Research Article

Gorenstein-Projective Modules over Upper Triangular Matrix Artin Algebras

Dadi Asefa

Department of Mathematics, College of Natural and Computational Sciences, Ambo University, Ambo, Ethiopia

Correspondence should be addressed to Dadi Asefa; dadi.asefa@ambou.edu.et

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Gorenstein-projective module is an important research topic in relative homological algebra, representation theory of algebras, triangulated categories, and algebraic geometry (especially in singularity theory). For a given algebra A , how to construct all the Gorenstein-projective A -modules is a fundamental problem in Gorenstein homological algebra. In this paper, we describe all complete projective resolutions over an upper triangular Artin algebra $\Lambda = \begin{pmatrix} A & {}_A M_B \\ 0 & B \end{pmatrix}$. We also give a necessary and sufficient condition for all finitely generated Gorenstein-projective modules over $\Lambda = \begin{pmatrix} A & {}_A M_B \\ 0 & B \end{pmatrix}$.

1. Introduction

Auslander and Bridger [1] generalized finitely generated projective modules to modules of Gorenstein dimension zero over two-sided Noetherian rings. After two decades, Enochs and Jenda [2] generalized it to an arbitrary ring and called it Gorenstein-projective modules. After that, this class of modules got special attention and has been studied by many authors (see, e.g., [3–15]). Given an algebra A , it is a fundamental problem to determine all the indecomposable finitely generated Gorenstein-projective A -modules, which is usually not easy.

Zhang [14] introduced the notion of compatible bimodules and then determined all the Gorenstein-projective modules over upper triangular matrix Artin algebra $\Lambda = \begin{pmatrix} A & {}_A M_B \\ 0 & B \end{pmatrix}$, in terms of Gorenstein-projective A and B -modules. This gives us a strong motivation to study Gorenstein-projective modules over upper triangular matrix Artin algebras.

The main aim of this paper is to describe all the complete projective resolutions and all finitely generated Gorenstein-projective modules over an upper triangular matrix Artin

sufficient and necessary conditions. Although these conditions ((GP1)–(GP5)) are long, they may be helpful, especially in some special cases.

Throughout the paper, $A - \text{Mod}$ denotes the category of left A -modules and $A - \text{mod}$ denotes the category of finitely generated left A -modules for a ring A .

2. Preliminaries

In this section, we recall some basic definitions and facts that will be used throughout the paper.

Following [16], an A -module M is said to be Gorenstein-projective, in $A - \text{Mod}$ (resp., in $A - \text{mod}$), if there is an exact sequence of projective modules in $A - \text{Mod}$ (resp., in $A - \text{mod}$)

$$\mathcal{P}^\bullet := \dots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow \dots, \quad (1)$$

with $\text{Hom}_A(P^\bullet, Q)$ being also exact for any projective module Q in $A - \text{Mod}$ (resp., in $A - \text{mod}$), such that $M \cong \text{Ker} d^0$. Such $\mathcal{P}^\bullet$ is called a complete projective resolution in $A - \text{Mod}$ (resp., $A - \text{mod}$).

2.1. Modules over Upper Triangular Matrix Algebra. Let ${}_A M_B$ be an $A - B$ bimodule and

algebra $\Lambda = \begin{pmatrix} A & {}_A M_B \\ 0 & B \end{pmatrix}$, by giving the corresponding

$$\Lambda = \begin{pmatrix} A & {}_A M_B \\ 0 & B \end{pmatrix} = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \mid a \in A, b \in B, m \in M \right\} \quad (2)$$

be an Artin algebra. Note that addition of Λ is the addition of matrix, and the multiplication is given by

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix} = \begin{pmatrix} aa' & am' + mb' \\ 0 & bb' \end{pmatrix}. \quad (3)$$

Left modules over Λ are described with a triple $\left(\begin{pmatrix} X \\ Y \end{pmatrix}_\phi \right)$, where X is an A -module and Y is a B -module, and $\phi: M \otimes_B Y \rightarrow X$ is an A -map. Λ -map is described with a pair $\left(\begin{pmatrix} f \\ g \end{pmatrix} \right): \left(\begin{pmatrix} X \\ Y \end{pmatrix}_\phi \right) \rightarrow \left(\begin{pmatrix} X' \\ Y' \end{pmatrix}_{\phi'} \right)$, where $f: X \rightarrow X'$ is an A -map and $g: Y \rightarrow Y'$ is a B -map, such that the diagram

$$\begin{array}{ccc} M \otimes_B Y & \xrightarrow{\phi} & X \\ \text{Id} \otimes g \downarrow & & \downarrow f \\ M \otimes_B Y' & \xrightarrow{\phi'} & X' \end{array}$$

is commutative.

A sequence

$$0 \rightarrow \left(\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}_{\phi_1} \right) \xrightarrow{\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}} \left(\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}_{\phi_2} \right) \xrightarrow{\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}} \left(\begin{pmatrix} X_3 \\ Y_3 \end{pmatrix}_{\phi_3} \right) \rightarrow 0 \quad (4)$$

in Λ -mod is exact if and only if a sequence

$$0 \rightarrow X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \rightarrow 0 \quad (5)$$

is exact in A -mod, and a sequence

$$0 \rightarrow Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} Y_3 \rightarrow 0 \quad (6)$$

is exact in B -mod.

2.2. Projective Modules. The indecomposable projective (left) Λ -modules are exactly $\begin{pmatrix} P \\ 0 \end{pmatrix}$ and $\begin{pmatrix} M \otimes_B Q \\ Q \end{pmatrix}_{\text{Id}}$, where P runs over indecomposable projective A -modules and Q runs over indecomposable projective B -modules.

