \text{IF}_n)$ is continuous; $L_1 \in C(I_c, \mathbb{R})$, $L_2 \in C(I_d, \mathbb{R})$, $f_1 \in C(I_c \times I_d, \text{IF}_n)$, and $f_2 \in C(I_c \times I_d, \text{IF}_n)$ are given functions.

For each $(\lambda, \theta) \in I_c \times I_d$, we define the state-dependent delays $\langle \phi_1, \phi_2 \rangle_{(\lambda, \theta)}(r_1, r_2)$ by

$$\langle \phi_1, \phi_2 \rangle_{(\lambda, \theta)}(r_1, r_2) = \langle \phi_1, \phi_2 \rangle(\lambda + r_1, \theta + r_2), \quad (r_1, r_2) \in I_0. \quad (20)$$

In the 2nd part of this section, we will prove the existence-uniqueness of intuitionistic fuzzy solution for the following state-dependent delay intuitionistic fuzzy functional partial differential equation of the general model:

$$\left\{ \begin{aligned} & \frac{\partial^2 \langle \phi_1, \phi_2 \rangle(\lambda, \theta)}{\partial \lambda \partial \theta} = \frac{\partial (q(\lambda, \theta) \langle \phi_1, \phi_2 \rangle(\lambda, \theta))}{\partial \theta} + G_2(\lambda, \theta, \langle \phi_1, \phi_2 \rangle_{(\lambda, \theta)}), & (\lambda, \theta) \in I_c \times I_d := [0, c] \times [0, d], \\ & \langle \phi_1, \phi_2 \rangle(\lambda, \theta) = \eta(\lambda, \theta), & (\lambda, \theta) \in \widehat{I}_r, \\ & \langle \phi_1, \phi_2 \rangle(\lambda, 0) + \int_0^\theta L_1(\lambda) \langle \phi_1, \phi_2 \rangle(\lambda, \sigma) d\sigma = f_1(\lambda, \theta), & (\lambda, \theta) \in I_c \times I_d, \\ & \langle \phi_1, \phi_2 \rangle(0, \theta) + \int_0^\lambda L_2(\theta) \langle \phi_1, \phi_2 \rangle(\zeta, \theta) d\zeta = f_2(\lambda, \theta), & (\lambda, \theta) \in I_c \times I_d, \end{aligned} \right. \quad (21)$$

where $G_2: I_r \times C(I_0, IF_n) \longrightarrow IF_n$, L_1 , L_2 , f_1 , and f_2 are as in problem (19), and $q \in C(I_c \times I_d, \mathbb{R})$.

Definition 10. A function $\langle \phi_1, \phi_2 \rangle \in C(I_r, IF_n)$ can be a solution of model (19) if $\langle \phi_1, \phi_2 \rangle$ verifies the following equation:

$$\begin{aligned} \langle \phi_1, \phi_2 \rangle(\lambda, \theta) = & P(\lambda, \theta) \int_0^\theta L_1(\lambda) \langle \phi_1, \phi_2 \rangle(\lambda, \sigma) d\sigma \\ & - \int_0^\lambda L_2(\theta) \langle \phi_1, \phi_2 \rangle(\zeta, \theta) d\zeta \\ & - L_1(0) \int_0^\theta \int_0^\lambda L_2(\sigma) \langle \phi_1, \phi_2 \rangle(\zeta, \sigma) d\zeta d\sigma \\ & + \int_0^\lambda \int_0^\theta G_1(\zeta, \sigma, \langle \phi_1, \phi_2 \rangle_{(\zeta, \sigma)}) d\sigma d\zeta, \end{aligned} \quad (22)$$

if $(\lambda, \theta) \in I_c \times I_d$ where $P(\lambda, \theta) = f_1(\lambda, \theta) + f_2(\lambda, \theta) - f_1(0, \theta) + L_1(0) \int_0^\theta f_2(\lambda, \sigma) d\sigma$ and $\langle \phi_1, \phi_2 \rangle(\lambda, \theta) = \eta(\lambda, \theta)$, if $(\lambda, \theta) \in \hat{I}_r$.

Let $L_1 = \sup_{\lambda \in I_c} |L_1(\lambda)|$ and $L_2 = \sup_{\theta \in I_d} |L_2(\theta)|$. By applying Banach fixed point theorem, we will prove the following results.

Theorem 3. Suppose that

- (1) $G_1: I_r \times C(I_0, IF_n) \longrightarrow IF_n$ is continuous
- (2) For $(\lambda, \theta) \in I_c \times I_d$, $\langle \phi_1, \phi_2 \rangle$ and $\langle \phi'_1, \phi'_2 \rangle \in C(I_r, IF_n)$, and $r_1, r_2 \in I_0$, we have

$$\begin{aligned} & d_\infty^n(G_1(\lambda, \theta, \langle \phi_1, \phi_2 \rangle_{(\lambda, \theta)}), G_1(\lambda, \theta, \langle \phi'_1, \phi'_2 \rangle_{(\lambda, \theta)})) \\ & \leq L d_\infty^n(\langle \phi_1, \phi_2 \rangle(\lambda + r_1, \theta + r_2), \langle \phi'_1, \phi'_2 \rangle(\lambda + r_1, \theta + r_2)), \end{aligned} \quad (23)$$

where $L > 0$ is a given constant. Moreover, for all $\beta > 0$ satisfying

$$\beta^2 - \beta(L_1 + L_2) - L_1 L_2 - L > 0. \quad (24)$$

Therefore, problem (19) has a unique intuitionistic fuzzy solution in $C(I_r, IF_n)$ with the metric D_β .

Proof. Consider the operator $T_1: C(I_r, IF_n) \longrightarrow C(I_r, IF_n)$ defined as follows:

$$T_1(\langle \phi_1, \phi_2 \rangle)(\lambda, \theta) := \begin{cases} \eta(\lambda, \theta), & (\lambda, \theta) \in \hat{I}_r, \\ P(\lambda, \theta) - \int_0^\theta L_1(\lambda) \langle \phi_1, \phi_2 \rangle(\lambda, \sigma) d\sigma - \int_0^\lambda L_2(\theta) \langle \phi_1, \phi_2 \rangle(\zeta, \theta) d\zeta - L_1(0) \int_0^\theta \int_0^\lambda L_2(\sigma) \langle \phi_1, \phi_2 \rangle(\zeta, \sigma) d\zeta d\sigma \\ + \int_0^\lambda \int_0^\theta G_1(\zeta, \sigma, \langle \phi_1, \phi_2 \rangle_{(\zeta, \sigma)}) d\sigma d\zeta, & \text{if } (\lambda, \theta) \in I_c \times I_d, \end{cases} \quad (25)$$

where $P(\lambda, \theta) = f_1(\lambda, \theta) + f_2(\lambda, \theta) - f_1(0, \theta) + L_1(0) \int_0^\theta f_2(\lambda, \sigma) d\sigma$.

We will prove that T_1 is a contraction operator.

Therefore, consider $\langle \phi_1, \phi_2 \rangle$ and $\langle \phi'_1, \phi'_2 \rangle \in C(I_r, IF_n)$, and for $(\lambda, \theta) \in I_c \times I_d$, we have

$$\begin{aligned} T_1(\langle \phi_1, \phi_2 \rangle)(\lambda, \theta) = & P(\lambda, \theta) - \int_0^\theta L_1(\lambda) \langle \phi_1, \phi_2 \rangle(\lambda, \sigma) d\sigma - \int_0^\lambda L_2(\theta) \langle \phi_1, \phi_2 \rangle(\zeta, \theta) d\zeta \\ & - L_1(0) \int_0^\theta \int_0^\lambda L_2(\sigma) \langle \phi_1, \phi_2 \rangle(\zeta, \sigma) d\zeta d\sigma + \int_0^\lambda \int_0^\theta G_1(\zeta, \sigma, \langle \phi_1, \phi_2 \rangle_{(\zeta, \sigma)}) d\sigma d\zeta, \\ T_1(\langle \phi'_1, \phi'_2 \rangle)(\lambda, \theta) = & P(\lambda, \theta) - \int_0^\theta L_1(\lambda) \langle \phi'_1, \phi'_2 \rangle(\lambda, \sigma) d\sigma - \int_0^\lambda L_2(\theta) \langle \phi'_1, \phi'_2 \rangle(\zeta, \theta) d\zeta \\ & - L_1(0) \int_0^\theta \int_0^\lambda L_2(\sigma) \langle \phi'_1, \phi'_2 \rangle(\zeta, \sigma) d\zeta d\sigma + \int_0^\lambda \int_0^\theta G_1(\zeta, \sigma, \langle \phi'_1, \phi'_2 \rangle_{(\zeta, \sigma)}) d\sigma d\zeta. \end{aligned} \quad (26)$$

