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Review Article

L^p Smoothness on Weighted Besov–Triebel–Lizorkin Spaces in terms of Sharp Maximal Functions

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It is known, in harmonic analysis theory, that maximal operators measure local smoothness of L^p functions. These operators are used to study many important problems of function theory such as the embedding theorems of Sobolev type and description of Sobolev space in terms of the metric and measure. We study the Sobolev-type embedding results on weighted Besov–Triebel–Lizorkin spaces via the sharp maximal functions. The purpose of this paper is to study the extent of smoothness on weighted function spaces under the condition $M_\alpha^\#(f) \in L^{p,\mu}$, where μ is a lower doubling measure, $M_\alpha^\#(f)$ stands for the sharp maximal function of f, and $0 \le \alpha \le 1$ is the degree of smoothness.

1. Introduction and Main Result

In this paper, we consider the some continuous embeddings on weighted Besov–Triebel–Lizorkin spaces via a general sharp maximal function introduced by Calderón and Scott [6]. Furthermore, we investigate the spaces introduced by Hajłasz [13] that are defined via pointwise inequalities and their connection with the Triebel–Lizorkin spaces. For more details, see [11, 12].

Now, let us begin by recalling some definitions and classical results in harmonic analysis on the n-dimensional Euclidean space $\mathbb{R}^n$ needed for later sections.

(1) A cube on $\mathbb{R}^n$ will always mean a cube with sides parallel to the axes and has nonempty interior. For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, we denote by Q_{jk} the dyadic cube $2^{-j}([0,1]^n + k)$, where $l(Q_{jk}) = 2^{-j}$ is its side length, $x_{Q_{jk}} = 2^{-j}k$ is its lower "left-corner," and $c_{Q_{jk}}$ is its center. We set $Q = \{Q_{jk}\colon\ j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ and $j_Q = -\log_2 l(Q)$ for all $Q \in Q$. When the dyadic cube Q appears as an index, such as $\sum_{Q\in\mathcal{Q}}$, it is understood that Q runs over all dyadic cubes in $\mathbb{R}^n$. For a function ν and dyadic cube $Q = Q_{jk}$, set

$$\nu_Q(x) = |Q|^{-(1/2)}\nu\left(2^j x - k\right) = |Q|^{(1/2)}\nu_j\left(x - x_Q\right), \qquad (1)$$

for all $x \in \mathbb{R}^n$, where $\nu_j(x) = 2^{nj}\nu(2^j x)$.

(2) Throughout the paper, w denotes a weight function, i.e., w is an almost every (a.e.) positive locally integrable function on $\mathbb{R}^n$. A function $f \in L^p(w)$, $0 < p < \infty \Longleftrightarrow$

$$\|f\|_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\mathrm{d}x\right)^{1/p} < \infty, \qquad (2)$$

and f belongs to the weak-L^p spaces, denoted by $L^{p,\infty}(w) \Longleftrightarrow$

$$\|f\|_{L^{p,\infty}(w)} = \sup_{\lambda>0} \lambda w\left(\{x \in \mathbb{R}^n\colon \quad f(x) > \lambda\}\right)^{1/p} < \infty. \qquad (3)$$

If $w = 1$, we do not write the subscript w.

A weight function w is said to be in the Muckenhoupt classes A_p, where $1 \le p < \infty$, if there exists a constant $C_p > 0$ such that for every cube Q,

$$\frac{1}{|Q|}\int_Q w\mathrm{d}y\left(\frac{1}{|Q|}\int_Q w^{1-p'}\mathrm{d}y\right)^{p-1} \le C_p. \qquad (4)$$

When $1 < p < \infty$, $(1/p) + (1/p') = 1$; for $p = 1$,

$$\frac{1}{|Q|}\int_Q w(y)\mathrm{d}y \leq C_1 w(x), \tag{5}$$

for a.e. $x \in Q$, or equivalently $Mw(x) \leq C_1 w(x)$ for a.e. $x \in \mathbb{R}^n$, where M is the Hardy–Littlewood maximal operator.

The class A_p was introduced by Muckenhoupt [16] in order to characterize the boundedness of the Hardy–Littlewood maximal operator M on the weighted Lebesgue spaces [8, 12]. The pioneering work of Muckenhoupt [16] showed that

$$M\colon L^p(w) \longrightarrow L^p(w), \tag{6}$$

$\Longleftrightarrow w \in A_p$ when $1 < p < \infty$ and

$$M\colon L^1(w) \longrightarrow L^{1,\infty}(w), \quad \Longleftrightarrow w \in A_1. \tag{7}$$

A weight function w is in Muckenhoupt's class $A_p(\mathbb{R}^n)$, $1 \leq p < \infty$, of weights if there exists a constant $C_p > 0$ such that for all cubes Q in $\mathbb{R}^n$,

$$\left(\frac{1}{|Q|}\int_Q w(y)\mathrm{d}y\right)\left(\frac{1}{|Q|}\int_Q w(y)^{1-p'}\mathrm{d}y\right)^{p-1} \leq C_p. \tag{8}$$

When $1 < p < \infty$, $(1/p) + (1/p') = 1$; well, for $p = 1$,

$$\frac{1}{|Q|}\int_Q w(y)\mathrm{d}y \leq C_1 w(x), \tag{9}$$

for a.e. $x \in B$, or equivalently $Mw(x) \leq C_1 w(x)$ for a.e. $x \in \mathbb{R}^n$, where M is the Hardy–Littlewood maximal operator.

