Research Article

Gorenstein-Projective Modules over Upper Triangular Matrix Artin Algebras

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Received 29 July 2021; Accepted 15 October 2021; Published 19 November 2021

1. Introduction

Auslander and Bridger [1] generalized finitely generated projective modules to modules of Gorenstein dimension zero over two-sided Noetherian rings. After two decades, Enochs and Jenda [2] generalized it to an arbitrary ring and called it Gorenstein-projective modules. After that, this class of modules got special attention and has been studied by many authors (see, e.g., [3–15]). Given an algebra $A$, it is a fundamental problem to determine all the indecomposable finitely generated Gorenstein-projective $A$-modules, which is usually not easy. Zhang [14] introduced the notion of compatible bimodules and then determined all the Gorenstein-projective modules over upper triangular matrix Artin algebra $\Lambda = \begin{pmatrix} A & M_B \\ 0 & B \end{pmatrix}$, in terms of Gorenstein-projective $A$ and $B$-modules. This gives us a strong motivation to study Gorenstein-projective modules over upper triangular matrix Artin algebras.

The main aim of this paper is to describe all the complete projective resolutions and all finitely generated Gorenstein-projective modules over an upper triangular matrix Artin algebra $\Lambda = \begin{pmatrix} A & M_B \\ 0 & B \end{pmatrix}$, by giving the corresponding sufficient and necessary conditions. Although these conditions ((GP1)–(GP5)) are long, they may be helpful, especially in some special cases.

Throughout the paper, $A - \text{Mod}$ denotes the category of left $A$-modules and $A - \text{mod}$ denotes the category of finitely generated left $A$-modules for a ring $A$.

2. Preliminaries

In this section, we recall some basic definitions and facts that will be used throughout the paper.

Following [16], an $A$-module $M$ is said to be Gorenstein-projective, in $A - \text{Mod}$ (resp., in $A - \text{Mod}$), if there is an exact sequence of projective modules in $A - \text{Mod}$ (resp., in $A - \text{Mod}$)

\[
\mathcal{P}^\ast = \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots,
\]

with $\text{Hom}_A(P^\ast, Q)$ being also exact for any projective module $Q$ in $A - \text{Mod}$ (resp., in $A - \text{mod}$), such that $M \cong \text{Ker} \partial^1$. Such $\mathcal{P}^\ast$ is called a complete projective resolution in $A - \text{Mod}$ (resp., $A - \text{mod}$).

2.1. Modules over Upper Triangular Matrix Algebra. Let $A M_B$ be an $A - B$ bimodule and
\[ \Lambda = \begin{pmatrix} A & M_B \\ 0 & B \end{pmatrix} \]  
(2)

be an Artin algebra. Note that addition of \( \Lambda \) is the addition of matrix, and the multiplication is given by

\[ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix} = \begin{pmatrix} aa' + mb' \\ 0 \end{pmatrix}. \]  
(3)

Left modules over \( \Lambda \) are described with a triple \( \left( X, Y, \phi \right) \), where \( X \) is an \( A \)-module and \( Y \) is a \( B \)-module, and \( \phi: M \otimes_B Y \rightarrow X \) is an \( A \)-map. \( \Lambda \)-map is described with a pair \( \left( \phi, \varphi \right) \), where \( \phi: X \rightarrow Y \) is an \( A \)-map and \( \varphi: Y \rightarrow Y' \) is a \( B \)-map, such that the diagram

\[ \begin{array}{ccc}
M \otimes_B Y & \xrightarrow{\phi} & X \\
\text{Id} \otimes \varphi & \downarrow & \downarrow f \\
M \otimes_B Y' & \xrightarrow{\varphi'} & X'
\end{array} \]

is commutative.

A sequence

\[ 0 \rightarrow \left( X_1, \frac{f_1}{g_1}, X_2, \frac{f_2}{g_2}, X_3 \right) \rightarrow X \]

in \( \Lambda \) - mod is exact if and only if a sequence

\[ 0 \rightarrow X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \rightarrow 0 \]

is exact in \( A \) - mod, and a sequence

\[ 0 \rightarrow Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} Y_3 \rightarrow 0 \]

is exact in \( B \) - mod.

2.2. Projective Modules. The indecomposable projective (left) \( \Lambda \)-modules are exactly \( P(0) \) and \( \left( M \otimes_B Q \right) \), where \( P \) runs over indecomposable projective \( A \)-modules and \( Q \) runs over indecomposable projective \( B \)-modules.

