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Research Article

Enumeration of the Edge Weights of Symmetrically Designed Graphs

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The idea of super $(a, 0)$-edge-antimagic labeling of graphs had been introduced by Enomoto et al. in the late nineties. This article addresses super $(a, 0)$-edge-antimagic labeling of a biparametric family of pancyclic graphs. We also present the aforesaid labeling on the disjoint union of graphs comprising upon copies of C_4 and different trees. Several problems shall also be addressed in this article.

1. Introduction

A graph $\Gamma(V, E)$ is a combination of two different sets, one is the set of vertices $V(\Gamma)$ and the other is the set of connections between these vertices, termed as set of edges $E(\Gamma)$. A graph Γ can either be connected or comprises upon connected parts known as graphs' components. The nonempty and simple graphs shall be considered here only all the way, consisting of $V(\Gamma)$, the set of vertices, and $E(\Gamma)$, the set of edges, having $|V(\Gamma)| = p$ and $|E(\Gamma)| = q$. In this case, the graph Γ is called a (p, q)-graph. Additionally, [1] can be cited for the comprehension of the graph theoretic terminologies.

A labeling is a function from the set of integers onto the components of a graph under certain constraints. The labeling is said to be total if it covers all components of the graph. If the labeling covers $V(\Gamma)$ or $E(\Gamma)$ only in the domain, then it is termed to be the vertex or edge labeling, respectively. The two important categories of labeling are magic and antimagic. The equal or unequal edge/vertex weights point towards, respectively, the magic and antimagic types of labeling.

Throughout the article, the abbreviations given in Table 1 are used.

Definition 1. On a (σ, ς)-graph $\Gamma = (V(\Gamma), E(\Gamma))$, a bijection γ from $V(\Gamma) \cup E(\Gamma) \overrightarrow{\text{onto}} \{1, 2, \ldots, \sigma + \varsigma\}$ is the notion of (a, d) – EAMT labeling if we keep the restriction upon the edge weights $\gamma(u) + \gamma(uv) + \gamma(v)$, $\forall uv \in E(G)$, which generates a consecutive integer sequence, with a being the minimum edge weight and d being the common difference. Γ is notioned as an (a, d) – EAMT graph, with the existence of such labeling.

Definition 2. If the smallest labels $1, 2, \ldots, \sigma$ are allocated to the points (vertices) of the (σ, ς)-graph Γ in an (a, d) – EAMT labeling, then this labeling is addressed as $S - (a, d)$ – EAMT labeling. And, Γ, in his scenario, is referred to be an $S - (a, d)$ – EAMT graph.

The edge weight (minimum) a (Definitions 1 and 2) becomes a constant at $d = 0, \forall uv \in E(\Gamma)$, and is called magic constant or magic sum of Γ.

Definition 3. A pancyclic graph $\Gamma(V, E)$ is a graph that contains the cycles of all orders up to $|V(\Gamma)|$.

The notion of magic labeling was highlighted by Sedlacek in the early sixties [2]. Later, Ringel and Hartsfield capitulated the idea of antimagic labeling with respect to vertex-

Table 1: Abbreviations being used onwards.

Terminology	Abbreviation
(a, d)-edge-antimagic total	(a, d) – EAMT
Super (a, d)-edge-antimagic total	$S-(a, d)$ – EAMT

sums of graphs in [3]. The idea of magic valuations of graphs had been brought by Kotzig and Rosa [4] which was indeed the graphs' $(a, 0)$ – EAMT labeling (introduced by Ringel et. al. in the nineties [5]). Enomoto and Llado introduced the idea of $S-(a, 0)$ – EAMT labeling of graphs using the term super edge-magic labeling in [6]. In the early 21st century, Bertault and Simanjuntak brought out the graphs' (a, d) – EAMT labeling [7]. The following notable and handy conjectures are included in the vicinity of graphs' $(a, 0)$ – EAMT labeling.

Conjecture 1 (see [4]). *Every tree admits an $(a, 0)$ – EAMT labeling.*

Conjecture 2 (see [6]). *Every tree admits an $S-(a, 0)$ – EAMT labeling.*

In the support of Conjecture 2, certain classes of trees have been sorted out by scientists. For trees having maximum of seventeen vertices, in [8], this conjecture has been verified. For instance, the $S-(a, 0)$ – EAMT labeling on a class of trees termed as w-trees can be observed in [9]. Similarly, $S-(a, 0)$ – EAMT labeling on various classes consisting of subdivisions of trees can be seen in [10, 11]. Some derivations on vertex-antimagicness of regular graphs have been discussed in [12]. In [13], the same labeling for the union of unicyclic graphs and isolated vertices has been provided. Enomoto et al. proved [6] that if a (p, q)-graph Γ (simple) is $S-(a, 0)$ – EAMT, it implies $q \leq 2p-3$. In addition, they derived that $K_{m,n}$ is $S-(a, 0)$ – EAMT only if either m or n is equal to 1. It is derived in [14] that the combination of graphs in the form of union of $K_{1,\theta}$ and $K_{1,\eta}$ is $S-(a, 0)$ – EAMT if $\theta = \chi(\eta+1)$ or $\eta = \chi(\theta+1)$. For only odd values of n, C_n is concreted to be $S-(a, 0)$ – EAMT [6]. The cycle of order 3 and cycle of order $n \geq 6$ are proven to be $S-(a, 0)$ – EAMT for even values of n. The generalized prism D_ξ is proven to be $S-(a, 0)$ – EAMT for all odd values of ξ in [15]. In [16, 17], $S-(a, 0)$ – EAMT labeling of maximum symmetric generalization of prism and special networks with magic constant $3p$ has been exhibited, respectively. An extremely important result on $S-(a, 0)$ – EAMT graphs is as follows.

Lemma 1 (see [15]). *A (p, q)-graph G is $S-(a, 0)$ – EAMT if and only if there exists a bijective function $\lambda\colon V(G) \longrightarrow \{1, 2, \ldots, p\}$ such that the set $S = \{\lambda(u) + \lambda(u) | uv \in E(G)\}$ consists of q consecutive integers. In such a case, λ extends to an $S-(a, 0)$ – EAMT labeling of G with magic constant $t = p + q + s$, where $s = \min(S)$ and $S = \{t-(p+1), t-(p+2), \ldots, t-(p+q)\}$.*

In Lemma 1, the sum $\zeta(u) + \zeta(v)$ is defined as edge sum for each edge $uv \in E(\Gamma)$. This lemma shall be used frequently in our findings, while it keeps this enough to allot the labels to merely the vertices of the network to capacitate the graph to be $S-(a, 0)$ – EAMT, where the edge-sums (consecutive) belong to $\mathbb{N}$. The given result is extremely relevant with regard to $S-(a, 0)$ – EAMT.

Lemma 2 (see [18]). *A simple graph Γ possesses an $S-(a, 0)$ – EAMT labeling $\Leftrightarrow \Gamma$ possesses an $S-(a-|E(G)|+1, 2)$ – EAMT labeling.*

2. Main Results

In this section, we shall address our main findings. In Section 2.1, we define an $S-(a, 0)$ – EAMT labeling on a pancyclic family of graphs, namely, Usmanian pancyclic graph UP_n^t. In Section 2.2, we design $S-(a, 0)$ – EAMT labeling on various disjoint unions of graphs comprising upon copies of C_4 and various trees/forests. Throughout, $i \in [a, b]$ represents $a \leq i \leq b$, for $i, a, b \in \mathbb{N}$, whereas $\mathbb{N}^{\mathrm{odd}}$ and $\mathbb{N}^{\mathrm{even}}$ are the representations of odd and even natural numbers, respectively.

2.1. Usmanian Pancyclic Graph UP_n^t. In computer science, there is a similar importance of the networks having no cycles and networks having a range of cycles. The importance is similar for a network containing cycles of all lengths from one to the number of systems connected within. In this situation, the role of programmers becomes prominent to avoid hackers halting of data as there is a closed path between any two arbitrary computers corresponding to such networks. The first kind of network is termed as a tree (connected and acyclic) and later is known as pancyclic network (connected and containing every order's cycle). This section deals with a family of pancyclic graphs denoted by UP_n^t, which is biparametric, and reveals that such complex structures are $S-(a, 0)$ – EAMT. We shall first introduce this structure as Definition 4.