We have the following inequality:

$$\begin{aligned}
d_{\infty}^n(T_1(\langle \phi_1, \phi_2 \rangle)(\lambda, \theta), T_1(\langle \phi'_1, \phi'_2 \rangle)(\lambda, \theta)) &\leq d_{\infty}^n\left(\int_0^{\theta} L_1(\lambda) \langle \phi_1, \phi_2 \rangle(\lambda, \sigma) d\sigma, \int_0^{\theta} L_1(\lambda) \langle \phi'_1, \phi'_2 \rangle(\lambda, \sigma) d\sigma\right) \\
&\quad + d_{\infty}^n\left(\int_0^{\lambda} L_2(\theta) \langle \phi_1, \phi_2 \rangle(\zeta, \theta) d\zeta, \int_0^{\lambda} L_2(\theta) \langle \phi'_1, \phi'_2 \rangle(\zeta, \theta) d\zeta\right) \\
&\quad + d_{\infty}^n\left(L_1(0) \int_0^{\theta} \int_0^{\lambda} L_2(\sigma) \langle \phi_1, \phi_2 \rangle(\zeta, \sigma) d\zeta d\sigma, L_1(0) \int_0^{\theta} \int_0^{\lambda} L_2(\sigma) \langle \phi'_1, \phi'_2 \rangle(\zeta, \sigma) d\zeta d\sigma\right) \\
&\quad + d_{\infty}^n\left(\int_0^{\lambda} \int_0^{\theta} G_1(\zeta, \sigma, \langle \phi_1, \phi_2 \rangle_{(\zeta, \sigma)}) d\sigma d\zeta, \int_0^{\lambda} \int_0^{\theta} G_1(\zeta, \sigma, \langle \phi'_1, \phi'_2 \rangle_{(\zeta, \sigma)}) d\sigma d\zeta\right) \\
&\leq L_1 \int_0^{\theta} d_{\infty}^n(\langle \phi_1, \phi_2 \rangle(\lambda, \sigma), \langle \phi'_1, \phi'_2 \rangle(\lambda, \sigma)) d\sigma \\
&\quad + L_2 \int_0^{\lambda} d_{\infty}^n(\langle \phi_1, \phi_2 \rangle(\zeta, \theta), \langle \phi'_1, \phi'_2 \rangle(\zeta, \theta)) d\zeta \\
&\quad + |L_1(0)| \sup_{\theta \in I_d} |L_2(\theta)| \int_0^{\theta} \int_0^{\lambda} d_{\infty}^n(\langle \phi_1, \phi_2 \rangle(\zeta, \sigma), \langle \phi'_1, \phi'_2 \rangle(\zeta, \sigma)) d\zeta d\sigma \\
&\quad + \int_0^{\lambda} \int_0^{\theta} d_{\infty}^n(G_1(\zeta, \sigma, \langle \phi_1, \phi_2 \rangle_{(\zeta, \sigma)}), G_1(\zeta, \sigma, \langle \phi'_1, \phi'_2 \rangle_{(\zeta, \sigma)})) d\sigma d\zeta.
\end{aligned} \tag{27}$$

Multiplying (27) by $e^{-\beta(\lambda+\theta)}$, then we get

$$\begin{aligned}
&e^{-\beta(\lambda+\theta)} d_{\infty}^n\left(\int_0^{\lambda} \int_0^{\theta} G_1(\zeta, \sigma, \langle \phi_1, \phi_2 \rangle(\zeta + r_1, \sigma + r_2)) d\sigma d\zeta, \int_0^{\lambda} \int_0^{\theta} G_1(\zeta, \sigma, \langle \phi'_1, \phi'_2 \rangle(\zeta + r_1, \sigma + r_2)) d\sigma d\zeta\right) e^{-\beta(\lambda+\theta)} \\
&\leq L \int_0^{\lambda} \int_0^{\theta} d_{\infty}^n(\langle \phi_1, \phi_2 \rangle(\zeta + r_1, \sigma + r_2), \langle \phi'_1, \phi'_2 \rangle(\zeta + r_1, \sigma + r_2)) d\sigma d\zeta e^{-\beta(\lambda+\theta)} \\
&\leq LD_{\beta}(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle) \int_0^{\lambda} \int_0^{\theta} e^{\beta(r_1+r_2+\zeta+\sigma)} d\sigma d\zeta e^{-\beta(\lambda+\theta)} \\
&\leq LD_{\beta}(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle) \frac{1}{\beta} e^{\beta(r_1+r_2)} \int_0^{\lambda} (e^{\beta(\zeta+\theta)} - e^{\beta\zeta}) d\zeta e^{-\beta(\lambda+\theta)} \\
&\leq \frac{L}{\beta^2} D_{\beta}(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle) e^{\beta(r_1+r_2)} (e^{\beta\lambda} - 1)(e^{\beta\theta} - 1) e^{-\beta(\lambda+\theta)} \\
&\leq \frac{L}{\beta^2} D_{\beta}(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle).
\end{aligned} \tag{28}$$

The same, we have

$$e^{-\beta(\lambda+\theta)} |L_1(0)| \sup_{y \in I_d} |L_2(\theta)| \int_0^{\theta} \int_0^{\lambda} d_{\infty}^n(\langle \phi_1, \phi_2 \rangle(\zeta, \sigma), \langle \phi'_1, \phi'_2 \rangle(\zeta, \sigma)) d\zeta d\sigma \leq \frac{L_1 L_2}{\beta^2} D_{\beta}(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle), \tag{29}$$

$$\begin{aligned}
&e^{-\beta(\lambda+\theta)} L_1 \int_0^{\theta} d_{\infty}^n(\langle \phi_1, \phi_2 \rangle(\lambda, \sigma), \langle \phi'_1, \phi'_2 \rangle(\lambda, \sigma)) d\sigma \leq L_1 D_{\beta}(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle) \int_0^{\theta} e^{\beta(\lambda+\sigma)} d\sigma e^{-\beta(\lambda+\theta)} \\
&\leq L_1 D_{\beta}(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle) (e^{-\beta(\lambda+\theta)} - e^{\beta\theta}) e^{-\beta(\lambda+\theta)} \\
&\leq \frac{L_1}{\beta} D_{\beta}(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle).
\end{aligned} \tag{30}$$

Similarly, we get

$$\begin{aligned} e^{-\beta(\lambda+\theta)} L_2 \int_0^\lambda d_\infty^n(\langle \phi_1, \phi_2 \rangle(\zeta, \theta), \langle \phi'_1, \phi'_2 \rangle(\zeta, \theta)) d\zeta \\ \leq \frac{L_2}{\beta} D_\beta(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle). \end{aligned} \quad (31)$$

From (28)–(31), we obtain

$$\begin{aligned} e^{-\beta(\lambda+\theta)} d_\infty^n(T_1(\langle \phi_1, \phi_2 \rangle)(\lambda, \theta), T_1(\langle \phi'_1, \phi'_2 \rangle)(\lambda, \theta)) \\ \leq \left(\frac{L_1 + L_2}{\beta} + \frac{L_1 L_2 + L}{\beta^2} \right) D_\beta(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle). \end{aligned} \quad (32)$$

If $(\lambda, \theta) \in \hat{I}_r$,

$$d_\infty^n(T_1(\langle \phi_1, \phi_2 \rangle)(\lambda, \theta), T_1(\langle \phi'_1, \phi'_2 \rangle)(\lambda, \theta)) = 0. \quad (33)$$

Hence, for each $(\lambda, \theta) \in I_r$,

$$\begin{aligned} D_\beta(T_1(\langle \phi_1, \phi_2 \rangle), T_1(\langle \phi'_1, \phi'_2 \rangle)) \\ \leq \left(\frac{L_1 + L_2}{\beta} + \frac{L_1 L_2 + L}{\beta^2} \right) D_\beta(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle). \end{aligned} \quad (34)$$

So, for all $\beta > 0$ satisfying (31), we have $((L_1 + L_2)/\beta) + ((L_1 L_2 + L)/\beta^2) < 1$. Then, T_1 is a contraction operator.

Therefore, T_1 has a unique fixed point, which is the solution of integral boundary problem (21). $\square$

Definition 11. A function $\langle \phi_1, \phi_2 \rangle \in C(I_r, IF_n)$ is called an intuitionistic fuzzy solution of problem (21) if $\langle \phi_1, \phi_2 \rangle$ verifies the following integral equation:

$$\begin{aligned} \langle \phi_1, \phi_2 \rangle(\lambda, \theta) = P(\lambda, \theta) - \int_0^\theta L_1(\lambda) \langle \phi_1, \phi_2 \rangle(\lambda, \sigma) d\sigma - \int_0^\lambda L_2(\theta) \langle \phi_1, \phi_2 \rangle(\zeta, \theta) d\zeta \\ + \int_0^\lambda q(\zeta, \theta) \langle \phi_1, \phi_2 \rangle(\zeta, \theta) d\zeta - L_1(0) \int_0^\theta \int_0^\lambda L_2(\sigma) \langle \phi_1, \phi_2 \rangle(\zeta, \sigma) d\zeta d\sigma \\ + \int_0^\lambda q(\zeta, 0) \langle \phi_1, \phi_2 \rangle(\zeta, 0) d\zeta + \int_0^\lambda \int_0^\theta G_2(\zeta, \sigma, \langle \phi_1, \phi_2 \rangle_{(\zeta, \sigma)}) d\sigma d\zeta, \end{aligned} \quad (35)$$

if $(\lambda, \theta) \in I_c \times I_{\hat{d}}$ where $P(\lambda, \theta) = f_1(\lambda, \theta) + f_2(\lambda, \theta) - f_1(0, \theta) + L_1(0) \int_0^\theta f_2(\lambda, \sigma) d\sigma$ and $\langle \phi_1, \phi_2 \rangle(\lambda, \theta) = \eta(\lambda, \theta)$ if $(\lambda, \theta) \in \hat{I}_r$.

Let $L_1 = \sup_{\lambda \in I_c} |L_1(\lambda)|$, $L_2 = \sup_{\theta \in I_d} |L_2(\theta)|$, and $q = \sup_{(\lambda, \theta) \in I_c \times I_d} |q(\lambda, \theta)|$. By applying the Banach fixed point theorem, we will prove the following result.

Theorem 4. Suppose that

- (1) A map $G_2: I_r \times C(I_0, IF_n) \longrightarrow IF_n$ is continuous
- (2) For $(\lambda, \theta) \in I_c \times I_d$, $\langle \phi_1, \phi_2 \rangle$ and $\langle \phi'_1, \phi'_2 \rangle \in C(I_r, IF_n)$, and $r_1, r_2 \in I_0$, we have

$$\begin{aligned} d_\infty^n(G_2(\lambda, \theta, \langle \phi_1, \phi_2 \rangle_{(\lambda, \theta)}), G_2(\lambda, \theta, \langle \phi'_1, \phi'_2 \rangle_{(\lambda, \theta)})) \\ \leq L d_\infty^n(\langle \phi_1, \phi_2 \rangle(\lambda + r_1, \theta + r_2), \langle \phi'_1, \phi'_2 \rangle(\lambda + r_1, \theta + r_2)), \end{aligned} \quad (36)$$

where $L > 0$ is a given constant. Then, for all $\beta > 0$ satisfying

$$\beta^2 - \beta(L_1 + L_2 + 2q) - L_1 L_2 - L > 0. \quad (37)$$

Therefore, problem (21) has a unique intuitionistic fuzzy solution in $C(I_r, IF_n)$ with the metric D_β .