3. Complete Projective Resolutions

The aim of this section is to describe all complete projective resolutions over an upper triangular matrix Artin algebra

$$\Lambda = \begin{pmatrix} A & {}_A M_B \\ 0 & B \end{pmatrix}.$$

3.1. Morphisms of Projective Λ -Modules. Note that any projective Λ -module is of the form

$$X = \begin{pmatrix} P \oplus M \otimes_B Q \\ Q \end{pmatrix}, \quad (7)$$

where P and Q are projective A -module and projective B -module, respectively. For each $i \in \mathbb{Z}$, we write

$$X^i = \begin{pmatrix} P^i \oplus M \otimes_B Q^i \\ Q^i \end{pmatrix}. \quad (8)$$

For such a projective Λ -module X^i as above and for any datum

$$(P^i, Q^i, \alpha^i, \beta^i, g^i), \quad (9)$$

where $\alpha^i: P^i \rightarrow P^{i+1}$ and $\beta^i: P^i \rightarrow M \otimes_B Q^{i+1}$ are A -maps and $g^i: Q^i \rightarrow Q^{i+1}$ is B -map $\forall i \in \mathbb{Z}$, we put

$$\begin{aligned} f^i &= \begin{pmatrix} \begin{pmatrix} \alpha^i & 0 \\ \beta^i & \text{Id}_{M \otimes_B Q^i} \end{pmatrix} \\ g^i \end{pmatrix}: \begin{pmatrix} P^i \oplus M \otimes_B Q^i \\ Q^i \end{pmatrix} \\ &\rightarrow \begin{pmatrix} P^{i+1} \oplus M \otimes_B Q^{i+1} \\ Q^{i+1} \end{pmatrix}. \end{aligned} \quad (10)$$

Lemma 1. Let X^i and $(P^i, Q^i, \alpha^i, \beta^i, g^i)$ be given as in (8) and (9). Then, $f^i: X^i \rightarrow X^{i+1}$ is a Λ -map if and only if f^i is given in (10).

Proof. First, assume that $f^i: X^i \rightarrow X^{i+1}$ is a Λ -map, where f^i is given in (10). Since the diagram

$$\begin{array}{ccc} M \otimes_B Q^i & \xrightarrow{\begin{pmatrix} 0 \\ \text{Id}_{M \otimes_B Q^i} \end{pmatrix}} & P^i \otimes M \otimes_B Q^i \\ \text{Id}_{M \otimes_B Q^i} \downarrow & & \downarrow \begin{pmatrix} \alpha^i & 0 \\ \beta^i & \text{Id}_{M \otimes_B Q^i} \end{pmatrix} \\ M \otimes_B Q^{i+1} & \xrightarrow{\begin{pmatrix} 0 \\ \text{Id}_{M \otimes_B Q^{i+1}} \end{pmatrix}} & P^{i+1} \otimes M \otimes_B Q^{i+1} \end{array}$$

is commutative, we see that $f^i: X^i \rightarrow X^{i+1}$ is a Λ -map, $\forall i \in \mathbb{Z}$.

Conversely, let $f^i: X^i \rightarrow X^{i+1}$ be an arbitrary Λ -map where $X^i = \begin{pmatrix} P^i \oplus M \otimes_B Q^i \\ Q^i \end{pmatrix}$. Then, f^i is of the form

$$\begin{pmatrix} \begin{pmatrix} \alpha^i & \omega^i \\ \beta^i & \delta^i \end{pmatrix} \\ g^i \end{pmatrix}: X^i \rightarrow X^{i+1}, \quad (11)$$

where $\alpha^i: P^i \rightarrow P^{i+1}$, $\beta^i: P^i \rightarrow M \otimes_B Q^{i+1}$, $\omega^i: M \otimes_B Q^i \rightarrow P^{i+1}$, and $\delta^i: M \otimes_B Q^i \rightarrow M \otimes_B Q^{i+1}$ are A -maps and $g^i: Q^i \rightarrow Q^{i+1}$ is B -map, $\forall i \in \mathbb{Z}$. By the definition of morphisms in Λ -mod, the diagram

$$\begin{array}{ccc} M \otimes_B Q^i & \xrightarrow{\begin{pmatrix} 0 \\ \text{Id}_{M \otimes_B Q^i} \end{pmatrix}} & P^i \otimes M \otimes_B Q^i \\ \text{Id}_{M \otimes_B Q^i} \downarrow & & \downarrow \begin{pmatrix} \alpha^i & \omega^i \\ \beta^i & \delta^i \end{pmatrix} \\ M \otimes_B Q^{i+1} & \xrightarrow{\begin{pmatrix} 0 \\ \text{Id}_{M \otimes_B Q^{i+1}} \end{pmatrix}} & P^{i+1} \otimes M \otimes_B Q^{i+1} \end{array}$$

is commutative. This implies that

$$\begin{aligned}\omega^i &= 0, \\ \delta^i &= \text{Id}_M \otimes g^i \quad \forall i \in \mathbb{Z}.\end{aligned}\quad (12)$$

Thus, f^i is given as in (10). This completes the proof. $\square$

3.2. The Exactness of Morphisms of Projective Modules. Consider the following conditions:

$$\text{(GP1)} \quad \mathcal{P}^\bullet = \dots \longrightarrow \alpha^{-2} P^{-1} \longrightarrow \alpha^{-1} P^0 \longrightarrow \alpha^0 P^1 \longrightarrow \alpha^1 \dots \text{ is a complex.}$$

$$\text{(GP2)} \quad \mathcal{Q}^\bullet = \dots \longrightarrow g^{-2} Q^{-1} \longrightarrow g^{-1} Q^0 \longrightarrow g^0 Q^1 \longrightarrow g^1 \dots \text{ is an exact sequence.}$$

$$\text{(GP3)} \quad \beta^{i+1} \alpha^i + (\text{Id}_M \otimes g^{i+1}) \beta^i = 0, \quad \forall i \in \mathbb{Z}.$$

$$\text{(GP4)} \quad \text{For any } (p^{i+1}, x^{i+1}) \in P^{i+1} \oplus M \otimes_B Q^{i+1} \text{ with } \alpha^{i+1}(p^{i+1}) = 0, \beta^{i+1}(p^{i+1}) + (\text{Id}_N \otimes g^{i+1})(x^{i+1}) = 0, \text{ there is } (p^i, x^i) \in P^i \oplus M \otimes_B Q^i \text{ such that } p^{i+1} = \alpha^i(p^i) \text{ and } x^{i+1} = \beta^i(p^i) + (\text{Id}_M \otimes g^i)(x^i), \forall i \in \mathbb{Z}.$$

Lemma 2. Let X^i be a projective Λ -module given as in (8) and $f^i: X^i \longrightarrow X^{i+1}$ be a Λ -map given as in (10). Then, $\text{Ker } f^{i+1} = \text{Im } f^i$ if and only if conditions (GP1)–(GP4) are satisfied.