3. Complete Projective Resolutions

The aim of this section is to describe all complete projective resolutions over an upper triangular matrix Artin algebra

\[ \Lambda = \begin{pmatrix} A & M_B \\ 0 & B \end{pmatrix}. \]

3.1. Morphisms of Projective \( \Lambda \)-Modules. Note that any projective \( \Lambda \)-module is of the form

\[ X = \begin{pmatrix} P \oplus M \otimes_B Q \\ Q \end{pmatrix}, \]  
(7)

where \( P \) and \( Q \) are projective \( A \)-module and projective \( B \)-module, respectively. For each \( i \in \mathbb{Z} \), we write

\[ X^i = \begin{pmatrix} P^i \oplus M \otimes_B Q^i \\ Q^i \end{pmatrix}. \]  
(8)

For such a projective \( \Lambda \)-module \( X^i \) as above and for any datum

\[ \left( P^i, Q^i, \alpha^i, \beta^i, \gamma^i \right), \]

where \( \alpha^i: P^i \rightarrow P^{i+1} \) and \( \beta^i: P^i \rightarrow M \otimes_B Q^i \) are \( A \)-maps and \( \gamma^i: Q^i \rightarrow Q^{i+1} \) is \( B \)-map \( \forall i \in \mathbb{Z} \), we put

\[ f^i = \begin{pmatrix} \alpha^i \\ \beta^i \text{Id} \otimes \gamma^i \end{pmatrix} : \begin{pmatrix} P^i \oplus M \otimes_B Q^i \\ Q^i \end{pmatrix} \rightarrow \begin{pmatrix} P^{i+1} \oplus M \otimes_B Q^{i+1} \\ Q^{i+1} \end{pmatrix}. \]  
(9)

Lemma 1. Let \( X^i \) and \( (P^i, Q^i, \alpha^i, \beta^i, \gamma^i) \) be given as in (8) and (9). Then, \( f^i: X^i \rightarrow X^{i+1} \) is a \( \Lambda \)-map if and only if \( f^i \) is given in (10).

Proof. First, assume that \( f^i: X^i \rightarrow X^{i+1} \) is a \( \Lambda \)-map, where \( f^i \) is given in (10). Since the diagram

\[ \begin{array}{ccc}
M \otimes_B Q^i & \xrightarrow{\text{Id} \otimes \gamma^i} & P^i \oplus M \otimes_B Q^i \\
\text{Id} \otimes \beta^i & \downarrow & \downarrow \alpha^i \\
M \otimes_B Q^{i+1} & \xrightarrow{\text{Id} \otimes \gamma^{i+1}} & P^{i+1} \oplus M \otimes_B Q^{i+1}
\end{array} \]

is commutative, we see that \( f^i: X^i \rightarrow X^{i+1} \) is a \( \Lambda \)-map, \( \forall i \in \mathbb{Z} \).

Conversely, let \( f^i: X^i \rightarrow X^{i+1} \) be an arbitrary \( \Lambda \)-map where \( X^i = \begin{pmatrix} P^i \oplus M \otimes_B Q^i \\ Q^i \end{pmatrix} \). Then, \( f^i \) is of the form

\[ \begin{pmatrix} \alpha^i \\ \beta^i \end{pmatrix} : X^i \rightarrow X^{i+1}, \]  
(11)

where \( \alpha^i: P^i \rightarrow P^{i+1} \), \( \beta^i: P^i \rightarrow M \otimes_B Q^i \), \( \omega^i: M \otimes_B Q^i \rightarrow P^{i+1} \), and \( \delta^i: M \otimes_B Q^i \rightarrow M \otimes_B Q^{i+1} \) are \( A \)-maps and \( \delta^i: Q^i \rightarrow Q^{i+1} \) is \( B \)-map, \( \forall i \in \mathbb{Z} \). By the definition of morphisms in \( \Lambda \)-mod, the diagram

\[ \begin{array}{ccc}
M \otimes_B Q^i & \xrightarrow{\text{Id} \otimes \gamma^i} & P^i \oplus M \otimes_B Q^i \\
\text{Id} \otimes \beta^i & \downarrow & \downarrow \alpha^i \\
M \otimes_B Q^{i+1} & \xrightarrow{\text{Id} \otimes \gamma^{i+1}} & P^{i+1} \oplus M \otimes_B Q^{i+1}
\end{array} \]

is commutative. This implies that
\[ \omega^i = 0, \]
\[ \delta^i = \text{Id}_M \otimes \delta^i \quad \forall i \in \mathbb{Z}. \]  
\[ (12) \]

Thus, \( f^i \) is given as in (10). This completes the proof.  