Proof. To prove the existence-uniqueness result, we will show that T_2 has a unique fixed point.

Define operator $T_2: C(I_r, IF_n) \longrightarrow C(I_r, IF_n)$ by

$$T_2(\langle \phi_1, \phi_2 \rangle)(\lambda, \theta) := \begin{cases} \eta(\lambda, \theta), & (\lambda, \theta) \in \hat{I}_r, \\ \begin{aligned} & P(\lambda, \theta) - \int_0^\theta L_1(\lambda) \langle \phi_1, \phi_2 \rangle(\lambda, \sigma) d\sigma - \int_0^\lambda L_2(\theta) \langle \phi_1, \phi_2 \rangle(\zeta, \theta) d\zeta + \int_0^\lambda q(\zeta, \theta) \langle \phi_1, \phi_2 \rangle(\zeta, \theta) d\zeta \\ & - L_1(0) \int_0^\theta \int_0^\lambda L_2(\sigma) \langle \phi_1, \phi_2 \rangle(\zeta, \sigma) d\zeta d\sigma + \int_0^\lambda q(\zeta, 0) \langle \phi_1, \phi_2 \rangle(\zeta, 0) d\zeta \\ & + \int_0^\lambda \int_0^\theta G_2(\zeta, \sigma, \langle \phi_1, \phi_2 \rangle_{(\zeta, \sigma)}) d\sigma d\zeta, \end{aligned} & \text{if } (\lambda, \theta) \in I_c \times I_d, \end{cases} \quad (38)$$

where

$$P(\lambda, \theta) = f_1(\lambda, \theta) + f_2(\lambda, \theta) - f_1(0, \theta) + L_1(0) \int_0^\theta f_2(\lambda, \sigma) d\sigma. \quad (39)$$

We prove that T_2 is a contraction operator. Then, consider $\langle \phi_1, \phi_2 \rangle$ and $\langle \phi'_1, \phi'_2 \rangle \in C(I_r, IF_n)$.

We can see that $d_\infty^n(T_2(\langle \phi_1, \phi_2 \rangle)(\lambda, \theta), T_2(\langle \phi'_1, \phi'_2 \rangle)(\lambda, \theta)) = 0$ if $(\lambda, \theta) \in I_0$. On the contrary, on $(\lambda, \theta) \in I_c \times I_d$, we have

$$\begin{aligned} T_2(\langle \phi_1, \phi_2 \rangle)(\lambda, \theta) &:= P(\lambda, \theta) - \int_0^\theta L_1(\lambda) \langle \phi_1, \phi_2 \rangle(\lambda, \sigma) d\sigma - \int_0^\lambda L_2(\theta) \langle \phi_1, \phi_2 \rangle(\zeta, \theta) d\zeta + \int_0^\lambda q(\zeta, \theta) \langle \phi_1, \phi_2 \rangle(\zeta, \theta) d\zeta \\ &\quad - L_1(0) \int_0^\theta \int_0^\lambda L_2(\sigma) \langle \phi_1, \phi_2 \rangle(\zeta, \sigma) d\zeta d\sigma + \int_0^\lambda q(\zeta, 0) \langle \phi_1, \phi_2 \rangle(\zeta, 0) d\zeta + \int_0^\lambda \int_0^\theta G_2(\zeta, \sigma, \langle \phi_1, \phi_2 \rangle_{(\zeta, \sigma)}) d\sigma d\zeta, \\ T_2(\langle \phi'_1, \phi'_2 \rangle)(\lambda, \theta) &:= P(\lambda, \theta) - \int_0^\theta L_1(\lambda) \langle \phi'_1, \phi'_2 \rangle(\lambda, \sigma) d\sigma - \int_0^\lambda L_2(\theta) \langle \phi'_1, \phi'_2 \rangle(\zeta, \theta) d\zeta + \int_0^\lambda q(\zeta, \theta) \langle \phi'_1, \phi'_2 \rangle(\zeta, \theta) d\zeta \\ &\quad - L_1(0) \int_0^\theta \int_0^\lambda L_2(\sigma) \langle \phi'_1, \phi'_2 \rangle(\zeta, \sigma) d\zeta d\sigma + \int_0^\lambda q(\zeta, 0) \langle \phi'_1, \phi'_2 \rangle(\zeta, 0) d\zeta + \int_0^\lambda \int_0^\theta G_2(\zeta, \sigma, \langle \phi'_1, \phi'_2 \rangle_{(\zeta, \sigma)}) d\sigma d\zeta. \end{aligned} \quad (40)$$

From the properties of supremum metric, we have

$$\begin{aligned} d_\infty^n(T_2(\langle \phi_1, \phi_2 \rangle)(\lambda, \theta), T_2(\langle \phi'_1, \phi'_2 \rangle)(\lambda, \theta)) &\leq d_\infty^n \left(\int_0^\theta L_1(\lambda) \langle \phi_1, \phi_2 \rangle(\lambda, \sigma) d\sigma, \int_0^\theta L_1(\lambda) \langle \phi'_1, \phi'_2 \rangle(\lambda, \sigma) d\sigma \right) \\ &\quad + d_\infty^n \left(\int_0^\lambda L_2(\theta) \langle \phi_1, \phi_2 \rangle(\zeta, \theta) d\zeta, \int_0^\lambda L_2(\theta) \langle \phi'_1, \phi'_2 \rangle(\zeta, \theta) d\zeta \right) \\ &\quad + d_\infty^n \left(L_1(0) \int_0^\theta \int_0^\lambda L_2(\sigma) \langle \phi_1, \phi_2 \rangle(\zeta, \sigma) d\zeta d\sigma, L_1(0) \int_0^\theta \int_0^\lambda L_2(\sigma) \langle \phi'_1, \phi'_2 \rangle(\zeta, \sigma) d\zeta d\sigma \right) \\ &\quad + d_\infty^n \left(\int_0^\lambda q(\zeta, \theta) \langle \phi_1, \phi_2 \rangle(\zeta, \theta) d\zeta, \int_0^\lambda q(\zeta, \theta) \langle \phi'_1, \phi'_2 \rangle(\zeta, \theta) d\zeta \right) \\ &\quad + d_\infty^n \left(\int_0^\lambda q(\zeta, 0) \langle \phi_1, \phi_2 \rangle(\zeta, 0) d\zeta, \int_0^\lambda q(\zeta, 0) \langle \phi'_1, \phi'_2 \rangle(\zeta, 0) d\zeta \right) \\ &\quad + d_\infty^n \left(\int_0^\lambda \int_0^\theta G_2(\zeta, \sigma, \langle \phi_1, \phi_2 \rangle_{(\zeta, \sigma)}) d\sigma d\zeta, \int_0^\lambda \int_0^\theta G_2(\zeta, \sigma, \langle \phi'_1, \phi'_2 \rangle_{(\zeta, \sigma)}) d\sigma d\zeta \right), \\ &\leq L_1 \int_0^\theta d_\infty^n(\langle \phi_1, \phi_2 \rangle(\lambda, \sigma), \langle \phi'_1, \phi'_2 \rangle(\lambda, \sigma)) d\sigma \\ &\quad + L_2 \int_0^\lambda d_\infty^n(\langle \phi_1, \phi_2 \rangle(\zeta, \theta), \langle \phi'_1, \phi'_2 \rangle(\zeta, \theta)) d\zeta \\ &\quad + |L_1(0)| \sup_{\theta \in I_d} |L_2(\theta)| \int_0^\theta \int_0^\lambda d_\infty^n(\langle \phi_1, \phi_2 \rangle(\zeta, \sigma), \langle \phi'_1, \phi'_2 \rangle(\zeta, \sigma)) d\zeta d\sigma \\ &\quad + \sup_{(\zeta, \sigma) \in I_c \times I_d} |q(\zeta, \sigma)| \int_0^\lambda d_\infty^n(\langle \phi_1, \phi_2 \rangle(\zeta, \theta), \langle \phi'_1, \phi'_2 \rangle(\zeta, \theta)) d\zeta \\ &\quad + \sup_{(\zeta, \sigma) \in I_c \times I_d} |q(\zeta, 0)| \int_0^\lambda d_\infty^n(\langle \phi_1, \phi_2 \rangle(\zeta, 0), \langle \phi'_1, \phi'_2 \rangle(\zeta, 0)) d\zeta \\ &\quad + \int_0^\lambda \int_0^\theta d_\infty^n(G_2(\zeta, \sigma, \langle \phi_1, \phi_2 \rangle_{(\zeta, \sigma)}), G_2(\zeta, \sigma, \langle \phi'_1, \phi'_2 \rangle_{(\zeta, \sigma)})) d\sigma d\zeta. \end{aligned} \quad (41)$$

Set $L_1 = \sup_{\lambda \in I_c} |L_1(\lambda)|$, $L_2 = \sup_{\theta \in I_d} |L_2(\theta)|$, and $q = \sup_{(\lambda, \theta) \in I_c \times I_d} |q(\lambda, \theta)|$. We get the following assessment:

$$\begin{aligned} & \sup_{(\zeta, \sigma) \in I_c \times I_d} |q(\zeta, \sigma)| \int_0^\lambda d_\infty^n(\langle \phi_1, \phi_2 \rangle(\zeta, \theta), \langle \phi'_1, \phi'_2 \rangle(\zeta, \theta)) d\zeta \\ &= q \int_0^\lambda d_\infty^n(\langle \phi_1, \phi_2 \rangle(\zeta, \theta), \langle \phi'_1, \phi'_2 \rangle(\zeta, \theta)) d\zeta, \\ &\leq q D_\beta(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle) \int_0^\lambda e^{\beta(\zeta+\theta)} d\zeta, \\ &\leq q D_\beta(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle) \frac{1}{\beta} (e^{\beta(\zeta+\theta)} - e^{\beta\theta}). \end{aligned} \quad (42)$$