Proof. By Lemma 1, f^i is of the form $f^i = \begin{pmatrix} \alpha^i & 0 \\ \beta^i \text{Id}_M \otimes g^i \end{pmatrix} : \begin{pmatrix} P^i \oplus M \otimes_B Q^i \\ Q^i \end{pmatrix} \longrightarrow \begin{pmatrix} P^{i+1} \oplus M \otimes_B Q^{i+1} \\ Q^{i+1} \end{pmatrix}$,

Where $\alpha^i: P^i \longrightarrow P^{i+1}$ and $\beta^i: P^i \longrightarrow M \otimes_B Q^{i+1}$ and $g^i: Q^i \longrightarrow Q^{i+1}$, $\forall i \in \mathbb{Z}$. We see that $\text{Ker } f^{i+1} = \text{Im } f^i$, $\forall i \in \mathbb{Z}$, if and only if

$$\begin{aligned}\text{Ker} \begin{pmatrix} \alpha^{i+1} & 0 \\ \beta^{i+1} \text{Id}_M \otimes g^{i+1} \end{pmatrix} &= \text{Im} \begin{pmatrix} \alpha^i & 0 \\ \beta^i \text{Id}_M \otimes g^i \end{pmatrix}, \\ \text{Ker } g^{i+1} &= \text{Im } g^i,\end{aligned}\quad (13)$$

where

$$\begin{aligned}\begin{pmatrix} \alpha^i & 0 \\ \beta^i \text{Id}_M \otimes g^i \end{pmatrix} : P^i \oplus M \otimes_B Q^i &\longrightarrow P^{i+1} \oplus M \otimes_B Q^{i+1}, \\ \forall i \in \mathbb{Z}.\end{aligned}\quad (14)$$

Note that $\text{Ker } g^{i+1} = \text{Im } g^i$, $\forall i \in \mathbb{Z}$, if and only if condition (GP2) is satisfied. Thus, it suffices to show that

$$\text{Ker} \begin{pmatrix} \alpha^{i+1} & 0 \\ \beta^{i+1} \text{Id}_M \otimes g^{i+1} \end{pmatrix} = \text{Im} \begin{pmatrix} \alpha^i & 0 \\ \beta^i \text{Id}_M \otimes g^i \end{pmatrix}. \quad (15)$$

$$\text{Now, } \text{Im} \begin{pmatrix} \alpha^i & 0 \\ \beta^i \text{Id}_N \otimes g^i \end{pmatrix} \subseteq \text{Ker} \begin{pmatrix} \alpha^{i+1} & 0 \\ \beta^{i+1} \text{Id}_N \otimes g^{i+1} \end{pmatrix} \text{ if and}$$

$$\text{only if } \begin{pmatrix} \alpha^{i+1} & 0 \\ \beta^{i+1} \text{Id}_N \otimes g^{i+1} \end{pmatrix} \begin{pmatrix} \alpha^i & 0 \\ \beta^i \text{Id}_N \otimes g^i \end{pmatrix} = 0, \text{ i.e.,}$$

$$\begin{pmatrix} \alpha^{i+1} \alpha^i & 0 \\ \beta^{i+1} \alpha^i + (\text{Id}_N \otimes g^{i+1}) \beta^i & \text{Id}_N \otimes g^{i+1} g^i \end{pmatrix} = 0, \quad (16)$$

if and only if conditions (GP1)–(GP3) are satisfied.

We also see that $\text{Ker} \begin{pmatrix} \alpha^{i+1} & 0 \\ \beta^{i+1} \text{Id}_N \otimes g^{i+1} \end{pmatrix} \subseteq \text{Im} \begin{pmatrix} \alpha^i & 0 \\ \beta^i \text{Id}_N \otimes g^i \end{pmatrix}$, $\forall i \in \mathbb{Z}$, if and only if condition (GP4) is satisfied.

Thus, $\text{Ker } f^{i+1} = \text{Im } f^i$, $\forall i \in \mathbb{Z}$, if and only if conditions (GP1)–(GP4) are satisfied.

Consider the following condition (GP5) $\forall i \in \mathbb{Z}$:

$$\text{(GP5)} \quad \text{For any } c^{i+1} \in \text{Hom}_A(P^{i+1}, A), d^{i+1} \in \text{Hom}_A(P^{i+1}, M), t^{i+1} \in \text{Hom}_B(Q^{i+1}, B), \text{ such that}$$

$$\begin{aligned}c^{i+1} \alpha^i &= 0, \\ d^{i+1} \alpha^i + (\text{Id}_M \otimes t^{i+1}) \beta^i &= 0, \\ t^{i+1} g^i &= 0,\end{aligned}\quad (17)$$

there are $c^{i+2} \in \text{Hom}_A(P^{i+2}, A)$, $d^{i+2} \in \text{Hom}_A(P^{i+2}, M)$, $t^{i+2} \in \text{Hom}_B(Q^{i+2}, B)$, such that

$$\begin{aligned}c^{i+1} &= c^{i+2} \alpha^{i+1}, \\ d^{i+1} &= d^{i+2} \alpha^{i+1} + (\text{Id}_M \otimes t^{i+2}) \beta^{i+1}, \\ t^{i+1} &= t^{i+2} g^{i+1}.\end{aligned}\quad (18)$$

$\square$

Lemma 3. Let X^i be a projective Λ -module given as in (8) and $f: X^i \longrightarrow X^{i+1}$ be a Λ -map given as in (10). Assume that $\text{Ker } f^{i+1} = \text{Im } f^i$, i.e., conditions (GP1)–(GP4) are satisfied. Consider the homomorphism of abelian groups $f^{i*} = \text{Hom}_\Lambda(f^i, \Lambda): \text{Hom}_\Lambda(X^{i+1}, \Lambda) \longrightarrow \text{Hom}_\Lambda(X^i, \Lambda)$. Then, $\text{Ker } f^{i*} = \text{Im } f^{(i+1)*}$, $\forall i \in \mathbb{Z}$, if and only if condition (GP5) is satisfied.

