Research Article

On the Upper Bounds of Fractional Metric Dimension of Symmetric Networks

Muhammad Javaid ¹, Muhammad Kamran Aslam,¹ and Jia-Bao Liu ²

¹Department of Mathematics, School of Science, University of Management and Technology, Lahore, Pakistan

²School of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, China

Correspondence should be addressed to Jia-Bao Liu; liujiabaoad@163.com

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Distance-based numeric parameters play a pivotal role in studying the structural aspects of networks which include connectivity, accessibility, centrality, clustering modularity, complexity, vulnerability, and robustness. Several tools like these also help to resolve the issues faced by the different branches of computer science and chemistry, namely, navigation, image processing, biometry, drug discovery, and similarities in chemical compounds. For this purpose, in this article, we are considering a family of networks that exhibits rotationally symmetric behaviour known as circular ladders consisting of triangular, quadrangular, and pentagonal faced ladders. We evaluate their upper bounds of fractional metric dimensions of the aforementioned networks.

1. Introduction

A network $\aleph$ comprises of two collections of distinct objects, which are set containing nodes/vertices $V(\aleph)$ and set of objects that forms connection among those nodes/vertices $E(\aleph)$, where $E(\aleph) \subseteq V(\aleph) \times V(\aleph)$. The cardinality of $V(\aleph)$ constitutes the order of $\aleph$ denoted by v , and size of $\aleph$ is the cardinality of $E(\aleph)$ denoted by e . For $a, b \in V(\aleph)$, $d(a, b)$ represents the distance of shortest path between a and b . In order to have an in depth study of graph theory's strata, please refer to [1–3].

Day by day technological advancements are restructuring our lives by replacing manpower with robots, devices, and machinery. In the same instance, the necessity of restructuring is also making it inevitable for us to utilize these machineries which are keeping in view cost effectiveness and serving the large amount of ques in production line, health services, and public dealing areas with minimum number of them. Here, distance-based parameters come to fulfill the aforementioned objectives. Consider $\aleph = \{m_1, m_2, \dots, m_k\} \subseteq V(\aleph)$, then $\aleph$ becomes an ordered set of vertices whenever some ordering is being imposed on them. The k -tuple metric form of $\aleph$ with respect to a is

represented by $r(a|\aleph) = (d(a, m_1), d(a, m_2), d(a, m_3), \dots, d(a, m_k))$, where $d(a, m_t)$ for $1 \leq t \leq k$ shows the distance of a from m_t . $\aleph$ changes into resolving set if for any pair of distinct vertices $a, b \in V(\aleph) - \aleph$ such that $r(a|\aleph) \neq r(b|\aleph)$. The resolving set having minimum number of vertices in $\aleph$ presents the metric dimension (MD) of $\aleph$ denoted by $\lambda(\aleph)$ [4, 5].

For any connected network, Slater [6, 7] laid the foundation of resolving set by calling it as locating set. After studying these terminologies at their own, Harary and Melter named it as the MD of a network. Since then, a number of researchers considered different families of networks for the computation of MD. The results regarding the families of networks with constant and bounded MD can be found in [8]. Chartrand et al. proved the results regarding the metric dimensions of path and cycle in [4]. Similarly, $\mathbb{P}(n, 2)$, $\mathbb{A}_m$, and $\mathbb{C}_m^2$ as networks bearing constant, MD has been shown in [8]. Moreover, in [9], the MD of the generalized Petersen network is proved to be bounded. Also the MD of the Jahangir network and infinite regular networks has been studied in [10, 11], respectively.

Chartrand et al. [4] employed MD to acquire the solution of integer programming problem (IPP). Later on, Currie and

Oellermann [12] established the concept of fractional metric dimension (FMD) and acquired the higher accuracy solution. Fehr et al. [13] utilized FMD to obtain the optimal solution of a certain linear programming relaxation problem. Arumugam and Mathew [14] uncovered some peculiar features of FMD. Afterwards, a bunch of results appeared related to the FMD of several networks that are the resultants of graph products, namely, Cartesian, hierarchial, corona, and lexicographic comb products (see [14–18]). Furthermore, the FMD of the generalized Jahangir network and permutation network can be found in [19, 20], respectively. In a recent time, Raza et al. [21] evaluated the FMD of metal organic networks.

Aisyah et al. defined the notion of local fractional metric dimension (LFMD) and calculated for corona product of two networks [22]. Similarly, Liu et al. [23] calculated the LFMD of rotationally symmetric and planar networks. More recently, Javaid et al. evaluated the sharp extremal values of LFMD of connected networks [24]. Results regarding the LFMD of cycle related networks can be found in [25]. In this article, we have considered a family of network bearing rotational symmetry known by circular ladders consisting of different triangular, quadrangular, and pentagonal faces. This article propels in the following manners: Section 1 is introduction, Section 2 covers the role of MD in Robotics and Chemistry, and Section 3 is of preliminaries. In Section 4, we will discuss the idea behind the creation of the circular ladders under consideration. Section 5, contains the main results, and Section 6 closes this article with some conclusions and future directions.

2. Applications

2.1. Robotics. The ever increasing demand of networking enlarged the study of distance-based dimensions. All such tools are used to allocate an appropriate region to an interpolator for its effective usage [6, 26]. The placement of robots at different sites such as restaurants, production units, and public health facilities come under study by using one such tool in [27]. The effective rectification of example and picture handling and handling of informational structures using the aforementioned parameters has been given in [28]. The parameter like these has not only made it possible to serve restaurants, factories, and public service areas with a minimum number of robots but also truncated the delay in their serving. To find more applications on this topic, we refer to [29, 30].

2.2. Chemistry. A chemical compound in graph theoretic form is regarded as a molecular graph having nodes as atoms and links between them as bonds between them [4]. In this picturesque form of a compound along with distance-based parameters give chemists an opportunity to remove discrepancies in a chemical structure and to find the sites showing similar properties in them. All such strategies can be found in [5, 6, 28, 31].

3. Preliminaries

A vertex $c \in \aleph$ is said to resolve a pair $\{a, b\}$ in $\aleph$ if $d(a, c) \neq d(b, c)$. For $a, b \in V(\aleph)$, then the resolving neighbourhood (RN) of the pair $\{a, b\}$ is given by $R\{a, b\} = \{c \in V(\aleph) | d(a, c) \neq d(b, c)\}$.

Assume a connected network $\aleph(V(\aleph), E(\aleph))$ bearing v as its order. A function $\kappa: V(\aleph) \rightarrow [0, 1]$ is called resolving function (RF) of $\aleph$ if each RN set $R\{a, b\}$, $\kappa(R\{a, b\}) = \sum_{c \in R\{a, b\}} \kappa(c) \geq 1$. An RF of $\aleph$ is known as minimal RF if any function $\phi: V(\aleph) \rightarrow [0, 1]$ such that $\phi \leq \kappa$ and $\phi(c) \neq \kappa(c)$ for at least one $c \in V(\aleph)$, which is not an RF of $\aleph$. The FMD of the network $\aleph$ is given by $\dim_f(\aleph) = \min\{|\kappa|: \kappa \text{ is minimal RF of } \aleph\}$, where $|\kappa| = \sum_{c \in V(\aleph)} \kappa(c)$ [14].

4. Construction of Networks

Here, we give the construction of networks covered in this article. A cycle C_3 is a network bearing the regularity of 2, 3 vertices, and 2 faces known as triangle. On the other hand, C_4 is a network having the regularity of 2 with 4 vertices and 2 faces. A 3-faced quadrangle is obtained after joining any pair of nonadjacent vertices of it. Figure 1(a) shows the planar quadrangles having 2 and 3 faces. A network (C_5) bearing the regularity of 2, 5 nodes, and 2 faces is known as pentagon. The 3-faced pentagons are formed by joining any two nonadjacent nodes of 2-faced pentagon, and 4-faced pentagons are formed by forming links between any two pairs of nonadjacent nodes of 3-faced pentagon. Figure 1(b) illustrates some of the possible 2-faced, 3-faced, and 4-faced planar pentagons.

4.1. Triangular Circular Ladder. The triangular circular ladder (TCL) $\aleph \cong \mathbb{T}_m$ is created by joining each edge $a_1^1 a_2^1, a_2^1 a_3^1, \dots, a_{m-1}^1 a_m^1, a_m^1 a_1^1$ of a cycle C_m with the corresponding nodes $a_1^2, a_2^2, \dots, a_m^2$, where $m \equiv 0 \pmod{2}$. Thus, each step of TCL consists of 2-faced triangle and its vertex and edge sets are as follows: $V(\aleph) = \{a_r^1, a_r^2 | 1 \leq r \leq m\}$, $E(\aleph) = \{a_r^1 a_{r+1}^1 | 1 \leq r \leq m \wedge a_{m+1}^1 = a_1^1\} \cup \{a_r^1 a_r^2, a_r^2 a_{r-1}^2 | 1 \leq r \leq m \wedge a_0^2 = a_m^2\}$ with $v = 2m$ and $e = 3m$ respectively. Figure 2 shows TCL.

4.2. Quadrangular Circular Ladders. For $m \geq 3$, the quadrangular circular ladder (QCL) $\mathbb{Q}_m^1$ (prism) is a cubic network which is the resultant of the Cartesian product $P_2 \times C_m$, with $v = 2m$ and $e = 3m$. Its sets of vertices and edges are, respectively, given by the following: $V(\mathbb{Q}_m^1) = \{a_r^1, a_r^2 | 1 \leq r \leq m\}$, $E(\mathbb{Q}_m^1) = \{a_r^1 a_{r+1}^1 | 1 \leq r \leq m \wedge a_{m+1}^1 = a_1^1\} \cup \{a_r^1 a_r^2, a_r^2 a_{r+1}^2 | 1 \leq r \leq m \wedge a_{m+1}^2 = a_1^2\}$.

The two possible QCLs $\aleph \cong \mathbb{Q}_m^2$ and $\aleph \cong \mathbb{Q}_m^3$ having each step of 3-faced pentagons are formed by forming edges either $a_r^1 a_{r-1}^2$ or $a_{r-1}^1 a_r^2$ in quadrangular circular ladder $\mathbb{Q}_m^1$. The sets of vertices and edges of these networks are, respectively, given as follows: $V(\mathbb{Q}_m^2) = \{a_r^1, a_r^2 | 1 \leq r \leq m\}$ and $E(\mathbb{Q}_m^2) = \{a_r^1 a_{r+1}^1, a_r^1 a_r^2, a_r^2 a_{r+1}^2 | 1 \leq r \leq m \wedge a_{m+1}^1 = a_1^1, a_{m+1}^2 = a_1^2\} \cup \{a_r^1 a_{r-1}^2 | 1 \leq r \leq m \wedge a_0^2 = a_m^2\}$ and $V(\mathbb{Q}_m^3) = \{a_r^1, a_r^2 | 1 \leq r \leq m\}$ and $E(\mathbb{Q}_m^3) = \{a_r^1 a_{r+1}^1, a_r^1 a_r^2, a_r^2 a_{r+1}^2 | 1 \leq r \leq m \wedge$

$a_{m+1}^1 = a_1^1, a_{m+1}^2 = a_1^2 \} \cup \{a_r^1 a_{r+1}^2 | 1 \leq r \leq m \wedge a_{m+1}^2 = a_1^2 \}$ with $v = 2m$ and $e = 4m$.

Figure 3 illustrates $\mathbb{Q}_m^1$, $\mathbb{Q}_m^2$, and $\mathbb{Q}_m^3$. It can be observed from Figure 3 that $\mathbb{Q}^3 \cong \mathbb{Q}_m^2$.

4.3. Pentagonal Circular Ladders. The each step of a pentagonal circular ladder (PCL) $\mathbb{P}_m^1$ consists of 2-faced pentagon, and it is created by inserting a new vertex a_r^3 between the vertices a_r^2 and a_{r+1}^2 of $\mathbb{Q}_m^1$, where $V(\mathbb{P}_m^1)$ and $E(\mathbb{P}_m^1)$ are given by $V(\mathbb{P}_m^1) = \{a_r^1, a_r^2, a_r^3 | 1 \leq r \leq m\}$ and $E(\mathbb{P}_m^1) = \{a_r^1 a_{r+1}^1 | 1 \leq r \leq m \wedge a_{m+1}^1 = a_1^1\} \cup \{a_r^1 a_r^2, a_r^2 a_r^3, a_r^2 a_{r+1}^3 | 1 \leq r \leq m \wedge a_0^3 = a_m^3\}$ with $v = 3m$ and $e = 4m$, respectively.

Figure 4 illustrates $\mathbb{P}_m^1$.

The three possible PCLs $\mathbb{N} \cong \mathbb{P}_m^2, \mathbb{P}_m^3, \mathbb{P}_m^4$, each step of which is having 3-faced pentagons are formed by creating edges either $a_r^2 a_{r+1}^2, a_r^1 a_{r+1}^2$, or $a_r^1 a_{r-1}^2$ (shown by magenta colour in Figure 5) in pentagonal circular ladder $\mathbb{P}_m^1$. Figure 5 illustrates $\mathbb{P}_m^2, \mathbb{P}_m^3$, and $\mathbb{P}_m^4$.

As seen from Figure 5, $\mathbb{P}_m^4 \cong \mathbb{P}_m^3$. The sets $V(\mathbb{P}_m^2)$ and $E(\mathbb{P}_m^2)$ are given as follows: $V(\mathbb{P}_m^2) = \{a_r^1, a_r^2, a_r^3 | 1 \leq r \leq m\}$ and $E(\mathbb{P}_m^2) = \{a_r^1 a_{r+1}^1, a_r^2 a_{r+1}^2 | 1 \leq r \leq m \wedge a_{m+1}^1 = a_1^1, a_{m+1}^2 = a_1^2\} \cup \{a_r^1 a_r^2, a_r^2 a_r^3, a_r^2 a_{r+1}^3 | 1 \leq r \leq m \wedge a_0^3 = a_m^3\}$, $V(\mathbb{P}_m^3) = \{a_r^1, a_r^2, a_r^3 | 1 \leq r \leq m\}$ and $E(\mathbb{P}_m^3) = \{a_r^1 a_{r+1}^1, a_r^2 a_{r-1}^2 | 1 \leq r \leq m \wedge a_{m+1}^1 = a_1^1, a_0^2 = a_m^2\} \cup \{a_r^1 a_r^2, a_r^2 a_r^3, a_r^2 a_{r-1}^3 | 1 \leq r \leq m \wedge a_0^3 = a_m^3\}$, $V(\mathbb{P}_m^4) = \{a_r^1, a_r^2, a_r^3 | 1 \leq r \leq m\}$ and $E(\mathbb{P}_m^4) = \{a_r^1 a_{r+1}^1, a_{r-1}^1 a_r^2 | 1 \leq r \leq m \wedge a_{m+1}^1 = a_1^1, a_0^1 = a_m^1\} \cup \{a_r^1 a_r^2, a_r^2 a_r^3, a_r^2 a_{r-1}^3 | 1 \leq r \leq m \wedge a_0^3 = a_m^3\}$ with $v = 3m$ and $e = 5m$.