Multiplying (42) by $\exp^{-\beta(\lambda+\theta)}$, we get

$$\begin{aligned} & e^{-\beta(\lambda+\theta)} \sup_{(\zeta, \sigma) \in I_c \times I_d} |q(\zeta, \sigma)| \int_0^\lambda d_\infty^n(\langle \phi_1, \phi_2 \rangle(\zeta, \theta), \langle \phi'_1, \phi'_2 \rangle(\zeta, \theta)) d\zeta \\ &\leq q D_\beta(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle) \frac{1}{\beta} (e^{\beta(\lambda+\theta)} - e^{\beta\theta}) e^{-\beta(\lambda+\theta)}, \\ &\leq \frac{q}{\beta} D_\beta(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle). \end{aligned} \quad (43)$$

Similarly, we obtain

$$\begin{aligned} & e^{-\beta(\lambda+\theta)} \sup_{(\zeta, 0) \in I_c \times I_d} |q(\zeta, 0)| \int_0^\lambda d_\infty^n(\langle \phi_1, \phi_2 \rangle(\zeta, 0), \langle \phi'_1, \phi'_2 \rangle(\zeta, 0)) d\sigma \\ &\leq \frac{q}{\beta} D_\beta(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle). \end{aligned} \quad (44)$$

From (30) and (31), we imply

$$\begin{aligned} & e^{-\beta(\lambda+\theta)} L_1 \int_0^\theta d_\infty^n(\langle \phi_1, \phi_2 \rangle(\lambda, \sigma), \langle \phi'_1, \phi'_2 \rangle(\lambda, \sigma)) d\sigma \\ &\leq \frac{L_1}{\beta} D_\beta(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle), \end{aligned} \quad (45)$$

$$\begin{aligned} & e^{-\beta(\lambda+\theta)} L_2 \int_0^\lambda d_\infty^n(\langle \phi_1, \phi_2 \rangle(\zeta, \theta), \langle \phi'_1, \phi'_2 \rangle(\zeta, \theta)) d\zeta \\ &\leq \frac{L_2}{\beta} D_\beta(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle). \end{aligned} \quad (46)$$

and from (28), we also get

$$\begin{aligned} & e^{-\beta(\lambda+\theta)} d_\infty^n \left(\int_0^\lambda \int_0^\theta G_2(\zeta, \sigma, \langle \phi_1, \phi_2 \rangle(\zeta + r_1, \sigma + r_2)) d\sigma d\zeta, \int_0^\lambda \int_0^\theta G_2(\zeta, \sigma, \langle \phi'_1, \phi'_2 \rangle(\zeta + r_1, \sigma + r_2)) d\sigma d\zeta \right) e^{-\beta(\lambda+\theta)} \\ &\leq \frac{L}{\beta^2} D_\beta(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle), \end{aligned} \quad (47)$$

$$e^{-\beta(\lambda+\theta)} |L_1(0)| \sup_{y \in I_d} |L_2(\theta)| \int_0^\theta \int_0^\lambda d_\infty^n(\langle \phi_1, \phi_2 \rangle(\zeta, \sigma), \langle \phi'_1, \phi'_2 \rangle(\zeta, \sigma)) d\zeta d\sigma \leq \frac{L_1 L_2}{\beta^2} D_\beta(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle). \quad (48)$$

From (43)–(48), we have

$$\begin{aligned} & e^{-\beta(\lambda+\theta)} d_\infty^n(T_2(\langle \phi_1, \phi_2 \rangle)(\lambda, \theta), T_2(\langle \phi'_1, \phi'_2 \rangle)(\lambda, \theta)) \\ &\leq \left(\frac{L_1 + L_2 + 2q}{\beta} + \frac{L_1 L_2 + L}{\beta^2} \right) D_\beta(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle). \end{aligned} \quad (49)$$

If $(\lambda, \theta) \in \widehat{I}_r$,

$$d_\infty^n(T_2(\langle \phi_1, \phi_2 \rangle)(\lambda, \theta), T_2(\langle \phi'_1, \phi'_2 \rangle)(\lambda, \theta)) = 0. \quad (50)$$

Hence, for each $(\lambda, \theta) \in I_r$,

$$\begin{aligned} & D_\beta(T_2(\langle \phi_1, \phi_2 \rangle), T_2(\langle \phi'_1, \phi'_2 \rangle)) \\ &\leq \left(\frac{L_1 + L_2 + 2q}{\beta} + \frac{L_1 L_2 + L}{\beta^2} \right) D_\beta(\langle \phi_1, \phi_2 \rangle, \langle \phi'_1, \phi'_2 \rangle). \end{aligned} \quad (51)$$

Then, for all $\beta > 0$ verifying (36), it implies $((L_1 + L_2 + 2q)/\beta) + ((L_1 L_2 + L)/\beta^2) < 1$; therefore, T_2 is a contraction operator.

Thus, T_2 has a unique fixed point, which is the solution of (21). $\square$

4. Examples

We propose the following two examples: to illustrate the usefulness of our main results and to find the intuitionistic fuzzy solution, we will use the method of steps proposed in [22, 23].

Example 1. Now, we present a first example to illustrate Theorem 3; consider the following state-dependent delay intuitionistic fuzzy partial functional differential equations for some $w > 0$ in the following form:

$$\frac{\partial^2 \langle \phi_1, \phi_2 \rangle (\lambda, \theta)}{\partial \lambda \partial \theta} = w^2 e^{w/2} \langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{4}, \theta - \frac{1}{4} \right), \quad (52)$$

$$(\lambda, \theta) \in [0, 1] \times [0, 1].$$

And, the integral boundary initial conditions are

$$\langle \phi_1, \phi_2 \rangle (\lambda, 0) + \frac{1}{8} \int_0^\theta \langle \phi_1, \phi_2 \rangle (\lambda, \sigma) d\sigma = \frac{C}{8w} e^{w\lambda} (e^{w\theta} + 8w - 1), \quad (53)$$

$$\langle \phi_1, \phi_2 \rangle (0, \theta) + \frac{1}{8} \int_0^\lambda \langle \phi_1, \phi_2 \rangle (\zeta, \theta) d\zeta = \frac{C}{8w} e^{w\theta} (e^{w\lambda} + 8w - 1), \quad (54)$$

where $C \in \text{IF}_1$ is a triangular intuitionistic fuzzy number, and the initial condition is

$$\langle \phi_1, \phi_2 \rangle (\lambda, \theta) = C(e^{w\lambda} + e^{w\theta} + \lambda\theta), \quad (55)$$

for $(\lambda, \theta) \in n[-(1/4), 1]_q \times h[-(1/4), 1]_x \setminus x(0, 1] \times C(0, 1]$.

Since

$$G_1(\lambda, \theta, \langle \phi_1, \phi_2 \rangle_{(\lambda, \theta)}) = w^2 e^{w/2} \langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{4}, \theta - \frac{1}{4} \right), \quad (56)$$

it satisfies hypotheses (1) and (2) of Theorem 3.

Indeed, it is easy to see that G_1 is continuous and

$$\begin{aligned} d_\infty(G_1(\lambda, \theta, \langle \phi_1, \phi_2 \rangle_{(\lambda, \theta)}), G_1(\lambda, \theta, \langle \phi'_1, \phi'_2 \rangle_{(\lambda, \theta)})) \\ \leq w^2 e^{w/2} d_\infty\left(\langle \phi_1, \phi_2 \rangle_{(\lambda, \theta)}\left(-\frac{1}{4}, -\frac{1}{4}\right), \langle \phi'_1, \phi'_2 \rangle_{(\lambda, \theta)}\left(-\frac{1}{4}, -\frac{1}{4}\right)\right). \end{aligned} \quad (57)$$

Therefore, G_1 satisfies 2 with an positive number $L = w^2 e^{w/2}$, $L_1 = L_2 = 1/8$, and $c = d = 1$.

Then, from Theorem 3, we see that if we choose positive weighted number β satisfying $\beta > (1/8) + \sqrt{(1/32) + w^2 e^{w/2}}$, Hence, all the assumptions of Theorem 3 are satisfied; therefore, we have a unique intuitionistic fuzzy solution of problems (45)–(47).

We try to calculate the intuitionistic fuzzy solution of this problem by using the method of steps proposed in [22, 23].

We write the deterministic solution of the crisp equation as follows: $v = ce^{w(\lambda+\theta)}$.

Assume that the parametric form of corresponding intuitionistic fuzzy number C is

$$\begin{aligned} [C]_\alpha &= [C_l^+(\alpha), C_r^+(\alpha)], \\ [C]^\alpha &= [C_l^-(\alpha), C_r^-(\alpha)], \end{aligned} \quad (58)$$

where it verifies the conditions of Lemma 1.

The function $G_1: [0, (\pi/2)] \times [0, 1] \times \text{IF}_1 \longrightarrow \text{IF}_1$ is defined as

$$G_1(\lambda, \theta, \langle \phi_1, \phi_2 \rangle_{(\lambda, \theta)}) = w^2 e^{w/2} \langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{4}, \theta - \frac{1}{4} \right). \quad (59)$$

The α -cuts of G_1 are

$$\begin{aligned} [G_1]_\alpha &= \left[w^2 e^{w/2} \left[\langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{4}, \theta - \frac{1}{4} \right) \right]_l^+(\alpha), w^2 e^{w/2} \right. \\ &\quad \cdot \left. \left[\langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{4}, \theta - \frac{1}{4} \right) \right]_r^+(\alpha) \right], \\ [G_1]^\alpha &= \left[w^2 e^{w/2} \left[\langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{4}, \theta - \frac{1}{4} \right) \right]_l^-(\alpha), w^2 e^{w/2} \right. \\ &\quad \cdot \left. \left[\langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{4}, \theta - \frac{1}{4} \right) \right]_r^-(\alpha) \right]. \end{aligned} \quad (60)$$

If we denote

$$\begin{aligned} [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_\alpha &= [[\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_l^+(\alpha), [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_r^+(\alpha)], \\ [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]^\alpha &= [[\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_l^-(\alpha), [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_r^-(\alpha)]. \end{aligned} \quad (61)$$