Definition 4. We are defining the Usmanian pancyclic graph, denoted by UP_n^t, being a graph with $|V(\mathrm{UP}_n^t)| = tn$ *and* $|E(\mathrm{UP}_n^t)| = 2tn - 3$, having the construction as follows ($n \geq 3$ being the order of the cycle C_n *and* $t \geq 2$ being the number of cycles):

(1) For even n:

For $n \equiv 0 \pmod 4$:

(i) For $n = 4$,

$$V\left(\mathrm{UP}_4^t\right) = \{y_i, z_i\colon i \in [1, t]\} \cup \{x_i\colon i \in [1, 2t]\},$$
$$E\left(\mathrm{UP}_4^t\right) = \{z_i z_{i+1}, y_i y_{i+1}\colon i \in [1, t-1]\} \cup \{y_i x_{2i}, z_i x_{2i}\colon i \in [1, t]\} \cup \{z_i x_{2i-1}, y_i x_{2i-1}\colon i \in [1, t]\} \cup \{x_i x_{i+1}\colon i \in [1, 2t-1]\}. \quad (1)$$

(ii) For $n = 8$,

$$\begin{aligned}
V\left(\mathrm{UP}_8^t\right) &= \{x_i, v_i\colon i \in [1,t]\} \cup \{w_i, y_i, z_i\colon i \in [1,2t]\}\\
E\left(\mathrm{UP}_8^t\right) &= \{w_i w_{i+1}, y_i y_{i+1}, z_i z_{i+1}\colon i \in [2, 2t-2], i \in \mathbb{N}^{\text{even}}\} \cup \{z_i w_i, y_i z_i\colon i \in [1,2t]\}\\
&\cup \{x_i y_{2i-1}, x_i y_{2i}, v_i w_{2i-1}, v_i w_{2i}\colon i \in [1,t]\}\\
&\cup \{y_i w_i\colon i \in [1,2t]\} \cup \{x_i v_i\colon i \in [1,t]\} \cup \left\{z_i w_{i+1}, y_i z_{i+1}\colon i \in [1, 2t-1], i \in \mathbb{N}^{\text{odd}}\right\}.
\end{aligned} \tag{2}$$

(iii) For $n = 12$,

$$\begin{aligned}
V\left(UP_{12}^t\right) &= \left\{x_i^j\colon i \in [1,2t], 2 \le j \le 6\right\} \cup \left\{x_i^j\colon i \in [1,t], j = 1,7\right\},\\
E\left(\mathrm{UP}_{12}^t\right) &= \left\{x_i^j x_{i+1}^j\colon i \in [2, 2(t-1)], i \in \mathbb{N}^{\text{even}}, 3 \le j \le 5\right\} \cup \left\{x_i^j x_i^{j+1}\colon i \in [1,2t], 2 \le j \le 5\right\}\\
&\cup \left\{x_i^1 x_{2i-1}^2, x_i^1 x_{2i}^2, x_i^7 x_{2i-1}^6, x_i^7 x_{2i}^6\colon i \in [1,t]\right\} \cup \left\{x_i^1 x_i^7\colon i \in [1,t]\right\} \cup \left\{x_i^j x_i^{8-j}\colon i \in [1,2t], 2 \le j \le 6\right\}\\
&\cup \left\{x_i^4 x_i^7\colon i \in [1, 2t-1], i \in \mathbb{N}^{\text{odd}}\right\} \cup \left\{x_i^2 x_{i+1}^5, x_i^3 x_{i+1}^6\colon i \in [1, 2t-1], i \in \mathbb{N}^{\text{odd}}\right\} \cup \left\{x_i^1 x_{2i}^4\colon i \in [1,t]\right\}.
\end{aligned} \tag{3}$$

(iv) For $n \ge 16$,

$$\begin{aligned}
V\left(\mathrm{UP}_n^t\right) &= \left\{x_i^j\colon i \in [1,2t], 2 \le j \le \frac{n}{2}\right\} \cup \left\{x_i^j\colon i \in [1,t], j = 1, \frac{n+2}{2}\right\},\\
E\left(\mathrm{UP}_n^t\right) &= \left\{x_i^j x_{i+1}^j\colon i \in [2, 2(t-1)], i \in \mathbb{N}^{\text{even}}, \frac{n}{4} \le j \le \frac{n+8}{4}\right\} \cup \left\{x_i^j x_i^{j+1}\colon i \in [1,2t], 2 \le j \le \frac{n-2}{2}\right\}\\
&\cup \left\{x_i^1 x_{2i-1}^2, x_i^1 x_{2i}^2, x_i^{(n+2)/2} x_{2i-1}^{n/2}, x_i^{(n+2)/2} x_{2i}^{n/2}\colon i \in [1,t]\right\} \cup \left\{x_i^1 x_i^{(n+2)/2}\colon i \in [1,t]\right\} \cup \left\{x_i^j x_i^{(n/2)+2-j}\colon i \in [1,2t], 2 \le j \le \frac{n}{2}\right\}\\
&\cup \left\{x_i^j x_i^{(n/2)+5-j}\colon i \in [1, 2t-1], i \in \mathbb{N}^{\text{odd}}, 4 \le j \le \frac{n+4}{4}\right\} \cup \left\{x_i^2 x_{i+1}^{(n-2)/2}, x_i^3 x_{i+1}^{n/2}\colon i \in [1, 2t-1], i \in \mathbb{N}^{\text{odd}}\right\}\\
&\cup \left\{x_i^1 x_{2i}^{n-4/2}\colon i \in [1,t]\right\} \cup \left\{x_i^j x_i^{((n-2)/2)-j}\colon i \in [2,2t], i \in \mathbb{N}^{\text{even}}, 2 \le j \le \frac{n-8}{4}\right\}.
\end{aligned} \tag{4}$$

For $n \equiv 2 \pmod 4$:

(i) For $n = 6$,

$$\begin{aligned}
V\left(\mathrm{UP}_6^t\right) &= \{x_i, y_i, z_i\colon i \in [1,2t]\},\\
E\left(\mathrm{UP}_6^t\right) &= \{z_i z_{i+1}, x_i x_{i+1}\colon i \in [1, 2t-1]\} \cup \{y_i z_i, x_i y_i\colon i \in [1,2t]\} \cup \{y_i y_{i+1}\colon i \in [1, 2t-1]\} \cup \{x_i z_i\colon i \in [1,2t]\}.
\end{aligned} \tag{5}$$

(ii) For $n = 10$,

$$V\left(\mathrm{UP}_{10}^{t}\right) = \left\{x_i^j \colon i \in [1, 2t], j \in [1, 5]\right\}$$
$$E\left(\mathrm{UP}_{10}^{t}\right) = \left\{x_i^j x_i^{j+1} \colon i \in [1, 2t], 1 \leq j \leq 4\right\} \cup \left\{x_i^1 x_{i+1}^1, x_i^5 x_{i+1}^5 \colon i \in [1, 2t-1], i \in \mathbb{N}^{\mathrm{odd}}\right\}$$
$$\cup \left\{x_i^j x_{i+1}^j \colon i \in [2, 2(t-1)], i \in \mathbb{N}^{\mathrm{even}}, 2 \leq j \leq 4\right\}$$
$$\cup \left\{x_i^1 x_i^5, x_i^2 x_i^4 \colon i \in [1, 2t]\right\} \cup \left\{x_i^j x_{i+1}^{2+j} \colon i \in [1, 2t-1], i \in \mathbb{N}^{\mathrm{odd}}, 1 \leq j \leq 3\right\}. \tag{6}$$

(iii) For $n \geq 14$,

$$V\left(\mathrm{UP}_{n}^{t}\right) = \left\{x_i^j \colon i \in [1, 2t], 1 \leq j \leq \frac{n}{2}\right\},$$
$$E\left(\mathrm{UP}_{n}^{t}\right) = \left\{x_i^j x_i^{j+1} \colon i \in [1, 2t], 1 \leq j \leq \frac{n-2}{2}\right\} \cup \left\{x_i^1 x_{i+1}^1, x_i^{n/2} x_{i+1}^{n/2} \colon i \in [1, 2t-1], i \in \mathbb{N}^{\mathrm{odd}}\right\}$$
$$\cup \left\{x_i^j x_{i+1}^j \colon i \in [2, 2(t-1)], i \in \mathbb{N}^{\mathrm{even}}, \frac{n-2}{4} \leq j \leq \frac{n+6}{4}\right\} \cup \left\{x_i^j x_i^{(n+2/2)-j} \colon i \in [1, 2t], 1 \leq j \leq \frac{n-2}{4}\right\} \tag{7}$$
$$\cup \left\{x_i^j x_{i+1}^{(n-6/2)+j} \colon i \in [1, 2t-1], i \in \mathbb{N}^{\mathrm{odd}}, 1 \leq j \leq 3\right\} \cup \left\{x_i^j x_i^{(n+8/2)-j} \colon i \in [1, 2t-1], i \in \mathbb{N}^{\mathrm{odd}}, 4 \leq j \leq \frac{n+2}{4}\right\}$$
$$\cup \left\{x_i^j x_i^{(n-4/2)-j} \colon i \in [2, 2t], i \in \mathbb{N}^{\mathrm{even}}, 1 \leq j \leq \frac{n-10}{4}\right\}.$$

(2) For odd n:
For $n \equiv 1 \pmod 4$:

(i) For $n = 5$,

$$V\left(\mathrm{UP}_{5}^{t}\right) = \left\{z_i, y_i \colon i \in [1, 2t]\right\} \cup \left\{x_i \colon i \in [1, t]\right\},$$
$$E\left(\mathrm{UP}_{5}^{t}\right) = \left\{z_i z_{i+1} \colon i \in [1, 2t-1], i \in \mathbb{N}^{\mathrm{odd}}\right\} \cup \left\{y_i z_i \colon i \in [1, 2t]\right\} \cup \left\{x_i y_{2i-1}, x_i y_{2i} \colon i \in [1, t]\right\}$$
$$\cup \left\{y_i y_{i+1} \colon i \in [2, 2t-2], i \in \mathbb{N}^{\mathrm{even}}\right\} \cup \left\{x_i z_{2i+1} \colon i \in [1, t-1]\right\} \cup \left\{x_i z_{2i-2} \colon i \in [2, t]\right\} \tag{8}$$
$$\cup \left\{x_i z_{2i-1}, x_i z_{2i} \colon i \in [1, t]\right\}.$$