3.2. The Exactness of Morphisms of Projective Modules.
Consider the following conditions:

- \((GP1)\) \( \varphi^* = \cdots \rightarrow a^2 \rightarrow \alpha^{-1} p^{-1} \rightarrow a^1 \rightarrow p^i \rightarrow a^i \rightarrow \cdots \) is a complex.
- \((GP2)\) \( \varphi^* = \cdots \rightarrow a^2 \rightarrow \alpha q^{-1} \rightarrow a^1 \rightarrow q^i \rightarrow a^i \rightarrow \cdots \) is an exact sequence.
- \((GP3)\) \( \beta^{i+1} \delta^i + (\text{Id}_M \otimes g^{i+1})\beta^i = 0, \forall i \in \mathbb{Z}. \)
- \((GP4)\) For any \((p^{i+1}, x^{i+1}) \in P^{i+1} \otimes M \otimes \alpha Q^{i+1}\) with \(\alpha^{i+1} (p^{i+1}) = 0\), \(\delta^{i+1} (p^{i+1}) + (\text{Id}_M \otimes g^{i+1})(x^{i+1}) = 0\), there is \((p^i, x^i) \in P^i \otimes M \otimes g^i\Pi\), such that \(p^{i+1} = \alpha^i (p^i)\) and \(x^{i+1} = \delta^i (p^i) + (\text{Id}_M \otimes g^i)(x^i), \forall i \in \mathbb{Z}. \)

**Lemma 2.** Let \( X^i \) be a projective \( \Lambda \)-module given as in (8) and \( f^i: X^i \rightarrow X^{i+1} \) be a \( \Lambda \)-map given as in (10). Then, Ker \( f^{i+1} = \text{Im} f^i \) if and only if conditions \((GP1)-(GP4)\) are satisfied.

**Proof.** By Lemma 1, \( f^i \) is of the form \( f^i = \left( \begin{array}{c}
\alpha^i \\
\beta^i \\
\text{Id}_M \otimes g^i
\end{array} \right) : \left( \begin{array}{c}
P^{i+1} \otimes M \otimes \alpha Q^{i+1}
\end{array} \right) \rightarrow \left( \begin{array}{c}
P^i \otimes M \otimes \alpha Q^i
\end{array} \right). \)

Where \( \alpha^i: P^i \rightarrow P^{i+1} \) and \( \beta^i: P^i \rightarrow M \otimes \alpha Q^i \) and \( g^i: Q^i \rightarrow Q^{i+1}, \forall i \in \mathbb{Z}. \) We see that Ker \( f^{i+1} = \text{Im} f^i, \forall i \in \mathbb{Z}, \) if and only if

\[ \text{Ker} \left( \begin{array}{c}
\alpha^{i+1} \\
\beta^{i+1} \\
\text{Id}_M \otimes g^{i+1}
\end{array} \right) = \text{Im} \left( \begin{array}{c}
\alpha^i \\
\beta^i \\
\text{Id}_M \otimes g^i
\end{array} \right), \]
\[ \text{Ker} g^{i+1} = \text{Im} g^i, \]

where

\[ \left( \begin{array}{c}
\alpha^i \\
\beta^i \\
\text{Id}_M \otimes g^i
\end{array} \right): P^i \otimes M \otimes \alpha Q^i \rightarrow P^{i+1} \otimes M \otimes \alpha Q^{i+1}, \]
\[ \forall i \in \mathbb{Z}. \]
\[ (13) \]

Note that Ker \( g^{i+1} = \text{Im} g^i, \forall i \in \mathbb{Z}, \) if and only if condition \((GP2)\) is satisfied. Thus, it suffices to show that

\[ \text{Ker} \left( \begin{array}{c}
\alpha^{i+1} \\
\beta^{i+1} \\
\text{Id}_M \otimes g^{i+1}
\end{array} \right) = \text{Im} \left( \begin{array}{c}
\alpha^i \\
\beta^i \\
\text{Id}_M \otimes g^i
\end{array} \right), \]
\[ (15) \]

Now, \( \text{Im} \left( \begin{array}{c}
\alpha^i \\
\beta^i \\
\text{Id}_M \otimes g^i
\end{array} \right) \subseteq \text{Ker} \left( \begin{array}{c}
\alpha^{i+1} \\
\beta^{i+1} \\
\text{Id}_M \otimes g^{i+1}
\end{array} \right) \) if and only if \( \left( \begin{array}{c}
\alpha^{i+1} \\
\beta^{i+1} \\
\text{Id}_M \otimes g^{i+1}
\end{array} \right) \left( \begin{array}{c}
\alpha^i \\
\beta^i \\
\text{Id}_M \otimes g^i
\end{array} \right) = 0, \) i.e.,

\[ \left( \begin{array}{c}
\alpha^{i+1} \alpha^i \\
\beta^{i+1} \beta^i \\
\text{Id}_M \otimes g^{i+1} \otimes g^i
\end{array} \right) = 0, \]
\[ (16) \]

if and only if conditions \((GP1)-(GP3)\) are satisfied.

We also see that \( \text{Ker} \left( \begin{array}{c}
\alpha^{i+1} \\
\beta^{i+1} \\
\text{Id}_M \otimes g^{i+1}
\end{array} \right) \subseteq \text{Im} \left( \begin{array}{c}
\alpha^i \\
\beta^i \\
\text{Id}_M \otimes g^i
\end{array} \right), \forall i \in \mathbb{Z}, \) if and only if condition \((GP4)\) is satisfied.

Thus, Ker \( f^{i+1} = \text{Im} f^i, \forall i \in \mathbb{Z}, \) if and only if conditions \((GP1)-(GP4)\) are satisfied.