Then,

$$\begin{aligned} \left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle (\lambda, \theta)}{\partial \lambda \partial \theta} \right]_\alpha &= \left[\left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle (\lambda, \theta)}{\partial \lambda \partial \theta} \right]_l^+(\alpha), \right. \\ &\quad \cdot \left. \left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle (\lambda, \theta)}{\partial \lambda \partial \theta} \right]_r^+(\alpha) \right], \\ \left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle (\lambda, \theta)}{\partial \lambda \partial \theta} \right]^\alpha &= \left[\left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle (\lambda, \theta)}{\partial \lambda \partial \theta} \right]_l^-(\alpha), \right. \\ &\quad \cdot \left. \left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle (\lambda, \theta)}{\partial \lambda \partial \theta} \right]_r^-(\alpha) \right]. \end{aligned} \quad (62)$$

Therefore, we solve the following state-dependent delay partial functional differential equations for $(\lambda, \theta) \in [0, 1] \times [0, 1]$:

$$\begin{cases} \left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle (\lambda, \theta)}{\partial \lambda \partial \theta} \right]_l^+(\alpha) = w^2 e^{w/2} \left[\langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{4}, \theta - \frac{1}{4} \right) \right]_l^+(\alpha), \\ \left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle (\lambda, \theta)}{\partial \lambda \partial \theta} \right]_r^+(\alpha) = w^2 e^{w/2} \left[\langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{4}, \theta - \frac{1}{4} \right) \right]_r^+(\alpha), \\ \left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle (\lambda, \theta)}{\partial \lambda \partial \theta} \right]_l^-(\alpha) = w^2 e^{w/2} \left[\langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{4}, \theta - \frac{1}{4} \right) \right]_l^-(\alpha), \\ \left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle (\lambda, \theta)}{\partial \lambda \partial \theta} \right]_r^-(\alpha) = w^2 e^{w/2} \left[\langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{4}, \theta - \frac{1}{4} \right) \right]_r^-(\alpha), \end{cases} \quad (63)$$

with initial conditions for $(\lambda, \theta) \in n[-(1/4), 1]_q \times h[-(1/4), 1]_x \setminus x(0, 1] \times C(0, 1]$:

$$\left\{ \begin{array}{l} [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_l^+ (\alpha) = C_l^+ (e^{w\lambda} + e^{w\theta} + \lambda\theta), \\ [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_r^+ (\alpha) = C_r^+ (e^{w\lambda} + e^{w\theta} + \lambda\theta), \\ [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_l^- (\alpha) = C_l^- (e^{w\lambda} + e^{w\theta} + \lambda\theta), \\ [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_r^- (\alpha) = C_r^- (e^{w\lambda} + e^{w\theta} + \lambda\theta), \end{array} \right. \quad (64)$$

and for $(\lambda, \theta) \in [0, 1] \times [0, 1]$, we have

$$\left\{ \begin{array}{l} [\langle \phi_1, \phi_2 \rangle (\lambda, 0)]_l^+ (\alpha) + \frac{1}{8} \int_0^\theta [\langle \phi_1, \phi_2 \rangle (\lambda, \sigma)]_l^+ (\alpha) d\sigma = C_l^+ (\alpha) \frac{1}{8w} e^{w\lambda} (e^{w\theta} + 8w - 1), \\ [\langle \phi_1, \phi_2 \rangle (\lambda, 0)]_r^+ (\alpha) + \frac{1}{8} \int_0^\theta [\langle \phi_1, \phi_2 \rangle (\lambda, \sigma)]_r^+ (\alpha) d\sigma = C_r^+ (\alpha) \frac{1}{8w} e^{w\lambda} (e^{w\theta} + 8w - 1), \\ [\langle \phi_1, \phi_2 \rangle (\lambda, 0)]_l^- (\alpha) + \frac{1}{8} \int_0^\theta [\langle \phi_1, \phi_2 \rangle (\lambda, \sigma)]_l^- (\alpha) d\sigma = C_l^- (\alpha) \frac{1}{8w} e^{w\lambda} (e^{w\theta} + 8w - 1), \\ [\langle \phi_1, \phi_2 \rangle (\lambda, 0)]_r^- (\alpha) + \frac{1}{8} \int_0^\theta [\langle \phi_1, \phi_2 \rangle (\lambda, \sigma)]_r^- (\alpha) d\sigma = C_r^- (\alpha) \frac{1}{8w} e^{w\lambda} (e^{w\theta} + 8w - 1), \\ [\langle \phi_1, \phi_2 \rangle (\lambda, 0)]_l^+ (\alpha) + \frac{1}{8} \int_0^\lambda [\langle \phi_1, \phi_2 \rangle (\zeta, \theta)]_l^+ (\alpha) d\zeta = C_l^+ (\alpha) \frac{1}{8w} e^{w\theta} (e^{w\lambda} + 8w - 1), \\ [\langle \phi_1, \phi_2 \rangle (\lambda, 0)]_r^+ (\alpha) + \frac{1}{8} \int_0^\lambda [\langle \phi_1, \phi_2 \rangle (\zeta, \theta)]_r^+ (\alpha) d\zeta = C_r^+ (\alpha) \frac{1}{8w} e^{w\theta} (e^{w\lambda} + 8w - 1), \\ [\langle \phi_1, \phi_2 \rangle (\lambda, 0)]_l^- (\alpha) + \frac{1}{8} \int_0^\lambda [\langle \phi_1, \phi_2 \rangle (\zeta, \theta)]_l^- (\alpha) d\zeta = C_l^- (\alpha) \frac{1}{8w} e^{w\theta} (e^{w\lambda} + 8w - 1), \\ [\langle \phi_1, \phi_2 \rangle (\lambda, 0)]_r^- (\alpha) + \frac{1}{8} \int_0^\lambda [\langle \phi_1, \phi_2 \rangle (\zeta, \theta)]_r^- (\alpha) d\zeta = C_r^- (\alpha) \frac{1}{8w} e^{w\theta} (e^{w\lambda} + 8w - 1). \end{array} \right. \quad (65)$$

We get

$$\left\{ \begin{array}{l} [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_l^+ (\alpha) = C_l^+ (\alpha) e^{w(\lambda+\theta)}, \quad (\lambda, \theta) \in [0, 1] \times [0, 1], \\ [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_r^+ (\alpha) = C_r^+ (\alpha) e^{w(\lambda+\theta)}, \quad (\lambda, \theta) \in [0, 1] \times [0, 1], \\ [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_l^- (\alpha) = C_l^- (\alpha) e^{w(\lambda+\theta)}, \quad (\lambda, \theta) \in [0, 1] \times [0, 1], \\ [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_r^- (\alpha) = C_r^- (\alpha) e^{w(\lambda+\theta)}, \quad (\lambda, \theta) \in [0, 1] \times [0, 1]. \end{array} \right. \quad (66)$$

Then,

$$\begin{aligned} [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_\alpha &= [C_l^+ (\alpha) e^{w(\lambda+\theta)}, C_r^+ (\alpha) e^{w(\lambda+\theta)}] [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]^\alpha \\ &= [C_l^- (\alpha) e^{w(\lambda+\theta)}, C_r^- (\alpha) e^{w(\lambda+\theta)}]. \end{aligned} \quad (67)$$

Now we denote the families N_α and N^α as follows:

$$\begin{aligned} [C_l^+ (\alpha) e^{w(\lambda+\theta)}, C_r^+ (\alpha) e^{w(\lambda+\theta)}] &= N_\alpha, \\ [C_l^- (\alpha) e^{w(\lambda+\theta)}, C_r^- (\alpha) e^{w(\lambda+\theta)}] &= N^\alpha, \end{aligned} \quad (68)$$

and the families N'_α and N'^α by

$$\begin{aligned} [w^2 e^{w/2} C_l^- (\alpha) e^{w(\lambda+\theta)}, w^2 e^{w/2} C_r^- (\alpha) e^{w(\lambda+\theta)}] &= N'_\alpha, \\ [w^2 e^{w/2} C_l^+ (\alpha) e^{w(\lambda+\theta)}, w^2 e^{w/2} C_r^+ (\alpha) e^{w(\lambda+\theta)}] &= N'^\alpha. \end{aligned} \quad (69)$$

It is easy to see that (N_α, N^α) and (N'_α, N'^α) satisfy the conditions of Proposition 1, and by applying Lemma 1, we can build the intuitionistic fuzzy solution $\langle \phi_1, \phi_2 \rangle$ $(\lambda, \theta) \in \text{IF}_1$ for equations (52)–(54) for every $\alpha \in [0, 1]$ as follows:

$$\begin{aligned} [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_\alpha &= [C_l^+(\alpha) e^{w(\lambda+\theta)}, C_r^+(\alpha) e^{w(\lambda+\theta)}], \\ [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]^\alpha &= [C_l^-(\alpha) e^{w(\lambda+\theta)}, C_r^-(\alpha) e^{w(\lambda+\theta)}]. \end{aligned} \quad (70)$$

Then, $\langle \phi_1, \phi_2 \rangle (x, y)$ is an intuitionistic fuzzy solution which satisfies the initial conditions (53)–(54) and is written as follows:

$$\langle \phi_1, \phi_2 \rangle (x, y) = C e^{w(\lambda+\theta)}. \quad (71)$$

Figures 1 and 2 show the membership and nonmembership functions of triangular intuitionistic fuzzy number $C = (-0.75, 0, 0.75; -2, 0, 2)$ and the simulation of α -cuts of

the intuitionistic fuzzy solution $\langle \phi_1, \phi_2 \rangle (x, y)$ at some values of (λ, θ) with $w = 0.45$.

In Figure 3 we present the surface of the intuitionistic fuzzy solution with triangular intuitionistic fuzzy numbers $C = (-0.75, 0, 0.75; -2, 0, 2)$ and $w = 0.45$.