(3) Note that if $w \in A_p$, then w is a doubling measure, i.e., there exists a constant $C \geq 1$ such that for all x and all $r > 0$,

$$w(B(x,2r)) \leq Cw(B(x,r)). \tag{10}$$

Another class of functions that plays an important role in harmonic analysis and in partial differential equation theory is the class of functions with bounded mean oscillation denoted by $\mathrm{BMO}(w)$, i.e., $\varphi \in \mathrm{BMO}(w)$, if there is a constant C:

$$\sup_Q \frac{1}{w(Q)}\int_Q \left|\varphi(y) - \varphi_Q\right| w(y)\mathrm{d}y < C, \tag{11}$$

where $\varphi_Q = (1/w(Q))\int_Q \varphi(y)w(y)\mathrm{d}y$ is the average of f on Q with respect to $\mathrm{d}w$. The smallest constant C for which (11) is satisfied is taken to be the norm of φ in the space $BMO(w)$ and is denoted by $\|\varphi\|_{BMO(w)}$.

(4) The sharp maximal function $M^{\#}f(x)$ of f is defined by

$$M^{\#}f(x) = \sup_Q \inf_{c\in\mathbb{R}} \frac{1}{|Q|}\int_Q |f(x) - c|\mathrm{d}x, \tag{12}$$

where Q is taken over all cubes in $\mathbb{R}^n$. Let $\alpha \geq 0$. The sharp fractional maximal function $M_\alpha^{\#}(f)$ of f is defined by

$$M_\alpha^{\#}(f)(x) = \sup_{x\in Q} \inf_{c\in\mathbb{R}} \frac{1}{|Q|^{(\alpha/n)}}\int_Q |f(x) - c|\mathrm{d}x. \tag{13}$$

(5) The space of Schwartz functions: let $\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions on $\mathbb{R}^n$ with the classical topology generated by the family of seminorms:

$$\|\nu\|_{k,N} = \sup_{x\in\mathbb{R}^n} \sup_{|\beta|\leq N} (1 + |x|)^k \left|\partial^\beta \nu(x)\right| \quad k, N \in \mathbb{N}_0, \nu \in \mathcal{S}(\mathbb{R}^n). \tag{14}$$

The topological dual space $\mathcal{S}'(\mathbb{R}^n)$ of $\mathcal{S}(\mathbb{R}^n)$ is the set of all continuous linear functional the space $\mathcal{S}(\mathbb{R}^n)$ is endowed with the weak $*$-topology. We denote by $\mathcal{S}_\infty(\mathbb{R}^n)$ the topological subspace of functions in $\mathcal{S}(\mathbb{R}^n)$ having all vanishing moments:

$$\mathcal{S}_\infty(\mathbb{R}^n) = \left\{\nu \in \mathcal{S}(\mathbb{R}^n)\colon \int_{\mathbb{R}^n} x^\beta \nu(x)\mathrm{d}x = 0, \quad \text{for every } \beta \in \mathbb{N}^n\right\}. \tag{15}$$

$\mathcal{S}_\infty'(\mathbb{R}^n)$ denotes the topological dual space of $\mathcal{S}_\infty(\mathbb{R}^n)$, namely, the set of all continuous linear functional on $\mathcal{S}_\infty'(\mathbb{R}^n)$. The space $\mathcal{S}_\infty'(\mathbb{R}^n)$ is also endowed with the weak $*$-topology. It is well known that $\mathcal{S}_\infty'(\mathbb{R}^n) = (\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n))$ as topological spaces, where $\mathcal{P}(\mathbb{R}^n)$denotes the set of all polynomials on $\mathbb{R}^n$; see, for example, ([21], Proposition 8.1). Similarly, for any $R \in \mathbb{N}$, the space $\mathcal{S}_R(\mathbb{R}^n)$ is defined to be the set of all Schwartz functions having vanishing moments of order R and $\mathcal{S}_R'(\mathbb{R}^n)$ is its topological dual space. We write $\mathcal{S}_{-1}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$.

The Fourier transform, $\mathcal{F}\nu = \widehat{\nu}$, of Schwartz function ν is defined by

$$\widehat{\nu}(\xi) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-i\xi\cdot x}\nu(x)\mathrm{d}y. \tag{16}$$

The convolution of two functions $\nu, \mu \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\nu * \mu(x) = \int_{\mathbb{R}^n} \nu(x - y)\mu(y)\mathrm{d}y \tag{17}$$

and still belongs to $\mathcal{S}(\mathbb{R}^n)$.

The convolution operator can be extended to $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ via $\nu * f(x) = \langle f, \mu(x - \cdot)\rangle$. It makes sense pointwise and is a C^∞ function on $\mathbb{R}^n$ of at most polynomial growth.

To simplify notation, we write often $\nu f = \nu * f$. In some other situations, to avoid confusion, we keep the notation $\nu * f$. As usual, ν_t denotes the function defined by $\nu_t(x) = t^{-n}\nu(x/t)$.

(6) In the rest of this paper, C expresses unspecified positive constant, possibly different at each occurrence; the symbol $A \leq B$ means that $A \leq CB$. If $A \leq B$ and $B \leq A$, then we write $A \simeq B$. The Greek letter χ_S denotes the characteristics function of a sphere S, where S is a measurable subset of $\mathbb{R}^n$ and $|S|$ represents its Lebesgue measure; p' and s' always denote the conjugate index of any $p > 1$ and $s > 1$, that is, $(1/p')\colon = 1 - (1/p)$ and $(1/s')\colon = 1 - (1/s)$.

Function spaces play a crucial role in the genesis of functional analysis and are widely used in the development of the modern analysis of partial differential equations. For instance, the classical Besov–Triebel–Lizorkin spaces are a class of function spaces containing many well-known classical function spaces and are more suitable in the treatment of a large type of partial differential equations (see for instance [5, 10]). A comprehensive treatment of these function spaces and their history can be found in Triebel's monographs [18, 19] and in the fundamental paper of Frazier and Jawerth [11].

In recent years, there has been increasing interest in a new family of function spaces, called new class of Besov–Triebel–Lizorkin spaces. These spaces unify and generalize many classical spaces including Besov spaces, Morrey spaces, and Triebel–Lizorkin spaces (see for instance [20]).