Consider the following condition \((GP5)\) \( \forall i \in \mathbb{Z}: \)

\[ \left( \begin{array}{c}
\alpha^{i+1} \\
\beta^{i+1} \\
\text{Id}_M \otimes g^{i+1}
\end{array} \right) : \left( \begin{array}{c}
P^{i+1} \otimes M \otimes \alpha Q^{i+1}
\end{array} \right) \rightarrow \left( \begin{array}{c}
P^i \otimes M \otimes \alpha Q^i
\end{array} \right). \]

There are \( c^{i+1} \in \text{Hom}_A(P^{i+1}, A), \) \( d^{i+2} \in \text{Hom}_A(P^{i+1}, M), t^{i+1} \in \text{Hom}_B(Q^{i+1}, B), \) such that

\[ c^{i+1} = c^{i+2} \alpha^{i+1}, \]
\[ d^{i+1} = d^{i+2} \alpha^{i+1} + (\text{Id}_M \otimes t^{i+2}) \beta^{i+1}, \]
\[ t^{i+1} = t^{i+2} \delta^{i+1}, \]
\[ (18) \]

**Lemma 3.** Let \( X^i \) be a projective \( \Lambda \)-module given as in (8) and \( f^i: X^i \rightarrow X^{i+1} \) be a \( \Lambda \)-map given as in (10). Assume that Ker \( f^{i+1} = \text{Im} f^i, \) i.e., conditions \((GP1)-(GP4)\) are satisfied. Consider the homomorphism of abelian groups \( f^i = \text{Hom}_A(f^i, A): \text{Hom}_A(X^{i+1}, A) \rightarrow \text{Hom}_A(X^i, A). \) Then, Ker \( f^i = \text{Im} (f^{i+1}), \) \( \forall i \in \mathbb{Z}, \) if and only if condition \((GP5)\) is satisfied.

**Proof.** Assume that Ker \( f^{i+1} = \text{Im} f^i. \) By Lemma 1, we know that \( f^i: X^i \rightarrow X^{i+1} \) is given as

\[ f^i = \left( \begin{array}{c}
\alpha^i \\
\beta^i \\
\text{Id}_M \otimes g^i
\end{array} \right) : \left( \begin{array}{c}
P^i \otimes M \otimes \alpha Q^i
\end{array} \right) \rightarrow \left( \begin{array}{c}
P^{i+1} \otimes M \otimes \alpha Q^{i+1}
\end{array} \right). \]
\[ (19) \]

Any element in the abelian group \( \text{Hom}_A(X^i, A) \) will be written as

\[ \text{Hom}_A\left( \left( \begin{array}{c}
P^i \otimes M \otimes \alpha Q^i \\
\frac{A \otimes M}{B}
\end{array} \right), \right) \]
\[ \forall i \in \mathbb{Z}. \]
\[ h^i = \left( \begin{array}{c} c^i \\ d^i \\ s^i \\ t^i \end{array} \right), \quad \left( P^i \otimes M \otimes B Q^i \right) \rightarrow \left( A \otimes M \right). \]  

(20)

By definition of morphisms in \( \Lambda \)-mod, the diagram

\[
\begin{array}{ccc}
M \otimes B Q^i & \xrightarrow{\left( \begin{array}{c} 0 \\ 0 \Id_M \otimes g^i \end{array} \right)} & P^i \otimes M \otimes B Q^i \\
\Id_M \otimes t^i & \downarrow & \downarrow \\
M \otimes B & \xrightarrow{\left( \begin{array}{c} 0 \\ \Id_M \end{array} \right)} & A \otimes M
\end{array}
\]

is commutative. This implies that

\[ f^i = \Hom_{\Lambda} \left( \left( \begin{array}{c} \alpha^i \\
\beta^i \Id_M \otimes g^i \\
g^i \end{array} \right), \Lambda \right); \quad \Hom_{\Lambda} \left( \left( P^{i+1} \otimes M \otimes B Q^{i+1} \right), \left( A \otimes M \right) \right) \rightarrow \Hom_{\Lambda} \left( \left( P^i \otimes M \otimes B Q^i \right), \left( A \otimes M \right) \right). \]

(23)

Thus, \( f^i \) maps

\[ h^{i+1} = \left( \begin{array}{c} c^{i+1} \\
\alpha^{i+1} \\
\beta^{i+1} \Id_M \otimes t^{i+1} \\
t^{i+1} \end{array} \right) \]  

(24)

to

\[ f^i \left( h^{i+1} \right) = h^{i+1} f^i = \left( \begin{array}{c} c^{i+1} \\
\alpha^{i+1} \\
\beta^{i+1} \Id_M \otimes g^{i+1} \\
t^{i+1} \end{array} \right), \]  