Example 2. Now, we present a second example to illustrate Theorem 4. Consider the following state-dependent delay intuitionistic fuzzy partial functional differential equations in the general form:

$$\begin{aligned} \frac{\partial^2 \langle \phi_1, \phi_2 \rangle (\lambda, \theta)}{\partial \lambda \partial \theta} &= \frac{\partial((1/2)e^{-(2/3)} \langle \phi_1, \phi_2 \rangle (\lambda, \theta))}{\partial \theta} - \frac{1}{2} \langle \phi_1, \phi_2 \rangle \\ &\cdot \left(\lambda - \frac{1}{3}, \theta - \frac{1}{3} \right) + c_1 e^{\lambda+\theta} + \frac{1}{2} c_2 \cos\left(\lambda - \frac{1}{3}\right). \end{aligned} \quad (72)$$

And, the integral boundary initial conditions are

$$\langle \phi_1, \phi_2 \rangle (\lambda, 0) + \frac{1}{2} \int_0^\theta \langle \phi_1, \phi_2 \rangle (\lambda, \sigma) d\sigma = \frac{c_1}{2} e^\lambda (1 + e^\theta) + c_2 \cos(\lambda) \left(1 + \frac{\theta}{2} \right), \quad (73)$$

$$\langle \phi_1, \phi_2 \rangle (0, \theta) + \frac{1}{2} \int_0^\lambda \langle \phi_1, \phi_2 \rangle (\zeta, \theta) d\zeta = \frac{c_1}{2} e^\theta (1 + e^\lambda) + c_2 \left(1 + \frac{\sin(\lambda)}{2} \right), \quad (74)$$

where c_1 and $c_2 \in \text{IF}_1$ are triangular intuitionistic fuzzy numbers and $(\lambda, \theta) \in [0, (\pi/2)] \times [0, 1]$, and for $(\lambda, \theta) \in nq - 1/3, \pi/2 h_{\times} x7 - 1/3, 1C; \setminus 0, \pi/2 \times 0, 1]$, the initial condition is

$$\langle \phi_1, \phi_2 \rangle (\lambda, \theta) = c_1 (e^\lambda + \lambda \theta) + c_2 (e^\theta + \cos(\lambda \theta)). \quad (75)$$

Since

$$\begin{aligned} G_2(\lambda, \theta, \langle \phi_1, \phi_2 \rangle_{(\lambda, \theta)}) &= -\frac{1}{2} \langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{3}, \theta - \frac{1}{3} \right) \\ &+ c_1 e^{\lambda+\theta} + \frac{1}{2} c_2 \cos\left(\lambda - \frac{1}{3}\right), \end{aligned} \quad (76)$$

it verifies conditions 1 and 2 of Theorem 1.

Indeed, it is easy to see that G_2 is continuous:

$$\begin{aligned} d_\infty(G_2(\lambda, \theta, \langle \phi_1, \phi_2 \rangle_{(\lambda, \theta)}), G_2(\lambda, \theta, \langle \phi'_1, \phi'_2 \rangle_{(\lambda, \theta)})) \\ \leq \frac{1}{2} d_\infty\left(\langle \phi_1, \phi_2 \rangle_{(\lambda, \theta)}\left(-\frac{1}{3}, -\frac{1}{3}\right), \langle \phi'_1, \phi'_2 \rangle_{(\lambda, \theta)}\left(-\frac{1}{3}, -\frac{1}{3}\right)\right), \end{aligned} \quad (77)$$

so the assumptions are satisfied with a positive number $L = 1/2$, $L_1 = L_2 = 1/2$, $c = \pi/2$, $d = 1$, and $q(\lambda, \theta) = (1/2)e^{-(2/3)}$.

If we choose the positive weighted number β satisfying $\beta^2 - \beta(1 + e^{-(2/3)}) - (3/4) > 0$, then $\beta > (1/2) + (e^{-(2/3)}/2) + \sqrt{1 + (1/4)e^{-(4/3)} + (1/2)e^{-(2/3)}}$; hence, all the hypothesis of

Theorem 4 hold; therefore, we have a unique intuitionistic fuzzy solution of equations (72)–(74).

We will use the method of steps proposed in [22, 23] to find an intuitionistic fuzzy solution of equations (72)–(74).

The classical solution of the crisp equation is $w = C_1 e^{\lambda+\theta} + C_2 \cos(\lambda)$.

Assume that the parametric forms of corresponding intuitionistic fuzzy number c_i ($i = 1, 2$) are

$$\begin{aligned} [c_i]_\alpha &= [c_{il}^+(\alpha), c_{ir}^+(\alpha)], \\ [c_i]^\alpha &= [c_{il}^-(\alpha), c_{ir}^-(\alpha)], \end{aligned} \quad (78)$$

where they verify the conditions of Lemma 1.

The function $G_2: [0, (\pi/2)] \times [0, 1] \times \text{IF}_1 \longrightarrow \text{IF}_1$ is defined by

$$\begin{aligned} G_2(\lambda, \theta, \langle \phi_1, \phi_2 \rangle_{(\lambda, \theta)}) &= -\frac{1}{2} \langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{3}, \theta - \frac{1}{3} \right) + c_1 e^{\lambda+\theta} \\ &+ \frac{1}{2} c_2 \cos\left(\lambda - \frac{1}{3}\right). \end{aligned} \quad (79)$$

The α -cuts of G_2 are

$$\begin{aligned} [G_2]_\alpha &= [G_{2l}^+(\alpha), G_{2r}^+(\alpha)], \\ [G_2]^\alpha &= [G_{2l}^-(\alpha), G_{2r}^-(\alpha)], \end{aligned} \quad (80)$$

where

FIGURE 1: $C = (-0.75, 0, 0.75; -2, 0, 2)$. FIGURE 2: The curve of intuitionistic fuzzy solutions $\langle \phi_1, \phi_2 \rangle(x, y)$ at different values of (λ, θ) .

FIGURE 3: The surface of the intuitionistic fuzzy solution.

$$\begin{aligned}
 G_{2l}^+(\alpha) &= -\frac{1}{2} \left[\langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{3}, \theta - \frac{1}{3} \right) \right]_l^+ (\alpha) + c_{1l}^+(\alpha) e^{\lambda+\theta} + \frac{1}{2} c_{2l}^+(\alpha) \cos \left(\lambda - \frac{1}{3} \right), \\
 G_{2r}^+(\alpha) &= -\frac{1}{2} \left[\langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{3}, \theta - \frac{1}{3} \right) \right]_r^+ (\alpha) + c_{1r}^+(\alpha) e^{\lambda+\theta} + \frac{1}{2} c_{2r}^+(\alpha) \cos \left(\lambda - \frac{1}{3} \right), \\
 G_{2l}^-(\alpha) &= -\frac{1}{2} \left[\langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{3}, \theta - \frac{1}{3} \right) \right]_l^- (\alpha) + c_{1l}^-(\alpha) e^{\lambda+\theta} + \frac{1}{2} c_{2l}^-(\alpha) \cos \left(\lambda - \frac{1}{3} \right), \\
 G_{2r}^-(\alpha) &= -\frac{1}{2} \left[\langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{3}, \theta - \frac{1}{3} \right) \right]_r^- (\alpha) + c_{1r}^-(\alpha) e^{\lambda+\theta} + \frac{1}{2} c_{2r}^-(\alpha) \cos \left(\lambda - \frac{1}{3} \right).
 \end{aligned}$$

(81)

If

$$\begin{aligned} [\langle \phi_1, \phi_2 \rangle(\lambda, \theta)]_\alpha &= [[\langle \phi_1, \phi_2 \rangle(\lambda, \theta)]_l^+(\alpha), [\langle \phi_1, \phi_2 \rangle(\lambda, \theta)]_r^+(\alpha)] \\ [\langle \phi_1, \phi_2 \rangle(\lambda, \theta)]^\alpha &= [[\langle \phi_1, \phi_2 \rangle(\lambda, \theta)]_l^-(\alpha), [\langle \phi_1, \phi_2 \rangle(\lambda, \theta)]_r^-(\alpha)], \end{aligned} \quad (82)$$

then

$$\begin{aligned} \left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle(\lambda, \theta)}{\partial \lambda \partial \theta} \right]_\alpha &= \left[\left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle(\lambda, \theta)}{\partial \lambda \partial \theta} \right]_l^+(\alpha) \right. \\ &\quad \cdot \left. \left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle(\lambda, \theta)}{\partial \lambda \partial \theta} \right]_r^+(\alpha) \right], \\ \left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle(\lambda, \theta)}{\partial \lambda \partial \theta} \right]^\alpha &= \left[\left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle(\lambda, \theta)}{\partial \lambda \partial \theta} \right]_l^-(\alpha) \right. \\ &\quad \cdot \left. \left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle(\lambda, \theta)}{\partial \lambda \partial \theta} \right]_r^-(\alpha) \right]. \end{aligned} \quad (83)$$

$$\begin{cases} \left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle(\lambda, \theta)}{\partial \lambda \partial \theta} \right]_l^+(\alpha) = \left[\frac{\partial((1/2)e^{-(2/3)} \langle \phi_1, \phi_2 \rangle(\lambda, \theta))}{\partial \theta} \right]_l^+(\alpha) - \frac{1}{2} \left[\langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{3}, \theta - \frac{1}{3} \right) \right]_l^+(\alpha) + c_{1l}^+(\alpha) e^{\lambda+\theta} + \frac{1}{2} c_{2l}^+(\alpha) \cos \left(\lambda - \frac{1}{3} \right), \\ \left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle(\lambda, \theta)}{\partial \lambda \partial \theta} \right]_r^+(\alpha) = \left[\frac{\partial((1/2)e^{-(2/3)} \langle \phi_1, \phi_2 \rangle(\lambda, \theta))}{\partial \theta} \right]_r^+(\alpha) - \frac{1}{2} \left[\langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{3}, \theta - \frac{1}{3} \right) \right]_r^+(\alpha) + c_{1r}^+(\alpha) e^{\lambda+\theta} + \frac{1}{2} c_{2r}^+(\alpha) \cos \left(\lambda - \frac{1}{3} \right), \\ \left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle(\lambda, \theta)}{\partial \lambda \partial \theta} \right]_l^-(\alpha) = \left[\frac{\partial((1/2)e^{-(3/4)} \langle \phi_1, \phi_2 \rangle(\lambda, \theta))}{\partial \theta} \right]_l^-(\alpha) - \frac{1}{2} \left[\langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{3}, \theta - \frac{1}{3} \right) \right]_l^-(\alpha) + c_{1l}^-(\alpha) e^{\lambda+\theta} + \frac{1}{2} c_{2l}^-(\alpha) \cos \left(\lambda - \frac{1}{3} \right), \\ \left[\frac{\partial^2 \langle \phi_1, \phi_2 \rangle(\lambda, \theta)}{\partial \lambda \partial \theta} \right]_r^-(\alpha) = \left[\frac{\partial((1/2)e^{-(2/3)} \langle \phi_1, \phi_2 \rangle(\lambda, \theta))}{\partial \theta} \right]_r^-(\alpha) - \frac{1}{2} \left[\langle \phi_1, \phi_2 \rangle \left(\lambda - \frac{1}{3}, \theta - \frac{1}{3} \right) \right]_r^-(\alpha) + c_{1r}^-(\alpha) e^{\lambda+\theta} + \frac{1}{2} c_{2r}^-(\alpha) \cos \left(\lambda - \frac{1}{3} \right), \end{cases} \quad (84)$$