In this paper, we study the extent of smoothness on weighted function spaces under the condition $M_\alpha^\# f \in L^{p,\mu}$, where μ is a lower doubling measure, $M_\alpha^\# f$ stands for the sharp maximal function of f, and $0 \le \alpha \le 1$ is the degree of smoothness. When $\alpha = 0$, $M_0^\# f = M^\# f$ is the classical sharp maximal function. It is well known that the Hardy–Littlewood maximal function Mf is controlled by the sharp maximal function $M^\# f$ via the celebrated Stein–Fefferman inequality: $\|\mathrm{Mf}\|_p \le \|M^\# f\|_p$ and in the case of $\alpha = 1$, it is shown that $\|M_1^\# f\|_p$: $\|f\|_{H_p}$ for some range of p. As a result, we extend the above results to the some general weighted spaces. Embedding results on weighted Besov–Triebel–Lizorkin spaces are obtained. Namely, $\|f\|_{\dot{F}^{\gamma,q}_{p_*,w}} \le \|M_\alpha^\# (f)\|_{p,w}$ (Theorem 1). As a consequence, we obtain $\|f\|_{\dot{F}^{\gamma,q}_{p_*,w}} \le \|f\|_{\dot{W}^{\alpha,p}(w)}$, where $\dot{W}^{\alpha,p}(w)$ stands for the fractional Sobolev space.

Now, we are ready to present the main theorem of this section.

Theorem 1. *Let α and γ be real numbers satisfying $0 \le \alpha \le 1$ and $\gamma < \alpha$, and w is the lower d–regular doubling measure. Suppose that $w\{f(x) > \varepsilon\} < \infty$ for every $\varepsilon > 0$ and $M_\alpha^\# (f) \in L^p(w)$ for $(d/(n+\alpha)) < p < (d/(\alpha-\gamma))$. Then, for each $0 < q \le \infty$,*

$$\|f\|_{\dot{F}^{\gamma,q}_{p_*,w}} \le \left\|M_\alpha^\# (f)\right\|_{p,w}, \quad \frac{1}{p_*} = \frac{1}{p} - \frac{\alpha-\gamma}{d}. \tag{18}$$

Remark 1. The condition that $w\{f(x) > \varepsilon\} < \infty$ for every $\varepsilon > 0$ is necessary. On the other hand, under this condition, $M_\alpha^\# (f) \in L^p(w)$ only if $p > (d/(n+\alpha))$.

Proof. If f is not a constant function, then there exists a ball $B(x_0, R)$ such that

$$\inf_{c\in\mathbb{R}} \int_{B(x_0,R)} |f(y) - c| \mathrm{d}y = c_0 > 0. \tag{19}$$

Therefore, for all $x \in \mathbb{R}^n$,

$$M_\alpha^\# (f)(x) \pm \left(R + |x - x_0|\right)^{-\alpha-n}. \tag{20}$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^n} \left|M_\alpha^\# (f)\right|^p w(x)\mathrm{d}y &\pm \int_{|x-x_0|>R} \left(R + |x - x_0|\right)^{(-\alpha-n)p} w(x)\mathrm{d}x \\ &\pm \sum_{k\ge 0} \left(R + 2^{k+1}R\right)^{(-\alpha-n)p} \int_{|x-x_0|<2^{k+1}R} w(x)\mathrm{d}x \\ &\pm R^{(-\alpha-n)p+d} \sum_{k\ge 0} 2^{k(-\alpha-n)p+kd} = \infty,\end{aligned} \tag{21}$$

if $p \le (d/(n+\alpha))$. □

Corollary 1. *Under the same conditions in Theorem 1, we have, for each $0 < q \le \infty$,*

$$\|f\|_{\dot{B}^{\gamma,q}_{p_*,w}} \le \left\|M_\alpha^\# (f)\right\|_{p,w}, \quad \frac{1}{p_*} = \frac{1}{p} - \frac{\alpha-\gamma}{d}, \tag{22}$$

for each $0 < p_* < q \le \infty$.

Proof. By Minkowski's inequality, we have

$$\|f\|_{\dot{B}^{\gamma,q}_{p_*,w}} \le \|f\|_{\dot{F}^{\gamma,q}_{p_*,w}}, \quad 0 < p_* < q \le \infty. \tag{23}$$

Then, applying Theorem 1, we obtain (22). This completes the proof. □

2. Preliminaries

In this section, we introduce some necessary and important definitions, notations, lemmas, and results.

Definition 1. Let ν be in the Schwartz space with supp $\widehat{\nu}$ contained in an annulus about the origin and

$$\sum_{j\in\mathbb{Z}} \widehat{\nu}\left(2^{-j}\xi\right) = 1 \quad \text{for all } \xi \ne 0. \tag{24}$$

Let μ be a doubling measure and $0 < p, q \le \infty$ and $\gamma \in \mathbb{R}$; the homogeneous Triebel–Lizorkin space $\dot{F}^{\gamma,q}_{p,\nu}$ is the set of all distributions f (modulo polynomials) such that

$$\begin{aligned}\|f\|_{\dot{F}^{\gamma,q}_{p,\mu}} &= \left\|\left(\sum_{j\in\mathbb{Z}} 2^{j\gamma q}\left|\nu_{2^{-j}} f\right|^q\right)^{1/q}\right\|_{p,\mu} < \infty; \quad 0 < p, q < \infty, \\ \|f\|_{\dot{F}^{\gamma,q}_{\infty,\mu}} &= \sup_Q \left\{\frac{1}{\mu(Q)} \int_Q \sum_{j=-\log_2 l(Q)}^{\infty} 2^{j\gamma q}\left|\nu_{2^{-j}} f\right|^q \mathrm{d}\mu(x)\right\}^{1/q} < \infty; \quad 0 < q \le \infty,\end{aligned} \tag{25}$$

with the interpretation that when $q = \infty$,

$$\|f\|_{\dot{F}^{\gamma,q}_{\infty,w}} = \sup_{Q} \sup_{j \geq -\log_2 l(Q)} \frac{1}{\mu(Q)} \int_Q 2^{j\gamma} \left|\nu_{2^{-j}} f\right| d\mu(x) < \infty. \tag{26}$$