(25)

which means that \( h^{i+1} = \left( \begin{array}{c} c^{i+1} \\
\alpha^{i+1} \\
\beta^{i+1} \Id_M \otimes t^{i+1} \\
t^{i+1} \end{array} \right) \in \text{Ker} f^i \) if and only if

\[ c^{i+1} = c^{i+2} \alpha^{i+1}, \]
\[ d^{i+1} = d^{i+2} \alpha^{i+1} + \left( \Id_M \otimes t^{i+2} \right) \beta^{i+1}, \]
\[ t^{i+1} \beta^{i+1} = 0, \]

(28)

By conditions (G1)–(G4), we have

\[ e^i = 0, \]
\[ s^i = \Id_M \otimes t^i \quad \forall i \in \mathbb{Z}. \]

(21)
Thus, \( f^i \circ f^{i+1} = 0 \), i.e., \( \text{Ker} f^i \subseteq \text{Ker} f^{i+1} \). Therefore, \( \text{Im} f^i \subseteq \text{Ker} f^{i+1} \) if and only if \( \text{Im} f^{i+1} \subseteq \text{Ker} f^i \), if and only if for any \( h^i = (X \oplus M_B) \) with

\[
\begin{align*}
\alpha_{i+1} &= 0, \\
\beta_{i+1} &= 0, \\
\gamma_{i+1} &= 0,
\end{align*}
\]

there is

\[
H^{i+2} = \begin{pmatrix}
\alpha_{i+2} \\
\beta_{i+2} \\
\gamma_{i+2}
\end{pmatrix}
\]

\( \begin{pmatrix}
\alpha_i \\
\beta_i \\
\gamma_i
\end{pmatrix}
\]

such that

\[
\begin{align*}
\alpha_{i+1} &= \alpha_i, \\
\beta_{i+1} &= \beta_i, \\
\gamma_{i+1} &= \gamma_i.
\end{align*}
\]

So, by the above arguments, we see that \( \text{Im} f^{i+1} \) is a complete \( \Lambda \)-module, and \( \text{Im} f^i \) is a complete \( \Lambda \)-module. We then get the following special kind of complete projective resolutions over \( \Lambda = (A \rightarrow M_B \rightarrow B) \).

**Proposition 1.** Let \( X^i \) be a projective \( \Lambda \)-module, given as in (8), and \( f^i: X^i \rightarrow X^{i+1} \) be a \( \Lambda \)-map, given as in (35). Then,

\[
\begin{array}{cccc}
X^* &=& \cdots & \longrightarrow X^{-1} & f^{-1} & X^0 & f^0 & X^1 & f^1 & \cdots \\
\end{array}
\]

is a complete \( \Lambda \)-projective resolution if and only if

\[
\begin{array}{cccc}
\alpha^* &=& \cdots & \longrightarrow \alpha^{-1} & f^{-1} & \alpha^0 & f^0 & \alpha^1 & f^1 & \cdots \\
\beta^* &=& \cdots & \longrightarrow \beta^{-1} & f^{-1} & \beta^0 & f^0 & \beta^1 & f^1 & \cdots \\
\gamma^* &=& \cdots & \longrightarrow \gamma^{-1} & f^{-1} & \gamma^0 & f^0 & \gamma^1 & f^1 & \cdots \\
\end{array}
\]

is a complete \( \Lambda \)-projective resolution, and \( M \oplus_B \alpha^* \) and \( \text{Hom}_A(\beta^*, M) \) are exact sequences.

**Proof.** Assume that (36) is a complete \( \Lambda \)-projective resolution. Then, by Theorem 1, we know that conditions (GP1), (GP2), (GP4), and (GP5) are satisfied.

Let \( p^{i+1} \in \text{Ker} f^{i+1} \). Since \( \beta^{i+1} = 0 \), it follows from (GP4) by taking \( x^{i+1} = 0 \) that \( p^{i+1} \in \text{Im} f^i \), i.e., \( \text{Ker} f^{i+1} \subseteq \text{Im} f^i \). Together with (GP1), we see that the sequence \( \alpha^* \) in (37) is exact.

Since \( \beta^i = 0 \), (GP4) means that \( M \oplus_B \beta^* \) is an exact sequence.

Since \( \beta^i = 0 \), (GP5) means that \( \text{Hom}_A(\beta^*, A) \), \( \text{Hom}_A(\beta^*, M) \), and \( \text{Hom}_B(\beta^*, B) \) are exact sequences.

Since \( \beta^* \) and \( \text{Hom}_A(\beta^*, A) \) are exact sequences, it follows that \( \beta^* \) is a complete \( \Lambda \)-projective resolution. Similarly, \( \alpha^* \) is a complete \( \beta \)-projective resolution.