where $(\lambda, \theta) \in [0, (\pi/2)] \times [0, 1]$ with initial conditions for $(\lambda, \theta) \in nq - 1/2, \pi/2h_{\times}x7 - 1/2, 1C; \setminus 0, \pi/2 \times 0, 1]$:

$$\begin{cases} [\langle \phi_1, \phi_2 \rangle(\lambda, \theta)]_l^+(\alpha) = c_{1l}^+(\alpha)(e^\lambda + \lambda\theta) + c_{2l}^+(\alpha)(e^\theta + \cos(\lambda\theta)), \\ [\langle \phi_1, \phi_2 \rangle(\lambda, \theta)]_r^+(\alpha) = c_{1r}^+(\alpha)(e^\lambda + \lambda\theta) + c_{2r}^+(\alpha)(e^\theta + \cos(\lambda\theta)), \\ [\langle \phi_1, \phi_2 \rangle(\lambda, \theta)]_l^-(\alpha) = c_{1l}^-(\alpha)(e^\lambda + \lambda\theta) + c_{2l}^-(\alpha)(e^\theta + \cos(\lambda\theta)), \\ [\langle \phi_1, \phi_2 \rangle(\lambda, \theta)]_r^-(\alpha) = c_{1r}^-(\alpha)(e^\lambda + \lambda\theta) + c_{2r}^-(\alpha)(e^\theta + \cos(\lambda\theta)), \end{cases} \quad (85)$$

and for $(\lambda, \theta) \in [0, (\pi/2)] \times [0, 1]$,

$$\left\{ \begin{aligned}
& [\langle \phi_1, \phi_2 \rangle (\lambda, 0)]_l^+ (\alpha) + \frac{1}{2} \int_0^\theta [\langle \phi_1, \phi_2 \rangle (\lambda, \sigma)]_l^+ (\alpha) d\sigma = \frac{1}{2} c_{1l}^+ (\alpha) e^\lambda (1 + e^\theta) + c_{2l}^+ (\alpha) \cos(\lambda) \left(1 + \frac{\theta}{2} \right), \\
& [\langle \phi_1, \phi_2 \rangle (\lambda, 0)]_r^+ (\alpha) + \frac{1}{2} \int_0^\theta [\langle \phi_1, \phi_2 \rangle (\lambda, \sigma)]_r^+ (\alpha) d\sigma = \frac{1}{2} c_{1r}^+ (\alpha) e^\lambda (1 + e^\theta) + c_{2r}^+ (\alpha) \cos(\lambda) \left(1 + \frac{\theta}{2} \right), \\
& [\langle \phi_1, \phi_2 \rangle (\lambda, 0)]_l^- (\alpha) + \frac{1}{2} \int_0^\theta [\langle \phi_1, \phi_2 \rangle (\lambda, \sigma)]_l^- (\alpha) d\sigma = \frac{1}{2} c_{1l}^- (\alpha) e^\lambda (1 + e^\theta) + c_{2l}^- (\alpha) \cos(\lambda) \left(1 + \frac{\theta}{2} \right), \\
& [\langle \phi_1, \phi_2 \rangle (\lambda, 0)]_r^- (\alpha) + \frac{1}{2} \int_0^\theta [\langle \phi_1, \phi_2 \rangle (\lambda, \sigma)]_r^- (\alpha) d\sigma = \frac{1}{2} c_{1r}^- (\alpha) e^\lambda (1 + e^\theta) + c_{2r}^- (\alpha) \cos(\lambda) \left(1 + \frac{\theta}{2} \right), \\
& [\langle \phi_1, \phi_2 \rangle (0, \theta)]_l^+ (\alpha) + \frac{1}{2} \int_0^\lambda [\langle \phi_1, \phi_2 \rangle (\zeta, \theta)]_l^+ (\alpha) d\zeta = \frac{1}{2} c_{1l}^+ (\alpha) e^\theta (1 + e^\lambda) + c_{2l}^+ (\alpha) \left(1 + \frac{\sin(\lambda)}{2} \right), \\
& [\langle \phi_1, \phi_2 \rangle (0, \theta)]_r^+ (\alpha) + \frac{1}{2} \int_0^\lambda [\langle \phi_1, \phi_2 \rangle (\zeta, \theta)]_r^+ (\alpha) d\zeta = \frac{1}{2} c_{1r}^+ (\alpha) e^\theta (1 + e^\lambda) + c_{2r}^+ (\alpha) \left(1 + \frac{\sin(\lambda)}{2} \right), \\
& [\langle \phi_1, \phi_2 \rangle (0, \theta)]_l^- (\alpha) + \frac{1}{2} \int_0^\lambda [\langle \phi_1, \phi_2 \rangle (\zeta, \theta)]_l^- (\alpha) d\zeta = \frac{1}{2} c_{1l}^- (\alpha) e^\theta (1 + e^\lambda) + c_{2l}^- (\alpha) \left(1 + \frac{\sin(\lambda)}{2} \right), \\
& [\langle \phi_1, \phi_2 \rangle (0, \theta)]_r^- (\alpha) + \frac{1}{2} \int_0^\lambda [\langle \phi_1, \phi_2 \rangle (\zeta, \theta)]_r^- (\alpha) d\zeta = \frac{1}{2} c_{1r}^- (\alpha) e^\theta (1 + e^\lambda) + c_{2r}^- (\alpha) \left(1 + \frac{\sin(\lambda)}{2} \right).
\end{aligned} \right. \quad (86)$$

We get

$$\left\{ \begin{aligned}
& [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_l^+ (\alpha) = c_{1l}^+ (\alpha) e^{\lambda+\theta} + c_{2l}^+ (\alpha) \cos(\lambda), \quad (\lambda, \theta) \in \left[0, \frac{\pi}{2} \right] \times [0, 1], \\
& [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_r^+ (\alpha) = c_{1r}^+ (\alpha) e^{\lambda+\theta} + c_{2r}^+ (\alpha) \cos(\lambda), \quad (\lambda, \theta) \in \left[0, \frac{\pi}{2} \right] \times [0, 1], \\
& [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_l^- (\alpha) = c_{1l}^- (\alpha) e^{\lambda+\theta} + c_{2l}^- (\alpha) \cos(\lambda), \quad (\lambda, \theta) \in \left[0, \frac{\pi}{2} \right] \times [0, 1], \\
& [\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_r^- (\alpha) = c_{1r}^- (\alpha) e^{\lambda+\theta} + c_{2r}^- (\alpha) \cos(\lambda), \quad (\lambda, \theta) \in \left[0, \frac{\pi}{2} \right] \times [0, 1].
\end{aligned} \right. \quad (87)$$

Then,

$$\begin{aligned}
[\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]_\alpha &= [c_{1l}^+ (\alpha) e^{\lambda+\theta} + c_{2l}^+ (\alpha) \cos(\lambda), c_{1r}^+ (\alpha) e^{\lambda+\theta} + c_{2r}^+ (\alpha) \cos(\lambda)], \\
[\langle \phi_1, \phi_2 \rangle (\lambda, \theta)]^\alpha &= [c_{1l}^- (\alpha) e^{\lambda+\theta} + c_{2l}^- (\alpha) \cos(\lambda), c_{1r}^- (\alpha) e^{\lambda+\theta} + c_{2r}^- (\alpha) \cos(\lambda)].
\end{aligned} \quad (88)$$

Now we denote

FIGURE 4: $c_1 = (-2, 0, 2; -2.75, 0, 2.75)$ and $c_2 = (-1.25, 0, 1.25; -3, 0, 3)$. FIGURE 5: $c_1 = (-2, 0, 2; -2.75, 0, 2.75)$ and $c_2 = (-1.25, 0, 1.25; -3, 0, 3)$.

FIGURE 6: The surface of the intuitionistic fuzzy solution.

$$\begin{aligned}
 [c_{1l}^+(\alpha)e^{\lambda+\theta} + c_{2l}^+(\alpha)\cos(\lambda), c_{1r}^+(\alpha)e^{\lambda+\theta} + c_{2r}^+(\alpha)\cos(\lambda)] &= N_\alpha, \\
 [c_{1l}^-(\alpha)e^{\lambda+\theta} + c_{2l}^-(\alpha)\cos(\lambda), c_{1r}^-(\alpha)e^{\lambda+\theta} + c_{2r}^-(\alpha)\cos(\lambda)] &= N^\alpha, \\
 [c_{1l}^+(\alpha)e^{\lambda+\theta}, c_{1r}^+(\alpha)e^{\lambda+\theta}] &= N'_\alpha, \\
 [c_{1l}^-(\alpha)e^{\lambda+\theta}, c_{1r}^-(\alpha)e^{\lambda+\theta}] &= N'^\alpha.
 \end{aligned} \tag{89}$$

It easy to see that (N_α, N^α) and (N'_α, N'^α) satisfy the conditions of Proposition 1, and by Lemma 1, we can build the intuitionistic fuzzy solution $\langle \phi_1, \phi_2 \rangle(\lambda, \theta) \in \text{IF}_1$ for

equations (52)–(54) by the form as follows for every $\alpha \in [0, 1]$:

$$\begin{aligned}
 [\langle \phi_1, \phi_2 \rangle(\lambda, \theta)]_\alpha &= [c_{1l}^+(\alpha)e^{\lambda+\theta} + c_{2l}^+(\alpha)\cos(\lambda), c_{1r}^+(\alpha)e^{\lambda+\theta} \\
 &\quad + c_{2r}^+(\alpha)\cos(\lambda)], \\
 [\langle \phi_1, \phi_2 \rangle(\lambda, \theta)]^\alpha &= [c_{1l}^-(\alpha)e^{\lambda+\theta} + c_{2l}^-(\alpha)\cos(\lambda), c_{1r}^-(\alpha)e^{\lambda+\theta} \\
 &\quad + c_{2r}^-(\alpha)\cos(\lambda)].
 \end{aligned} \tag{90}$$

Then, $\langle \phi_1, \phi_2 \rangle(x, y)$ is an intuitionistic fuzzy solution which verifies the initial conditions (53)–(54) and can be written as follows:

$$\langle \phi_1, \phi_2 \rangle(x, y) = c_1 e^{\lambda+\theta} + c_2 \cos(\lambda). \quad (91)$$

Figures 4 and 5 represent the membership and non-membership functions of triangular intuitionistic fuzzy numbers $c_1 = (-2, 0, 2; -2.75, 0, 2.75)$ and $c_2 = (-1.25, 0, 1.25; -3, 0, 3)$ and the simulation of α -cuts of the intuitionistic fuzzy solution $\langle \phi_1, \phi_2 \rangle(x, y)$ at some values of (λ, θ) .