The three possible PCLs $\mathbb{N} \cong \mathbb{P}_m^5, \mathbb{P}_m^6$ and $\mathbb{P}_m^7$ whose each step is having 4-faced pentagons are created by forming edge $a_r^1 a_{r-1}^2$ (shown by cyan colour in Figure 6) in $\mathbb{P}_m^2$, edge $a_r^1 a_{r-1}^3$ in $\mathbb{P}_m^2$, edges $a_{r-1}^1 a_r^2$ in $\mathbb{P}_m^2$, or edges $a_r^1 a_r^2$ and $a_r^1 a_{r-1}^2$ in the plane network $\mathbb{P}_m^1$. Figure 6(a) illustrates $\mathbb{P}_m^5$, 6(b) illustrates $\mathbb{P}_m^6$, and 6(c) illustrates $\mathbb{P}_m^7$, respectively.

It can be seen from Figure 6 that $\mathbb{P}_m^6 \cong \mathbb{P}_m^5$.

The sets of vertices and edges of these networks are $V(\mathbb{P}_m^5) = \{a_r^1, a_r^2, a_r^3 | 1 \leq r \leq m\}$ $E(\mathbb{P}_m^5) = \{a_r^1 a_{r+1}^1, a_r^2 a_{r+1}^2, a_r^1 a_{r-1}^2 | 1 \leq r \leq m \wedge a_{m+1}^1 = a_1^1, a_{m+1}^2 = a_1^2, a_0^2 = a_m^2\} \cup \{a_r^1 a_r^2, a_r^2 a_r^3, a_r^2 a_{r+1}^3 | 1 \leq r \leq m\}$, $V(\mathbb{P}_m^6) = \{a_r^1, a_r^2, a_r^3 | 1 \leq r \leq m\}$, $E(\mathbb{P}_m^6) = \{a_r^1 a_{r+1}^1, a_r^2 a_{r+1}^2, a_r^1 a_{r-1}^2 | 1 \leq r \leq m\} \cup \{a_r^1 a_r^2, a_r^2 a_r^3, a_r^2 a_{r-1}^3 | 1 \leq r \leq m \wedge a_0^3 = a_m^3\}$, $V(\mathbb{P}_m^7) = \{a_r^1, a_r^2, a_r^3 | 1 \leq r \leq m\}$, $E(\mathbb{P}_m^7) = \{a_r^1 a_{r+1}^1, a_r^1 a_r^2, a_r^1 a_{r-1}^2 | 1 \leq r \leq m \wedge a_0^3 = a_m^3\} \cup \{a_r^1 a_r^2, a_r^2 a_r^3, a_r^2 a_{r-1}^3 | 1 \leq r \leq m \wedge a_0^3 = a_m^3\}$ with $v = 3m$ and $e = 6m$, respectively.

5. Main Results

5.1. Triangular Circular Ladder

Lemma 1. Let $\mathbb{N} \cong \mathbb{T}_m$ be TCL with $m \geq 6$ and $m \equiv 0 \pmod{2}$. For $1 \leq r, l \leq m$, $p \geq 1$, $s \geq 2$ $p \equiv 1 \pmod{2}$ and $s \equiv 0 \pmod{2}$, we have

- $|R_l| = |R\{a_l^1, a_l^2\}| = |R_r| = |R\{a_r^1, a_{r-1}^2\}| = m + 1$ and $|\cup_{l=1}^m R_l| \cup |\cup_{r=1}^m R_r| = 2m$, where $a_0^2 = a_m^2$.
- $|R_r| < |R_u| 2(m-1)$ for $1 \leq u \leq 8$, where $\bar{R}_1 = R\{a_r^1, a_{r+p}^1\}$, $\bar{R}_2 = R\{a_r^1, a_{r+s}^1\}$, $\bar{R}_3 = R\{a_r^2, a_{r+s}^2\}$, $\bar{R}_4 = R\{a_r^2, a_{r+p}^2\}$, $\bar{R}_5 = R\{a_r^1, a_{r+p}^2\}$, $\bar{R}_6 = R\{a_r^1, a_{r+s}^2\}$, $\bar{R}_7 =$

$R\{a_r^1, a_{r-p}^2\}$, $\bar{R}_8 = R\{a_r^1, a_{r-s}^2\}$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$ with $a_{m+1}^1 = a_1^1$.

Proof

- The RNs of $\{a_r^1, a_r^2\}$ and $\{a_r^1, a_{r-1}^2\}$ are $R_l = R\{a_l^1, a_l^2\} = V(\mathbb{T}_m) - \{a_h^1 | h \equiv l+1, l+2, \dots, l+(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv l+1, l+2, \dots, l+(m/2)-1 \pmod{m}\}$ and $R_r = R\{a_r^1, a_{r-1}^2\} = V(\mathbb{T}_m) - \{a_h^1 | h \equiv r-1, r-2, \dots, r-(m/2)+1 \pmod{m}\} \cup \{a_h^2 | h \equiv r-2, r-3, \dots, r-(m/2) \pmod{m}\}$. We note that $\cup_{l=1}^m R_l = V(\mathbb{T}_m)$, $\cup_{r=1}^m R_r = V(\mathbb{T}_m)$ and $|\cup_{l=1}^m R_l| \cup |\cup_{r=1}^m R_r| = 2m$.
- The RNs of $\bar{R}_u$ for $1 \leq u \leq 8$ are $\bar{R}_1 = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{T}_m) - \{a_h^2 | h \equiv r + ((p-1)/2), r + ((p+m-1)/2) \pmod{m}\} = \bar{R}_4 = R\{a_r^2, a_{r+p}^2\}$, $\bar{R}_2 = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{T}_m) - \{a_h^2 | h \equiv r + (s/2), r + ((m+s)/2) \pmod{m}\} = \bar{R}_3 = R\{a_r^2, a_{r+s}^2\}$, $\bar{R}_5 = R\{a_r^1, a_{r+p}^2\} = V(\mathbb{T}_m) - \{a_h^1 | h \equiv r + ((p+1)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((m+p-1)/2) \pmod{m}\}$, $\bar{R}_6 = R\{a_r^1, a_{r+s}^2\} = V(\mathbb{T}_m) - \{a_h^1 | h \equiv r + ((m+s)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + (s/2) \pmod{m}\}$, $\bar{R}_7 = R\{a_r^1, a_{r-p}^2\} = V(\mathbb{T}_m) - \{a_h^1 | h \equiv r - ((p+1)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((m-p-1)/2) \pmod{m}\}$ and $\bar{R}_8 = R\{a_r^1, a_{r-s}^2\} = V(\mathbb{T}_m) - \{a_h^1 | h \equiv r - (s/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((m+s-2)/2) \pmod{m}\}$, respectively.

Clearly, $|\bar{R}_u| = 2(m-1)$. Since $|R_r| = m+1 < |\bar{R}_u|$, then $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 2(m-1) \geq |R_r|$. $\square$

Theorem 1. If $\mathbb{N} \cong \mathbb{T}_m$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{T}_m) \leq (2m/(m+1))$.

Proof

Case 1. When $m = 6$.

The RNs are given as follows.

We have seen from the above that Table 1 shows the RNs with maximum cardinality that is 10 and Table 2 shows the RNs with minimum cardinality that is 7, which is less than the cardinalities of all other RNs $\bar{R}_r$ for $1 \leq t \leq 6$ that is 10. Moreover, $\cup_{r=1}^{12} R_r = V(\mathbb{T}_6)$ which implies $|\cup_{r=1}^{12} R_r| = 12$ and $|\bar{R}_u \cap \cup_{r=1}^{12} R_r| \geq |R_r|$, where $1 \leq u \leq 24$.

Now, we define a function $\kappa: V(\mathbb{T}_6) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = (1/7)$. As we can see, R_r for $1 \leq r \leq 12$ of $\mathbb{T}_6$ are pairwise overlapping; hence, $\exists$ another minimal resolving function $\bar{\kappa}$ of $\mathbb{T}_6$ such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(\mathbb{T}_6) \leq \sum_{r=1}^{12} (1/7) = (12/7)$.

Case 2. $m \geq 8$.

According to Lemma 1 the RNs with minimum cardinality are $R\{a_r^1, a_r^2\}$ and $R\{a_r^1, a_{r-1}^2\}$ having the cardinality of $m+1$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$, where $1 \leq u \leq 8$.

Let $\eta = |\cup_{r=1}^m R_r| = 2m$ and $\lambda = |R_r| = m+1$. Now, we define a function $\kappa: V(\mathbb{T}_m) \rightarrow [0, 1]$ such that

FIGURE 1: Quadrangles (a) and pentagons (b).

FIGURE 2: TCL $\mathbb{T}_m$. FIGURE 3: QCLs: (a) $\mathbb{Q}_m^1$, (b) $\mathbb{Q}_m^2$, and (c) $\mathbb{Q}_m^3$. FIGURE 4: $\mathbb{P}_m^1$.

$\kappa(c) = \begin{cases} 1/\lambda & \text{for } c \in \cup_{r=1}^m R_r, \\ 0 & \text{for } c \in V(\mathbb{T}_m) - \cup_{r=1}^m R_r. \end{cases}$ We can see that κ is a resolving function for $\mathbb{T}_m$ because $\kappa(R\{a, b\}) \geq 1 \forall \{a, b\} \subset V(\mathbb{N})$. Suppose that there is another resolving function ρ , such that $\rho(x) \leq \kappa(x)$, for at least one $x \in V(\mathbb{T}_m)$ $\rho(x) \neq \kappa(x)$. As a result, $\rho(R\{a, b\}) < 1$, where $R\{a, b\}$ is an RN of $\mathbb{T}_m$ having the minimum cardinality of λ . This shows that ρ is not a resolving function which is contradiction. Hence, κ is a minimal resolving function that achieves minimum $|\kappa|$ for $\mathbb{T}_m$. As a matter of fact R_r for $1 \leq r \leq m$ of $\mathbb{T}_m$ are pairwise overlapping; hence, $\exists$ another minimal resolving function $\bar{\kappa}$ of $\mathbb{T}_m$ such that $|\bar{\kappa}| \leq |\kappa|$. As a result, we arrive at the following: $\dim_f(\mathbb{T}_m) \leq \sum_{r=1}^m (1/\lambda) = (2m/(m+1))$.

5.2. Quadrangular Circular Ladder

Lemma 2. Let $\mathbb{N} \cong \mathbb{Q}_m^1$ be the 2-faced QCL with $m \geq 6$ and $m \equiv 0 \pmod{2}$. For $1 \leq l, r \leq m$, $p \geq 3$, $s \geq 2$ $p \equiv 1 \pmod{2}$ and $s \equiv 0 \pmod{2}$. Then,

- (a) $|R_l| = |R\{a_l^1, a_{l+1}^2\}| = |R_r| = |R\{a_r^1, a_{r-1}^2\}| = m$ and $|\cup_{l=1}^m R_l| \cup |\cup_{r=1}^m R_r| = 2m$.
- (b) For $p \geq 3$, $s \geq 2$, $|R_r| < |\bar{R}_1| = |R\{a_r^1, a_{r+s}^1\}| = |\bar{R}_2| = |R\{a_r^2, a_{r+s}^2\}| = |\bar{R}_3| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_4| = |R\{a_r^1, a_{r-p}^2\}| = 2(m-2)$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 2m$, where $1 \leq u \leq 4$.
- (c) For $p \geq 3$, $s \geq 0$, $|R_r| < |\bar{R}_5| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_6| = |R\{a_r^2, a_{r+p}^2\}| = |\bar{R}_7| = |R\{a_r^1, a_{r+s}^2\}| = |\bar{R}_8| = |R\{a_r^1, a_{r-s}^2\}| = 2m$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 2m$, where $5 \leq u \leq 8$.

Proof

- (a) The RNs of a_r^1, a_{r+1}^2 and a_r^1, a_{r-1}^2 are given by $R_l = R\{a_l^1, a_{l+1}^2\} = V(\mathbb{Q}_m^1) - \{a_h^1 | h \equiv l+1, l+2, \dots, l+(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv l, l-1, \dots, l-(m/2)+1 \pmod{m}\}$ and $R_r = R\{a_r^1, a_{r-1}^2\} = V(\mathbb{Q}_m^1) - \{a_h^1 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r, r-1, \dots, r+(m/2)-1 \pmod{m}\}$, respectively. We note

FIGURE 5: PCLs: (a) $\mathbb{P}_m^2$, (b) $\mathbb{P}_m^3$, and (c) $\mathbb{P}_m^4$. FIGURE 6: PCLs: (a) $\mathbb{P}_m^5$, (b) $\mathbb{P}_m^6$, and (c) $\mathbb{P}_m^7$.TABLE 1: $\bar{R}_u$ for $1 \leq u \leq 49$.