(ii) For $n = 9$,

$$V\left(UP_{9}^{t}\right) = \left\{w_i, v_i, y_i, z_i \colon i \in [1, 2t]\right\} \cup \left\{x_i \colon i \in [1, t]\right\},$$
$$E\left(\mathrm{UP}_{9}^{t}\right) = \left\{v_i v_{i+1} \colon i \in [1, 2t-1], i \in \mathbb{N}^{\mathrm{odd}}\right\} \cup \left\{y_i z_i, z_i w_i, w_i v_i \colon i \in [1, 2t]\right\} \cup \left\{x_i y_{2i-1}, x_i y_{2i} \colon i \in [1, t]\right\}$$
$$\cup \left\{y_i w_{i+1}, z_i z_{i+1}, w_i y_{i+1} \colon i \in [2, 2t-2], i \in \mathbb{N}^{\mathrm{even}}\right\} \cup \left\{y_i w_i, y_i v_i \colon i \in [1, 2t]\right\} \cup \left\{x_i v_{2i-1}, x_i v_{2i} \colon i \in [1, t]\right\}. \tag{9}$$

(iii) For $n \geq 13$,

$$\begin{aligned}
V\left(\mathrm{UP}_n^t\right) &= \left\{x_i^j\colon i \in [1,2t],\, j \in \left[2,\frac{n+1}{2}\right]\right\} \cup \left\{x_i^1\colon i \in [1,t]\right\},\\
E\left(\mathrm{UP}_n^t\right) &= \left\{x_i^j x_i^{j+1}\colon i \in [1,2t],\, 2 \leq j \leq \frac{n-1}{2}\right\} \cup \left\{x_i^1 x_{2i-1}^2, x_i^1 x_{2i}^2\colon i \in [1,t]\right\}\\
&\cup \left\{x_i^{n+1/2} x_{i+1}^{n+1/2}\colon 1 \leq i \leq 2t-1,\, i \in \mathbb{N}^{\mathrm{odd}}\right\}\\
&\cup \left\{x_i^{(n-1)/4} x_{i+1}^{(n+7)/4}, x_i^{(n+7)/4} x_{i+1}^{(n-1)/4}, x_i^{(n+3)/4} x_{i+1}^{(n+3)/4}\colon 2 \leq i \leq 2(t-1),\, i \in \mathbb{N}^{\mathrm{even}}\right\}\\
&\cup \left\{x_i^j x_i^{((n+3)/2)-j}\colon i \in [1,2t],\, 2 \leq j \leq \frac{n-1}{4}\right\}\\
&\cup \left\{x_i^j x_i^{(n+5/2)-j}\colon i \in [1,2t],\, 2 \leq j \leq \frac{n-1}{4}\right\} \cup \left\{x_i^1 x_{2i-1}^{(n+1)/2}, x_i^1 x_{2i}^{(n+1)/2}\colon i \in [1,t]\right\}.
\end{aligned} \tag{10}$$

For $n \equiv 3 \pmod 4$:

(i) For $n = 3$:

Define $\mathrm{UP}_3^t \cong P_t \times C_3$ as follows:

$$\begin{aligned}
V\left(\mathrm{UP}_3^t\right) &= \left\{z_i, x_i, y_i\colon i \in [1,t]\right\},\\
E\left(\mathrm{UP}_3^t\right) &= \left\{z_i z_{i+1}, x_i x_{i+1}, y_i y_{i+1}\colon i \in [1,t-1]\right\} \cup \left\{x_i z_i, x_i y_i, y_i z_i\colon i \in [1,t]\right\}.
\end{aligned} \tag{11}$$

(ii) For $n = 7$,

$$\begin{aligned}
V\left(\mathrm{UP}_7^t\right) &= \left\{z_i, w_i, y_i\colon i \in [1,2t]\right\} \cup \left\{x_i\colon i \in [1,t]\right\},\\
E\left(\mathrm{UP}_7^t\right) &= \left\{w_i w_{i+1}\colon i \in [1,2t-1],\, i \in \mathbb{N}^{\mathrm{odd}}\right\} \cup \left\{z_i w_i, y_i z_i\colon i \in [1,2t]\right\} \cup \left\{x_i y_{2i-1}, x_i y_{2i}\colon i \in [1,t]\right\}\\
&\cup \left\{z_i z_{i+1}, y_i w_{i+1}, w_i y_{i+1}\colon i \in [2,2t-2],\, i \in \mathbb{N}^{\mathrm{even}}\right\} \cup \left\{y_i w_i\colon i \in [1,2t]\right\} \cup \left\{x_i w_{2i-1}, x_i w_{2i}\colon i \in [1,2t]\right\}.
\end{aligned} \tag{12}$$

(iii) For $n \geq 11$,

$$\begin{aligned}
V\left(\mathrm{UP}_n^t\right) &= \left\{x_i^j\colon i \in [1,2t],\, 2 \leq j \leq \frac{n+1}{2}\right\} \cup \left\{x_i^1\colon i \in [1,t]\right\},\\
E\left(\mathrm{UP}_n^t\right) &= \left\{x_i^j x_i^{j+1}\colon i \in [1,2t],\, 2 \leq j \leq \frac{n-1}{2}\right\} \cup \left\{x_i^1 x_{2i-1}^2, x_i^1 x_{2i}^2\colon i \in [1,t]\right\} \cup \left\{x_i^{(n+1)/2} x_{i+1}^{(n+1)/2}\colon i \in [1,2t-1],\, i \in \mathbb{N}^{\mathrm{odd}}\right\}\\
&\cup \left\{x_i^{(n+1)/4} x_{i+1}^{(n+9)/4}, x_i^{(n+5)/4} x_{i+1}^{(n+5)/4}, x_i^{(n+9)/4} x_{i+1}^{(n+1)/4}\colon i \in [2,2(t-1)],\, i \in \mathbb{N}^{\mathrm{even}}\right\}\\
&\cup \left\{x_i^j x_i^{((n+5)/2)-j}\colon i \in [1,2t],\, 2 \leq j \leq \frac{n+1}{4}\right\} \cup \left\{x_i^j x_i^{((n+3)/2)-j}\colon i \in [1,2t],\, 2 \leq j \leq \frac{n-3}{4}\right\}\\
&\cup \left\{x_i^1 x_{2i-1}^{(n+1)/2}, x_i^1 x_{2i}^{(n+1)/2}\colon i \in [1,t]\right\}.
\end{aligned} \tag{13}$$

Theorem 1. *The pancyclic graph UP_n^t is $S-(a,0)-EAMT$ admitting $a = 3tn$, $\forall\, t$ and $n \equiv 0 \pmod 2$.*

Proof. We discuss here two cases as per Definition 4.

For $n \equiv 0 \pmod 4$:

(i) For $n = 4$:
We define a labeling $\psi_1'\colon V(\mathrm{UP}_4^t) \longrightarrow \{1,2,\ldots,4t\}$ as follows:

$$\psi_1'(x_i) = \begin{cases} 2i-1\colon i \in [1,2t-1], & i \in \mathbb{N}^{\mathrm{odd}}; \\ 2i\colon 2 \le i \le 2t, & i \in \mathbb{N}^{\mathrm{even}}; \end{cases}$$
$$\psi_1'(y_i) = 4i-2\colon i \in [1,t],$$
$$\psi_1'(z_i) = 4i-1\colon i \in [1,t]. \tag{14}$$

The edge-sums' set generated as per the labeling design constitutes a consecutive sequence of positive integers $3,4,\ldots,8t-1$. As per Lemma 1, ψ_1' is extendable to an $S-(a,0)-\mathrm{EAMT}$ labeling of UP_4^t with magic constant $a = 12t$.

(ii) For $n = 8$:
We define a labeling $\psi_1''\colon V(\mathrm{UP}_8^t) \longrightarrow \{1,2,\ldots,8t\}$ as follows:

$$\psi_1''(x_i) = 8i-3\colon 1 \le i \le t,$$
$$\psi_1''(y_i) = 4i-1\colon 1 \le i \le 2t,$$
$$\psi_1''(z_i) = \begin{cases} 4i-3\colon i \in [1,2t-1], & i \in \mathbb{N}^{\mathrm{odd}}; \\ 4i\colon 2 \le i \le 2t, & i \in \mathbb{N}^{\mathrm{even}}; \end{cases} \tag{15}$$
$$\psi_1''(w_i) = 4i-2\colon i \in [1,2t],$$
$$\psi_1''(v_i) = 8i-4\colon i \in [1,t].$$

The edge-sums' set generated as per the labeling design constitutes a consecutive sequence of positive integers $3,4,\ldots,16t-1$. In the light of Lemma 1, ψ_1'' constitutes an $S-(a,0)-\mathrm{EAMT}$ labeling of UP_8^t admitting $a = 24t$.