Conversely, assume that \( \beta^* \) is a complete \( \beta \)-projective resolution, \( \alpha^* \) is a complete \( \beta \)-projective resolution, and \( M \oplus_B \alpha^* \) and \( \text{Hom}_A(\beta^*, M) \) are exact sequences. Then, conditions (GP1), (GP2), (GP4), and (GP5) are satisfied,

3.4. Special Complete Projective Resolutions. We consider the special case of \( f: X^i \rightarrow X^{i+1} \) where

\[
\begin{array}{cccc}
f^i &=& \begin{pmatrix}
\alpha_i \\
\beta_i \\
\gamma_i
\end{pmatrix} \\
\end{array}
\]

and we take \( \beta_i = 0 \) \( \forall i \in \mathbb{Z} \), i.e.,

\[
\begin{array}{cccc}
f^i &=& \begin{pmatrix}
\alpha_i \\
\beta_i \\
\gamma_i
\end{pmatrix} \\
\end{array}
\]

Then the condition (GP3) holds for each \( i \in \mathbb{Z} \). We then get the following special kind of complete projective resolutions over \( \Lambda = (A \rightarrow M_B \rightarrow B) \).

**Theorem 1.** Let \( X^i \) be a projective \( \Lambda \)-module, given as in (8), and \( f^i: X^i \rightarrow X^{i+1} \) be a \( \Lambda \)-map, given as in (10). Then,

\[
\begin{array}{cccc}
X^* &=& \cdots & \longrightarrow X^{-1} & f^{-1} & X^0 & f^0 & X^1 & f^1 & \cdots \\
\end{array}
\]

is a complete \( \Lambda \)-projective resolution if and only if conditions (GP1)–(GP5) are satisfied.
where $\beta^i = 0$, $\forall i \in \mathbb{Z}$. It follows from Theorem 1 that (36) is a complete $\Lambda$-projective resolution. \hfill \Box

### 4. Gorenstein-Projective Modules

In this section, we describe all the Gorenstein-projective modules over $\Lambda = \left( \begin{array}{c|c} A & M_B \\ \hline 0 & B \end{array} \right)$.

#### 4.1. Main Result.

Keeping the notations in Section 3, we put

$\overset{\dagger}{N} = \{ L \in A - \text{mod} | \text{Ext}^1_{\Lambda}(L, N) = 0, \ \forall i \geq 1 \}$. \hspace{1cm} (39)

In the next result, we give sufficient and necessary conditions for all Gorenstein-projective modules over $\Lambda = \left( \begin{array}{c|c} A & M_B \\ \hline 0 & B \end{array} \right)$.

**Theorem 2.** Let $\Lambda = \left( \begin{array}{c|c} A & M_B \\ \hline 0 & B \end{array} \right)$ be an Artin algebra and $N$ be an $\Lambda$-module. Then, $N$ is a Gorenstein-projective $\Lambda$-module if and only if one of the following cases holds:

$$N = \text{Ker} f^0 = \left\{ (p^0, x^0) \in P^0 \oplus M \oplus Q^0 | \text{Id}^0(p^0) + (\text{Ker}^0(g^0))(x^0) = 0 \right\}.$$ \hspace{1cm} (42)

where $(P^i, Q^i, \alpha^i, g^i, \beta^i)$ is as in (9) with $P^i \neq 0 \neq Q^i$ for some $i$ and $f^i$ is given as in (10), such that conditions $(GP1) – (GP5)$ are satisfied.

**Proof.** First, we prove the sufficiency.

Suppose that case (1) holds. So, $N = \left( \begin{array}{c} L \\ 0 \end{array} \right)$ with $L \in \text{Gproj}\ A$ with $L \in \text{Gproj}\ A$, where $L = \text{Ker}^0 \in \text{Gproj}\ A$ with a complete $A$-projective resolution

$$U^* = \ldots \xrightarrow{\alpha^{-2}} (\begin{array}{c} 0 \\ 0 \\ 0 \end{array}) \xrightarrow{\alpha^{-1}} (\begin{array}{c} (p^0)^{-1} \\ 0 \\ 0 \\ 0 \end{array}) \xrightarrow{\alpha^{-1}} (\begin{array}{c} (p^0)^{-1} \\ 0 \\ 0 \end{array}) \xrightarrow{\alpha^{-1}} (\begin{array}{c} (p^0)^{-1} \\ 0 \\ 0 \end{array}) \xrightarrow{\alpha^{-1}} \ldots \hspace{1cm} (44)$$

is exact.

We know that $\text{Hom}_\Lambda(U^*, \left( \begin{array}{c|c} M \\ \hline 0 \end{array} \right)) \equiv \text{Hom}_\Lambda(\mathcal{P}^*, M)$. By assumption, the sequence $\text{Hom}_\Lambda(\mathcal{P}^*, M)$ is exact. Thus, $\text{Hom}_\Lambda(U^*, \left( \begin{array}{c|c} M \\ \hline 0 \end{array} \right))$ is exact sequence.