Proof. Assume that $\text{Ker } f^{i+1} = \text{Im } f^i$. By Lemma 1, we know that $f^i: X^i \longrightarrow X^{i+1}$ is given as

$$\begin{aligned}f^i &= \begin{pmatrix} \begin{pmatrix} \alpha^i & 0 \\ \beta^i \text{Id}_M \otimes g^i \end{pmatrix} \\ g^i \end{pmatrix} : \begin{pmatrix} P^i \oplus M \otimes_B Q^i \\ Q^i \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} P^{i+1} \oplus M \otimes_B Q^{i+1} \\ Q^{i+1} \end{pmatrix}.\end{aligned}\quad (19)$$

Any element in the abelian group $\text{Hom}_\Lambda(X^i, \Lambda) = \text{Hom}_\Lambda\left(\begin{pmatrix} P^i \oplus M \otimes_B Q^i \\ Q^i \end{pmatrix}, \begin{pmatrix} A \oplus M \\ B \end{pmatrix}\right)$, $\forall i \in \mathbb{Z}$, will be written as

$$h^i = \left(\begin{pmatrix} c^i & e^i \\ d^i & s^i \\ t^i \end{pmatrix} \right): \begin{pmatrix} P^i \oplus M \otimes_B Q^i \\ Q^i \end{pmatrix} \longrightarrow \begin{pmatrix} A \oplus M \\ B \end{pmatrix}. \quad (20)$$

By definition of morphisms in $\Lambda\text{-mod}$, the diagram

$$\begin{array}{ccc} M \otimes_B Q^i & \xrightarrow{\begin{pmatrix} 0 \\ \text{Id}_M \otimes Q^i \end{pmatrix}} & P^i \oplus M \otimes_B Q^i \\ \text{Id}_M \otimes t^i \downarrow & & \downarrow \begin{pmatrix} c^i & e^i \\ d^i & s^i \end{pmatrix} \\ M \otimes_B B & \xrightarrow{\begin{pmatrix} 0 \\ \text{Id}_M \end{pmatrix}} & A \oplus M \end{array}$$

is commutative. This implies that

$$\begin{aligned} f^{i*} &= \text{Hom}_\Lambda \left(\begin{pmatrix} \alpha^i & 0 \\ \beta^i & \text{Id}_M \otimes g^i \\ g^i \end{pmatrix}, \Lambda \right): \\ &\text{Hom}_\Lambda \left(\begin{pmatrix} P^{i+1} \oplus M \otimes_B Q^{i+1} \\ Q^{i+1} \end{pmatrix}, \begin{pmatrix} A \oplus M \\ B \end{pmatrix} \right) \longrightarrow \text{Hom}_\Lambda \left(\begin{pmatrix} P^i \oplus M \otimes_B Q^i \\ Q^i \end{pmatrix}, \begin{pmatrix} A \oplus M \\ B \end{pmatrix} \right). \end{aligned} \quad (23)$$

Thus, f^{i*} maps

$$h^{i+1} = \begin{pmatrix} \begin{pmatrix} c^{i+1} & 0 \\ d^{i+1} & \text{Id}_M \otimes t^{i+1} \\ t^{i+1} \end{pmatrix} \end{pmatrix} \quad (24)$$

to

$$\begin{aligned} f^{i*}(h^{i+1}) &= h^{i+1} f^i = \begin{pmatrix} \begin{pmatrix} c^{i+1} & 0 \\ d^{i+1} & \text{Id}_M \otimes t^{i+1} \\ t^{i+1} \end{pmatrix} \begin{pmatrix} \alpha^i & 0 \\ \beta^i & \text{Id}_M \otimes g^i \end{pmatrix} \\ t^{i+1} g^i \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} c^{i+1} \alpha^i & 0 \\ d^{i+1} \alpha^i + (\text{Id}_M \otimes t^{i+1}) \beta^i & \text{Id}_M \otimes t^{i+1} g^i \\ t^{i+1} g^i \end{pmatrix} \end{pmatrix}. \end{aligned} \quad (25)$$

This means that $h^{i+1} = \begin{pmatrix} \begin{pmatrix} c^{i+1} & 0 \\ d^{i+1} & \text{Id}_M \otimes t^{i+1} \\ t^{i+1} \end{pmatrix} \end{pmatrix} \in \text{Ker } f^{i*}$ if and only if

$$\begin{aligned} c^{i+1} \alpha^i &= 0, \\ d^{i+1} \alpha^i + (\text{Id}_M \otimes t^{i+1}) \beta^i &= 0, \\ t^{i+1} g^i &= 0. \end{aligned} \quad (26)$$

$$\begin{aligned} e^i &= 0, \\ s^i &= \text{Id}_M \otimes t^i \quad \forall i \in \mathbb{Z}. \end{aligned} \quad (21)$$

Thus,

$$h^i = \begin{pmatrix} \begin{pmatrix} c^i & 0 \\ d^i & \text{Id}_M \otimes t^i \\ t^i \end{pmatrix} \end{pmatrix}: \begin{pmatrix} P^i \oplus M \otimes_B Q^i \\ Q^i \end{pmatrix} \longrightarrow \begin{pmatrix} A \oplus M \\ B \end{pmatrix}. \quad (22)$$

Hence, $f^{i*} = \text{Hom}_\Lambda(f^i, \Lambda): \text{Hom}_\Lambda(X^{i+1}, \Lambda) \longrightarrow \text{Hom}_\Lambda(X^i, \Lambda)$, $\forall i \in \mathbb{Z}$, is written as

$$\text{We see that } h^{i+1} = \begin{pmatrix} \begin{pmatrix} c^{i+1} & 0 \\ d^{i+1} & \text{Id}_M \otimes t^{i+1} \\ t^{i+1} \end{pmatrix} \end{pmatrix} \in \text{Im } f^{i+1*} \text{ if}$$

and only if there is $h^{i+2} = \begin{pmatrix} \begin{pmatrix} c^{i+2} & 0 \\ d^{i+2} & \text{Id}_M \otimes t^{i+2} \\ t^{i+2} \end{pmatrix} \end{pmatrix}$ such that

$$\begin{aligned} &\begin{pmatrix} \begin{pmatrix} c^{i+2} & 0 \\ d^{i+2} & \text{Id}_M \otimes t^{i+2} \\ t^{i+2} \end{pmatrix} \begin{pmatrix} \alpha^{i+1} & 0 \\ \beta^{i+1} & \text{Id}_M \otimes g^{i+1} \end{pmatrix} \\ t^{i+2} g^{i+1} \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} c^{i+1} & 0 \\ d^{i+1} & \text{Id}_M \otimes t^{i+1} \\ t^{i+1} \end{pmatrix} \end{pmatrix}, \end{aligned} \quad (27)$$

which means that

$$\begin{aligned} c^{i+1} &= c^{i+2} \alpha^{i+1}, \\ d^{i+1} &= d^{i+2} \alpha^{i+1} + (\text{Id}_M \otimes t^{i+2}) \beta^{i+1}, \\ t^{i+1} &= t^{i+2} g^{i+1}. \end{aligned} \quad (28)$$