(iii) For $n \ge 12$:
We define a labeling $\psi_1\colon V(\mathrm{UP}_n^t) \longrightarrow \{1,2,\ldots,tn\}$ as follows:

$$\psi_1(x_i^j) = \begin{cases} \dfrac{2ni-n+2}{2}\colon & i \in [1,t] \text{ and } j = 1; \\ \dfrac{ni-4j+6}{2}\colon & i \in [1,2t-1],\ j \in \left[2,\dfrac{n+4}{4}\right], i \in \mathbb{N}^{\mathrm{odd}}; \\ \dfrac{ni+4j-n-2}{2}\colon & i \in [2,2t],\ j \in \left[2,\dfrac{n}{4}\right], i \in \mathbb{N}^{\mathrm{even}}; \\ \dfrac{ni+4j-n-4}{2}\colon & i \in [2,2t],\ j = \dfrac{n+4}{4}, i \in \mathbb{N}^{\mathrm{even}}; \\ \dfrac{ni+4j-2n-4}{2}\colon & i \in [1,2t-1],\ j \in \left[\dfrac{n+8}{4},\dfrac{n}{2}\right], i \in \mathbb{N}^{\mathrm{odd}}; \\ \dfrac{ni-4j+n+4}{2}\colon & i \in [2,2t],\ j \in \left[\dfrac{n+8}{4},\dfrac{n}{2}\right], i \in \mathbb{N}^{\mathrm{even}}; \\ \dfrac{2ni+4j-3n-4}{2}\colon & i \in [1,t] \text{ and } j = \dfrac{n+2}{2}. \end{cases} \tag{16}$$

With the abovementioned scheme, the edge-sums being generated form a consecutive integer sequence set $\Delta = \{3,4,\ldots,2tn-1\}$. ψ_1 is extendible to $S-(a,0)-\mathrm{EAMT}$ labeling of $\mathrm{UP}_n^t, n \ge 12$, according to Lemma 1, with $a = 3tn$.

For $n \equiv 2 \pmod 4$:

(i) For $n = 6$:
The labeling $\psi_2'\colon V(\mathrm{UP}_6^t) \longrightarrow \{1,2,\ldots,6t\}$ is being defined as follows:

$$\psi_2'(x_i) = \begin{cases} 3i\colon 1 \le i \le 2t-1, & i \in \mathbb{N}^{\mathrm{odd}}; \\ 3i-1\colon 2 \le i \le 2t, & i \in \mathbb{N}^{\mathrm{even}}; \end{cases}$$
$$\psi_2'(y_i) = \begin{cases} 3i-2\colon 1 \le i \le 2t-1, & i \in \mathbb{N}^{\mathrm{odd}}; \\ 3i\colon 2 \le i \le 2t, & i \in \mathbb{N}^{\mathrm{even}}; \end{cases} \tag{17}$$
$$\psi_2'(z_i) = \begin{cases} 3i-1\colon 1 \le i \le 2t-1, & i \in \mathbb{N}^{\mathrm{odd}}; \\ 3i-2\colon 2 \le i \le 2t, & i \in \mathbb{N}^{\mathrm{even}}. \end{cases}$$

The edge-sums' set generated as per the labeling design constitutes a consecutive natural numbers' sequence $3, 4, \ldots, 12t-1$. As per Lemma 1, ψ_2' is extendable towards $S-(a,0)-$EAMT labeling of UP_6^t with $a = 18t$.

(ii) For $n \geq 10$:
Define a labeling $\psi_2\colon V(UP_n^t) \longrightarrow \{1, 2, \ldots, tn\}$ as the following function:

$$\psi_2\left(x_i^j\right) = \begin{cases} \dfrac{ni-4j+4}{2}: & i \in [1, 2t-1], j \in \left[1, \dfrac{n+2}{4}\right], \quad i \in \mathbb{N}^{\text{odd}}; \\ \dfrac{ni+4j-n}{2}: & i \in [2, 2t], j \in \left[1, \dfrac{n-2}{4}\right], \quad i \in \mathbb{N}^{\text{even}}; \\ \dfrac{ni+4j-2n-2}{2}: & i \in [1, 2t-1], j \in \left[\dfrac{n+6}{4}, \dfrac{n}{2}\right], \quad i \in \mathbb{N}^{\text{odd}}; \\ \dfrac{ni-4j+n+2}{2}: & i \in [2, 2t], j \in \left[\dfrac{n+2}{4}, \dfrac{n}{2}\right], \quad i \in \mathbb{N}^{\text{even}}. \end{cases} \tag{18}$$

With the abovementioned scheme, the edge-sums being generated form a consecutive integer sequence set $\{3, 4, \ldots, 2tn-1\}$. ψ_2 constitutes $S-(a,0)-$EAMT labeling of $UP_n^t, n \geq 10$, according to Lemma 1, admitting $a = 3tn$. □

Theorem 2. *The pancyclic graph UP_n^t is $S-(a,0)-$EAMT with magic sum $a = 3tn$, for all t and $n \equiv 1\,(\text{mod}\,2)$.*

Proof. We discuss here two cases.

For $n \equiv 1\,(\text{mod}\,4)$:

(i) For $n = 5$:
Define a labeling $\psi_3'\colon V(UP_5^t) \longrightarrow \{1, 2, \ldots, 5t\}$ as follows:

$$\psi_3'(x_i) = 5i-2\colon i \in [1, t],$$

$$\psi_3'(y_i) = \begin{cases} \dfrac{1}{2}(5i-3)\colon i \in [1, 2t-1], & i \in \mathbb{N}^{\text{odd}}; \\ \dfrac{1}{2}(5i)\colon i \in [2, 2t], & i \in \mathbb{N}^{\text{even}}; \end{cases}$$

$$\psi_3'(z_i) = \begin{cases} \dfrac{1}{2}(5i-1)\colon i \in [1, 2t-1], & i \in \mathbb{N}^{\text{odd}}; \\ \dfrac{1}{2}(5i-2)\colon i \in [2, 2t], & i \in \mathbb{N}^{\text{even}}. \end{cases} \tag{19}$$

The edge-sums' set generated as per the labeling design constitutes a consecutive natural numbers' sequence $3, 4, \ldots, 10t-1$. ψ_3' is extendable to $S-(a,0)-$EAMT labeling of UP_5^t having $a = 15t$ (as per Lemma 1).

(ii) For $n = 9$:
We construct a labeling $\psi_3'\colon V(UP_9^t) \longrightarrow \{1, 2, \ldots, 9t\}$ as follows:

$$\psi_3''(x_i) = 9i-4\colon 1 \leq i \leq t,$$

$$\psi_3''(y_i) = \begin{cases} \dfrac{1}{2}(9i-3)\colon i \in [1, 2t-1], & i \in \mathbb{N}^{\text{odd}}; \\ \dfrac{1}{2}(9i-4)\colon i \in [2, 2t], & i \in \mathbb{N}^{\text{even}}; \end{cases}$$

$$\psi_3'(z_i) = \begin{cases} \dfrac{1}{2}(9i-7)\colon i \in [1, 2t-1], & i \in \mathbb{N}^{\text{odd}}; \\ \dfrac{1}{2}(9i)\colon i \in [2, 2t], & i \in \mathbb{N}^{\text{even}}; \end{cases}$$

$$\psi_3''(w_i) = \begin{cases} \dfrac{1}{2}(9i-5)\colon i \in [1, 2t-1], & i \in \mathbb{N}^{\text{odd}}; \\ \dfrac{1}{2}(9i-2)\colon i \in [2, 2t], & i \in \mathbb{N}^{\text{even}}; \end{cases}$$

$$\psi_3''(v_i) = \begin{cases} \dfrac{1}{2}(9i-1)\colon i \in [1, 2t-1], & i \in \mathbb{N}^{\text{odd}}; \\ \dfrac{1}{2}(9i-6)\colon i \in [2, 2t], & i \in \mathbb{N}^{\text{even}}. \end{cases} \tag{20}$$

The edge-sums' set generated as per the labeling design constitutes a natural numbers' sequence $3, 4, \ldots, 18t-1$. ψ_3'' is extendable to $S-(a,0)-$EAMT labeling of UP_9^t, by Lemma 1, with the admittance of $a = 27t$.

(iii) For $n \geq 13$:
We are going to construct a labeling $\psi_3\colon V(UP_n^t) \longrightarrow \{1, 2, \ldots, tn\}$ as follows:

$$\psi_3\left(x_i^j\right) = \begin{cases} \dfrac{2ni-n+1}{2}: & i \in [1,t],\ j=1, \\ \dfrac{ni-4j+5}{2}: & i \in [1,2t-1],\ j \in \left[2, \dfrac{n+3}{4}\right], i \in \mathbb{N}^{\text{odd}}, \\ \dfrac{ni+4j-n-3}{2}: & i \in [2,2t],\ j \in \left[2, \dfrac{n+3}{4}\right], i \in \mathbb{N}^{\text{even}}, \\ \dfrac{ni+4j-2n-3}{2}: & i \in [1,2t-1],\ j \in \left[\dfrac{n+7}{4}, \dfrac{n+1}{2}\right], i \in \mathbb{N}^{\text{odd}}, \\ \dfrac{ni-4j+n+5}{2}: & i \in [2,2t],\ j \in \left[\dfrac{n+7}{4}, \dfrac{n+1}{2}\right], i \in \mathbb{N}^{\text{even}}. \end{cases} \tag{21}$$

With the abovementioned scheme, the edge-sums being generated forms a consecutive integer sequence set $\{3,4,\ldots,2tn-1\}$. Once again, ψ_3 extends to a $S-(a,0)-\text{EAMT}$ labeling of $\text{UP}_n^t, n \geq 13$ having $a = 3tn$ by Lemma 1.