Note that $\text{Hom}_\Lambda(U^*, \left( \begin{array}{c|c} A \\ \hline 0 \end{array} \right)) \equiv \text{Hom}_\Lambda(\mathcal{P}^*, A)$. Since $\mathcal{P}^*$ is complete $A$-projective resolution, we have that the sequence $\text{Hom}_\Lambda(\mathcal{P}^*, A)$ is exact. Thus, $\text{Hom}_\Lambda(U^*, \left( \begin{array}{c} A \\ 0 \end{array} \right))$ is exact sequence.

(1) $N = \left( \begin{array}{c} L \\ 0 \end{array} \right)$ with $L \in \text{Gproj}\ A$ and the sequence $\text{Hom}_\Lambda(\mathcal{P}^*, M)$ is exact, where $L$ admits a complete $A$-projective resolution

$$\ldots \xrightarrow{\alpha^{-2}} p^0 \xrightarrow{\alpha^{-1}} p^0 \xrightarrow{\alpha^{-1}} p^0 \xrightarrow{\alpha^{-1}} \ldots.$$ \hspace{1cm} (40)

(2) $N = \left( \begin{array}{c|c} M \oplus K \\ \hline K \end{array} \right)$ with $K \in \text{Gproj}\ B$, where $K$ admits a complete $B$-projective resolution

$$\ldots \xrightarrow{g^1} Q^0 \xrightarrow{g^0} Q^0 \xrightarrow{g^0} \ldots,$$ \hspace{1cm} (41)

such that $\text{Ker}(\text{Id}_M \oplus g^0) = M \oplus \text{Ker} g^0 \forall i \in \mathbb{Z}$.

(3) $N$ is the $\Lambda$-module:

$\overset{\dagger}{\mathcal{P}^*} = \ldots \xrightarrow{\alpha^{-2}} p^0 \xrightarrow{\alpha^{-1}} p^0 \xrightarrow{\alpha^{-1}} p^0 \xrightarrow{\alpha^{-1}} \ldots,$ \hspace{1cm} (43)

such that the sequence $\text{Hom}_\Lambda(\mathcal{P}^*, M)$ is exact. Thus, the sequence

Since $\Lambda = \left( \begin{array}{c|c} A & M \\ \hline 0 & B \end{array} \right)$ as left $\Lambda$-modules,

$\text{Hom}_\Lambda(U^*, \left( \begin{array}{c|c} A \\ \hline 0 \end{array} \right))$ is exact. Thus, $U^*$ is a complete $\Lambda$-projective resolution. Thus,

$$\text{Ker}(\alpha^0) = (\text{Ker}^0) = \left( \begin{array}{c} L \\ 0 \end{array} \right) = N.$$ \hspace{1cm} (45)

is a Gorenstein-projective $\Lambda$-module.
Suppose that case (2) holds. That is, $N = \left( \begin{array}{c} M \otimes_B K \\ K \end{array} \right)$
with $K \in \text{G proj } B$, where $K$ admits a complete $B$-projective resolution

\[
\cdots \rightarrow \text{Id}_N \otimes g^{-2} M \otimes_B Q^{-1} \rightarrow \text{Id}_M \otimes g^{-1} M \otimes_B Q^0 \rightarrow \text{Id}_M \otimes g^0 M \otimes_B Q^1 \rightarrow \text{Id}_M \otimes g^1 \cdots
\]  

(47)
is exact. Thus, the sequence

\[
\mathcal{P} = \cdots \xrightarrow{g^{-2}} M \otimes_B Q^{-1} \xrightarrow{g^{-1}} M \otimes_B Q^0 \xrightarrow{g^0} M \otimes_B Q^1 \xrightarrow{g^1} \cdots
\]  

(48)
is a complete $A$-projective resolution, and (GP5) means that $\text{Hom}_A (\mathcal{P}, M)$ is exact. Thus, $N = \text{Ker } f^0 = \left( \begin{array}{c} \text{Ker } g^0 \\ 0 \end{array} \right)$ is of form (1).

If $P^i = 0$, $\forall i \in \mathbb{Z}$, in (8), then $a^i = 0$ and $\beta^i = 0$, $\forall i \in \mathbb{Z}$. By (GP2) and (GP5), we have that

\[
Q^* : \cdots \xrightarrow{\sigma^{-2}} Q^{-1} \xrightarrow{\sigma^{-1}} Q^0 \xrightarrow{\sigma^0} Q^1 \xrightarrow{\sigma^1} \cdots
\]  

(53)
is a complete $B$-projective resolution, and (GP5) means that $\text{Ker } (\text{Id}_N \otimes g^i) = N \otimes \text{Ker } g^i \forall i \in \mathbb{Z}$. Thus, $\text{Ker } g^0 \in \text{G proj } B$, and $N = \text{Ker } f^0 = \left( \begin{array}{c} \text{N} \otimes_B \text{Ker } g^0 \\ \text{Ker } g^0 \end{array} \right)$ is of form (2).