The homogeneous Besov–Lipschitz space $\dot{B}^{\gamma,q}_{p}$ is the set of all distributions f (modulo polynomials) such that

$$\|f\|_{\dot{B}^{\gamma,q}_{p}} = \left(\sum_{j \in \mathbb{Z}} 2^{j\gamma q} \|\nu_{2^{-j}} f\|^q_{p,w} \right)^{1/q} < \infty; \quad 0 < p, q < \infty. \tag{27}$$

The supremum is taken over all dyadic cubes Q, and $l(Q)$ denotes the length of sides of the cube Q.

Moreover, it is well known that the Besov–Lipschitz spaces and the Triebel–Lizorkin spaces are independent of the choices of ν (see, for example [2–4, 11]). Throughout this paper, ν will be taken as in Definition 1. It is well known that many classical smoothness spaces are covered by the Besov and Triebel–Lizorkin spaces. We recall some examples in the case when $d\mu = w dx$ and $w \in A_\infty$:

(1) $\dot{F}^{0,2}_{p,w} = H_{p,w}$, $0 < p < \infty$.

(2) $F^{0,2}_{p,w} = h_{p,w}$, $0 < p < \infty$, where $H_{p,w}$ denotes the weighted Hardy spaces of $f \in \mathcal{S}'$ for which

$$\|f\|_{H_{p,w}} = \left\| \sup_{t>0} \mu_t * f \right\|_{p,w} < \infty \tag{28}$$

and $h_{p,w}$ is the local weighted Hardy space of $f \in \mathcal{S}'$ for which

$$\|f\|_{h_{p,w}} = \left\| \sup_{0<t<1} \mu_t * f \right\|_{p,w} < \infty, \tag{29}$$

where μ is a fixed function in $\mathcal{S}$ with $\int_{\mathbb{R}^n} \mu(x) dx \neq 0$. By the fundamental work of Fefferman and Stein [9] adapted to the weighted case, $H_{p,w}$ or $h_{p,w}$ does not depend on the choices of μ. In particular,

$$\dot{F}^{0,2}_{p,w} = L^p(w), \quad 1 < p < \infty. \tag{30}$$

(3) $\dot{F}^{\gamma,2}_{p,w} = H^{\gamma}_{p,w}$, $1 < p < \infty$, where $H^{\gamma}_{p,w}$ denotes the weighted Bessel potential space defined by

$$\|f\|_{H^{\gamma}_{p,w}} = \left\| \mathcal{F}^{-1} \left(1 + |\xi|^2\right)^{\gamma/2} \mathcal{F} f \right\|_{p,w}. \tag{31}$$

In particular, when the exponent is a natural number, say $\gamma = N \in \mathbb{N}$, then the weighted Bessel potential space can be identified with the classical Sobolev space:

$$W^N_{p,w} = \left\{ f \in L^{p,w}: \left\| \sum_{|\sigma| \leq N} \partial^\sigma f \right\|_{L^{p,w}} < \infty \right\}, \quad 1 < p < \infty. \tag{32}$$

(4) $\dot{F}^{0,2}_{\infty,w} = \mathrm{BMO}(w)$.

All the above identities have to be understood in the sense of equivalent quasi-norms.

Definition 2. We say that a doubling measure μ is lower d–regular, where $d \geq n$, if there is some constant $C > 0$ such that

$$\mu(B(x,t)) \geq C t^d \tag{33}$$

holds for all ball $B(x,t) \subset \mathbb{R}^n$.

Remark 2. An example of measure μ lower d–regular is $d\mu = w dx$,where

$$w(x) = |x|^\alpha \log^\beta\left(2 + |x|^{-1}\right). \tag{34}$$

In fact, $w \in A_p$ if $-n < \alpha < n(p-1)$ and $\beta \in \mathbb{R}$; hence, w is doubling. Moreover, if $0 \leq \alpha < n(p-1)$ and $\beta \geq 0$, then w satisfies $w(B(x,t)) \geq C t^{n+\alpha}$ for all $0 < t < \infty$ and all x.

Lemma 1. *Let $w \in A_p$ and d–regular. Then, we have*

$$\left\| M^{\#}_{\alpha}(f) \right\|_{p,w} \leq \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{\alpha p + d}} w(x) w(y) dx dy \right)^{1/p}, \quad 1 \leq p < \infty. \tag{35}$$

Proof. Let Q be a cube and $x, y \in Q$. Then,

$$\begin{aligned} \left|f(y) - f_Q\right| &= \left| f(y) - \frac{1}{|Q|} \int_Q f(z) \right| dz \leq \frac{1}{|Q|} \int_Q |f(y) - f(z)| dz \\ &\leq |f(y) - f(x)| + \frac{1}{|Q|} \int_Q |f(z) - f(x)| dz. \end{aligned} \tag{36}$$

Integrating over the cube Q with respect to y, we get

$$\int_Q \left|f(y) - f_Q\right| dy \leq \int_Q |f(y) - f(x)| dy. \tag{37}$$

If $w \in A_1$, then we have for almost all $x \in Q$, $w(x) \succcurlyeq (1/|Q|) \int_Q w(y) dy \succcurlyeq |Q|^{-1+(d/n)}$. Hence,

$$\int_Q \left|f(y) - f_Q\right| dy \leq |Q|^{1-(d/n)} \int_Q |f(y) - f(x)| w(y) dy. \tag{38}$$