For $n \equiv 3 \pmod 4$:

(i) For $n = 3$:
We define a labeling ψ_4': $V(\text{UP}_3^t) \longrightarrow \{1,2,\ldots,3t\}$ as follows:

$$\begin{aligned} \psi_4'(x_i) &= \begin{cases} 3i-2: 1 \leq i \leq t-1, & i \in \mathbb{N}^{\text{odd}}, \\ 3i-1: 2 \leq i \leq t, & i \in \mathbb{N}^{\text{even}}, \end{cases} \\ \psi_4'(y_i) &= \begin{cases} 3i: 1 \leq i \leq t-1, & i \in \mathbb{N}^{\text{odd}}, \\ 3i-2: 2 \leq i \leq t, & i \in \mathbb{N}^{\text{even}}, \end{cases} \\ \psi_4'(z_i) &= \begin{cases} 3i-1: 1 \leq i \leq t-1, & i \in \mathbb{N}^{\text{odd}}, \\ 3i: 2 \leq i \leq t, & i \in \mathbb{N}^{\text{even}}. \end{cases} \end{aligned} \tag{22}$$

The edge-sums' set generated as per the labeling design constitutes a consecutive sequence of positive integers $3,4,\ldots,6t-1$. Now, ψ_4' is extendable to $S-(a,0)-\text{EAMT}$ labeling of $\text{UP}_3^t \cong P_t \times C_3$ [15] with $a = 9t$ according to Lemma 1.

(ii) For $n = 7$:
A labeling ψ_4'': $V(\text{UP}_7^t) \longrightarrow \{1,2,\ldots,7t\}$ is being defined as follows:

$$\begin{aligned} \psi_4''(x_i) &= 7i-3: 1 \leq i \leq t, \\ \psi_4''(y_i) &= \begin{cases} \dfrac{1}{2}(7i-3): & i \in [1,2t-1], i \in \mathbb{N}^{\text{odd}}, \\ \dfrac{1}{2}(7i-2): & i \in [2,2t], i \in \mathbb{N}^{\text{even}}, \end{cases} \\ \psi_4''(z_i) &= \begin{cases} \dfrac{1}{2}(7i-5): & i \in [1,2t-1], i \in \mathbb{N}^{\text{odd}}, \\ \dfrac{1}{2}(7i): & i \in [2,2t], i \in \mathbb{N}^{\text{even}}, \end{cases} \\ \psi_4''(w_i) &= \begin{cases} \dfrac{1}{2}(7i-1): & i \in [1,2t-1], i \in \mathbb{N}^{\text{odd}}, \\ \dfrac{1}{2}(7i-4): & i \in [2,2t], i \in \mathbb{N}^{\text{even}}. \end{cases} \end{aligned} \tag{23}$$

The edge-sums' set generated as per the labeling design constitutes a consecutive natural numbers' sequence $3,4,\ldots,14t-1$. Now, ψ_4'' is extendable to $S-(a,0)-\text{EAMT}$ labeling of UP_7^t, according to Lemma 1 having $a = 21t$.

(iii) For $n \geq 11$:
The labeling ψ_4: $V(\text{UP}_n^t) \longrightarrow \{1,2,\ldots,tn\}$ is constructed as follows:

$$\psi_4\left(x_i^j\right) = \begin{cases} \dfrac{2ni-n+1}{2}: & i \in [1,t],\ j=1, \\ \dfrac{ni-4j+5}{2}: & i \in [1,2t-1],\ j \in \left[2, \dfrac{n+1}{4}\right], i \in \mathbb{N}^{\text{odd}}, \\ \dfrac{ni+4j-n-3}{2}: & i \in [2,2t],\ j \in \left[2, \dfrac{n+1}{4}\right], i \in \mathbb{N}^{\text{even}}, \\ \dfrac{ni+4j-2n-3}{2}: & i \in [1,2t-1],\ j \in \left[\dfrac{n+5}{4}, \dfrac{n+1}{2}\right], i \in \mathbb{N}^{\text{odd}}, \\ \dfrac{ni-4j+n+5}{2}: & i \in [2,2t],\ j \in \left[\dfrac{n+5}{4}, \dfrac{n+1}{2}\right], i \in \mathbb{N}^{\text{even}}. \end{cases} \tag{24}$$

With the abovementioned scheme, the edge-sums being generated form a consecutive integer sequence set $\{3,4,\ldots,2tn-1\}$. ψ_4 is extendable to an $S-(a,0)-$EAMT labeling of $UP_n^t, n\geq 11$, under the light of Lemma 1, admitting $a=3tn$.

A direct outcome of Lemma 2 is as follows. □

Theorem 3. *The pancyclic graph UP_n^t is $S-(tn+4,2)-EAMT$, for all t and n.*

2.2. $S-(a,0)-$EAMT Labeling of Disjoint Union of C_4 with Trees. It is a well-known fact that the graph C_4 is not $S-(a,0)-$EAMT [6], and work is still in progress in order to determine if its disjoint copies are $S-(a,0)-$EAMT. In this section, we shall provide an $S-(a,0)-$EAMT labeling of disjoint copies of C_4 with various trees in the form of several results. This will give a support to researchers to carry out their work to determine the aforesaid labeling of the disjoint copies of C_4. Throughout this section, the union will represent a disjoint union of graphs only.

Theorem 4. *For odd m, the graph $mC_4\cup 2K_{1,m}\cup((7m-3)/2)K_1$ acquires an $S-(a,0)-EAMT$ labeling admitting $a=21m+2$.*

Proof. Consider a graph $mC_4\cup 2K_{1,m}\cup((7m-3)/2)K_1$ with vertex and edge sets:

$$\begin{aligned}V(\Lambda_1)&=\left\{x_1^i,x_2^i\colon i\in[1,m]\right\}\cup\{y_i,z_i\colon i\in[1,m]\}\\&\cup\{k_i\colon 1\leq i\leq 2m\}\cup\left\{l_i\colon 1\leq i\leq\frac{7m-3}{2}\right\}\cup\{c_1,c_2\},\\E(\Lambda_1)&=\left\{y_ix_1^i,y_ix_2^i\colon 1\leq i\leq m\right\}\cup\left\{z_ix_1^i,z_ix_2^i\colon 1\leq i\leq m\right\}\\&\cup\{c_1k_i\colon 1\leq i\leq m\}\cup\{c_2k_i\colon m+1\leq i\leq 2m\}.\end{aligned}\tag{25}$$

If $p=|V(\Lambda_1)|=(19m+1)/2$ and $q=|E(\Lambda_1)|=6m$, we sketch a labeling $f_1\colon V(\Lambda_1)\longrightarrow\{1,2,\ldots,(19m+1)/2\}$ as follows:

$$f_1(x_1^i)=\begin{cases}\frac{1}{2}(7m+i), & 1\leq i\leq m; i\equiv 1,\ (\mathrm{mod}\ 2),\\ \frac{1}{2}(6m+i), & 2\leq i\leq m-1; i\equiv 0,\ (\mathrm{mod}\ 2),\end{cases}$$

$$f_1(x_2^i)=5m-(i-1)\colon 1\leq i\leq m,$$

$$f_1(y_i)=\begin{cases}\frac{i+4m+1}{2}, & 1\leq i\leq m; i\equiv 1,\ (\mathrm{mod}\ 2),\\ \frac{i+5m+1}{2}, & 2\leq i\leq m-1; i\equiv 0,\ (\mathrm{mod}\ 2),\end{cases}$$

$$\begin{aligned}f_1(z_i)&=\begin{cases}\frac{1}{2}(i+10m+1), & i\in[1,m]; i\in\mathbb{N}^{\text{odd}},\\ \frac{1}{2}(i+11m+1), & i\in[2,m-1]; i\in\mathbb{N}^{\text{even}},\end{cases}\\f_1(k_i)&=i\colon i\in[1,2m],\\f_1(c_1)&=\frac{15m+1}{2},\\f_1(c_2)&=\frac{19m+1}{2},\\f_1(l_i)&=\begin{cases}6m+i, & 1\leq i\leq\frac{3m-1}{2},\\ i+6m+1, & i\in\left[\frac{3m+1}{2},\frac{7m-3}{2}\right].\end{cases}\end{aligned}\tag{26}$$

The edge-sums' sets constituted by the abovementioned design generates a consecutive sequence of integer $(11m+3/2),(11m+5/2),\ldots,23m+1/2$. Under the shadow of Lemma 1, f_1 accredits an $S-(a,0)-$EAMT labeling of Λ_1 having $a=21m+2$. □

Theorem 5. *For odd m, the graph $mC_4\cup 2mK_2\cup 2mK_1$ acquires an $S-(a,0)-EAMT$ labeling having $a=(43m+3)/2$.*