RNs	Elements	Equality
$R\{a_1^1, a_2^1\}$	$V(\mathbb{T}_6) - \{a_1^2, a_4^2\}$	$R\{a_4^1, a_5^1\}, R\{a_1^2, a_2^2\}$
$R\{a_2^1, a_3^1\}$	$V(\mathbb{T}_6) - \{a_2^2, a_5^2\}$	$R\{a_5^1, a_6^1\}, R\{a_2^2, a_3^2\}$
$R\{a_3^1, a_4^1\}$	$V(\mathbb{T}_6) - \{a_3^2, a_6^2\}$	$R\{a_1^1, a_6^1\}, R\{a_3^2, a_4^2\}$
$R\{a_1^1, a_3^1\}$	$V(\mathbb{T}_6) - \{a_1^2, a_5^1\}$	$R\{a_1^2, a_3^2\}$
$R\{a_1^1, a_4^1\}$	$V(\mathbb{T}_6) - \{a_1^2, a_6^1\}$	$R\{a_2^2, a_4^2\}$
$R\{a_3^1, a_5^1\}$	$V(\mathbb{T}_6) - \{a_1^2, a_4^1\}$	$R\{a_3^2, a_5^2\}$
$R\{a_4^1, a_6^1\}$	$V(\mathbb{T}_6) - \{a_2^2, a_5^1\}$	$R\{a_4^2, a_6^2\}$
$R\{a_1^1, a_5^1\}$	$V(\mathbb{T}_6) - \{a_1^2, a_6^1\}$	$R\{a_1^2, a_5^2\}$
$R\{a_1^1, a_6^1\}$	$V(\mathbb{T}_6) - \{a_2^2, a_6^2\}$	$R\{a_1^2, a_6^2\}$
$R\{a_2^1, a_5^1\}$	$V(\mathbb{T}_6) - \{a_3^2, a_6^2\}$	$R\{a_2^2, a_5^2\}$
$R\{a_3^1, a_6^1\}$	$V(\mathbb{T}_6) - \{a_1^2, a_4^1\}$	$R\{a_3^2, a_6^2\}$
$R\{a_1^1, a_6^1\}$	$V(\mathbb{T}_6) - \{a_4^2, a_6^2\}$	$R\{a_2^2, a_6^2\}$
$R\{a_1^1, a_2^2\}$	$V(\mathbb{T}_6) - \{a_1^1\} \cup \{a_4^2\}$	
$R\{a_2^1, a_3^2\}$	$V(\mathbb{T}_6) - \{a_3^1\} \cup \{a_5^2\}$	$R\{a_1^1, a_4^2\}$
$R\{a_3^1, a_4^2\}$	$V(\mathbb{T}_6) - \{a_4^1\} \cup \{a_6^2\}$	$R\{a_2^1, a_5^2\}$
$R\{a_4^1, a_5^2\}$	$V(\mathbb{T}_6) - \{a_5^1\} \cup \{a_1^2\}$	$R\{a_3^2, a_6^2\}$
$R\{a_5^1, a_6^2\}$	$V(\mathbb{T}_6) - \{a_6^1\} \cup \{a_2^2\}$	$R\{a_5^2, a_6^2\}$
$R\{a_6^1, a_1^2\}$	$V(\mathbb{T}_6) - \{a_1^1\} \cup \{a_3^2\}$	$R\{a_3^2, a_6^2\}$
$R\{a_1^1, a_2^2\}$	$V(\mathbb{T}_6) - \{a_5^1\} \cup \{a_4^2\}$	
$R\{a_2^1, a_3^2\}$	$V(\mathbb{T}_6) - \{a_6^1\} \cup \{a_5^2\}$	$R\{a_1^1, a_4^2\}$
$R\{a_3^1, a_4^2\}$	$V(\mathbb{T}_6) - \{a_1^1\} \cup \{a_6^2\}$	$R\{a_2^1, a_5^2\}$
$R\{a_4^1, a_5^2\}$	$V(\mathbb{T}_6) - \{a_2^1\} \cup \{a_1^2\}$	$R\{a_3^2, a_6^2\}$
$R\{a_1^1, a_6^2\}$	$V(\mathbb{T}_6) - \{a_3^1\} \cup \{a_2^2\}$	$R\{a_2^2, a_6^2\}$
$R\{a_2^1, a_1^2\}$	$V(\mathbb{T}_6) - \{a_4^1\} \cup \{a_3^2\}$	

that $\cup_{l=1}^m R_l = V(\mathbb{Q}_m^1)$, $\cup_{r=1}^m R_r = V(\mathbb{Q}_m^1)$ and $|(\cup_{l=1}^m R_l) \cup (\cup_{r=1}^m R_r)| = 2m$.

- (b) The RNs of $\bar{R}_u$ for $1 \leq u \leq 4$ are as follows: $\bar{R}_1 = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{Q}_m^1) - \{a_h^1 | h \equiv r - (s/2), r + (s/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r - (s/2), r + (s/2) \pmod{m}\} = \bar{R}_2 = R\{a_r^2, a_{r+s}^2\}$, $\bar{R}_3 = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{Q}_m^1) - \{a_h^1 | h \equiv r + ((p+1)/2), r + ((p+m-1)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((p-1)/2), r + ((m-p+1)/2) \pmod{m}\}$ and $\bar{R}_4 = R\{a_r^1, a_{r-p}^1\} = V(\mathbb{Q}_m^1) - \{a_h^1 | h \equiv$

$r+p, r - ((p+1)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((p+1)/2), r - ((p-1)/2) \pmod{m}\}$ respectively. Clearly, $|\bar{R}_u| = 2(m-2)$. Since $|R_r| = 2(m-2) < |\bar{R}_u|$, then $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 2(m-2) \geq |R_r|$, it can be easily seen that $\cup_{r=1}^m R_r = V(\mathbb{Q}_m^1)$ and $|\cup_{r=1}^m R_r| = 2m$.

- (c) The RNs of $a_r^1, a_{r+p}^1, a_r^2, a_{r+p}^2, a_r^1, a_{r+s}^1$ and a_r^1, a_{r-p}^1 are $\bar{R}_5 = R\{a_r^1, a_{r+p}^1\} = \bar{R}_6 = R\{a_r^2, a_{r+p}^2\} = \bar{R}_7 = R\{a_r^1, a_{r+s}^1\} = \bar{R}_8 = R\{a_r^1, a_{r-p}^1\} = V(\mathbb{G})$. We can see that $|\bar{R}_u| = 2m$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 2n \geq |R_r|$. $\square$

TABLE 2: RNs having minimum cardinality.

RNs	Elements
$R_1 = R\{a_1^1, a_1^2\}$	$V(\mathbb{T}_6) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_2^2, a_3^2\}$
$R_3 = R\{a_3^1, a_3^2\}$	$V(\mathbb{T}_6) - \{a_4^1, a_5^1, a_6^1\} \cup \{a_4^2, a_5^2\}$
$R_5 = R\{a_5^1, a_5^2\}$	$V(\mathbb{T}_6) - \{a_1^1, a_2^1, a_6^1\} \cup \{a_1^2, a_2^2\}$
$R_7 = R\{a_7^1, a_7^2\}$	$V(\mathbb{T}_6) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_5^2, a_6^2\}$
$R_9 = R\{a_9^1, a_9^2\}$	$V(\mathbb{T}_6) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_1^2, a_2^2\}$
$R_{11} = R\{a_{11}^1, a_{11}^2\}$	$V(\mathbb{T}_6) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_3^2, a_4^2\}$
$R_2 = R\{a_2^1, a_2^2\}$	$V(\mathbb{T}_6) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_3^2, a_4^2\}$
$R_4 = R\{a_4^1, a_4^2\}$	$V(\mathbb{T}_6) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_5^2, a_6^2\}$
$R_6 = R\{a_6^1, a_6^2\}$	$V(\mathbb{T}_6) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_1^2, a_2^2\}$
$R_8 = R\{a_8^1, a_8^2\}$	$V(\mathbb{T}_6) - \{a_1^1, a_2^1, a_6^1\} \cup \{a_1^2, a_2^2\}$
$R_{10} = R\{a_{10}^1, a_{10}^2\}$	$V(\mathbb{T}_6) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_2^2, a_3^2\}$
$R_{12} = R\{a_{12}^1, a_{12}^2\}$	$V(\mathbb{T}_6) - \{a_4^1, a_5^1, a_6^1\} \cup \{a_4^2, a_5^2\}$

Theorem 2. If $\mathbb{N} \cong \mathbb{Q}_m^1$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{Q}_m^1) \leq 2$.

Proof

Case 1. When $m = 6$.

The RNs are given as follows.

We can see that Tables 3 and 4 show the RNs with maximum cardinalities, that is, 12 and 8, respectively, whereas, Table 5 shows the RNs with minimum cardinality, that is, 6, and $\bigcup_{r=1}^{12} R_r = V(\mathbb{Q}_6^1)$ which implies $|\bigcup_{r=1}^{12} R_r| = 12$ and $|\bar{R}_u \cap \bigcup_{r=1}^{12} R_r| \geq |R_r|$, where $1 \leq u \leq 36$.

Here, we define a function $\phi: V(\mathbb{Q}_6^1) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = (1/6)$. Also R_r for $1 \leq r \leq 12$ of $\mathbb{Q}_6^1$ are pairwise overlapping; hence, $\exists$ another minimal resolving function $\bar{\kappa}$ of $\mathbb{Q}_6^1$ such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(\mathbb{Q}_6^1) \leq \sum_{r=1}^{12} (1/6) = 2$.

Case 2. When $m \geq 6$.

We have seen from Lemma 2 that the RNs with minimum cardinality of m are $R\{a_r^1, a_r^2\}$ and $R\{a_r^1, a_{r-1}^2\}$, respectively.

Let $\eta = |\bigcup_{r=1}^m R_r| = 2m$ and $\lambda = |R_r| = 2m$. Then, we define a function $\kappa: V(\mathbb{Q}_m^1) \rightarrow [0, 1]$ such that $\kappa(c) =$

$\begin{cases} 1/\lambda & \text{for } c \in \bigcup_{r=1}^m R_r, \\ 0 & \text{for } c \in V(\mathbb{Q}_m^1) - \bigcup_{r=1}^m R_r. \end{cases}$ We see that κ is a resolving function for $\mathbb{Q}_m^1$ because $\kappa(R\{a, b\}) \geq 1 \forall \{a, b\} \subset V(\mathbb{N})$. Now, suppose on contrary that there is another resolving function ρ , such that $\rho(u) \leq \phi(u)$, for at least one $u \in V(\mathbb{Q}_m^1)$ $\rho(u) \neq \kappa(u)$. As a result, $\rho(R\{a, b\}) < 1$, where $R\{a, b\}$ is an RN of $\mathbb{Q}_m^1$ having minimum cardinality λ . This shows that ρ is not a resolving function which is a contradiction to our supposition. Therefore, κ is a minimal resolving function that achieves minimum $|\kappa|$ for $\mathbb{Q}_m^1$. Since all the RNs of $\mathbb{Q}_m^1$ are equal, hence $\exists$ another minimal resolving function $\bar{\kappa}$ of $\mathbb{Q}_m^1$ such that $|\bar{\kappa}| \leq |\kappa|$. Assigning $(1/\lambda)$ to all the vertices of $\mathbb{Q}_m^1$ and calculating their sum, we get: $\dim_f(\mathbb{Q}_m^1) \leq \sum_{r=1}^\eta (1/m) = 2$. $\square$

TABLE 3: $\bar{R}_u$ for $1 \leq u \leq 21$.

Resolving neighbourhoods			Set/elements
$R\{a_1^2, a_2^2\}$	$R\{a_4^2, a_5^2\}$	$R\{a_1^2, a_4^2\}$	$V(\mathbb{Q}_6^1)$
$R\{a_2^2, a_3^2\}$	$R\{a_4^2, a_5^2\}$	$R\{a_1^2, a_5^2\}$	$V(\mathbb{Q}_6^1)$
$R\{a_3^2, a_4^2\}$	$R\{a_4^2, a_5^2\}$	$R\{a_2^2, a_5^2\}$	$V(\mathbb{Q}_6^1)$
$R\{a_3^2, a_6^2\}$	$R\{a_4^2, a_5^2\}$	$R\{a_2^2, a_6^2\}$	$V(\mathbb{Q}_6^1)$
$R\{a_1^1, a_3^1\}$	$R\{a_4^2, a_5^2\}$	$R\{a_5^1, a_6^1\}$	$V(\mathbb{Q}_6^1)$
$R\{a_1^1, a_6^1\}$	$R\{a_4^2, a_5^2\}$	$R\{a_2^1, a_4^1\}$	$V(\mathbb{Q}_6^1)$
$R\{a_2^1, a_6^1\}$	$R\{a_4^2, a_5^2\}$	$R\{a_1^1, a_5^1\}$	$V(\mathbb{Q}_6^1)$

Lemma 3. Let $\mathbb{N} \cong \mathbb{Q}_m^2$ be the 3-faced QCL with $m \geq 6$ and $m \equiv 0 \pmod{2}$. For $1 \leq l, r \leq m$, $p \geq 3$, $s \geq 2$ $p \equiv 1 \pmod{2}$ and $s \equiv 0 \pmod{2}$. Then

- (a) $|R_l| = |R\{a_l^1, a_{l-1}^2\}| = |R_r| = |R\{a_r^1, a_r^2\}| = m$ with $a_0^2 = a_m^2$, $|\bigcup_{l=1}^m R_l| = 2m$, $|\bigcup_{r=1}^m R_r| = 2m$ and $|\bigcup_{l=1}^m R_l \cup \bigcup_{r=1}^m R_r| = 2m$.
- (b) $|R_r| < |\bar{R}_1| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_2| = |R\{a_r^1, a_{r+s}^1\}| = |\bar{R}_3| = |R\{a_r^2, a_{r+s}^2\}| = |\bar{R}_4| = |R\{a_r^1, a_{r+p}^2\}| = |\bar{R}_5| = |R\{a_r^1, a_{r+s}^2\}| = |\bar{R}_6| = |R\{a_r^1, a_{r-p}^1\}| = |\bar{R}_7| = |R\{a_r^1, a_{r-s}^1\}|$ and $|\bar{R}_u \cap \bigcup_{r=1}^m R_r| \geq |R_r|$ with $a_{m+1}^1 = a_1^1$ and $1 \leq u \leq 8$.

Proof

- (a) The RNs of a_r^1, a_{r-1}^2 and a_r^1, a_r^2 are $R\{a_r^1, a_{r-1}^2\} = V(\mathbb{Q}_m^2) - \{a_h^1 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r, r+1, \dots, r+(m/2)-1 \pmod{m}\}$ and $R\{a_r^1, a_r^2\} = V(\mathbb{Q}_m^2) - \{a_h^1 | h \equiv r+1, r+2, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\}$ respectively.

We note that $\bigcup_{r=1}^m R_r = V(\mathbb{Q}_m^2)$ and $|\bigcup_{r=1}^m R_r| = 3m$.

- (b) The RNs of $\{a_r^1, a_{r+p}^1\}, \{a_r^1, a_{r+s}^1\}, \{a_r^2, a_{r+s}^2\}, \{a_r^1, a_{r+p}^2\}, \{a_r^1, a_{r+s}^2\}, \{a_r^1, a_{r-p}^1\}$ and $\{a_r^1, a_{r-s}^1\}$ are given by: $\bar{R}_1 = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{Q}_m^2) - \{a_h^1 | h \equiv r + ((p-1)/2), r + ((m+p-1)/2) \pmod{m}\}$, $\bar{R}_2 = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{Q}_m^2) - \{a_h^1 | h \equiv r + (s/2), r + ((m+s)/2) \pmod{m}\}$, $\bar{R}_3 = R\{a_r^2, a_{r+s}^2\} = V(\mathbb{Q}_m^2) - \{a_h^2 | h \equiv r + (s/2), r + ((m+s)/2) \pmod{m}\}$, $\bar{R}_4 = R\{a_r^1, a_{r+p}^2\} = V(\mathbb{Q}_m^2) - \{a_h^1 | h \equiv r + ((p+1)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r - ((p-1)/2) \pmod{m}\}$, $\bar{R}_5 = R\{a_r^1, a_{r+s}^2\} = V(\mathbb{Q}_m^2) - \{a_h^1 | h \equiv r + ((m+s)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((m+s)/2) \pmod{m}\}$, $\bar{R}_6 = R\{a_r^1, a_{r-p}^1\} = V(\mathbb{Q}_m^2) - \{a_h^1 | h \equiv r - ((p+1)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((m+s)/2) \pmod{m}\}$, $\bar{R}_7 = R\{a_r^1, a_{r-s}^1\} = V(\mathbb{Q}_m^2) - \{a_h^1 | h \equiv r - (s/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r - (s/2) \pmod{m}\}$, $\bar{R}_8 = R\{a_r^2, a_{r+p}^2\} = V(\mathbb{Q}_m^2) - \{a_h^2 | h \equiv r + ((p+1)/2), r + ((m+p+1)/2) \pmod{m}\}$. Clearly, $|\bar{R}_u| = 2(m-1)$. Since $|R\{a_r^1, a_{r-1}^2\}| = |R\{a_r^1, a_r^2\}| = n < |R\{a_r^1, a_{r+1}^1\}|$ then $|R\{a_r^1, a_{r+1}^1\} \cap \bigcup_{r=1}^m R_r| \geq |R_r|$. $\square$

Theorem 3. Let $\mathbb{N} \cong \mathbb{Q}_m^2$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{Q}_m^2) \leq 2$.