If $P^i \neq 0$ and $Q^i \neq 0$ for some $i \in \mathbb{Z}$ in (9), then $N = \text{Ker } f^0$ is of form (3) by Theorem 1. This completes the proof. □

4.2. Special Gorenstein-Projective Modules

Corollary 1. Suppose that $L$ is a Gorenstein-projective $A$-module with respect to the complete $A$-projective resolution

\[
\mathcal{P}^* : \cdots \xrightarrow{\sigma^{-2}} P^{-1} \xrightarrow{\sigma^{-1}} P^0 \xrightarrow{\sigma^0} P^1 \xrightarrow{\sigma^1} \cdots
\]  

(54)
and suppose that $K$ is a Gorenstein-projective $B$-module with respect to the complete $B$-projective resolution

\[
\mathcal{Q}^* : \cdots \xrightarrow{\sigma^{-2}} Q^{-1} \xrightarrow{\sigma^{-1}} Q^0 \xrightarrow{\sigma^0} Q^1 \xrightarrow{\sigma^1} \cdots
\]  

(55)
Assume that $M \otimes_B \mathcal{P}^*$ and $\text{Hom}_A (\mathcal{P}^*, M)$ are exact sequences. Then, $\left( \begin{array}{c} L \\ \text{N} \otimes_B K \end{array} \right)$ is a Gorenstein-projective $A$-module with respect to the complete $A$-projective resolution
we get following exact sequence of complexes:

\[ \mathcal{X}^* = \cdots \to \left( \begin{array}{c} P_0 \\ 0 \end{array} \right) \oplus \left( \begin{array}{c} M \otimes Q^0 \\ Q^0 \end{array} \right) \to \left( \begin{array}{c} P_1 \\ 0 \end{array} \right) \oplus \left( \begin{array}{c} M \otimes Q^1 \\ Q^1 \end{array} \right) \to \cdots \]

where \( \Lambda \)-map is \( f^i = \left( \begin{array}{cc} \alpha^i & 0 \\ 0 & \text{Id}_M \otimes g^i \end{array} \right) \):

\[ \left( \begin{array}{c} P^i \\ 0 \end{array} \right) \oplus \left( \begin{array}{c} M \otimes BQ^i \\ Q^i \end{array} \right) \to \left( \begin{array}{c} P^{i+1} \\ 0 \end{array} \right) \oplus \left( \begin{array}{c} M \otimes BQ^{i+1} \\ Q^{i+1} \end{array} \right). \]

Proof. Since \( M \otimes_B \Omega^* \) is exact, we have that \( \mathcal{X}^* \) is exact.

Since each term in the complex \( \mathcal{X}^* \) is projective, applying \( \text{Hom}_\Lambda(\mathcal{X}^*,-) \) to the exact sequence

\[ 0 \to \left( \begin{array}{c} M \\ 0 \end{array} \right) \to \left( \begin{array}{c} M \\ B \end{array} \right) \to \left( \begin{array}{c} 0 \\ B \end{array} \right) \to 0, \]

we get following exact sequence of complexes:

\[ 0 \to \text{Hom}_\Lambda\left( \mathcal{X}^*, \left( \begin{array}{c} M \\ 0 \end{array} \right) \right) \to \text{Hom}_\Lambda\left( \mathcal{X}^*, \left( \begin{array}{c} M \\ B \end{array} \right) \right) \to \text{Hom}_\Lambda\left( \mathcal{X}^*, \left( \begin{array}{c} 0 \\ B \end{array} \right) \right) \to 0. \]

Since \( \mathcal{G}^* \) is a complete \( B \)-projective resolution, it follows that \( \text{Hom}_\Lambda\left( \mathcal{X}^*, \left( \begin{array}{c} 0 \\ B \end{array} \right) \right) \equiv \text{Hom}_B(\mathcal{G}^*, B) \) is exact. By assumption, \( \text{Hom}_\Lambda\left( \mathcal{G}^*, M \right) \) is exact. Thus, it follows that \( \text{Hom}_\Lambda\left( \mathcal{X}^*, \left( \begin{array}{c} M \\ B \end{array} \right) \right) \) is also exact.

Since \( \text{Hom}_\Lambda\left( \mathcal{X}^*, \left( \begin{array}{c} A \\ 0 \end{array} \right) \right) \equiv \text{Hom}_\Lambda(\mathcal{P}^*, A) \) and \( \text{Hom}_\Lambda(\mathcal{P}^*, A) \) is exact, it follows that \( \text{Hom}_\Lambda\left( \mathcal{X}^*, \left( \begin{array}{c} A \\ 0 \end{array} \right) \right) \) is exact.

All together, we see that \( \mathcal{X}^* \) is a complete \( \Lambda \)-projective resolution. Thus, \( \text{Ker} f^0 = \left( \begin{array}{c} L \\ 0 \end{array} \right) \oplus \left( M \otimes B \right) \) is a Gorenstein-projective \( \Lambda \)-module. \( \square \)

Data Availability

No datasets were generated or analyzed during the current study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

References