Proof. Consider the graph $\Lambda_2\cong mC_4\cup 2mK_2\cup 2mK_1$, for odd m, with the following vertex-edge connections:

$$\begin{aligned}V(\Lambda_2)&=\left\{x_1^i,x_2^i\colon i\in[1,m]\right\}\cup\{z_i,y_i\colon 1\leq i\leq m\}\\&\cup\{q_i,p_i\colon i\in[1,2m]\}\cup\{l_i\colon i\in[1,2m]\},\\E(\Lambda_2)&=\left\{y_ix_1^i,y_ix_2^i\colon 1\leq i\leq m\right\}\cup\left\{z_ix_1^i,z_ix_2^i\colon 1\leq i\leq m\right\}\\&\cup\{q_ip_i\colon i\in[1,2m]\}.\end{aligned}\tag{27}$$

Here, order is $p=10m$ and size is $q=6m$. Now, we design a labeling $f_2\colon V(\Lambda_2)\longrightarrow\{1,2,\ldots,10m\}$ as follows:

$$f_2(x_1^i)=\begin{cases}\frac{1}{2}(7m+i), & i\in[1,m]; i\in\mathbb{N}^{\text{odd}},\\ \frac{1}{2}(6m+i), & i\leq[2,m-1]; i\in\mathbb{N}^{\text{even}},\end{cases}$$

$$f_2(x_2^i)=5m-(i-1)\colon i\in[1,m],$$

$$f_2(y_i)=\begin{cases}\frac{1}{2}(i+4m+1), & i\in\{1,\ldots,m\}; i\in\mathbb{N}^{\text{odd}},\\ \frac{1}{2}(i+5m+1), & i\in\{2,\ldots,m-1\}; i\in\mathbb{N}^{\text{even}},\end{cases}$$

$$f_2(z_i) = \begin{cases} \frac{1}{2}(i + 10m + 1), & i \in [1, m]; i \in \mathbb{N}^{\text{odd}}, \\ \frac{1}{2}(i + 11m + 1), & i \in [2, m-1]; i \in \mathbb{N}^{\text{even}}, \end{cases}$$

$$f_2(p_i) = i\colon\ i \in [1, 2m],$$

$$f_2(q_i) = \begin{cases} \frac{1}{2}(15m - i + 2)\colon\ i \in [1, m]; i \in \mathbb{N}^{\text{odd}}, \\ \frac{1}{2}(16m - i + 2)\colon\ i \in [2, m-1]; i \in \mathbb{N}^{\text{even}}, \\ \frac{1}{2}(21m - i + 2)\colon\ i \in [m+2, 2m-1]; i \in \mathbb{N}^{\text{odd}}, \\ \frac{1}{2}(20m - i + 2)\colon\ i \in [m+1, 2m]; i \in \mathbb{N}^{\text{even}}, \end{cases}$$

$$f_2(l_i) = \begin{cases} i + 6m\colon\ i \in [1, m], \\ i + 7m\colon\ i \in [m+1, 2m]. \end{cases} \tag{28}$$

The edge-sums' set constituted by the scheme f_2 generates a sequence consisting of consecutive integer $(11m + 3)/2, (11m + 5)/2, \ldots, (23m + 1)/2$. Under the shadow of Lemma 1, f_2 constitutes an $S - (a, 0) -$ EAMT labeling of Λ_2 with magic sum $a = (43m + 3)/2$. □

Theorem 6. *For odd m, the graph* $mC_4 \cup 2P_{m+1} \cup ((5m - 1)/2)K_1$ *acquires an* $S - (a, 0) - EAMT$ *labeling having* $a = 18m + 5$.

Proof. Let $\Lambda_3 \cong mC_4 \cup 2P_{m+1} \cup ((5m - 1)/2)K_1$, where m is odd, having vertex and edge sets interlinked:

$$\begin{aligned} V(\Lambda_3) &= \{x_1^i, x_2^i\colon\ i \in [1, m]\} \cup \{z_i, y_i\colon\ i \in [1, m]\} \cup \{p_i\colon\ i \in [1, m+1]\} \\ &\quad \cup \{q_i\colon\ i \in [1, m+1]\} \cup \left\{l_i\colon\ i \in \left[1, \frac{5m-1}{2}\right]\right\}, \\ E(\Lambda_3) &= \{y_i x_1^i, y_i x_2^i\colon\ i \in [1, m]\} \cup \{z_i x_1^i, z_i x_2^i\colon\ i \in [1, m]\} \\ &\quad \cup \{p_i p_{i+1}\colon\ i \in [1, m]\} \cup \{q_i q_{i+1}\colon\ i \in [1, m]\}. \end{aligned} \tag{29}$$

We have $p = (17m + 3)/2$ and $q = 6m$. A labeling $f_3\colon V(\Lambda_3) \longrightarrow \{1, 2, \ldots, (17m + 3)/2\}$ is being defined as follows:

$$f_3(x_1^i) = \begin{cases} \frac{1}{2}(5m + i + 2), & i \in [1, m]; i \in \mathbb{N}^{\text{odd}}, \\ \frac{1}{2}(4m + i + 2), & i \in [2, m-1]; i \in \mathbb{N}^{\text{even}}, \end{cases}$$

$$f_3(x_2^i) = 4m - i + 2\colon\ 1 \le i \le m,$$

$$f_3(y_i) = \begin{cases} \frac{2m + i + 3}{2}, & i \in [1, m]; i \in \mathbb{N}^{\text{odd}}, \\ \frac{3m + i + 3}{2}, & i \in [2, m-1]; i \in \mathbb{N}^{\text{even}}, \end{cases}$$

$$f_3(z_i) = \begin{cases} \frac{8m + i + 3}{2}, & i \in \{1, \ldots, m\}; i \in \mathbb{N}^{\text{odd}}, \\ \frac{9m + i + 3}{2}, & i \in \{2, \ldots, m-1\}; i \in \mathbb{N}^{\text{even}}, \end{cases}$$

$$f_3(p_i) = \begin{cases} \frac{i + 1}{2}, & i \in [1, m]; i \in \mathbb{N}^{\text{odd}}, \\ \frac{11m + i + 3}{2}, & i \in [2, m+1]; i \in \mathbb{N}^{\text{even}}, \end{cases}$$

$$f_3(q_i) = \begin{cases} \frac{1}{2}(i + m + 2), & i \in [1, m]; i \in \mathbb{N}^{\text{odd}}, \\ \frac{1}{2}(16m + i + 2), & i \in [2, m+1]; i \in \mathbb{N}^{\text{even}}, \end{cases}$$

$$f_3(l_i) = \begin{cases} i + 5m + 1\colon\ 1 \le i \le \frac{m + 1}{2}, \\ \frac{2i + 11m + 3}{2}\colon\ \frac{m + 3}{2} \le i \le \frac{5m - 1}{2}. \end{cases} \tag{30}$$

The edge-sums' set constituted by the scheme f_3 generates a sequence consisting of consecutive integer $(7m + 7)/2, (7m + 9)/2, \ldots, (19m + 5)/2$. Under the shadow of Lemma 1, f_3 constitutes an $S - (a, 0) -$ EAMT of the graph Λ_3 with magic constant $a = 18m + 5$. □

Theorem 7. *For odd* m, $mC_4 \cup (2m-2)K_2 \cup P_4 \cup 2mK_1$ *possesses* $S-(a,0)-EAMT$ *labeling having* $a = (43m+5)/2$.

Proof. With m taken odd, consider $\Lambda_4 \cong mC_4 \cup (2m-2)K_2 \cup P_4 \cup 2mK_1$, having vertex-edge connections:

$$\begin{aligned} V(\Lambda_4) &= \{x_1^i, x_2^i\colon i \in [1,m]\} \cup \{y_i, z_i\colon i \in [1,m]\} \cup \{l_i\colon 1 \le i \le 2m\} \\ &\quad \cup \{q_i, p_i\colon i \in [1, 2(m-1)]\} \cup \{t_i\colon i \in [1,4]\}, \\ E(\Lambda_4) &= \{y_i x_1^i, y_i x_2^i\colon 1 \le i \le m\} \cup \{z_i x_1^i, z_i x_2^i\colon i \in [1,m]\} \\ &\quad \cup \{p_i q_i\colon i \in [1, 2(m-1)]\} \cup \{t_i t_{i+1}\colon 1 \le i \le 3\}. \end{aligned} \tag{31}$$