TABLE 4: $\bar{R}_u$ for $21 \leq u \leq 36$.

RNs	Elements	Equality
$R\{a_1^2, a_2^2\}$	$V(Q_6^1) - \{a_2^1, a_5^1\} \cup \{a_3^2, a_6^2\}$	$R\{a_1^1, a_3^1\}, R\{a_2^2, a_6^2\}, R\{a_4^1, a_6^1\}$
$R\{a_2^2, a_4^2\}$	$V(Q_6^1) - \{a_3^1, a_6^1\} \cup \{a_2^2, a_5^2\}$	$R\{a_2^1, a_5^1\}, R\{a_1^2, a_5^2\}, R\{a_1^1, a_5^1\}$
$R\{a_3^2, a_5^2\}$	$V(Q_6^1) - \{a_1^1, a_4^1\} \cup \{a_2^2, a_4^2\}$	$R\{a_3^1, a_5^1\}, R\{a_2^2, a_6^2\}, R\{a_2^1, a_6^1\}$
$R\{a_1^1, a_4^1\}$	$V(Q_6^1) - \{a_3^1, a_5^1\} \cup \{a_2^2, a_6^2\}$	
$R\{a_2^1, a_5^1\}$	$V(Q_6^1) - \{a_4^1, a_6^1\} \cup \{a_1^2, a_3^2\}$	
$R\{a_3^1, a_6^1\}$	$V(Q_6^1) - \{a_1^1, a_5^1\} \cup \{a_2^2, a_4^2\}$	

TABLE 5: RNs with minimum cardinality.

RNs	Elements
$R_1 = R\{a_1^1, a_2^2\}$	$V(Q_6^1) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_2^2, a_5^2, a_6^2\}$
$R_3 = R\{a_3^1, a_4^2\}$	$V(Q_6^1) - \{a_4^1, a_5^1, a_6^1\} \cup \{a_1^2, a_2^2, a_3^2\}$
$R_5 = R\{a_5^1, a_6^2\}$	$V(Q_6^1) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_2^2, a_4^2, a_5^2\}$
$R_7 = R\{a_2^1, a_1^2\}$	$V(Q_6^1) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_2^2, a_3^2, a_4^2\}$
$R_9 = R\{a_4^1, a_3^2\}$	$V(Q_6^1) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_2^2, a_3^2, a_5^2\}$
$R_{11} = R\{a_6^1, a_5^2\}$	$V(Q_6^1) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_1^2, a_2^2, a_6^2\}$
$R_2 = R\{a_2^1, a_3^2\}$	$V(Q_6^1) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_1^2, a_2^2, a_6^2\}$
$R_4 = R\{a_4^1, a_5^2\}$	$V(Q_6^1) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_2^2, a_3^2, a_4^2\}$
$R_6 = R\{a_6^1, a_5^2\}$	$V(Q_6^1) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_2^2, a_4^2, a_5^2\}$
$R_8 = R\{a_3^1, a_2^2\}$	$V(Q_6^1) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_2^2, a_4^2, a_5^2\}$
$R_{10} = R\{a_5^1, a_4^2\}$	$V(Q_6^1) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_1^2, a_5^2, a_6^2\}$
$R_{12} = R\{a_1^1, a_6^2\}$	$V(Q_6^1) - \{a_4^1, a_5^1, a_6^1\} \cup \{a_1^2, a_2^2, a_3^2\}$

TABLE 6: $\bar{R}_u$ for $1 \leq u \leq 52$.

RNs	Elements	Equality
$R\{a_1^2, a_2^2\}$	$V(Q_6^2) - \{a_2^1, a_5^1\}$	$R\{a_2^2, a_6^2\}$
$R\{a_2^2, a_3^2\}$	$V(Q_6^2) - \{a_3^1, a_6^1\}$	$R\{a_4^2, a_5^2\}$
$R\{a_3^2, a_4^2\}$	$V(Q_6^2) - \{a_1^1, a_4^1\}$	$R\{a_2^2, a_6^2\}$
$R\{a_1^2, a_5^2\}$	$V(Q_6^2) - \{a_2^1, a_5^1\}$	$R\{a_4^2, a_5^2\}$
$R\{a_2^2, a_6^2\}$	$V(Q_6^2) - \{a_2^1, a_5^1\}$	$R\{a_2^2, a_6^2\}$
$R\{a_3^2, a_5^2\}$	$V(Q_6^2) - \{a_2^1, a_4^1\}$	$R\{a_2^2, a_6^2\}$
$R\{a_1^2, a_4^2\}$	$V(Q_6^2) - \{a_3^1, a_6^1\}$	
$R\{a_2^2, a_5^2\}$	$V(Q_6^2) - \{a_1^1, a_4^1\}$	
$R\{a_3^2, a_6^2\}$	$V(Q_6^2) - \{a_2^1, a_5^1\}$	
$R\{a_1^1, a_2^1\}$	$V(Q_6^2) - \{a_2^1, a_4^1\}$	$R\{a_4^1, a_5^1\}$
$R\{a_2^1, a_3^1\}$	$V(Q_6^2) - \{a_2^1, a_5^1\}$	$R\{a_5^1, a_6^1\}$
$R\{a_3^1, a_4^1\}$	$V(Q_6^2) - \{a_2^1, a_6^1\}$	$R\{a_1^1, a_6^1\}$
$R\{a_1^1, a_3^1\}$	$V(Q_6^2) - \{a_2^1, a_5^1\} \cup \{a_2^2, a_6^2\}$	$R\{a_2^1, a_3^1\}$
$R\{a_2^1, a_4^1\}$	$V(Q_6^2) - \{a_3^1, a_6^1\} \cup \{a_2^2, a_6^2\}$	$R\{a_2^1, a_4^1\}$
$R\{a_3^1, a_5^1\}$	$V(Q_6^2) - \{a_1^1, a_4^1\} \cup \{a_2^2, a_4^2\}$	$R\{a_2^1, a_5^1\}$
$R\{a_4^1, a_6^1\}$	$V(Q_6^2) - \{a_2^1, a_5^1\} \cup \{a_2^2, a_6^2\}$	$R\{a_2^1, a_6^1\}$
$R\{a_1^1, a_5^1\}$	$V(Q_6^2) - \{a_3^1, a_6^1\} \cup \{a_2^2, a_6^2\}$	$R\{a_1^1, a_5^1\}$
$R\{a_2^1, a_6^1\}$	$V(Q_6^2) - \{a_1^1, a_4^1\} \cup \{a_2^2, a_4^2\}$	$R\{a_2^1, a_6^1\}$
$R\{a_1^1, a_4^1\}$	$V(Q_6^2) - \{a_2^1, a_5^1\}$	
$R\{a_2^1, a_5^1\}$	$V(Q_6^2) - \{a_2^1, a_6^1\}$	
$R\{a_3^1, a_6^1\}$	$V(Q_6^2) - \{a_2^1, a_5^1\}$	
$R\{a_1^1, a_2^1\}$	$V(Q_6^2) - \{a_2^1\} \cup \{a_2^2\}$	
$R\{a_2^1, a_3^1\}$	$V(Q_6^2) - \{a_2^1\} \cup \{a_2^2\}$	
$R\{a_3^1, a_4^1\}$	$V(Q_6^2) - \{a_4^1\} \cup \{a_3^2\}$	
$R\{a_4^1, a_5^1\}$	$V(Q_6^2) - \{a_5^1\} \cup \{a_4^2\}$	
$R\{a_5^1, a_6^1\}$	$V(Q_6^2) - \{a_6^1\} \cup \{a_5^2\}$	
$R\{a_6^1, a_1^1\}$	$V(Q_6^2) - \{a_1^1\} \cup \{a_6^2\}$	
$R\{a_1^1, a_2^1\}$	$V(Q_6^2) - \{a_5^1\} \cup \{a_5^2\}$	
$R\{a_2^1, a_3^1\}$	$V(Q_6^2) - \{a_6^1\} \cup \{a_6^2\}$	
$R\{a_3^1, a_4^1\}$	$V(Q_6^2) - \{a_1^1\} \cup \{a_1^2\}$	
$R\{a_4^1, a_5^1\}$	$V(Q_6^2) - \{a_2^1\} \cup \{a_2^2\}$	
$R\{a_5^1, a_6^1\}$	$V(Q_6^2) - \{a_3^1\} \cup \{a_3^2\}$	
$R\{a_1^1, a_6^1\}$	$V(Q_6^2) - \{a_4^1\} \cup \{a_4^2\}$	
$R\{a_2^1, a_5^1\}$	$V(Q_6^2) - \{a_5^1\} \cup \{a_5^2\}$	
$R\{a_3^1, a_6^1\}$	$V(Q_6^2) - \{a_6^1\} \cup \{a_6^2\}$	
$R\{a_4^1, a_1^1\}$	$V(Q_6^2) - \{a_1^1\} \cup \{a_1^2\}$	
$R\{a_5^1, a_2^1\}$	$V(Q_6^2) - \{a_2^1\} \cup \{a_2^2\}$	
$R\{a_6^1, a_3^1\}$	$V(Q_6^2) - \{a_3^1\} \cup \{a_3^2\}$	

Proof

Case 1. When $m = 6$.

The RNs are given as follows.

Tables 6 and 7 given above shows the RNs with cardinalities of 10 and 6 respectively. Clearly, the table shows the RNs bearing the minimum cardinality of 6 and $\cup_{r=1}^{12} R_r = V(Q_6^2)$ which implies $|\cup_{r=1}^{12} R_r| = 12$ and $|\bar{R}_u \cap \cup_{r=1}^{12} R_r| \geq |R_r|$, where $1 \leq u \leq 52$.

Now, we define a function $\kappa: V(Q_6^2) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = (1/6)$. Moreover, R_r for $1 \leq r \leq 12$ of Q_6^2 are pairwise overlapping; thus, $\exists$ is another minimal resolving function $\bar{\kappa}$ of Q_6^2 such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(Q_6^2) \leq \sum_{r=1}^{12} (1/6) = 2$.

Case 2. $m \geq 8$

It is clear from Lemma 3 that the RNs bearing the minimum cardinality of m in Q_m^2 are $R_l = R\{a_l^1, a_l^2\}$ and $R_r = R\{a_r^1, a_{r-1}^2\}$ respectively, with $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 2n \geq |R_r|$.

Let $\eta = |\cup_{r=1}^m R_r| = 2m$ and $\lambda = |R_r| = m$. Then we define a mapping $\kappa: V(Q_m^2) \rightarrow [0, 1]$ such that $\kappa(c) =$

$$\begin{cases} 1/\lambda & \text{for } c \in \cup_{r=1}^m R_r, \\ 0 & \text{for } c \in V(Q_m^2) - \cup_{r=1}^m R_r. \end{cases}$$

We see that κ is a resolving function for Q_m^1 because $\kappa(R\{a, b\}) \geq 1 \forall \{a, b\} \subset V(N)$. Now suppose on contrary that there is another resolving function ρ , such that $\rho(u) \leq \phi(u)$, for at least one $u \in V(Q_m^1)$ $\rho(u) \neq \kappa(u)$. As a result, $\rho(R\{a, b\}) < 1$, where $R\{a, b\}$ is a RN of

Q_m^1 having minimum cardinality λ . This shows that ρ is not a resolving function which is a contradiction to our supposition. Therefore, κ is a minimal resolving function that achieves minimum $|\kappa|$ for Q_m^1 . Since all the RNs of Q_m^2 are equal, hence $\exists$ another minimal resolving function $\bar{\kappa}$ of Q_m^2 such that $|\bar{\kappa}| \leq |\kappa|$. Assigning $(1/\lambda)$ to all the vertices of Q_m^2 and calculating their sum, we get $\dim_f(Q_m^2) \leq \sum_{r=1}^{\eta} (1/m) = 2$. $\square$

TABLE 7: R_r with minimum cardinalities for $1 \leq r \leq 12$.

RNs	Elements
$R_1 = R\{a_1^1, a_2^2\}$	$V(\mathbb{Q}_6^2) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_2^2, a_3^2, a_4^2\}$
$R_3 = R\{a_3^1, a_4^2\}$	$V(\mathbb{Q}_6^2) - \{a_4^1, a_5^1, a_6^1\} \cup \{a_4^2, a_5^2, a_6^2\}$
$R_5 = R\{a_5^1, a_6^2\}$	$V(\mathbb{Q}_6^2) - \{a_6^1, a_1^1, a_2^1\} \cup \{a_6^2, a_1^2, a_2^2\}$
$R_7 = R\{a_1^2, a_2^3\}$	$V(\mathbb{Q}_6^2) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_2^2, a_3^2, a_4^2\}$
$R_9 = R\{a_1^3, a_2^4\}$	$V(\mathbb{Q}_6^2) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_4^2, a_5^2, a_6^2\}$
$R_{11} = R\{a_3^2, a_4^3\}$	$V(\mathbb{Q}_6^2) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_2^2, a_3^2, a_4^2\}$
$R_2 = R\{a_1^2, a_2^3\}$	$V(\mathbb{Q}_6^2) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_2^2, a_3^2, a_4^2\}$
$R_4 = R\{a_1^4, a_2^5\}$	$V(\mathbb{Q}_6^2) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_2^2, a_3^2, a_4^2\}$
$R_6 = R\{a_1^5, a_2^6\}$	$V(\mathbb{Q}_6^2) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_4^2, a_5^2, a_6^2\}$
$R_8 = R\{a_3^5, a_4^6\}$	$V(\mathbb{Q}_6^2) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_4^2, a_5^2, a_6^2\}$
$R_{10} = R\{a_1^6, a_2^7\}$	$V(\mathbb{Q}_6^2) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_4^2, a_5^2, a_6^2\}$
$R_{12} = R\{a_1^7, a_2^8\}$	$V(\mathbb{Q}_6^2) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_4^2, a_5^2, a_6^2\}$

Corollary 1. Let $\aleph \cong \mathbb{Q}_m^3$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{Q}_m^3) \leq 2$.