By conditions (G1)–(G4), we have

$$\begin{aligned}
f^{i+1} f^i &= \begin{pmatrix} \begin{pmatrix} \alpha^{i+1} & 0 \\ \beta^{i+1} & \text{Id}_M \otimes g^{i+1} \end{pmatrix} & \begin{pmatrix} \alpha^i & 0 \\ \beta^i & \text{Id}_M \otimes g^i \end{pmatrix} \\ g^{i+1} g^i & \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} \alpha^{i+1} \alpha^i & 0 \\ \beta^{i+1} \alpha^i + (\text{Id}_M \otimes g^{i+1}) \beta^i & \text{Id}_M \otimes g^{i+1} g^i \end{pmatrix} \\ g^{i+1} g^i & \end{pmatrix} = 0.
\end{aligned} \tag{29}$$

Thus, $f^{i*} f^{i+1*} (h^{i+2}) = h^{i+2} f^{i+1} f^i = 0$, i.e., $\text{Im } f^{i+1*} \subseteq \text{Ker } f^{i*}$. Therefore, $\text{Im } f^{i+1*} = \text{Ker } f^{i*}$ if and only if $\text{Im } f^{i+1*} \supseteq \text{Ker } f^{i*}$, if and only if for any

$$\begin{aligned}
h^i &= \begin{pmatrix} \begin{pmatrix} c^{i+1} & 0 \\ d^{i+1} & \text{Id}_M \otimes t^{i+1} \end{pmatrix} \\ t^{i+1} & \end{pmatrix} : \begin{pmatrix} P^{i+1} \oplus M \otimes_B Q^{i+1} \\ Q^{i+1} \end{pmatrix} \rightarrow \\
&\begin{pmatrix} A \oplus M \\ B \end{pmatrix} \text{ with} \\
&\quad c^{i+1} \alpha^i = 0, \\
&\quad d^{i+1} \alpha^i + (\text{Id}_M \otimes t^{i+1}) \beta^i = 0, \\
&\quad t^{i+1} g^i = 0,
\end{aligned} \tag{30}$$

there is

$$\begin{aligned}
h^{i+2} &= \begin{pmatrix} \begin{pmatrix} c^{i+2} & 0 \\ d^{i+2} & \text{Id}_M \otimes t^{i+2} \end{pmatrix} \\ t^{i+2} & \end{pmatrix} : \begin{pmatrix} P^{i+2} \oplus M \otimes_B Q^{i+2} \\ Q^{i+2} \end{pmatrix} \\
&\rightarrow \begin{pmatrix} A \oplus M \\ B \end{pmatrix},
\end{aligned} \tag{31}$$

such that

$$\begin{aligned}
c^{i+1} &= c^{i+2} \alpha^{i+1}, \\
d^{i+1} &= d^{i+2} \alpha^{i+1} + (\text{Id}_M \otimes t^{i+2}) \beta^{i+1}, \\
t^{i+1} &= t^{i+2} g^{i+1}.
\end{aligned} \tag{32}$$

So, by the above arguments, we see that $\text{Im } f^{i+1*} = \text{Ker } f^{i*}$, $\forall i \in \mathbb{Z}$, if and only if condition (GP5) is satisfied. This completes the proof. $\square$

3.3. Main Result. Now, by Lemmas 2 and 3, we get the following sufficient and necessary conditions of any complete projective resolutions over $\Lambda = \begin{pmatrix} A & A M_B \\ 0 & B \end{pmatrix}$.

Theorem 1. Let X^i be a projective Λ -module, given as in (8), and $f^i: X^i \rightarrow X^{i+1}$ be a Λ -map, given as in (10). Then,

$$\mathcal{X}^\bullet = \dots \xrightarrow{f^{-2}} X^{-1} \xrightarrow{f^{-1}} X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \dots \tag{33}$$

is a complete Λ -projective resolution if and only if conditions (GP1)–(GP5) are satisfied.

3.4. Special Complete Projective Resolutions. We consider the special case of $f: X^i \rightarrow X^{i+1}$ where

$$\begin{aligned}
f^i &= \begin{pmatrix} \begin{pmatrix} \alpha^i & 0 \\ \beta^i & \text{Id}_M \otimes g^i \end{pmatrix} \\ g^i & \end{pmatrix} : \begin{pmatrix} P^i \oplus M \otimes_B Q^i \\ Q^i \end{pmatrix} \\
&\rightarrow \begin{pmatrix} P^{i+1} \oplus M \otimes_B Q^{i+1} \\ Q^{i+1} \end{pmatrix},
\end{aligned} \tag{34}$$

and we take $\beta^i = 0 \forall i \in \mathbb{Z}$, i.e.,

$$\begin{aligned}
f^i &= \begin{pmatrix} \begin{pmatrix} \alpha^i & 0 \\ 0 & \text{Id}_M \otimes g^i \end{pmatrix} \\ g^i & \end{pmatrix} : \begin{pmatrix} P^i \oplus M \otimes_B Q^i \\ Q^i \end{pmatrix} \\
&\rightarrow \begin{pmatrix} P^{i+1} \oplus M \otimes_B Q^{i+1} \\ Q^{i+1} \end{pmatrix}.
\end{aligned} \tag{35}$$

Then the condition (GP3) holds for each $i \in \mathbb{Z}$. We then get the following special kind of complete projective resolutions over $\Lambda = \begin{pmatrix} A & A M_B \\ 0 & B \end{pmatrix}$.