Here, we have $p = 10m$ and $q = 6m+1$. A labeling function $f\colon V(\Lambda_4) \longrightarrow \{1, 2, \ldots, 10m\}$ is being defined as follows:

$$f_4(x_1^i) = \begin{cases} \frac{1}{2}(7m+i)\colon & i \in [1,m]; i \in \mathbb{N}^{\text{odd}}, \\ \frac{1}{2}(6m+i)\colon & i \in [2, m-1]; i \in \mathbb{N}^{\text{even}}, \end{cases}$$

$$f_4(x_2^i) = 5m - (i-1)\colon 1 \le i \le m,$$

$$f_4(y_i) = \begin{cases} \frac{i+4m+1}{2}\colon & i \in \{1, \ldots, m\}; i \in \mathbb{N}^{\text{odd}}, \\ \frac{i+5m+1}{2}\colon & i \in \{2, \ldots, m-1\}; i \in \mathbb{N}^{\text{even}}, \end{cases}$$

$$f_4(z_i) = \begin{cases} \frac{i+10m+1}{2}\colon & i \in \{1, \ldots, m\}; i \in \mathbb{N}^{\text{odd}}, \\ \frac{i+11m+1}{2}\colon & i \in \{2, \ldots, m-1\}; i \in \mathbb{N}^{\text{even}}, \end{cases}$$

$$f_4(p_i) = i\colon i \in [1, 2(m-1)],$$

$$f_4(q_i) = \begin{cases} \frac{15m-i+2}{2}\colon i \in [1,m]; i \in \mathbb{N}^{\text{odd}}; \\ \frac{16m-i+2}{2}\colon i \in [2, m-1]; i \in \mathbb{N}^{\text{even}}; \\ \frac{21m-i+2}{2}\colon i \in [m+2, 2m-3]; i \in \mathbb{N}^{\text{odd}}; \\ \frac{20m-i+2}{2}\colon \frac{i}{\mathrm{i}\{m+1, \ldots, 2(m-1)\}}; i \in \mathbb{N}^{\text{even}}; \end{cases}$$

$$f_4(t_i) = \begin{cases} 2m-1\colon i = 1; \\ \frac{19m+3}{2}\colon i = 2; \\ 2m\colon i = 3; \\ 9m+1\colon i = 4; \end{cases} \tag{32}$$

$$f_4(l_i) = \begin{cases} i+6m\colon 1 \le i \le m; \\ i+7m\colon i \in [m+1, 2m]. \end{cases}$$

The edge-sums' set constituted by the scheme f_4 generates a sequence consisting of consecutive integer $(11m+3)/2, (11m+5)/2, \ldots, (23m+3)/2$. Lemma 1 implies that f_4 extends to an $S-(a,0)-$EAMT labeling of Λ_4 with $a = (43m+5)/2$. □

Theorem 8. *For odd* m, $mC_4 \cup S_{m-1,m} \cup ((5m-1)/2)K_1$ *possesses an* $S-(a,0)-EAMT$ *labeling with* $a = 2(9m+1)$.

Proof. Consider the graph $\Lambda_5 \cong mC_4 \cup \cup S_{m-1,m} \cup ((5m-1)/2)K_1$ with vertex-edge connections as follows:

$$\begin{aligned} V(\Lambda_5) &= \{x_1^i, x_2^i\colon i \in [1,m]\} \cup \{z_i, y_i\colon i \in [1,m]\} \\ &\quad \cup \{s_i, t_i\colon i \in [1,m]\} \cup \left\{l_i\colon i \in \left[1, \frac{5m-1}{2}\right]\right\} \cup \{c\}, \\ E(\Lambda_5) &= \{y_i x_1^i, y_i x_2^i\colon i \in [1,m]\} \cup \{z_i x_1^i, z_i x_2^i\colon i \in [1,m]\} \\ &\quad \cup \{ct_i\colon i \in [1,m]\} \cup \{t_m s_i\colon i \in [1,m]\}. \end{aligned} \tag{33}$$

Then, $p = (17m+1)/2$ and $q = 6m$. Again, a labeling $f_5\colon V(\Lambda_5) \longrightarrow \{1, 2, \ldots, (17m+1)/2\}$ is being defined as follows:

$$f_5(x_1^i) = \begin{cases} \frac{1}{2}(5m+i)\colon & i \in \{1, \ldots, m\}; i \in \mathbb{N}^{\text{odd}}; \\ \frac{1}{2}(4m+i)\colon & i \in [2, m-1]; i \in \mathbb{N}^{\text{even}}, \end{cases}$$

$$f_5(x_2^i) = 4m - (i-1)\colon i \in [1,m],$$

$$f_5(y_i) = \begin{cases} \frac{i+2m+1}{2}\colon & i \in \{1, \ldots, m\}; i \in \mathbb{N}^{\text{odd}}; \\ \frac{i+3m+1}{2}\colon & i \in \{2, \ldots, m-1\}; i \in \mathbb{N}^{\text{even}}, \end{cases}$$

$$f_5(z_i) = \begin{cases} \frac{i+8m+1}{2}\colon & i \in [1,m]; i \in \mathbb{N}^{\text{odd}}; \\ \frac{i+9m+1}{2}\colon & i \in [2, m-1]; i \in \mathbb{N}^{\text{even}}, \end{cases}$$

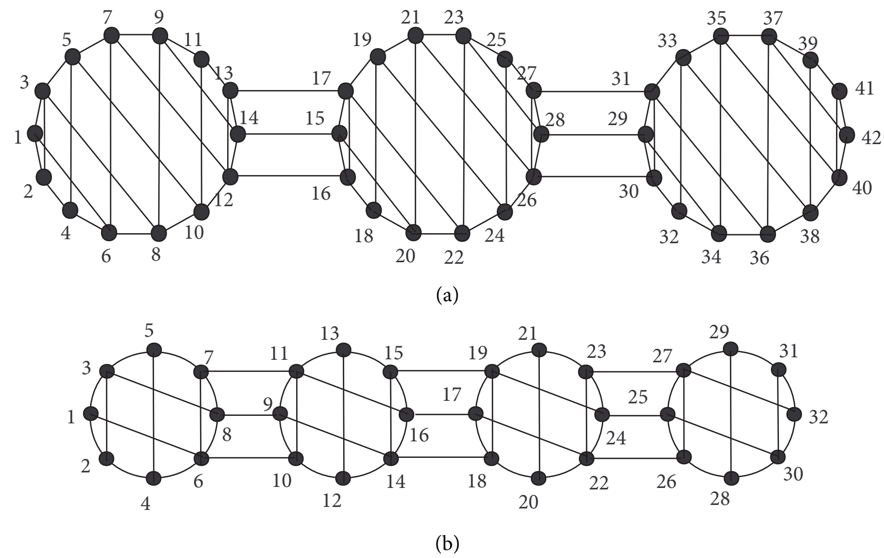

Figure 1: An $S-(126,0)-$EAMT and $S-(96,0)-$EAMT labeling of the pancyclic graphs (a)UP^3_{14} and (b)UP^4_8.

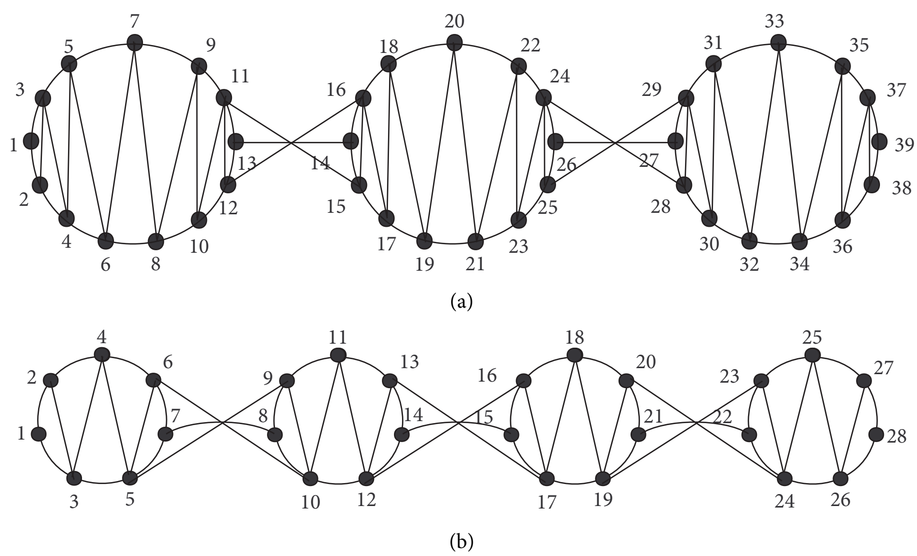

Figure 2: An $S-(117,0)-$EAMT and $S-(84,0)-$EAMT labeling of the pancyclic graphs (a)UP^3_{13} and (b)UP^4_7.

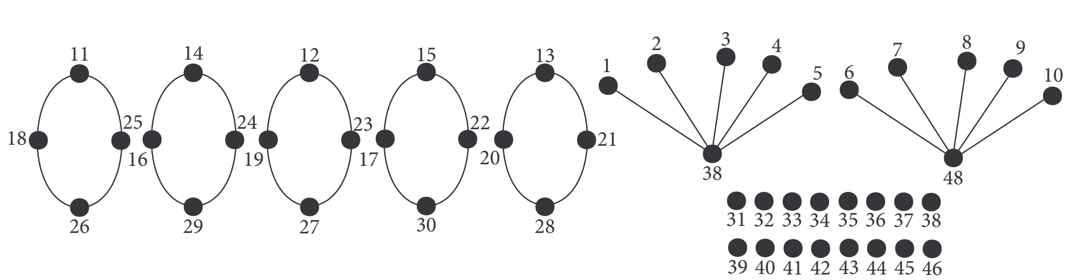

Figure 3: An $S-(107,0)-$EAMT labeling of the graph $5C_4 \cup 2K_{1,5} \cup 16K_1$.