The last inequality implies that

$$\begin{aligned} \frac{1}{|Q|^{(\alpha/n)+1}} \int_Q \left|f(y) - f_Q\right| dy &\leq |Q|^{(-\alpha-d)/n} \int_Q |f(y) - f(x)| w(y) dy \\ &\leq \int_{\mathbb{R}^n} \frac{|f(y) - f(x)|}{|y-x|^{\alpha+d}} w(y) dy. \end{aligned} \tag{39}$$

On the other hand, if $w \in A_p$, $p > 1$; then using (37) and Hölder's inequality, we obtain

$$\begin{aligned}\int_Q |f(y)-f_Q|\,\mathrm{d}y &\le \left(\int_Q |f(y)-f(x)|^p w(y)\mathrm{d}y\right)^{1/p}\left(\int_Q w^{1-p'}(y)\mathrm{d}y\right)^{(p-1)/p}\\ &\le |Q|^{1-(d/np)}\left(\int_Q |f(y)-f(x)|^p w(y)\mathrm{d}y\right)^{1/p}\\ &\le |Q|^{1-(d/np)}\left(\int_Q |f(y)-f(x)|^p w(y)\mathrm{d}y\right)^{1/p}\\ &\le |Q|^{1+(\alpha/n)}\left(\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{\alpha p+d}}w(y)\mathrm{d}y\right)^{1/p}.\end{aligned}\tag{40}$$

Therefore, we conclude that

$$M_\alpha^{\#}(f)(x)\le\left(\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{\alpha p+d}}w(y)\mathrm{d}y\right)^{1/p},\quad 1\le p<\infty.\tag{41}$$

This completes the proof. □

Lemma 2. *We say that f is in the fractional Sobolev space $\dot{W}^{\alpha,p}(w)$, $0<\alpha<1$, $1\le p<\infty$, if*

$$\|f\|_{\dot{W}^{\alpha,p}(w)}=\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{\alpha p+d}}w(x)w(y)\mathrm{d}x\mathrm{d}y\right)^{1/p}<\infty.\tag{42}$$

Corollary 2. *Let $w\in A_p$ and lower d–regular, $0<\alpha<1$, $1\le p<\infty$, and $f\in\dot{W}^{\alpha,p}(w)$. Then,*

$$\left\|M_\alpha^{\#}(f)\right\|_{p,w}\le\|f\|_{\dot{W}^{\alpha,p}(w)}.\tag{43}$$

One can immediately obtain the following corollary.

Corollary 3. *Let α and γ be real numbers satisfying $0<\alpha<1$ and $\gamma<\alpha$. Assume $w\in A_p$ and $f\in\dot{W}^{\alpha,p}(w)$ with $(d/(n+\alpha))<p<(d/(\alpha-\gamma))$. Then, for each $0<q\le\infty$,*

$$\|f\|_{\dot{F}^{\gamma,q}_{p_*,w}}\le\|f\|_{\dot{W}^{\alpha,p}(w)},\tag{44}$$

where p_* is given by $(1/p_*)=(1/p)-((\alpha-\gamma)/d)$.

Recall that for $0<\alpha<1$, $1<p<\infty$ and $w\in A_\infty$; we have (see [17])

$$\|f\|_{\dot{F}^{\alpha,\infty}_{p,w}}\approx\left\|M_\alpha^{\#}(f)\right\|_{p,w}.\tag{45}$$

In particular,

$$\|f\|_{\dot{F}^{\gamma,q}_{p_*,w}}\le\|f\|_{\dot{F}^{\alpha,\infty}_{p,w}},\tag{46}$$

with $0<\alpha<1$, $1<p<\infty$, and p_* is as before.

Lemma 3. *Let $f\in W^{1,1}_{loc}(\mathbb{R}^n)$ and $0\le\alpha\le1$. Then, for every $x\in\mathbb{R}^n$, there is a constant $C(n)$ such that*

$$M_\alpha^{\#}(f)(x)\le C(n)M_{1-\alpha}(|\nabla f|)(x).\tag{47}$$

Proof. The proof is an immediate consequence of the well-known Poincaré inequality.

For all ball $B(x,R)$ and all $f\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, there is a constant $C(n)$ such that

$$\int_{B(x,R)}|f(y)-f_{B(x,R)}|\mathrm{d}y\le C(n)R\int_{B(x,R)}|\nabla f(y)|\mathrm{d}y\tag{48}$$

holds. □

Corollary 4. *Let f be a locally integrable function such that $|\nabla f|\in H_r(w)$ and $0<p_*\le\infty$ are determined by*

$$\frac{1}{p_*}=\frac{1}{r}+\frac{\gamma-1}{d}.\tag{49}$$

Then, f is in $\dot{F}^{\gamma,q}_{p_*,w}$. Moreover, we have

$$\|f\|_{\dot{F}^{\gamma,q}_{p_*,w}}\le\||\nabla f|\|_{H_r(w)}.\tag{50}$$

Proof. Let $P_t(x)=(c_n t/(t^2+|x|^2)^{(n+1)/2})$ be the Poisson kernel with the constant $C(n)$ such that $\int_{\mathbb{R}^n}P_t(x)\mathrm{d}x=1$. Then, there exists a constant $C=C(n)$ such that $\mathrm{M}f(x)\le C\sup_{t>0}P_t*|f|$. In fact, if Q is a cube with diam $(Q)=t$ and $x\in Q$, then we have

$$\begin{aligned}\frac{1}{|Q|}\int_Q|f(y)|\mathrm{d}y&=\frac{1}{|Q|}\int_Q\frac{\left(t^2+|x-y|^2\right)^{(n+1)/2}}{C(n)t}\\ &\quad\cdot P_t(x-y)|f(y)|\mathrm{d}y\\ &\le C(n)\frac{t^n}{|Q|}\int_Q P_t(x-y)|f(y)|\mathrm{d}y\\ &\le C(n)P_t*|f|.\end{aligned}\tag{51}$$