Proposition 1. Let X^i be a projective Λ -module, given as in (8), and $f^i: X^i \rightarrow X^{i+1}$ be a Λ -map, given as in (35). Then,

$$\mathcal{X}^\bullet = \dots \xrightarrow{f^{-2}} X^{-1} \xrightarrow{f^{-1}} X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \dots \tag{36}$$

is a complete Λ -projective resolution if and only if

$$\mathcal{P}^\bullet = \dots \xrightarrow{\alpha^{-2}} P^{-1} \xrightarrow{\alpha^{-1}} P^0 \xrightarrow{\alpha^0} P^1 \xrightarrow{\alpha^1} \dots \tag{37}$$

is a complete A -projective resolution,

$$\mathcal{Q}^\bullet = \dots \xrightarrow{g^{-2}} Q^{-1} \xrightarrow{g^{-1}} Q^0 \xrightarrow{g^0} Q^1 \xrightarrow{g^1} \dots \tag{38}$$

is a complete B -projective resolution, and $M \otimes_B \mathcal{Q}^\bullet$ and $\text{Hom}_A(\mathcal{P}^\bullet, M)$ are exact sequences.

Proof. Assume that (36) is a complete Λ -projective resolution. Then, by Theorem 1, we know that conditions (GP1), (GP2), (GP4), and (GP5) are satisfied.

Let $p^{i+1} \in \text{Ker } \alpha^{i+1}$. Since $\beta^{i+1} = 0$, it follows from (GP4) (by taking $x^{i+1} = 0$) that $p^{i+1} \in \text{Im } \alpha^i$, i.e., $\text{Ker } \alpha^{i+1} \subseteq \text{Im } \alpha^i$. Together with (GP1), we see that the sequence $\mathcal{P}^\bullet$ in (37) is exact.

Since $\beta^i = 0$, (GP4) means that $M \otimes_B \mathcal{Q}^\bullet$ is an exact sequence.

Since $\beta^i = 0$, (GP5) means that $\text{Hom}_A(\mathcal{P}^\bullet, A)$, $\text{Hom}_A(\mathcal{P}^\bullet, M)$, and $\text{Hom}_B(\mathcal{Q}^\bullet, B)$ are exact sequences.

Since $\mathcal{P}^\bullet$ and $\text{Hom}_A(\mathcal{P}^\bullet, A)$ are exact sequences, it follows that $\mathcal{P}^\bullet$ is a complete A -projective resolution. Similarly, $\mathcal{Q}^\bullet$ is a complete B -projective resolution.

Conversely, assume that $\mathcal{P}^\bullet$ is a complete A -projective resolution, $\mathcal{Q}^\bullet$ is a complete B -projective resolution, and $M \otimes_B \mathcal{Q}^\bullet$ and $\text{Hom}_A(\mathcal{P}^\bullet, M)$ are exact sequences. Then, conditions (GP1), (GP2), (GP4), and (GP5) are satisfied,

where $\beta^i = 0, \forall i \in \mathbb{Z}$. It follows from Theorem 1 that (36) is a complete Λ -projective resolution. $\square$

4. Gorenstein-Projective Modules

In this section, we describe all the Gorenstein-projective modules over $\Lambda = \begin{pmatrix} A & {}_A M_B \\ 0 & B \end{pmatrix}$.

4.1. Main Result. Keeping the notations in Section 3, we put

$${}^\perp N = \{L \in A - \text{mod} \mid \text{Ext}_A^i(L, N) = 0, \quad \forall i \geq 1\}. \quad (39)$$

In the next result, we give sufficient and necessary conditions for all Gorenstein-projective modules over $\Lambda = \begin{pmatrix} A & {}_A M_B \\ 0 & B \end{pmatrix}$.

Theorem 2. Let $\Lambda = \begin{pmatrix} A & {}_A M_B \\ 0 & B \end{pmatrix}$ be an Artin algebra and N be an Λ -module. Then, N is a Gorenstein-projective Λ -module if and only if one of the following cases holds:

(1) $N = \begin{pmatrix} L \\ 0 \end{pmatrix}$ with $L \in G \text{proj } A$ and the sequence $\text{Hom}_A(\mathcal{P}^\bullet, M)$ is exact, where L admits a complete A -projective resolution

$$\dots \xrightarrow{\alpha^{-2}} P^{-1} \xrightarrow{\alpha^{-1}} P^0 \xrightarrow{\alpha^0} P^1 \xrightarrow{\alpha^1} \dots, \quad (40)$$

(2) $N = \begin{pmatrix} M \otimes_B K \\ K \end{pmatrix}$ with $K \in G \text{proj } B$, where K admits a complete B -projective resolution

$$\dots \xrightarrow{g^{-2}} Q^{-1} \xrightarrow{g^{-1}} Q^0 \xrightarrow{g^0} Q^1 \xrightarrow{g^1} \dots, \quad (41)$$

such that $\text{Ker}(\text{Id}_M \otimes g^i) = M \otimes_B \text{Ker } g^i \forall i \in \mathbb{Z}$.

(3) N is the Λ -module:

$$N = \text{Ker } f^0 = \left(\begin{array}{c} \{(p^0, x^0) \in P^0 \oplus M \otimes_B Q^0 \mid \alpha^0(p^0) = 0, \beta^0(p^0) + (\text{Id}_M \otimes g^0)(x^0) = 0\} \\ \text{Ker } g^0 \end{array} \right), \quad (42)$$

where $(P^i, Q^i, \alpha^i, g^i, \beta^i)$ is as in (9) with $P^i \neq 0 \neq Q^i$ for some i and f^i is given as in (10), such that conditions (GP1)–(GP5) are satisfied.

Proof. First, we prove the sufficiency.

Suppose that case (1) holds. So, $N = \begin{pmatrix} L \\ 0 \end{pmatrix}$ with

$L \in G \text{proj } A$ with $L \in G \text{proj } A$, where $L = \text{Ker } \alpha^0 \in G \text{proj } A$ with a complete A -projective resolution

$$\mathcal{P}^\bullet = \dots \xrightarrow{\alpha^{-2}} P^{-1} \xrightarrow{\alpha^{-1}} P^0 \xrightarrow{\alpha^0} P^1 \xrightarrow{\alpha^1} \dots, \quad (43)$$

such that the sequence $\text{Hom}_A(\mathcal{P}^\bullet, M)$ is exact.

Thus, the sequence

$$\mathcal{U}^\bullet = \dots \xrightarrow{\begin{pmatrix} \alpha^{-2} \\ 0 \end{pmatrix}} \begin{pmatrix} P^{-1} \\ 0 \end{pmatrix} \xrightarrow{\begin{pmatrix} \alpha^{-1} \\ 0 \end{pmatrix}} \begin{pmatrix} P^0 \\ 0 \end{pmatrix} \xrightarrow{\begin{pmatrix} \alpha^0 \\ 0 \end{pmatrix}} \begin{pmatrix} P^1 \\ 0 \end{pmatrix} \xrightarrow{\begin{pmatrix} \alpha^1 \\ 0 \end{pmatrix}} \dots \quad (44)$$

is exact.