Proof. The result is evident as $\mathbb{Q}_m^3 \cong \mathbb{Q}_m^2$. $\square$

5.3. Pentagonal Circular Ladders

Lemma 4. Let $\aleph \cong \mathbb{P}_m^1$ be the 2-faced QCL with $m \geq 10$ and $m \equiv 0 \pmod{2}$. For $1 \leq l, r \leq m$, $p \geq 3$, $s \geq 2$, $p \equiv 1 \pmod{2}$ and $s \equiv 0 \pmod{2}$. Then,

- $|R_l| = |R\{a_l^1, a_{l-2}^3\}| = |R_r| = |R\{a_r^1, a_{r+1}^3\}| = (5m/2) - 2$ with $a_{m+1}^3 = a_1^3$, $|\cup_{l=1}^m R_l| = |\cup_{r=1}^m R_r| = 3m$ and $|\cup_{l=1}^m R_l \cup \cup_{r=1}^m R_r| = 3m$.
- $|R_r| < |\bar{R}_1| = |R\{a_r^2, a_r^3\}| = |\bar{R}_2| = |R\{a_r^2, a_{r-1}^3\}| = (5m/2)$ with $a_0^3 = a_m^3$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_3| = |R\{a_r^1, a_{r+1}^1\}| = |\bar{R}_4| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_5| = |R\{a_r^2, a_{r+1}^2\}| = |\bar{R}_6| = |R\{a_r^2, a_{r+p}^2\}| = |\bar{R}_7| = |R\{a_r^3, a_{r+s}^3\}| = |\bar{R}_8| = |R\{a_r^1, a_{r+s}^2\}| = 3m - 2$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$ with $a_{m+1}^1 = a_1^1$.
- $|R_r| < |\bar{R}_9| = |R\{a_r^1, a_r^2\}| = |\bar{R}_{10}| = |R\{a_r^1, a_{r+s}^1\}| = |\bar{R}_{11}| = |R\{a_r^3, a_{r+1}^3\}| = |\bar{R}_{12}| = |R\{a_r^3, a_{r+p}^3\}| = |\bar{R}_{13}| = |R\{a_r^2, a_{r+s}^2\}| = |\bar{R}_{14}| = |R\{a_r^1, a_r^3\}|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_{15}| = |R\{a_r^1, a_{r+1}^2\}| = |\bar{R}_{16}| = |R\{a_r^1, a_{r-1}^2\}|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.

Proof

- The RNs of $\{a_r^1, a_{r+1}^3\}$ and $\{a_1^1, a_{l-2}^3\}$ are $R_l = R\{a_r^1, a_{r+1}^3\} = V(\mathbb{P}_m^1) - \{a_h^1 | h \equiv r+2, r+3, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r, r+4 \pmod{m}\}$ and $R_2 = R\{a_1^1, a_{l-2}^3\} = V(\mathbb{P}_m^1) - V(\mathbb{P}_m^1) - \{a_h^1 | h \equiv r+2, r+3, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r, r-5 \pmod{m}\}$, respectively. We can see that $|R_r| = (5m/2) - 2$ and $|\cup_{r=1}^m R_r| = |\cup_{l=1}^m R_l| = V(\mathbb{P}_m^1)$ and $|\cup_{l=1}^m R_l \cup \cup_{r=1}^m R_r| = 3m$.
- The RNs of $\{a_r^2, a_{r-1}^3\}$ and $\{a_r^2, a_r^3\}$ are $\bar{R}_1 = R\{a_r^2, a_{r-1}^3\} = V(\mathbb{P}_m^1) - \{a_h^1 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\}$ and $\bar{R}_1 = R\{a_r^2, a_r^3\} = V(\aleph) -$

$\{a_h^1 | h \equiv r+1, r+2, \dots, r+(m/2) \pmod{m}\} \pmod{m}$, respectively.

Since for $1 \leq u \leq 2$, $|\bar{R}_u| > |R_r| = (5m/2) - 1$, thus $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 3m$.

- The RNs of $\{a_r^1, a_{r+1}^1\}$, $\{a_r^1, a_{r+p}^1\}$, $\{a_r^2, a_{r+1}^2\}$, $\{a_r^2, a_{r+p}^2\}$, $\{a_r^3, a_{r+s}^3\}$ and $\{a_r^1, a_{r+s}^2\}$ are $\bar{R}_3 = R\{a_r^1, a_{r+1}^1\} = V(\mathbb{P}_m^1) - \{a_h^3 | h \equiv r, r+(m/2) \pmod{m}\} = \bar{R}_5 = R\{a_r^2, a_{r+1}^2\}$, $\bar{R}_4 = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{P}_m^1) - \{a_h^3 | h \equiv r + ((p-1)/2), r + ((p+m-1)/2) \pmod{m}\} = \bar{R}_6 = R\{a_r^2, a_{r+p}^2\}$, $\bar{R}_7 = R\{a_r^3, a_{r+s}^3\} = V(\mathbb{P}_m^1) - \{a_h^3 | h \equiv r + (s/2), r + ((m+s)/2) \pmod{m}\}$ and $\bar{R}_8 = R\{a_r^1, a_{r+s}^2\} = V(\mathbb{P}_m^1) - \{a_h^1 | h \equiv r + (s/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r + ((m+s-2)/2) \pmod{m}\}$, respectively.

Clearly, for $3 \leq u \leq 8$, $|\bar{R}_u| = 3m - 2$. Since $|R\{a_r^2, a_r^3\}| = (5m/2) < |R\{a_r^1, a_{r+1}^1\}|$, then $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 3m - 2 \geq |R_r|$.

- The RN of $\{a_r^1, a_r^2\}$ is $\bar{R}_9 = R\{a_r^1, a_r^2\} = V(\mathbb{P}_m^1) - \{a_h^2 | h \equiv r-1, r+1 \pmod{m}\} \cup \{a_h^3 | h \equiv r-2, r+1 \pmod{m}\}$. $\bar{R}_{10} = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{P}_m^1) - \{a_h^1 | h \equiv r + (s/2)r + ((m+s)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + (s/2), r + ((m+s)/2) \pmod{m}\}$, $\bar{R}_{11} = R\{a_r^3, a_{r+1}^3\} = \{a_r^1, a_{r+2}^1\}$, $\bar{R}_{12} = R\{a_r^3, a_{r+p}^3\} = \bar{R}_{13} = R\{a_r^2, a_{r+s}^2\} = R\{a_r^1, a_{r+s}^3\}$, $\bar{R}_{14} = R\{a_r^1, a_r^3\} = V(\mathbb{P}_m^1) - \{a_h^2 | h \equiv r, r+2 \pmod{m}\} \cup \{a_h^3 | h \equiv r-1, r+2 \pmod{m}\}$.

Clearly, for $9 \leq u \leq 14$, $|R_r| = (5m/2) < |\bar{R}_u|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.

- The RNs of $\{a_r^1, a_{r+1}^2\}$ and $\{a_r^1, a_{r-1}^2\}$ are $\bar{R}_{15} = R\{a_r^1, a_{r+1}^2\} = V(\mathbb{P}_m^1) - \{a_h^1 | h \equiv r+1, r+2, \dots, r+(m/2) \pmod{5}\} \cup \{a_h^2 | h \equiv r+(m/2) \pmod{m}\}$ and $\bar{R}_{16} = R\{a_r^1, a_{r-1}^2\} = V(\mathbb{P}_m^1) - \{a_h^1 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r-(m/2) \pmod{m}\}$, respectively. We can see that $|R_r| = (5m/2)$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$. $\square$

Theorem 4. If $\aleph \cong \mathbb{P}_m^1$ with $m \geq 8$ and $m \equiv 0 \pmod{2}$. Then,

$$\dim_f(\mathbb{P}_m^1) \leq \begin{cases} 6m/(5m-2) & \text{for } 6 \leq m \leq 8, \\ 6m/(5m-4) & \text{for } m \geq 10 \end{cases}.$$

Proof

Case 1. (i) When $m = 6$.

The RNs are given as follows.

We can see that Tables 8 and 9 show the RNs with maximum cardinality of 16 and 15, respectively, whereas both Tables 10 and 11 show the RNs with minimum cardinality of 14. Moreover, $|\cup_{r=1}^{45} R_r| = V(\mathbb{P}_6^1)$ which implies $|\cup_{r=1}^{45} R_r| = 18$ and $|\bar{R}_r \cap \cup_{r=1}^{24} R_r| \geq |R_r|$.

Now, we define a mapping $\kappa: V(\mathbb{P}_6^1) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = \kappa(a_r^3) = (1/14)$. It is observed that R_r for $1 \leq r \leq 45$ of $\mathbb{P}_6^1$ are pairwise overlapping; hence, $\exists$ another minimal resolving function $\bar{\kappa}$ of $\mathbb{P}_6^1$

TABLE 8: $\bar{R}_u$ for $1 \leq u \leq 23$.

RNs	Elements	Equality
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_6^1) - \{a_1^3, a_4^3\}$	$R\{a_1^1, a_5^1\}, R\{a_2^1, a_5^1\}, R\{a_4^1, a_5^1\}, R\{a_1^2, a_6^2\}, R\{a_3^2, a_6^2\}$
$R\{a_1^1, a_3^1\}$	$V(\mathbb{P}_6^1) - \{a_2^3, a_3^3\}$	$R\{a_5^1, a_6^1\}, R\{a_2^2, a_3^2\}, R\{a_5^1, a_6^1\}, R\{a_1^1, a_4^1\}, R\{a_2^1, a_4^1\}, R\{a_1^1, a_4^1\}, R\{a_2^1, a_4^1\}, R\{a_1^3, a_3^3\}, R\{a_5^2, a_6^2\}$
$R\{a_1^1, a_4^1\}$	$V(\mathbb{P}_6^1) - \{a_3^3, a_6^3\}$	$R\{a_1^1, a_6^1\}, R\{a_3^2, a_4^2\}, R\{a_2^1, a_6^2\}, R\{a_1^1, a_5^1\}, R\{a_2^1, a_5^1\}, R\{a_1^1, a_6^1\}$

TABLE 9: $\bar{R}_u$ for $24 \leq u \leq 35$.

RNs	Elements
$R\{a_1^2, a_3^3\}$	$V(\mathbb{P}_6^1) - \{a_2^1, a_3^1, a_4^1\}$
$R\{a_2^2, a_3^3\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_4^1, a_5^1\}$
$R\{a_3^2, a_3^3\}$	$V(\mathbb{P}_6^1) - \{a_4^1, a_5^1, a_6^1\}$
$R\{a_4^2, a_4^3\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_5^1, a_6^1\}$
$R\{a_5^2, a_5^3\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_2^1, a_6^1\}$
$R\{a_6^2, a_6^3\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_2^1, a_3^1\}$
$R\{a_2^2, a_1^3\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_5^1, a_6^1\}$
$R\{a_3^2, a_2^3\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_2^1, a_6^1\}$
$R\{a_4^2, a_3^3\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_2^1, a_3^1\}$
$R\{a_5^2, a_4^3\}$	$V(\mathbb{P}_6^1) - \{a_2^1, a_3^1, a_4^1\}$
$R\{a_6^2, a_5^3\}$	$V(\mathbb{P}_6^1) - \{a_3^1, a_4^1, a_5^1\}$
$R\{a_2^2, a_6^3\}$	$V(\mathbb{P}_6^1) - \{a_4^1, a_5^1, a_6^1\}$

such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(\mathbb{P}_6^1) \leq \sum_{r=1}^{16} (1/14) = (18/14) = (9/7)$.

(ii) When $m = 8$.

The RNs are given as follows.

We can see that Tables 12–14 shows the RNs with maximum cardinalities that is 22 and 20, respectively, whereas Table 15 shows the RNs with minimum cardinality of 19. Moreover, $\cup_{r=1}^{16} R_r = V(\mathbb{P}_8^1)$ which implies $|\cup_{r=1}^{16} R_r| = 19$ and $|\bar{R}_r \cap \cup_{r=1}^{24} R_r| \geq |R_r|$.

Now, we define a mapping $\kappa: V(\mathbb{P}_8^1) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = \kappa(a_r^3) = (1/19)$. It is observed that R_r for $1 \leq r \leq 16$ of $\mathbb{P}_8^1$ are pairwise overlapping; hence, $\exists$ another minimal resolving function $\bar{\kappa}$ of $\mathbb{P}_8^1$ such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(\mathbb{P}_8^1) \leq \sum_{r=1}^{16} (1/19) = (24/19)$.

(iii) For $6 \leq m \leq 8$.

We have seen that $\dim_f(\mathbb{P}_6^1) \leq (9/7)$ whereas $\dim_f(\mathbb{P}_8^1) \leq (24/19)$. In generic form, we can write it as $\dim_f \leq (6m/(5m-2))$.

Case 2. When $m = 10$.

The RNs are given as follows.

We can see that Tables 16–18 show the RNs with maximum cardinalities, that is, 28 and 26 and 25, respectively, whereas Table 19 shows the RNs with minimum cardinality of 23. Moreover, $\cup_{r=1}^{16} R_r = V(\mathbb{P}_{10}^1)$ which implies $|\cup_{r=1}^{16} R_r| = 23$ and $|\bar{R}_r \cap \cup_{r=1}^{24} R_r| \geq |R_r|$.

Now, we define a mapping $\kappa: V(\mathbb{P}_{10}^1) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = \kappa(a_r^3) = (1/23)$. It is observed that R_r for $1 \leq r \leq 16$ of $\mathbb{P}_{10}^1$ are pairwise

overlapping; hence, $\exists$ another minimal resolving function $\bar{\kappa}$ of $\mathbb{P}_{10}^1$ such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(\mathbb{P}_{10}^1) \leq \sum_{r=1}^{23} (1/19) \leq (30/23)$.

Case 3. $m \geq 12$.

From Lemma 4, we can see that the RNs with minimum cardinality of $(5m/2) - 2$ are $R\{a_r^1, a_{r+1}^3\}$ and $R\{a_r^1, a_{r-2}^3\}$, respectively. Moreover, for $1 \leq u \leq 15$, $|R_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.