Thus, we have

$$Mf(x)\le C\sup_{t>0}P_t*|f(x)|.\tag{52}$$

Using Lemma 3 with $\alpha = 1$ and Proposition 2 (see below), we obtain

$$\|f\|_{\dot F^{\gamma,q}_{p_*,w}} \le \left\|M_1^{\#}(f)\right\|_{r,w} \le \left\|\sup_{t>0} P_t * |\nabla f|\right\|_{r,w} \le \||\nabla f|\|_{H_r(w)}. \tag{53}$$

□

Remark 3. If we take $\gamma = 0$, $q = 2$, $r > 1$, and $w = 1$, in Corollary 4, we obtain the classical Sobolev–Gagliardo–Nirenberg inequality:

$$\|f\|_{p_*} \le \||\nabla f|\|_r, \tag{54}$$

with

$$\frac{1}{p_*} = \frac{1}{r} - \frac{1}{n}. \tag{55}$$

3. Some Useful Lemmas

We start this section with some useful lemmas that will be helpful in proving our main result.

Lemma 4 (see [7]). *Provided $\gamma < 1$, $\lambda > 0$, and $0 < q \le 1$, there exist Schwartz functions ν and μ on $\mathbb{R}^n$ such that*

(1) supp $\nu \subset B(0,1)$ and $\widehat{\nu}(0) = 0$

(2) supp $\widehat{\mu} \subset \{(1/2) \le |\xi| \le 2\}$ and $\widehat{\mu}(\xi) \ge c > 0$ on $\{(3/5) \le |\xi| \le (25/3)\}$

(3) $\sum_{j\in\mathbb{Z}} 2^{j\gamma q} |\mu^*_{2^{-j}} f|^q \le C \sum_{j\in\mathbb{Z}} 2^{j\gamma q} |\nu^*_{2^{-j}} f|^q$

Lemma 5. *Assume that $w(B(x,t)) \ge C_w t^d$ for each $x \in \mathbb{R}^n$ and each $t > 0$, and let $\nu \in S$ supported on $B(0,1)$ such that*

$$\int_{\mathbb{R}^n} \nu(x) \mathrm{d}x = 0. \tag{56}$$

Fix a large $\lambda > 0$, and define

$$\nu_t^* f(x) = \sup_{y\in\mathbb{R}^n} \frac{|\nu_t * f(y)|}{(1 + (|x - y|/t))^{\lambda}}. \tag{57}$$

Then,

$$\nu_t^* f(x) \le \min\left(s^{\alpha} M_{\alpha}^{\#}(f)(x), s^{\alpha-(d/p)} \left\|M_{\alpha}^{\#}(f)\right\|_{p,w}\right). \tag{58}$$

Proof. We adapt here the proof given in [7] in the unweighted case. Use the well-known estimate

$$\left(1 + \frac{|x - z|}{s}\right)^{-\lambda} \le \sum_{k=1}^{\infty} 2^{-k\lambda} \chi\left(\frac{|x - z|}{2^k s}\right), \quad \lambda > 0, \tag{59}$$

where χ denotes the characteristic function of the interval $[0,1]$, to obtain, for any $\lambda > 0$,

$$\nu_s^* f(x) \le \sum_{k=1}^{\infty} 2^{-k\lambda} \sup_{z\in\mathbb{R}^n} |\nu_s * f(z)| \chi\left(\frac{|x - z|}{2^k s}\right). \tag{60}$$

By taking any $z \in B(x, 2^k s)$ and using the fact that ν is supported in the unit ball and has mean equal zero, we obtain

$$\begin{aligned} |\nu_s * f(z)| &\le \int_{B(z,(1+2^k)s)} \left|\nu_s(z - y)\left(f(y) - f_{B(x,(1+2^k)s)}\right)\right| \mathrm{d}y \\ &\le s^{\alpha}\left(1 + 2^k\right)^{n+\alpha} M_{\alpha}^{\#}(f)(x), \end{aligned} \tag{61}$$

which holds. Hence,

$$\nu_s^* f(x) \le s^{\alpha} \sum_{k=1}^{\infty} 2^{-k\lambda}\left(1 + 2^k\right)^{n+\alpha} M_{\alpha}^{\#}(f)(x). \tag{62}$$

If we choose λ large enough, we obtain

$$\nu_s^* f(x) \le s^{\alpha} M_{\alpha}^{\#}(f)(x). \tag{63}$$

On the other hand, by (61), we have for any fixed $x \in B(z,s)$,

$$|\nu_s * f(z)| \le s^{\alpha} M_{\alpha}^{\#}(f)(x). \tag{64}$$

Rising (65) to the pth power and integrating over the ball $B(z,s)$ with respect to $w(x)\mathrm{d}x$, one has that

$$\begin{aligned} |\nu_s * f(z)| &\le w\left((B(z,s)^{-(1/p)} s^{\alpha} \left\|M_{\alpha}^{\#}(f)\right\|_{L_p(w)}\right. \\ &\le s^{\alpha-(d/p)} \left\|M_{\alpha}^{\#}(f)\right\|_{p,w}. \end{aligned} \tag{65}$$

By using (60), we obtain

$$\nu_s^* f(x) \le s^{\alpha - d/p} \left\|M_{\alpha}^{\#}(f)\right\|_{p,w}. \tag{66}$$

□

Proof. Proof of Theorem 1. □

Proof. We consider only the case when $0 < q \le 1$. In the case when $1 < q \le \infty$, estimate (18) follows from the case $q = 1$ by the embedding

$$\dot F^{\gamma,q_0}_{p,w} \subset \dot F^{\gamma,q_1}_{p,w}, \quad 0 < q_0 \le q_1 \le \infty. \tag{67}$$