We know that $\text{Hom}_\Lambda(\mathcal{U}^\bullet, \begin{pmatrix} M \\ B \end{pmatrix}) \cong \text{Hom}_A(\mathcal{P}^\bullet, M)$.

By assumption, the sequence $\text{Hom}_A(\mathcal{P}^\bullet, M)$ is exact. Thus,

$\text{Hom}_\Lambda(\mathcal{U}^\bullet, \begin{pmatrix} M \\ B \end{pmatrix})$ is exact sequence.

Note that $\text{Hom}_\Lambda(\mathcal{U}^\bullet, \begin{pmatrix} A \\ 0 \end{pmatrix}) \cong \text{Hom}_A(\mathcal{P}^\bullet, A)$. Since

$\mathcal{P}^\bullet$ is complete A -projective resolution, we have that the sequence $\text{Hom}_A(\mathcal{P}^\bullet, A)$ is exact. Thus, $\text{Hom}_\Lambda(\mathcal{U}^\bullet, \begin{pmatrix} A \\ 0 \end{pmatrix})$

is exact sequence.

Since $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \cong \begin{pmatrix} A \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M \\ B \end{pmatrix}$ as left Λ -modules,

$\text{Hom}_\Lambda(\mathcal{U}^\bullet, \begin{pmatrix} A & M \\ 0 & B \end{pmatrix})$ is exact. Thus, $\mathcal{U}^\bullet$ is a complete

Λ -projective resolution. Thus,

$$\text{Ker} \begin{pmatrix} \alpha^0 \\ 0 \end{pmatrix} = \begin{pmatrix} \text{Ker } \alpha^0 \\ 0 \end{pmatrix} = \begin{pmatrix} L \\ 0 \end{pmatrix} = N \quad (45)$$

is a Gorenstein-projective Λ -module.

Suppose that case (2) holds. That is, $N = \begin{pmatrix} M \otimes_B K \\ K \end{pmatrix}$ with $K \in G \text{ proj } B$, where K admits a complete B -projective resolution

$$\dots \xrightarrow{g^{-2}} Q^{-1} \xrightarrow{g^{-1}} Q^0 \xrightarrow{g^0} Q^1 \xrightarrow{g^1} \dots, \quad (46)$$

such that $\text{Ker}(\text{Id}_M \otimes g^i) = M \otimes_B \text{Ker } g^i \forall i \in \mathbb{Z}$.

Since $\text{Ker}(\text{Id}_M \otimes g^i) = M \otimes_B \text{Ker } g^i \forall i \in \mathbb{Z}$, the sequence

$$\dots \longrightarrow \text{Id}_N \otimes g^{-2} M \otimes_B Q^{-1} \longrightarrow \text{Id}_M \otimes g^{-1} M \otimes_B Q^0 \longrightarrow \text{Id}_M \otimes g^0 M \otimes_B Q^1 \longrightarrow \text{Id}_M \otimes g^1 \dots \quad (47)$$

is exact. Thus, the sequence

$$\mathcal{W}^\bullet = \dots \xrightarrow{\begin{pmatrix} \text{Id}_N \otimes g^{-2} \\ g^{-2} \end{pmatrix}} \begin{pmatrix} M \otimes_B Q^{-1} \\ Q^{-1} \end{pmatrix} \xrightarrow{\begin{pmatrix} \text{Id}_M \otimes g^{-1} \\ g^{-1} \end{pmatrix}} \begin{pmatrix} N \otimes_B Q^0 \\ Q^0 \end{pmatrix} \xrightarrow{\begin{pmatrix} \text{Id}_M \otimes g^0 \\ g^0 \end{pmatrix}} \begin{pmatrix} N \otimes_B Q^1 \\ Q^1 \end{pmatrix} \xrightarrow{\begin{pmatrix} \text{Id}_M \otimes g^1 \\ g^1 \end{pmatrix}} \dots \quad (48)$$

is exact.

Note that $\text{Hom}_\Lambda(\mathcal{W}^\bullet, \begin{pmatrix} A \\ 0 \end{pmatrix}) = 0$. We have that

$$\text{Hom}_\Lambda(\mathcal{W}^\bullet, \begin{pmatrix} M \\ B \end{pmatrix}) \cong \text{Hom}_B(Q^\bullet, B) \quad (49)$$

is exact, i.e., the sequence $\text{Hom}_\Lambda(\mathcal{W}^\bullet, \begin{pmatrix} M \\ B \end{pmatrix})$ is exact.

Since $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \cong \begin{pmatrix} A \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M \\ B \end{pmatrix}$ as left Λ -module, we have that $\mathcal{W}^\bullet$ is a complete Λ -projective resolution. Thus,

$$\text{Ker} \begin{pmatrix} \text{Id}_M \otimes g^0 \\ g^0 \end{pmatrix} = \begin{pmatrix} \text{Ker}(\text{Id}_M \otimes g^0) \\ \text{Ker } g^0 \end{pmatrix} = \begin{pmatrix} M \otimes_B K \\ K \end{pmatrix} = N \quad (50)$$

is a Gorenstein-projective Λ -module.

Suppose that case (3) holds. Then, by Theorem 1, the sequence

$$\mathcal{X}^\bullet := \dots \xrightarrow{f^{-2}} X^{-1} \xrightarrow{f^{-1}} X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \dots \quad (51)$$

is a complete Λ -projective resolution, where X^i is as given in (8). Thus, $G = \text{Ker } f^0$ is a Gorenstein-projective Λ -module.

Now, we show the necessity. Let N be a Gorenstein-projective Λ -module. Then, $N = \text{Ker } f^0$, where f^i is the Λ -map occurring in a complete Λ -projective resolution of $\mathcal{X}^\bullet$. By Theorem 1, $\mathcal{X}^\bullet$ is of form (33), where X^i and f^i are given in (8) and (10), respectively, satisfying conditions (GP1)–(GP5).