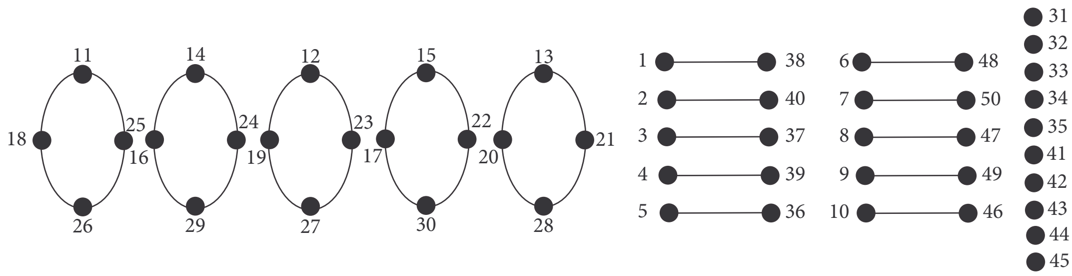

Figure 4: An $S-(109,0)$ – EAMT labeling of the graph $5C_4 \cup 10K_2 \cup 10K_1$.

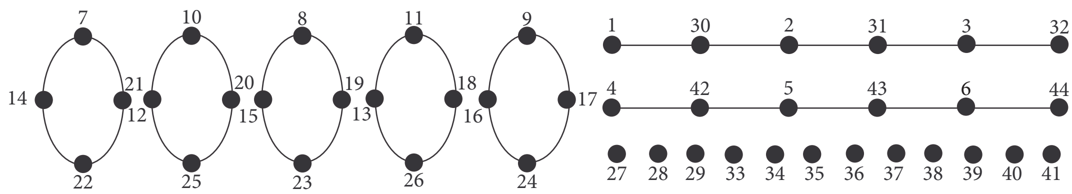

Figure 5: An $S-(95,0)$ – EAMT labeling of the graph $5C_4 \cup 2P_6 \cup 12K_1$.

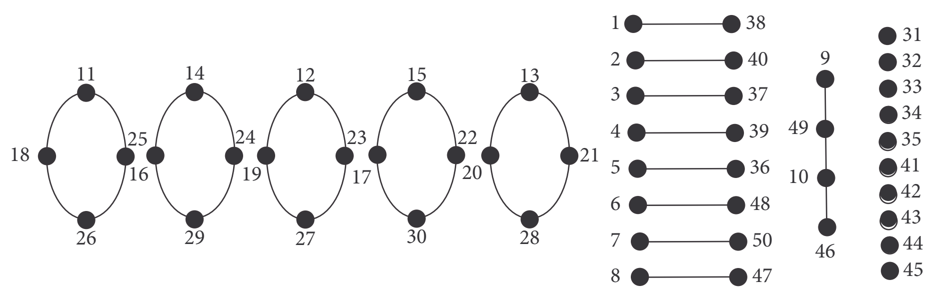

Figure 6: An $S-(109,0)$ – EAMT labeling of the graph $5C_4 \cup 8K_2 \cup P_4 \cup 10K_1$.

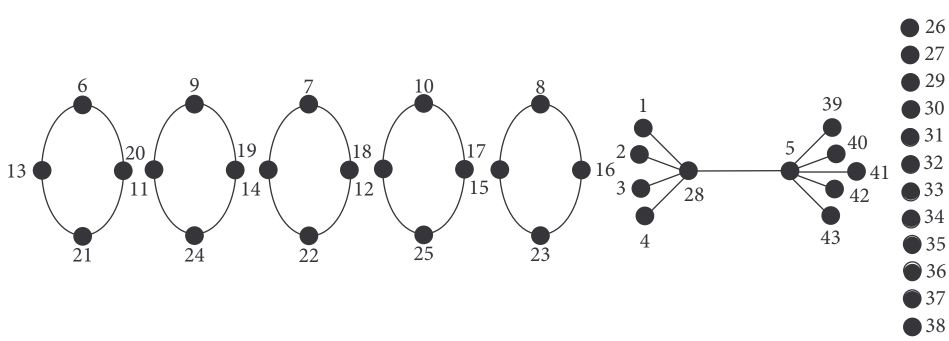

Figure 7: An $S-(92,0)$ – EAMT labeling of the graph $5C_4 \cup S_{4,5} \cup 12K_1$.

$$
\begin{aligned}
f_5(t_i) &= i\colon\ 1 \le i \le m;\\
f_5(s_i) &= \frac{2i + 15m + 1}{2}\colon\ 1 \le i \le m;\\
f(c) &= \frac{11m + 1}{2},\\
f_5(l_i) &= \begin{cases} i + 5m\colon & i \in \left[1, \dfrac{m-1}{2}\right];\\ i + 5m + 1\colon & i \in \left[\dfrac{m+1}{2}, \dfrac{5m-1}{2}\right]. \end{cases}
\end{aligned}
\tag{34}
$$

The edge-sums' set constituted by the scheme f_2 generates a sequence consisting of consecutive integer $(7m + 3)/2, (7m + 5)/2, \ldots, (19m + 1)/2$. Under the shadow of Lemma 1, f_5 constitutes to an $S-(a, 0)-$EAMT labeling of Λ_1 admitting $a = 2(9m + 1)$.

The following results are direct consequences of Lemma 2, from Theorems 4–8. □

Theorem 9. *For odd m, the graph* $mC_4 \cup 2K_{1,m} \cup ((7m-3)/2)K_1$ *admits an* $S-(15m + 3, 2)-$*EAMT labeling.*

Theorem 10. *For odd m, the graph* $mC_4 \cup 2mP_2 \cup 2mK_1$ *admits an* $S-((31m + 5)/2, 2)-$*EAMT labeling.*

Theorem 11. *For odd m, the graph* $mC_4 \cup 2P_{m+1} \cup ((5m-1)/2)K_1$ *admits an* $S-(12m + 6, 2)-$*EAMT labeling.*

Theorem 12. *For odd m,* $mC_4 \cup (2m-2)P_2 \cup P_4 \cup 2mK_1$ *admits an* $S-((31m + 5)/2, 2)-$*EAMT labeling.*

Theorem 13. *For odd m,* $mC_4 \cup S_{m-1,m} \cup ((5m-1)/2)K_1$ *admits an* $S-(12m + 3, 2)-$*EAMT labeling.*

2.3. Examples and Proposed Open Problems. An $S-(126, 0)-$EAMT labeling of the graph UP_n^t is being presented in Figure 1(a), corresponding to the parameters $t = 3$ and $n = 14$. Furthermore, Figure 1(b) presents an $S-(96, 0)-$EAMT labeling of UP_n^t corresponding to $t = 4$ and $n = 8$. Here, it can be observed that the value of the magic constant is perfect as per our depiction in Theorem 1.

Figures 2(a) and 2(b) illustrate Theorem 2 by providing $S-(117, 0)-$EAMT and $S-(84, 0)-$EAMT labeling of the graph UP_n^t.

Figures 3–7 are the illustrations of Theorems 4–8, respectively, for particular values of the parameters involved.

The open problems related to Section 2.2 are as follows:

(i) For even m, determine any $S-(a, 0)-$EAMT labeling of $mC_4 \cup 2K_{1,m} \cup (7m-3/2)K_1$
(ii) For even m, determine any $S-(a, 0)-$EAMT labeling of $mC_4 \cup 2mK_2 \cup 2mK_1$
(iii) For even m, determine any $S-(a, 0)-$EAMT labeling of $mC_4 \cup 2P_{m+1} \cup ((5m-1)/2)K_1$
(iv) For even m, determine any $S-(a, 0)-$EAMT labeling of $mC_4 \cup (2m-2)K_2 \cup P_4 \cup 2mK_1$
(v) For even m, determine any $S-(a, 0)-$EAMT labeling of $mC_4 \cup S_{m-1,m} \cup ((5m-1)/2)K_1$

3. Conclusion

In this article,

(i) We have obtained $S-(a, 0)-$EAMT and $S-(a', 2)-$EAMT labeling of a pancyclic class of graphs, namely, Usmanian pancyclic graph, denoted by UP_n^t.
(ii) We have exhibited the existence of $S-(a, 0)-$EAMT and $S-(a\prime, 2)-$EAMT labeling on disjoint copies of C_4 with various trees. Specifically, $mC_4 \cup 2K_{1,m} \cup (7m-3/2)K_1$, $mC_4 \cup 2mK_2 \cup 2mK_1$, $mC_4 \cup 2P_{m+1} \cup (5m-1/2)\ K_1$, $mC_4 \cup (2m-2)K_2 \cup P_4 \cup 2mK_1$ and $mC_4 \cup S_{m-1,m} \cup (5m-1/2)K_1$, whereas C_4 itself is not $S-(a, 0)-$EAMT. These obtained results open a new direction for researchers to derive $S-(a, 0)-$EAMT labeling of disjoint copies of C_4.
(iii) A few open problems have also been proposed for future work in this area.

Data Availability

The whole data are included within this article. However, the reader may contact the corresponding author for more details on the data.

Conflicts of Interest

The authors declare no conflicts of interest.

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