Let $k > 0$ be chosen later and let μ and ν be as in Lemma 4. Assume $0 < ((\alpha - \gamma < d)/p)$ and $0 < q \le 1$. Then, using (58), we get

$$\begin{aligned}\sum_{j\in\mathbb{Z}} 2^{j\gamma q}\left(\mu_{2^{-j}}^{*} f(x)\right)^{q} &\le \sum_{j\in\mathbb{Z}} 2^{j\gamma q}\left(\nu_{2^{-j}}^{*} f(x)\right)^{q}\\ &\le M_{\alpha}^{\#}(f)(x)\sum_{j\le k} 2^{-j(\alpha-\gamma)q} + \left\|M_{\alpha}^{\#}(f)\right\|_{p,w}\sum_{j>k} 2^{-j((\alpha-\gamma-d)/p)q}\\ &\le 2^{-k(\alpha-\gamma)q} M_{\alpha}^{\#}(f)(x) + 2^{-k((\alpha-\gamma-d)/p)q}\left\|M_{\alpha}^{\#}(f)\right\|_{p,w}.\end{aligned} \tag{68}$$

Choose $2^{-k} = (M_{\alpha}^{\#}(f)(x)/\|M_{\alpha}^{\#}(f)\|_{p,w})$ to deduce that

$$\left(\sum_{j\in\mathbb{Z}} 2^{j\gamma q}\left(\mu_{2^{-j}}^{*} f(x)\right)^{q}\right)^{1/q} \le \left(M_{\alpha}^{\#}(f)(x)\right)^{p/p_*} \cdot \left(\left\|M_{\alpha}^{\#}(f)\right\|_{p,w}\right)^{1-(p/p_*)}, \tag{69}$$

where p_* is given by $(1/p_*) = (1/p) - ((\alpha-\gamma)/d)$. Thus, we have

$$\|f\|_{\dot{F}^{\gamma,q}_{p_*,w}} \le \left\|M_{\alpha}^{\#}(f)\right\|_{p,w}. \tag{70}$$

□

4. Some Extensions

In this section, we will assume that μ is a nonnegative Borel doubling measure on $\mathbb{R}^n$; there exists $\beta = \beta(\mu) > 0$ such that

$$\mu(B_{2r}) \le 2^{\beta n}\mu(B_r), \tag{71}$$

for all ball B_r. The smallest such β is called a doubling constant of μ.

For each $N \in \mathbb{N}\cup\{-1\}$, $m \in \mathbb{N}_0$, and $l \in \mathbb{N}$, we set

$$\mathscr{A} = \mathscr{A}^{l}_{N,m} = \left\{\nu \in \mathcal{S}_N(\mathbb{R}^n) : \|\nu\|_{m,N+l+1} \le 1\right\}. \tag{72}$$

Definition 3. Let $\gamma \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \le \infty$. The homogeneous grand Tribel–Lizorkin space is the set of all tempered functions f such that when $0 < q < \infty$,

$$\|f\|_{\mathscr{A}\dot{F}^{\gamma,q}_{p,\mu}} = \left\|\left(\sum_{j\in\mathbb{Z}} 2^{j\gamma q}\sup_{\nu\in\mathscr{A}}\left|\nu_{2^{-j}} f\right|^{q}\right)^{1/q}\right\|_{p,\mu} < \infty \tag{73}$$

and when $q = \infty$,

$$\|f\|_{\mathscr{A}\dot{F}^{\gamma,\infty}_{p,\mu}} = \left\|\sup_{j\in\mathbb{Z}} 2^{j\gamma}\sup_{\nu\in\mathscr{A}}\left|\nu_{2^{-j}} f\right|\right\|_{p,\mu} < \infty. \tag{74}$$

Proposition 1. *Let $\gamma \in \mathbb{R}$, $0 < p \le \infty$, and $0 < q \le \infty$, and μ is the doubling measure with a constant equal to β. Set $J = n\beta\max(1, (1/p), (1/q))$. If $\mathscr{A} = \mathscr{A}^{l}_{N,m}$ with $l \in \mathbb{N}$, $N + 1 > \max(\gamma, J - n - \gamma)$, and $m > \max(J, n + N + 1)$, then*

$$\|f\|_{\mathscr{A}\dot{F}^{\gamma,q}_{p,\mu}} = \|f\|_{\dot{F}^{\gamma,q}_{p,\mu}}. \tag{75}$$

Proof. Arguing as in the proof in ([15], Theorem 1.2) and using the almost-diagonality theorem (see [1], Theorem 4.2), we obtain the desired result. □

Proposition 2. *Let α and γ be real numbers satisfying $0 \le \alpha \le 1$ and $\gamma < \alpha$ and μ be a lower d−regular doubling measure. Assume f is a smooth function and $M_{\alpha}^{\#}(f) \in L^p(w)$ with $(d/(n+\alpha)) < p < (d/(\alpha-\gamma))$. Then, for each $0 < q \le \infty$,*

(1)

$$\|f\|_{\mathscr{A}\dot{F}^{\gamma,q}_{p_*,\mu}} \le \left\|M_{\alpha}^{\#}(f)\right\|_{p,\mu}, \tag{76}$$

where p_* is given by $(1/p_*) = (1/p) - ((\alpha-\gamma)/d)$

(2) For all $(n/(n+\alpha)) < p \le \infty$ and $0 \le \alpha < \infty$,

$$\|f\|_{\mathscr{A}\dot{F}^{\alpha,\infty}_{p,\mu}} \le \left\|M_{\alpha}^{\#}(f)\right\|_{p,\mu}. \tag{77}$$

(3) For all $(n/(n+\alpha)) < p \le \infty$, $0 \le \alpha < \infty$,

$$\|f\|_{\mathscr{A}\dot{F}^{((\alpha-d)/p),\infty}_{\infty,\mu}} \le \left\|M_{\alpha}^{\#}(f)\right\|_{p,\mu}. \tag{78}$$