If $Q^i = 0, \forall i \in \mathbb{Z}$, in (8), then $\beta^i = 0$ and $g^i = 0, \forall i \in \mathbb{Z}$. By (GP1) and (GP4), we see that

$$\mathcal{P}^\bullet: \dots \xrightarrow{\alpha^{-2}} P^{-1} \xrightarrow{\alpha^{-1}} P^0 \xrightarrow{\alpha^0} P^1 \xrightarrow{\alpha^1} \dots \quad (52)$$

is a complete A -projective resolution, and (GP5) means that $\text{Hom}_A(\mathcal{P}^\bullet, M)$ is exact. Thus, $N = \text{Ker } f^0 = \begin{pmatrix} \text{Ker } \alpha^0 \\ 0 \end{pmatrix}$ is of form (1).

If $P^i = 0, \forall i \in \mathbb{Z}$, in (8), then $\alpha^i = 0$ and $\beta^i = 0, \forall i \in \mathbb{Z}$. By (GP2) and (GP5), we have that

$$\mathcal{Q}^\bullet: \dots \xrightarrow{g^{-2}} Q^{-1} \xrightarrow{g^{-1}} Q^0 \xrightarrow{g^0} Q^1 \xrightarrow{g^1} \dots \quad (53)$$

is a complete B -projective resolution, and (GP5) means that $\text{Ker}(\text{Id}_N \otimes g^i) = N \otimes_B \text{Ker } g^i \forall i \in \mathbb{Z}$. Thus, $\text{Ker } g^0 \in G \text{ proj } B$, and $N = \text{Ker } f^0 = \begin{pmatrix} N \otimes_B \text{Ker } g^0 \\ \text{Ker } g^0 \end{pmatrix}$ is of form (2).

If $P^i \neq 0$ and $Q^i \neq 0$ for some $i \in \mathbb{Z}$ in (9), then $N = \text{Ker } f^0$ is of form (3) by Theorem 1. This completes the proof. $\square$

4.2. Special Gorenstein-Projective Modules

Corollary 1. Suppose that L is a Gorenstein-projective A -module with respect to the complete A -projective resolution

$$\mathcal{P}^\bullet = \dots \xrightarrow{\alpha^{-2}} P^{-1} \xrightarrow{\alpha^{-1}} P^0 \xrightarrow{\alpha^0} P^1 \xrightarrow{\alpha^1} \dots, \quad (54)$$

and suppose that K is a Gorenstein-projective B -module with respect to the complete B -projective resolution

$$\mathcal{Q}^\bullet = \dots \xrightarrow{g^{-2}} Q^{-1} \xrightarrow{g^{-1}} Q^0 \xrightarrow{g^0} Q^1 \xrightarrow{g^1} \dots \quad (55)$$

Assume that $M \otimes_B \mathcal{Q}^\bullet$ and $\text{Hom}_A(\mathcal{P}^\bullet, M)$ are exact sequences. Then, $\begin{pmatrix} L \\ 0 \end{pmatrix} \oplus \begin{pmatrix} N \otimes_B K \\ K \end{pmatrix}$ is a Gorenstein-projective Λ -module with respect to the complete Λ -projective resolution

$$\mathcal{X}^\bullet = \dots \xrightarrow{f^{-1}} \begin{pmatrix} P^0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M \otimes_B Q^0 \\ Q^0 \end{pmatrix} \xrightarrow{f^0} \begin{pmatrix} P^1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M \otimes_B Q^1 \\ Q^1 \end{pmatrix} \xrightarrow{f^1} \dots, \quad (56)$$

where Λ -map is $f^i = \begin{pmatrix} \begin{pmatrix} \alpha^i \\ 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \text{Id}_M \otimes g^i \\ g^i \end{pmatrix} \end{pmatrix}$:

$$\begin{pmatrix} P^i \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M \otimes_B Q^i \\ Q^i \end{pmatrix} \longrightarrow \begin{pmatrix} P^{i+1} \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M \otimes_B Q^{i+1} \\ Q^{i+1} \end{pmatrix}.$$

Proof. Since $M \otimes_B Q^\bullet$ is exact, we have that $\mathcal{X}^\bullet$ is exact.

Since each term in the complex $\mathcal{X}^\bullet$ is projective, applying $\text{Hom}_\Lambda(\mathcal{X}^\bullet, -)$ to the exact sequence

$$0 \longrightarrow \begin{pmatrix} M \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} M \\ B \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ B \end{pmatrix} \longrightarrow 0, \quad (57)$$

we get following exact sequence of complexes:

$$\begin{aligned} 0 &\longrightarrow \text{Hom}_\Lambda\left(\mathcal{X}^\bullet, \begin{pmatrix} M \\ 0 \end{pmatrix}\right) \\ &\longrightarrow \text{Hom}_\Lambda\left(\mathcal{X}^\bullet, \begin{pmatrix} M \\ B \end{pmatrix}\right) \\ &\longrightarrow \text{Hom}_\Lambda\left(\mathcal{X}^\bullet, \begin{pmatrix} 0 \\ B \end{pmatrix}\right) \longrightarrow 0. \end{aligned} \quad (58)$$

Since $Q^\bullet$ is a complete B -projective resolution, it follows that $\text{Hom}_\Lambda\left(\mathcal{X}^\bullet, \begin{pmatrix} 0 \\ B \end{pmatrix}\right) \cong \text{Hom}_B(Q^\bullet, B)$ is exact. By assumption, $\text{Hom}_A(Q^\bullet, M)$ is exact. Thus, it follows that $\text{Hom}_\Lambda\left(\mathcal{X}^\bullet, \begin{pmatrix} M \\ B \end{pmatrix}\right)$ is also exact.

Since $\text{Hom}_\Lambda\left(\mathcal{X}^\bullet, \begin{pmatrix} A \\ 0 \end{pmatrix}\right) \cong \text{Hom}_A(\mathcal{P}^\bullet, A)$ and $\text{Hom}_A(\mathcal{P}^\bullet, A)$ is exact, it follows that $\text{Hom}_\Lambda\left(\mathcal{X}^\bullet, \begin{pmatrix} A \\ 0 \end{pmatrix}\right)$ is exact.

All together, we see that $\mathcal{X}^\bullet$ is a complete Λ -projective resolution. Thus, $\text{Ker} f^0 = \begin{pmatrix} L \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M \otimes_B K \\ K \end{pmatrix}$ is a Gorenstein-projective Λ -module. $\square$

Data Availability

No datasets were generated or analyzed during the current study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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