Let $\eta = |\cup_{r=1}^m R_r| = 3m$ and $\lambda = |R_r| = (5m/2) - 2$. Then, we define a mapping $\kappa: V(\mathbb{P}_m^1) \rightarrow [0, 1]$

such that $\kappa(c) = \begin{cases} 1/\lambda & \text{for } c \in \cup_{r=1}^m R_r, \\ 0 & \text{for } c \in V(\mathbb{P}_m^1) - \cup_{r=1}^m R_r. \end{cases}$

We note that κ is a resolving function for $\mathbb{P}_m^1$ because $\kappa(R\{a, b\}) \geq 1 \forall \{a, b\} \in V(\mathbb{P}_m^1)$. Now, assume that there is another resolving function ρ , such that $\rho(u) \leq \kappa(u)$, for at least one $u \in V(\mathbb{P}_m^1)$ $\rho(u) \neq \kappa(u)$. As a result, $\rho(R\{a, b\}) < 1$, where $R\{a, b\}$ is a RN of $\mathbb{P}_m^1$ bearing minimum cardinality λ . This shows that ρ is not a resolving function a contradiction. Therefore, κ is a minimal resolving function for $\mathbb{P}_m^1$ that achieves minimum $|\kappa|$. As a matter of fact R_r for $1 \leq t \leq 2m$ of $\mathbb{P}_m^1$ are pairwise overlapping; hence, $\exists$ another minimal resolving function $\bar{\kappa}$ of $\mathbb{P}_m^1$ such that $|\bar{\kappa}| \leq |\kappa|$. Therefore, assigning $(1/\lambda)$ to all the vertices of $\mathbb{P}_m^1$ and calculating their sum, we get: $\dim_f(\mathbb{P}_m^1) \leq \sum_{r=1}^{\eta} 1/\lambda = (3m/((5m/2) - 2)) = (6m/(5m-4))$. $\square$

Lemma 5. Let $\aleph \cong \mathbb{P}_m^2$ be the 3-faced QCL with $m \geq 6$ and $m \equiv 0 \pmod{2}$. For $1 \leq l, r \leq m$, $p \geq 3$, $s \geq 2$ $p \equiv 1 \pmod{2}$ and $s \equiv 0 \pmod{2}$. Then,

- $|R_l| = |R\{a_l^2, a_l^3\}| = |R_r| = |R\{a_r^2, a_{r-1}^3\}| = (3m/2) + 1$, $|\cup_{l=1}^m R_l|$, $|\cup_{r=1}^m R_r| = 3m$ and $|\{(\cup_{l=1}^m R_l) \cup (\cup_{r=1}^m R_r)\}| = 3m$.
- $|R_r| < |\bar{R}_1| = |R\{a_r^1, a_{r+1}^3\}| = |\bar{R}_2| = |R\{a_r^1, a_{r-1}^3\}|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_3| = |R\{a_r^1, a_{r+1}^1\}| = |\bar{R}_4| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_5| = |R\{a_r^2, a_{r+1}^2\}| = |\bar{R}_6| = |R\{a_r^2, a_{r+p}^2\}| = |\bar{R}_7| = |R\{a_r^3, a_{r+s}^3\}| = |\bar{R}_8| = |R\{a_r^1, a_{r+s}^2\}| = |\bar{R}_9| = |R\{a_r^1, a_{r+s}^3\}| = |\bar{R}_{10}| = |R\{a_r^1, a_{r+p}^3\}|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$ with $a_{m+1}^1 = a_1^1$ and $a_{m+1}^2 = a_1^2$.
- $|R_r| < |\bar{R}_{11}| = |R\{a_r^2, a_{r+p}^3\}| = |\bar{R}_{12}| = |R\{a_r^2, a_{r+s}^3\}|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_{13}| = |R\{a_r^1, a_r^2\}|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.

TABLE 10: R_r for $1 \leq r \leq 27$.

RNs	Elements	Equality
$R\{a_1^1, a_3^1\}$	$V(\mathbb{P}_6^1) - \{a_2^1, a_5^1\} \cup \{a_2^2, a_5^2\}$	$R\{a_1^1, a_6^1\}, R\{a_2^1, a_5^1\}, R\{a_2^2, a_5^2\}, R\{a_1^3, a_3^3\}, R\{a_4^3, a_5^3\}, R\{a_3^3, a_6^3\}$
$R\{a_2^1, a_4^1\}$	$V(\mathbb{P}_6^1) - \{a_3^1, a_6^1\} \cup \{a_3^2, a_6^2\}$	$R\{a_1^1, a_5^1\}, R\{a_2^1, a_4^1\}, R\{a_2^2, a_5^2\}, R\{a_1^1, a_5^1\}, R\{a_2^1, a_5^1\}, R\{a_3^3, a_3^3\}, R\{a_5^3, a_6^3\}, R\{a_1^3, a_4^3\}, R\{a_3^3, a_6^3\}$
$R\{a_3^1, a_5^1\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_4^1\} \cup \{a_1^2, a_4^2\}$	$R\{a_2^1, a_6^1\}, R\{a_3^2, a_5^2\}, R\{a_2^2, a_6^2\}, R\{a_1^1, a_6^1\}, R\{a_2^1, a_6^1\}, R\{a_3^3, a_3^3\}, R\{a_1^3, a_6^3\}, R\{a_3^3, a_5^3\}, R\{a_2^3, a_5^3\}$

TABLE 11: R_r for $28 \leq r \leq 45$.

RNs	Elements
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_6^1) - \{a_2^2, a_6^2\} \cup \{a_2^3, a_5^3\}$
$R\{a_3^1, a_5^1\}$	$V(\mathbb{P}_6^1) - \{a_2^2, a_6^2\} \cup \{a_3^3, a_5^3\}$
$R\{a_1^1, a_5^1\}$	$V(\mathbb{P}_6^1) - \{a_2^2, a_6^2\} \cup \{a_3^3, a_6^3\}$
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_6^1) - \{a_3^1, a_4^1\} \cup \{a_1^3, a_5^3\}$
$R\{a_3^1, a_5^1\}$	$V(\mathbb{P}_6^1) - \{a_5^1, a_6^1\} \cup \{a_3^3, a_5^3\}$
$R\{a_5^1, a_6^1\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_2^1\} \cup \{a_2^3, a_5^3\}$
$R\{a_2^1, a_6^1\}$	$V(\mathbb{P}_6^1) - \{a_5^1, a_6^1\} \cup \{a_3^3, a_4^3\}$
$R\{a_4^1, a_2^1\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_2^1\} \cup \{a_3^3, a_6^3\}$
$R\{a_6^1, a_3^1\}$	$V(\mathbb{P}_6^1) - \{a_3^1, a_4^1\} \cup \{a_2^3, a_5^3\}$
$R\{a_2^1, a_2^1\}$	$V(\mathbb{P}_6^1) - \{a_2^1, a_3^1\} \cup \{a_3^3, a_5^3\}$
$R\{a_4^1, a_4^1\}$	$V(\mathbb{P}_6^1) - \{a_2^1, a_5^1\} \cup \{a_2^3, a_5^3\}$
$R\{a_6^1, a_6^1\}$	$V(\mathbb{P}_6^1) - \{a_2^1, a_5^1\} \cup \{a_1^3, a_4^3\}$
$R\{a_2^1, a_3^1\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_5^1\} \cup \{a_2^3, a_5^3\}$
$R\{a_4^1, a_5^1\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_6^1\} \cup \{a_3^3, a_4^3\}$
$R\{a_1^1, a_6^1\}$	$V(\mathbb{P}_6^1) - \{a_2^1, a_3^1\} \cup \{a_3^3, a_6^3\}$
$R\{a_3^1, a_3^1\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_6^1\} \cup \{a_3^3, a_5^3\}$
$R\{a_5^1, a_3^1\}$	$V(\mathbb{P}_6^1) - \{a_2^1, a_3^1\} \cup \{a_3^3, a_4^3\}$
$R\{a_1^1, a_5^1\}$	$V(\mathbb{P}_6^1) - \{a_4^1, a_5^1\} \cup \{a_3^3, a_6^3\}$

Proof

- (a) The RNs of $a_r^2 a_r^3$ and $a_r^2 a_{r-1}^3$ are $R\{a_r^2 a_r^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r+1, r+2, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r+1, r+2, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r+1, r+2, \dots, r+(m/2)-1 \pmod{m}\}$ and $R\{a_r^2 a_{r-1}^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r-2, r-3, \dots, r-(m/2)+1 \pmod{m}\}$. We note that $\cup_{r=1}^m R_r = V(\mathbb{P}_m^2)$ and $|\cup_{r=1}^m R_r| = 3m$.
- (b) The RNs of $\{a_r^1, a_{r-1}^3\}$ and $\{a_r^1, a_{r-1}^3\}$ are $\bar{R}_1 = R\{a_r^1, a_{r+1}^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r+2, r+3, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r, r+(m/2) \pmod{m}\}$ and $\bar{R}_2 = R\{a_r^1, a_{r-1}^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r-2, r-3, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r \pmod{m}\}$ respectively. It is clear from the above that $|R_r| < |\bar{R}_u|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- (c) The RNs of $\{a_r^1, a_{r+1}^1\}, \{a_r^1, a_{r+p}^1\}, \{a_r^2, a_{r+1}^2\}, \{a_r^2, a_{r+p}^2\}, \{a_r^3, a_{r+s}^3\}, \{a_r^1, a_{r+s}^2\}, \{a_r^1, a_{r+s}^3\}$ and $\{a_r^1, a_{r+p}^3\}$ are $\bar{R}_3 = R\{a_r^1, a_{r+1}^1\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r, r+(m/2) \pmod{m}\} = \bar{R}_5 = R\{a_r^2, a_{r+1}^2\}, \bar{R}_4 = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r+((p-1)/2), r+((p+m-1)/2) \pmod{m}\} = \bar{R}_6 = R\{a_r^2, a_{r+p}^2\}, \bar{R}_7 = R\{a_r^3, a_{r+s}^3\} = V(\mathbb{P}_m^2) - \{a_h^3 | h \equiv r+(s/2), r+((m+s)/2) \pmod{m}\}, \bar{R}_8 = R\{a_r^1, a_{r+s}^2\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r+(s/2), r+((s+m)/2) \pmod{m}\}, \bar{R}_9 = R\{a_r^1, a_{r+s}^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r+((s+2)/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r+((s+m)/2) \pmod{m}\}, \bar{R}_{10} = R\{a_r^1, a_{r+p}^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r+((p+m+1)/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r+((p-1)/2) \pmod{m}\}. Clearly, $|\bar{R}_u| = 3m-2$. Since $|R_r| = (3m/2)+1 < |\bar{R}_u|$, then $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 3m-2 \geq |R_r|$.$

- (d) The RNs of $\{a_r^2, a_{r+p}^3\}$ and $\{a_r^2, a_{r+s}^3\}$ are $\bar{R}_{11} = R\{a_r^2, a_{r+p}^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r+((p+1)/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r+((p+m+1)/2) \pmod{m}\}$ and $\bar{R}_{12} = R\{a_r^2, a_{r+s}^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r+((m+s)/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r+(s/2) \pmod{m}\}$. Clearly, $|\bar{R}_u| = 3(m-1)$. Since $|R_r| = (3m/2)+1 < |\bar{R}_u|$, then $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- (e) The RN of $\{a_r^1, a_r^2\}$ is $\bar{R}_{13} = R\{a_r^1, a_r^2\} = V(\mathbb{P}_m^2)$. Clearly, $|R\{a_r^2 a_r^3\}| = (3m/2)+1 < |R\{a_r^1 a_r^2\}|$ and $|R\{a_r^1 a_r^2\} \cap \cup_{r=1}^m R_r| = 3n \geq |R_r|$. $\square$

Theorem 5. If $\mathbb{N} \cong \mathbb{P}_m^2$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{P}_m^2) \leq (6m/(3m+2))$.

Proof

Case 1. When $m = 6$.

The RNs are given as follows.

Tables 20–22 show the RNs with maximum cardinality of 18, 16, and 14, respectively. On the other hand, Table 23 bears RNs with the minimum cardinality of 10. Also, it is observed that $\cup_{r=1}^{12} R_r = V(\mathbb{P}_6^2)$ which implies $|\cup_{r=1}^{12} R_r| = 18$ and $|\bar{R}_r \cap \cup_{r=1}^6 R_r| \geq |R_r|$.

Now, we define a function $\kappa: V(\mathbb{P}_6^2) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = \kappa(a_r^3) = (1/10)$. As R_r for $1 \leq t \leq 12$ of $\mathbb{P}_6^2$ are pairwise overlapping; hence, $\exists$ another minimal resolving function $\bar{\kappa}$ of $\mathbb{P}_6^2$ such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(\mathbb{P}_6^2) \leq \sum_{r=1}^{18} (1/10) = (9/5)$.

Case 2. $m \geq 8$.

As Lemma 5 clears the fact that among the RNs of $\mathbb{P}_m^2$, those that have the minimum cardinality of $(3m/2)+1$ are $R\{a_r^2, a_r^3\}$ and $R\{a_r^2, a_{r-1}^3\}$, respectively. Also, $|\bar{R}_u \cap \cup_{r=1}^m R_r| \leq |R_r|$, where $1 \leq u \leq 17$.

Let $\eta = |\cup_{r=1}^m R_r| = 3m$ and $\lambda = |R_r| = (3m/2)+1$. Then, we define a function $\kappa: V(\mathbb{P}_m^2) \rightarrow [0, 1]$ such that $\kappa(c) = \begin{cases} 1/\lambda & \text{for } c \in \cup_{r=1}^m R_r, \\ 0 & \text{for } c \in V(\mathbb{P}_m^2) - \cup_{r=1}^m R_r. \end{cases}$ It is seen that κ is a resolving function for $\mathbb{P}_m^2$ because $\kappa(R\{a, b\}) \geq 1 \forall \{a, b\} \subset V(\mathbb{P}_m^2)$. Now, suppose on contrary that there is another resolving function ρ , such that $\rho(u) \leq \kappa(u)$, for at least one $u \in V(\mathbb{P}_m^2) \rho(u) \neq \kappa(u)$. As a result, $\rho(R\{a, b\}) < 1$, where $R\{a, b\}$ is an RN of $\mathbb{P}_m^2$ having minimum cardinality λ . This shows that ρ is not a resolving function, a contradiction. Thus, κ is a minimal resolving function for $\mathbb{P}_m^2$ which attains minimum $|\kappa|$. Also, R_r for $1 \leq t \leq 2m$ of $\mathbb{P}_m^2$ are pairwise overlapping; hence, $\exists$ another minimal resolving function $\bar{\kappa}$ of $\mathbb{P}_m^2$ such that

TABLE 12: $\bar{R}_u$ for $1 \leq u \leq 30$.

RNs	Elements	Equality
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_8^1) - \{a_3^1, a_5^1\}$	$R\{a_1^1, a_6^1\}, R\{a_2^1, a_7^1\}, R\{a_3^1, a_8^1\}, R\{a_4^1, a_1^1\}, R\{a_5^1, a_2^1\}$
$R\{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_2^1, a_6^1\}$	$R\{a_6^1, a_7^1\}, R\{a_2^1, a_3^1\}, R\{a_1^1, a_8^1\}, R\{a_5^1, a_7^1\}, R\{a_1^1, a_4^1\}, R\{a_2^1, a_1^1\}, R\{a_3^1, a_3^1\}, R\{a_5^1, a_2^1\}$
$R\{a_1^1, a_4^1\}$	$V(\mathbb{P}_8^1) - \{a_3^1, a_7^1\}$	$R\{a_7^1, a_1^1\}, R\{a_3^1, a_4^1\}, R\{a_2^1, a_8^1\}, R\{a_2^1, a_5^1\}, R\{a_2^1, a_2^1\}, R\{a_1^1, a_6^1\}, R\{a_2^1, a_6^1\}$
$R\{a_1^1, a_5^1\}$	$V(\mathbb{P}_8^1) - \{a_4^1, a_8^1\}$	$R\{a_1^1, a_8^1\}, R\{a_4^1, a_5^1\}, R\{a_2^1, a_8^1\}, R\{a_1^1, a_6^1\}, R\{a_2^1, a_6^1\}$

TABLE 13: $\bar{R}_u$ for $28 \leq u \leq 38$.