Proof. We have from (58) that if μ is a lower d-regular measure, then

$$\sup_{\nu\in\mathscr{A}}\left|\nu_{2^{-j}} f\right| \le \min\left(2^{j\alpha}M_{\alpha}^{\#}(f)(x), 2^{j((\alpha-d)/p)}\left\|M_{\alpha}^{\#}(f)\right\|_{p,w}\right). \tag{79}$$

Arguing as in the proof of Proposition 1, we obtain the desired result easily. □

Definition 4. Let μ be a doubling measure $0 < p < \infty$ and $0 < \alpha \le 1$. The homogeneous fractional Hajłasz–Sobolev space $\dot{M}^{\alpha,p}_{\mu}(\mathbb{R}^n)$ is the set of all measurable functions $L^p_{\mu,\mathrm{loc}}$ for which there exists a nonnegative function $g \in L^p_{\mu}$ such that

$$|f(x) - f(y)| \le |x-y|^{\alpha}[g(x) + g(y)], \tag{80}$$

for μ − a.e. $x, y \in \mathbb{R}^n$.

$\dot{M}^{\alpha,p}_{\mu}(\mathbb{R}^n)$ is equipped with the seminorm

$$\|f\|_{\dot{M}^{\alpha,p}_{\mu}} = \inf_{g\in D(f)}\|g\|_{p,\mu}, \tag{81}$$

where $D(f)$ denotes the class of all nonnegative Borel functions g satisfying (80). Thus, Lemma 4.1 in [15] implies the following Sobolev embedding.

Lemma 6. *Let* $0<\alpha\leq 1$, $0<\delta<(n/\alpha)$, *and* p_* *be given by* $(1/p_*) = (1/\delta)-(\alpha/n)$. *Then, for all* $x\in\mathbb{R}^n$, $0<r<\infty$, $f\in \dot{M}_\mu^{\alpha,p}(\mathbb{R}^n)$, *and* $g\in D(f)$,

$$\begin{aligned}&\inf_{c\in\mathbb{R}}\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|f(y)-c|^{p_*}\mathrm{d}\mu\right)^{1/p_*}\\&\leq r^\alpha\left(\frac{1}{\mu(B(x,2r))}\int_{B(x,2r)}g(y)^\delta\mathrm{d}\mu\right)^{1/\delta}.\end{aligned}\tag{82}$$

Remark 4. Lemma 6 is due to Hajłasz ([13], Theorem 8.7) when $\alpha=1$.

Corollary 5. *Let* α,γ, *and* δ *be real numbers satisfying* $0\leq\alpha\leq 1$, $\gamma<\alpha$, *and* $(n/(n+\alpha))<p<(n/(\alpha-\gamma))$. *Assume* $f\in\dot{M}_\mu^{\alpha,p}(\mathbb{R}^n)$. *Then, for each* $0<q\leq\infty$,

$$\|f\|_{\mathscr{A}\dot{F}^{\gamma,q}_{p_*,\mu}}\leq\|f\|_{\dot{M}_\mu^{\alpha,p}},\tag{83}$$

where $(1/p_*)=(1/p)-((\alpha-\gamma)/n)$.

Proof. Fix a ball $B(x,2r)$. Then, using Lemma 6 and by taking $\delta=(n/(n+\alpha))$ and Hölder's inequality, we obtain

$$\begin{aligned}&\inf_{c\in\mathbb{R}}\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|f(y)-c|\mathrm{d}\mu\\&\leq r^\alpha\left(\frac{1}{\mu(B(x,2r))}\int_{B(x,2r)}|g(y)|^\delta\mathrm{d}\mu\right)^{1/\delta}.\end{aligned}\tag{84}$$

Hence,

$$M_\alpha^{\#}(f)(x)\leq\left(M_\mu\left(g^\delta\right)\right)^{1/\delta}(x)\tag{85}$$

holds, where $M_\mu(g)$ is the maximal function with respect to the measure μ. The $L^{p/\delta}$-boundedness of M_μ when $\delta<p<\infty$ and Proposition 2 lead to estimate (83). □

Also, recall that $\dot{M}_\mu^{\alpha,p}(\mathbb{R}^n)=\mathscr{A}\dot{F}^{\alpha,\infty}_{p,\mu}(\mathbb{R}^n)$ for $0<\alpha\leq 1$ and $(n/(n+\alpha))<p<\infty$ in [15] and $\dot{M}_\mu^{1,p}(\mathbb{R}^n)=\dot{F}^{1,2}_{p,\mu}(\mathbb{R}^n)\approx\dot{H}_\mu^p$ for $(n/(n+1))<p<\infty$ in [14]. Here, $\dot{H}_\mu^p$ denotes, for $p>0$, the homogeneous Hardy–Sobolev space, i.e., the space of tempered distributions f on $\mathbb{R}^n$, such that $\partial_j f\in H_\mu^p$ for each $j=1,\ldots,n$ and

$$\|f\|_{\dot{H}_\mu^p}=\sum_{j=1}^n\left\|\partial_j f\right\|_{H_{p,\mu}}.\tag{86}$$

Consequently, if $f\in\dot{H}_\mu^p$ with $(n/n+1)<p<(n/(1-\gamma))$, then

$$\|f\|_{\dot{F}^{\gamma,q}_{p_*}}\leq\|f\|_{\dot{H}_\mu^p},\quad\frac{1}{p_*}=\frac{1}{p}-\frac{1-\gamma}{n}.\tag{87}$$

In particular, we have, for $(n/(n+1))<p<n$, the following well-known result:

$$\|f\|_{p_*}\leq\|f\|_{\dot{H}^p},\quad\frac{1}{p_*}=\frac{1}{p}-\frac{1}{n}.\tag{88}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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