In Figure 6, we show the surface of the intuitionistic fuzzy solution with triangular intuitionistic fuzzy numbers $c_1 = (-2, 0, 2; -2.75, 0, 2.75)$ and $c_2 = (-1.25, 0, 1.25; -3, 0, 3)$.

5. Conclusion

This research work proposed an integral boundary value problem for an intuitionistic fuzzy partial functional differential equation with state-dependent delay. We have proved the existence-uniqueness of intuitionistic fuzzy solutions for these problems with integral boundary conditions by applying the Banach fixed point theorem. These results are illustrated by numerical examples. Studying local and nonlocal intuitionistic fuzzy delay partial differential equations is the next step that will be considered.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors are responsible for study conception and design, data collection, analysis and interpretation of results, and manuscript preparation.

References

- [1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.
- [2] K. T. Atanassov, *Intuitionistic Fuzzy Sets*, Academic Press, New York, NY, USA, 1983.
- [3] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, pp. 87–96, 1986.
- [4] P. Melo-Pinto, T. Kim, K. Atanassov, E. Sotirova, A. Shannon, and M. Krawczak, "Generalized net model of e-learning evaluation with intuitionistic fuzzy estimations," in *Issues in the Representation and Processing of Uncertain and Imprecise Information*, pp. 241–249, Warszawa, Poland, 2005.
- [5] S. Sotirov, E. Sotirova, P. Melin, O. Castillo, and K. Atanassov, "Modular neural network preprocessing procedure with intuitionistic fuzzy intercriteria analysis method," in *Flexible Query Answering Systems 2015*, T. Andreasen, Ed., pp. 175–186, Springer, Cham, Switzerland, 2016.
- [6] S. K. De, R. Biswas, and A. R. Roy, "An application of intuitionistic fuzzy sets in medical diagnosis," *Fuzzy Sets Systems*, vol. 117, pp. 209–213, 2001.
- [7] A. Kharal, "Homeopathic drug selection using intuitionistic fuzzy sets," *Homeopathy*, vol. 98, pp. 35–39, 2009.
- [8] D. F. Li and C. T. Cheng, "New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions," *Pattern Recognition Letters*, vol. 23, pp. 221–225, 2002.
- [9] M. H. Shu, C. H. Cheng, and J. R. Chang, "Using intuitionistic fuzzy sets for fault-tree analysis on printed circuit board assembly," *Microelectronics Reliability*, vol. 46, no. 12, pp. 2139–2148, 2006.
- [10] S. Wan and J. Dong, *Decision Making Theories and Methods Based on Interval-Valued Intuitionistic Fuzzy Set*, Springer, Singapore, 2020.
- [11] S. Wan, G. Xu, and J. Dong, "An Atanassov intuitionistic fuzzy programming method for group decision making with interval-valued Atanassov intuitionistic fuzzy preference relations," *Applied Soft Computing*, vol. 95, Article ID 106556, 2020.
- [12] S. Melliani and L. S. Chadli, "Intuitionistic fuzzy differential equation," *Notes on Intuitionistic Fuzzy Sets*, vol. 6, pp. 37–41, 2000.
- [13] B. Ben Amma, S. Melliani, and L. S. Chadli, "The Cauchy problem of intuitionistic fuzzy differential equations," *Notes on Intuitionistic Fuzzy Sets*, vol. 24, no. 1, pp. 37–47, 2018.
- [14] S. Melliani, M. Elomari, M. Atroui, and L. S. Chadli, "Intuitionistic fuzzy differential equation with nonlocal condition," *Notes on Intuitionistic Fuzzy Sets*, vol. 21, no. 4, pp. 58–68, 2015.
- [15] B. Ben Amma, S. Melliani, and L. S. Chadli, *Intuitionistic Fuzzy Functional Differential Equations, Fuzzy Logic in Intelligent System Design: Theory and Applications*, pp. 335–357, Springer International Publishing, Cham, Switzerland, 2018.
- [16] B. Ben Amma, S. Melliani, and L. S. Chadli, "Numerical solution of intuitionistic fuzzy differential equations by Euler and Taylor methods," *Notes on Intuitionistic Fuzzy Sets*, vol. 22, no. 2, pp. 71–86, 2016.
- [17] B. Ben Amma, S. Melliani, and L. S. Chadli, "Numerical solution of intuitionistic fuzzy differential equations by Adams three order predictor-corrector method," *Notes on Intuitionistic Fuzzy Sets*, vol. 22, no. 3, pp. 47–69, 2016.
- [18] M. Mary Jansi Rani and K. Jamshida, "Numerical solution of intuitionistic fuzzy differential equations using Runge-Kutta method," *International Journal of Scientific Research and Modern Education*, vol. 5, no. 2, pp. 2455–2460, 2020.
- [19] S. Melliani, H. Atti, B. Ben Amma, and L. S. Chadli, "Solution of n -th order intuitionistic fuzzy differential equation by variational iteration method," *Notes on Intuitionistic Fuzzy Sets*, vol. 24, pp. 92–105, 2018.
- [20] V. Nirmala, V. Parimala, and P. Rajarajeswari, "Multiple numerical solutions for intuitionistic fuzzy differential equations," *International Journal of Advanced Research Trends in Engineering and Technology*, vol. 5, no. 12, pp. 477–483, 2018.
- [21] S. Melliani and L. S. Chadli, "Introduction to intuitionistic fuzzy partial differential equations," *Notes on Intuitionistic Fuzzy Sets*, vol. 7, pp. 39–42, 2001.
- [22] B. Ben Amma, S. Melliani, and L. S. Chadli, "Integral boundary value problem for intuitionistic fuzzy partial hyperbolic differential equations," *Nonlinear Analysis and Boundary Value Problems*, Springer, vol. 292, Berlin, Germany, 2019.

- [23] B. Ben Amma, S. Melliani, and L. S. Chadli, "The existence and uniqueness of intuitionistic fuzzy solutions for intuitionistic fuzzy partial functional differential equations," *International Journal of Differential Equations*, vol. 2019, Article ID 9210641, 13 pages, 2019.
- [24] S. Bernard, "Moving the boundaries of granulopoiesis modelling," *Bulletin of Mathematical Biology*, vol. 78, pp. 2358–2363, 2016.
- [25] M. Craig, A. R. Humphries, and M. C. Mackey, "A mathematical model of granulopoiesis incorporating the negative feedback dynamics and kinetics of G-CSF/neutrophil binding and internalization," *Bulletin of Mathematical Biology*, vol. 78, pp. 2304–2357, 2016.
- [26] F. A. Davidson and S. A. Gourley, "The effects of temporal delays in a model for a food-limited diffusion population," *Journal of Mathematical Analysis and Applications*, vol. 261, pp. 633–648, 2001.
- [27] F. Hartung, T. Krisztin, H.-O. Walther, and J. Wu, "Chapter 5 functional differential equations with state-dependent delays: theory and applications," *Handbook of Differential Equations: Ordinary Differential Equations*, Elsevier, vol. 4, pp. 435–545, Amsterdam, Netherlands, 2006.
- [28] T. Heil, I. Fischer, W. Elsaer, B. Krauskopf, K. Green, and A. Gavrielides, "Delay dynamics of semiconductor lasers with short external cavities: bifurcation scenarios, and mechanisms," *Physical Review E*, vol. 67, Article ID 066214, 2003.
- [29] J. M. Mahaffy, J. Bélair, and M. C. Mackey, "Hematopoietic model with moving boundary condition and state dependent delay: applications in erythropoiesis," *Journal of Theoretical Biology*, vol. 190, pp. 135–146, 1998.
- [30] A. Otto and G. Radons, "Transformations from variable delays to constant delays with applications in engineering and biology," in *Advances in Delays and Dynamics*, vol. 7, pp. 169–183, Springer International Publishing, Cham, Switzerland, 2017.
- [31] A. V. Rezounenko, "A condition on delay for differential equations with discrete state-dependent delay," *Journal of Mathematical Analysis and Applications*, vol. 385, pp. 506–516, 2012.
- [32] J. Sieber and B. Krauskopf, "Complex balancing motions of an inverted pendulum subject to delayed feedback control," *Physica D: Nonlinear Phenomena*, vol. 197, no. 3, pp. 332–345, 2004.
- [33] L. Yongkun, "Periodic solutions for delay Lotka Volterra competition systems," *Journal of Mathematical Analysis and Applications*, vol. 246, no. 1, pp. 230–244, 2000.
- [34] S. Melliani, M. Elomari, L. S. Chadli, and R. Ettoussi, "Intuitionistic fuzzy metric space," *Notes on Intuitionistic Fuzzy Sets*, vol. 21, no. 1, pp. 43–53, 2015.