RNs	Elements	Equality
$R\{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_2^1, a_5^1\} \cup \{a_2^2, a_5^2\}$	$R\{a_1^1, a_1^1\}, R\{a_2^1, a_2^1\}, R\{a_3^1, a_3^1\}, R\{a_4^1, a_4^1\}, R\{a_5^1, a_5^1\}, R\{a_6^1, a_6^1\}, R\{a_7^1, a_7^1\}, R\{a_8^1, a_8^1\}$
$R\{a_1^1, a_4^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_6^1\} \cup \{a_3^2, a_6^2\}$	$R\{a_1^1, a_8^1\}, R\{a_2^1, a_4^1\}, R\{a_2^1, a_2^1\}, R\{a_1^1, a_1^1\}, R\{a_2^1, a_5^1\}, R\{a_3^1, a_3^1\}, R\{a_3^1, a_6^1\}, R\{a_1^1, a_6^1\}$
$R\{a_1^1, a_5^1\}$	$V(\mathbb{P}_8^1) - \{a_4^1, a_7^1\} \cup \{a_4^2, a_7^2\}$	$R\{a_1^1, a_7^1\}, R\{a_3^1, a_5^1\}, R\{a_1^1, a_7^1\}, R\{a_1^1, a_6^1\}, R\{a_2^1, a_6^1\}, R\{a_3^1, a_4^1\}, R\{a_3^1, a_7^1\}, R\{a_3^1, a_3^1\}, R\{a_3^1, a_3^1\}$
$R\{a_1^1, a_6^1\}$	$V(\mathbb{P}_8^1) - \{a_5^1, a_8^1\} \cup \{a_5^2, a_8^2\}$	$R\{a_2^1, a_8^1\}, R\{a_3^1, a_5^1\}, R\{a_2^1, a_8^1\}, R\{a_3^1, a_7^1\}, R\{a_3^1, a_7^1\}, R\{a_4^1, a_5^1\}, R\{a_1^1, a_8^1\}, R\{a_3^1, a_7^1\}, R\{a_3^1, a_8^1\}$

TABLE 14: $\bar{R}_u$ for $39 \leq u \leq 54$.

RNs	Elements
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_8^1) - \{a_2^1, a_2^1\} \cup \{a_3^1, a_7^1\}$
$R\{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_2^1, a_2^1\} \cup \{a_1^1, a_3^1\}$
$R\{a_1^1, a_4^1\}$	$V(\mathbb{P}_8^1) - \{a_2^1, a_2^1\} \cup \{a_3^1, a_6^1\}$
$R\{a_1^1, a_5^1\}$	$V(\mathbb{P}_8^1) - \{a_2^1, a_2^1\} \cup \{a_5^1, a_8^1\}$
$R\{a_1^1, a_6^1\}$	$V(\mathbb{P}_8^1) - \{a_2^1, a_2^1\} \cup \{a_1^1, a_1^1\}$
$R\{a_1^1, a_7^1\}$	$V(\mathbb{P}_8^1) - \{a_2^1, a_2^1\} \cup \{a_1^1, a_1^1\}$
$R\{a_1^1, a_8^1\}$	$V(\mathbb{P}_8^1) - \{a_2^1, a_2^1\} \cup \{a_1^1, a_1^1\}$
$R\{a_2^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_6^1, a_7^1, a_8^1\}$
$R\{a_2^1, a_4^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_2^1, a_5^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_6^1, a_7^1, a_8^1\}$
$R\{a_2^1, a_6^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_2^1, a_7^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_2^1, a_8^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_3^1, a_4^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_3^1, a_5^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_3^1, a_6^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_3^1, a_7^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_3^1, a_8^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_4^1, a_5^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_4^1, a_6^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_4^1, a_7^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_4^1, a_8^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_5^1, a_6^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_5^1, a_7^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_5^1, a_8^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_6^1, a_7^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_6^1, a_8^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_7^1, a_8^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$

TABLE 15: R_r with minimum cardinalities for $1 \leq r \leq 16$.

RNs	Elements
$R_1 = \{a_1^1, a_2^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_3^1, a_4^1\} \cup \{a_1^1, a_3^1\}$
$R_3 = \{a_1^1, a_4^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_3^1, a_6^1\} \cup \{a_1^1, a_3^1\}$
$R_5 = \{a_1^1, a_6^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_6^1\} \cup \{a_1^1, a_3^1\}$
$R_7 = \{a_1^1, a_8^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_3^1, a_4^1\} \cup \{a_1^1, a_3^1\}$
$R_9 = \{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_7^1, a_8^1\} \cup \{a_1^1, a_3^1\}$
$R_{11} = \{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_8^1\} \cup \{a_1^1, a_3^1\}$
$R_{13} = \{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_3^1, a_4^1\} \cup \{a_1^1, a_3^1\}$
$R_{15} = \{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_1^1, a_3^1\}$
$R_2 = \{a_2^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_4^1, a_5^1\} \cup \{a_1^1, a_3^1\}$
$R_4 = \{a_4^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_1^1, a_3^1\}$
$R_6 = \{a_6^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_1^1, a_3^1\}$
$R_8 = \{a_8^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_4^1, a_5^1\} \cup \{a_1^1, a_3^1\}$
$R_{10} = \{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_7^1, a_8^1\} \cup \{a_1^1, a_3^1\}$
$R_{12} = \{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_1^1, a_3^1\}$
$R_{14} = \{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_3^1, a_4^1\} \cup \{a_1^1, a_3^1\}$
$R_{16} = \{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_5^1, a_6^1, a_7^1\} \cup \{a_1^1, a_3^1\}$

$|\bar{\kappa}| \leq |\kappa|$. Therefore, assigning $(1/\lambda)$ to all the vertices of $\mathbb{P}_m^2$ and calculating their sum, we get $\dim_f(\mathbb{P}_m^2) \leq \sum_{r=1}^n (1/\lambda) = (3m / ((3m+2)/2)) = (6m / (3m+2))$. $\square$

Lemma 6. Let $\mathbb{N} \cong \mathbb{P}_m^3$ be the 3-faced QCL with $m \geq 6$ and $m \equiv 0 \pmod{2}$. For $1 \leq r \leq m$, $p \geq 3$, $s \geq 2$, $p \equiv 1 \pmod{2}$ and $s \equiv 0 \pmod{2}$. Then,

- $|R_r| = |R\{a_r^1, a_r^2\}| = |R_r| = 2m - 1$, $|\cup_{r=1}^m R_r| = 3m$ and $|\cup_{r=1}^m R_r| = 3m$.
- $|R_r| < |\bar{R}_1| = |R\{a_r^1, a_{r+1}^1\}| = |\bar{R}_2| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_3| = |R\{a_r^2, a_{r+1}^2\}| = |\bar{R}_4| = |R\{a_r^2, a_{r+p}^2\}| = |\bar{R}_5| = |R\{a_r^3, a_{r+1}^3\}| = |\bar{R}_6| = |R\{a_r^3, a_{r+p}^3\}| = |\bar{R}_7| = |R\{a_r^1, a_{r+s}^1\}| = |\bar{R}_8| = |R\{a_r^1, a_{r+1}^1\}| = |\bar{R}_9| = |R\{a_r^1, a_{r-1}^1\}| = |\bar{R}_{10}| = |R\{a_r^1, a_r^3\}|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$ with $a_{m+1}^1 = a_1^1$ and $a_{m+1}^2 = a_1^2$.

Proof

- The RN of $\{a_r^1, a_r^2\}$ is $R\{a_r^1, a_r^2\} = V(\mathbb{P}_m^3) - \{a_h^1 | h \equiv r + 1, r + 2, \dots, r + (m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + 1, r + 2, \dots, r + (m/2) - 1 \pmod{m}\} \cup \{a_h^3 | h \equiv r + 1, r + 2 \pmod{m}\}$.

We note that $\cup_{r=1}^m R_r = V(\mathbb{P}_m^3)$ and $|\cup_{r=1}^m R_r| = 3m$.

- The RNs of $\{a_r^1, a_{r+1}^1\}$, $\{a_r^1, a_{r+p}^1\}$, $\{a_r^2, a_{r+1}^2\}$, $\{a_r^2, a_{r+p}^2\}$, $\{a_r^3, a_{r+1}^3\}$, $\{a_r^3, a_{r+p}^3\}$, $\{a_r^1, a_{r+s}^1\}$, $\{a_r^1, a_{r-1}^1\}$, and $\{a_r^1, a_r^3\}$ are $\bar{R}_1 = R\{a_r^1, a_{r+1}^1\} = V(\mathbb{P}_m^3) -$

TABLE 16: $\bar{R}_u$ for $1 \leq u \leq 57$.

RNs	Elements	Equality
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_{10}^1) - \{a_1^3, a_6^3\}$	$R\{a_6^1, a_7^1\}, R\{a_7^1, a_2^1\}, R\{a_6^2, a_7^2\}, R\{a_5^1, a_8^1\}, R\{a_5^2, a_8^2\}, R\{a_3^1, a_{10}^1\}, R\{a_3^2, a_{10}^2\}, R\{a_4^1, a_9^1\}, R\{a_4^2, a_9^2\}, R\{a_5^3, a_7^3\}$
$R\{a_2^1, a_3^1\}$	$V(\mathbb{P}_{10}^1) - \{a_2^3, a_7^3\}$	$R\{a_7^1, a_8^1\}, R\{a_2^2, a_3^2\}, R\{a_7^2, a_8^2\}, R\{a_1^1, a_4^1\}, R\{a_6^2, a_7^2\}, R\{a_1^2, a_4^2\}, R\{a_6^1, a_9^1\}, R\{a_6^2, a_9^2\}, R\{a_1^3, a_3^3\}, R\{a_6^3, a_9^3\}, R\{a_6^1, a_9^1\}, R\{a_6^2, a_9^2\}$
$R\{a_3^1, a_4^1\}$	$V(\mathbb{P}_{10}^1) - \{a_3^3, a_8^3\}$	$R\{a_8^1, a_9^1\}, R\{a_2^2, a_4^2\}, R\{a_8^2, a_9^2\}, R\{a_1^1, a_5^1\}, R\{a_2^2, a_5^2\}, R\{a_1^2, a_5^2\}, R\{a_1^1, a_{10}^1\}, R\{a_7^2, a_{10}^2\}, R\{a_2^3, a_4^3\}, R\{a_7^3, a_9^3\}, R\{a_1^2, a_6^2\}$
$R\{a_4^1, a_5^1\}$	$V(\mathbb{P}_{10}^1) - \{a_4^3, a_9^3\}$	$R\{a_9^1, a_{10}^1\}, R\{a_4^2, a_5^2\}, R\{a_9^2, a_{10}^2\}, R\{a_1^1, a_6^1\}, R\{a_3^2, a_6^2\}, R\{a_1^1, a_8^1\}, R\{a_2^2, a_8^2\}, R\{a_3^3, a_5^3\}, R\{a_8^3, a_{10}^3\}$
$R\{a_4^1, a_5^1\}$	$V(\mathbb{P}_{10}^1) - \{a_5^3, a_{10}^3\}$	$R\{a_1^1, a_{10}^1\}, R\{a_4^2, a_5^2\}, R\{a_1^2, a_{10}^2\}, R\{a_1^1, a_7^1\}, R\{a_3^2, a_6^2\}, R\{a_1^2, a_9^2\}, R\{a_2^2, a_9^2\}, R\{a_4^3, a_6^3\}, R\{a_1^3, a_9^3\}$

TABLE 17: $\bar{R}_u$ for $58 \leq u \leq 122$.

RNs	Elements	Equality
$R\{a_1^1, a_3^1\}$	$V(\mathbb{P}_{10}^1) - \{a_1^2, a_7^2\} \cup \{a_2^2, a_7^2\}$	$R\{a_6^1, a_8^1\}, R\{a_1^2, a_2^2\}, R\{a_2^2, a_2^2\}, R\{a_5^1, a_9^1\}, R\{a_5^2, a_9^2\}, R\{a_1^3, a_2^3\}, R\{a_6^3, a_7^3\}, R\{a_5^3, a_8^3\}, R\{a_3^3, a_{10}^3\}$
$R\{a_2^1, a_4^1\}$	$V(\mathbb{P}_{10}^1) - \{a_1^3, a_8^3\} \cup \{a_2^3, a_8^3\}$	$R\{a_7^1, a_9^1\}, R\{a_2^2, a_4^2\}, R\{a_7^2, a_9^2\}, R\{a_1^1, a_5^1\}, R\{a_1^2, a_5^2\}, R\{a_6^1, a_{10}^1\}, R\{a_6^2, a_{10}^2\}, R\{a_3^3, a_3^3\}, R\{a_7^3, a_8^3\}, R\{a_1^3, a_4^3\}, R\{a_6^3, a_9^3\}$
$R\{a_3^1, a_5^1\}$	$V(\mathbb{P}_{10}^1) - \{a_4^1, a_9^1\} \cup \{a_4^2, a_9^2\}$	$R\{a_8^1, a_{10}^1\}, R\{a_3^2, a_5^2\}, R\{a_2^2, a_{10}^2\}, R\{a_1^2, a_6^2\}, R\{a_2^2, a_6^2\}, R\{a_1^1, a_7^1\}, R\{a_1^2, a_7^2\}, R\{a_3^3, a_4^3\}, R\{a_3^3, a_9^3\}, R\{a_3^3, a_{10}^3\}, R\{a_3^3, a_{10}^3\}$
$R\{a_4^1, a_6^1\}$	$V(\mathbb{P}_{10}^1) - \{a_5^1, a_{10}^1\} \cup \{a_5^2, a_{10}^2\}$	$R\{a_1^1, a_9^1\}, R\{a_4^2, a_6^2\}, R\{a_1^2, a_9^2\}, R\{a_1^3, a_7^3\}, R\{a_3^2, a_7^2\}, R\{a_1^2, a_8^2\}, R\{a_2^2, a_2^2\}, R\{a_4^3, a_5^3\}, R\{a_9^3, a_{10}^3\}, R\{a_1^3, a_8^3\}$
$R\{a_5^1, a_7^1\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_6^1\} \cup \{a_1^2, a_6^2\}$	$R\{a_2^1, a_{10}^1\}, R\{a_5^2, a_7^2\}, R\{a_2^2, a_{10}^2\}, R\{a_4^1, a_8^1\}, R\{a_4^2, a_8^2\}, R\{a_5^3, a_6^3\}, R\{a_1^3, a_{10}^3\}, R\{a_4^3, a_7^3\}, R\{a_2^3, a_9^3\}$
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_{10}^1) - \{a_2^2, a_{10}^2\} \cup \{a_3^3, a_9^3\}$	
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_{10}^1) - \{a_1^2, a_3^2\} \cup \{a_3^3, a_{10}^3\}$	
$R\{a_3^1, a_2^1\}$	$V(\mathbb{P}_{10}^1) - \{a_2^2, a_2^2\} \cup \{a_3^3, a_3^3\}$	
$R\{a_1^1, a_4^1\}$	$V(\mathbb{P}_{10}^1) - \{a_2^2, a_5^2\} \cup \{a_3^3, a_3^3\}$	
$R\{a_1^1, a_5^1\}$	$V(\mathbb{P}_{10}^1) - \{a_2^2, a_5^2\} \cup \{a_3^3, a_3^3\}$	
$R\{a_1^1, a_6^1\}$	$V(\mathbb{P}_{10}^1) - \{a_2^2, a_7^2\} \cup \{a_3^3, a_3^3\}$	
$R\{a_1^1, a_7^1\}$	$V(\mathbb{P}_{10}^1) - \{a_2^2, a_7^2\} \cup \{a_3^3, a_3^3\}$	
$R\{a_1^1, a_8^1\}$	$V(\mathbb{P}_{10}^1) - \{a_2^2, a_8^2\} \cup \{a_3^3, a_3^3\}$	
$R\{a_1^1, a_9^1\}$	$V(\mathbb{P}_{10}^1) - \{a_2^2, a_{10}^2\} \cup \{a_3^3, a_{10}^3\}$	
$R\{a_{10}^1, a_{10}^1\}$	$V(\mathbb{P}_{10}^1) - \{a_2^2, a_{10}^2\} \cup \{a_3^3, a_{10}^3\}$	

TABLE 18: $\bar{R}_u$ for $123 \leq u \leq 142$.

RNs	Elements
$R\{a_1^2, a_3^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_1^1, a_1^1, a_1^1, a_6^1\}$
$R\{a_2^2, a_3^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_1^1, a_1^1, a_1^1, a_8^1\}$
$R\{a_5^2, a_5^3\}$	$V(\mathbb{P}_{10}^1) - \{a_6^1, a_7^1, a_8^1, a_9^1, a_{10}^1\}$
$R\{a_7^2, a_7^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_8^1, a_9^1, a_{10}^1\}$
$R\{a_9^2, a_9^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_2^1, a_4^1, a_{10}^1\}$
$R\{a_2^2, a_1^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_7^1, a_8^1, a_9^1, a_{10}^1\}$
$R\{a_4^2, a_3^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_3^1, a_9^1, a_{10}^1\}$
$R\{a_6^2, a_5^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_3^1, a_4^1, a_5^1\}$
$R\{a_8^2, a_7^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_4^1, a_5^1, a_6^1, a_7^1\}$
$R\{a_{10}^2, a_9^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_6^1, a_7^1, a_8^1, a_9^1\}$
$R\{a_2^2, a_2^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_1^1, a_1^1, a_6^1, a_7^1\}$
$R\{a_2^2, a_2^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_1^1, a_1^1, a_6^1, a_7^1\}$
$R\{a_2^2, a_2^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_1^1, a_1^1, a_6^1, a_7^1\}$
$R\{a_6^2, a_6^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_7^1, a_8^1, a_9^1, a_{10}^1\}$
$R\{a_8^2, a_8^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_3^1, a_9^1, a_{10}^1\}$
$R\{a_{10}^2, a_{10}^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_3^1, a_9^1, a_{10}^1\}$
$R\{a_3^2, a_3^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_5^1, a_9^1, a_{10}^1\}$
$R\{a_5^2, a_4^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_3^1, a_4^1, a_{10}^1\}$
$R\{a_7^2, a_6^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_1^1, a_1^1, a_1^1, a_6^1\}$
$R\{a_9^2, a_8^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_1^1, a_1^1, a_7^1, a_8^1\}$
$R\{a_{10}^2, a_{10}^3\}$	$V(\mathbb{P}_{10}^1) - \{a_6^1, a_7^1, a_8^1, a_9^1, a_{10}^1\}$

TABLE 19: R_r with minimum cardinalities for $1 \leq r \leq 20$.

RNs	Elements
$R_1 = \{a_1^1, a_2^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_1^1, a_1^1, a_6^1\} \cup \{a_2^2\} \cup \{a_1^3, a_5^3\}$
$R_3 = \{a_3^1, a_4^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_6^1, a_7^1, a_8^1\} \cup \{a_7^2\} \cup \{a_3^3, a_7^3\}$
$R_5 = \{a_5^1, a_6^3\}$	$V(\mathbb{P}_{10}^1) - \{a_7^1, a_8^1, a_9^1, a_{10}^1\} \cup \{a_9^2\} \cup \{a_5^3, a_9^3\}$
$R_7 = \{a_7^1, a_8^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_9^1, a_{10}^1\} \cup \{a_1^2\} \cup \{a_7^3, a_7^3\}$
$R_9 = \{a_9^1, a_{10}^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_3^1, a_4^1\} \cup \{a_2^2\} \cup \{a_3^3, a_9^3\}$
$R_{11} = \{a_1^2, a_{10}^3\}$	$V(\mathbb{P}_{10}^1) - \{a_7^1, a_8^1, a_9^1, a_{10}^1\} \cup \{a_8^2\} \cup \{a_1^3, a_7^3\}$
$R_{13} = \{a_4^1, a_2^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_9^1, a_{10}^1\} \cup \{a_{10}^2\} \cup \{a_3^3, a_9^3\}$
$R_{15} = \{a_6^1, a_3^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_3^1, a_4^1\} \cup \{a_2^2\} \cup \{a_1^3, a_3^3\}$
$R_{17} = \{a_8^1, a_6^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_5^1, a_6^1\} \cup \{a_4^2\} \cup \{a_3^3, a_7^3\}$
$R_{19} = \{a_{10}^1, a_8^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_6^1, a_7^1, a_8^1\} \cup \{a_6^2\} \cup \{a_5^3, a_9^3\}$
$R_2 = \{a_2^1, a_3^3\}$	$V(\mathbb{P}_{10}^1) - \{a_4^1, a_5^1, a_6^1, a_7^1\} \cup \{a_6^2\} \cup \{a_2^3, a_6^3\}$
$R_4 = \{a_4^1, a_5^3\}$	$V(\mathbb{P}_{10}^1) - \{a_6^1, a_7^1, a_8^1, a_9^1\} \cup \{a_8^2\} \cup \{a_4^3, a_8^3\}$
$R_6 = \{a_6^1, a_7^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_1^1, a_9^1, a_{10}^1\} \cup \{a_{10}^2\} \cup \{a_6^3, a_{10}^3\}$
$R_8 = \{a_8^1, a_9^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_3^1, a_{10}^1\} \cup \{a_2^2\} \cup \{a_2^3, a_8^3\}$
$R_{10} = \{a_1^1, a_{10}^3\}$	$V(\mathbb{P}_{10}^1) - \{a_2^1, a_3^1, a_4^1, a_5^1\} \cup \{a_4^2\} \cup \{a_4^3, a_{10}^3\}$
$R_{12} = \{a_3^1, a_3^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_8^1, a_9^1, a_{10}^1\} \cup \{a_2^2\} \cup \{a_2^3, a_8^3\}$
$R_{14} = \{a_5^1, a_3^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_3^1, a_{10}^1\} \cup \{a_1^2\} \cup \{a_4^3, a_{10}^3\}$
$R_{16} = \{a_7^1, a_5^3\}$	$V(\mathbb{P}_{10}^1) - \{a_2^1, a_3^1, a_4^1, a_5^1\} \cup \{a_2^2\} \cup \{a_2^3, a_6^3\}$
$R_{18} = \{a_9^1, a_7^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_5^1, a_6^1, a_7^1\} \cup \{a_2^2\} \cup \{a_4^3, a_8^3\}$
$R_{20} = \{a_1^1, a_2^3\}$	$V(\mathbb{P}_{10}^1) - \{a_6^1, a_7^1, a_8^1, a_9^1\} \cup \{a_7^2\} \cup \{a_6^3, a_{10}^3\}$

$$\{a_h^2 | h \equiv r, r + (m/2)(\text{mod } m)\} \cup \{a_h^3 | h \equiv r, r + (m/2)(\text{mod } m)\}, \bar{R}_2 = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{P}_m^3) - \{a_h^2 | h \equiv r + ((p-1)/2), r + ((p+m-1)/2)(\text{mod } m)\} \cup \{a_h^3 | h \equiv r + ((p-1)/2), r + ((p+m-1)/2)(\text{mod } m)\}$$

$$(\text{mod } m)), \bar{R}_3 = R\{a_r^2, a_{r+1}^2\} = V(\mathbb{P}_m^3) - \{a_h^1 | h \equiv r + 1, r + ((m+2)/2)(\text{mod } m)\} \cup \{a_h^3 | h \equiv r, r + (m/2)(\text{mod } m)\}, \bar{R}_4 = R\{a_r^2, a_{r+p}^2\} = V(\mathbb{P}_m^3) - \{a_h^1 | h \equiv r + ((p+1)/2), r + ((p+m+1)/2)(\text{mod } m)\} \cup \{a_h^3 | h$$

TABLE 20: $\bar{R}_u$ for $1 \leq u \leq 6$.

Resolving neighbourhoods			Set/elements
$R\{a_1^1, a_2^1\}$	$R\{a_1^1, a_2^1\}$	$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_6^2)$
$R\{a_4^1, a_5^1\}$	$R\{a_5^1, a_6^1\}$	$R\{a_6^1, a_7^1\}$	$V(\mathbb{P}_6^2)$

TABLE 21: $\bar{R}_u$ for $7 \leq u \leq 24$.

RNs	Elements	Equality
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\}$	$R\{a_1^1, a_2^1\}, R\{a_2^1, a_3^1\}, R\{a_2^1, a_5^1\}, R\{a_4^1, a_7^1\}, R\{a_4^1, a_7^1\}$
$R\{a_2^1, a_3^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\}$	$R\{a_1^1, a_2^1\}, R\{a_2^1, a_3^1\}, R\{a_2^1, a_5^1\}, R\{a_4^1, a_7^1\}, R\{a_4^1, a_7^1\}$
$R\{a_3^1, a_4^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\}$	$R\{a_1^1, a_2^1\}, R\{a_2^1, a_3^1\}, R\{a_2^1, a_5^1\}, R\{a_4^1, a_7^1\}, R\{a_4^1, a_7^1\}$

TABLE 22: $\bar{R}_u$ for $25 \leq u \leq 29$.

RNs	Elements	Equality
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\}$	$R\{a_1^1, a_2^1\}, R\{a_2^1, a_3^1\}, R\{a_2^1, a_5^1\}, R\{a_4^1, a_7^1\}, R\{a_4^1, a_7^1\}$
$R\{a_2^1, a_3^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\}$	$R\{a_1^1, a_2^1\}, R\{a_2^1, a_3^1\}, R\{a_2^1, a_5^1\}, R\{a_4^1, a_7^1\}, R\{a_4^1, a_7^1\}$
$R\{a_3^1, a_4^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\}$	$R\{a_1^1, a_2^1\}, R\{a_2^1, a_3^1\}, R\{a_2^1, a_5^1\}, R\{a_4^1, a_7^1\}, R\{a_4^1, a_7^1\}$

TABLE 23: R_r with minimum cardinalities for $1 \leq r \leq 12$.

RNs	Elements
$R_1 = R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_2 = R\{a_2^1, a_3^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_3 = R\{a_3^1, a_4^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_4 = R\{a_4^1, a_5^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_5 = R\{a_5^1, a_6^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_6 = R\{a_6^1, a_7^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_7 = R\{a_7^1, a_8^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_8 = R\{a_8^1, a_9^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_9 = R\{a_9^1, a_{10}^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_{10} = R\{a_{10}^1, a_{11}^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_{11} = R\{a_{11}^1, a_{12}^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_{12} = R\{a_{12}^1, a_{13}^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$

$\equiv r + ((p+1)/2), r + ((p+m+1)/2) \pmod{m}$,
 $\bar{R}_5 = R\{a_r^1, a_{r+1}^1\} = V(\mathbb{P}_m^3) - \{a_h^1 | h \equiv r+1, r + ((m+2)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r+1, r + ((m+2)/2) \pmod{m}\}$,
 $\bar{R}_6 = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{P}_m^3) - \{a_h^1 | h \equiv r + ((p+1)/2), r + ((p+m+1)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r + ((p+1)/2), r + ((p+m+1)/2) \pmod{m}\}$,
 $\bar{R}_7 = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{P}_m^3) - \{a_h^1 | h \equiv r + (s/2), r + ((s+m)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r + ((s-2)/2), r + ((s+m-2)/2) \pmod{m}\} = \bar{R}_8 = R\{a_r^1, a_{r+1}^1\}$,
 $\bar{R}_9 = R\{a_r^1, a_{r-1}^1\} = V(\mathbb{P}_m^3) - \{a_h^1 | h \equiv r - (s/2), r - ((s+m)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r - ((s-2)/2), r - ((s+m-2)/2) \pmod{m}\}$,
 $\bar{R}_{10} = R\{a_r^1, a_r^1\} = V(\mathbb{P}_m^3) - \{a_h^1 | h \equiv r, r + 2 \pmod{m}\} \cup \{a_h^1 | h \equiv r - 1, r + 2 \pmod{m}\}$,
 Clearly, $|\bar{R}_u| = 3m - 4$. Since $|R_r| = (3m/2) + 1 < |\bar{R}_u|$, then $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 3m - 2 \geq |R_r|$.

- (c) The RN of $\{a_r^1, a_{r+p}^1\}$, $\{a_r^1, a_{r+s}^1\}$, $\{a_r^1, a_{r-p}^1\}$, and $\{a_r^1, a_{r-s}^1\}$ are $\bar{R}_{11} = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{G}) - \{a_h^1 | h \equiv r + ((p+1)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r + ((p+m-1)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r + ((p-1)/2) \pmod{m}\}$,
 $\bar{R}_{12} = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{G}) - \{a_h^1 | h \equiv r + ((m+s)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r + (s/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r +$

$((m+s-2)/2) \pmod{m}\}$, $\bar{R}_{13} = R\{a_r^1, a_{r-p}^1\} = V(\mathbb{G}) - \{a_h^1 | h \equiv r - ((p+1)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r - ((p+m-1)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r - ((p-1)/2) \pmod{m}\}$,
 $\bar{R}_{14} = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{G}) - \{a_h^1 | h \equiv r - ((m+s)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r - (s/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r - ((m+s-2)/2) \pmod{m}\}$,
 Clearly, $|\bar{R}_u| = 3(m-1)$. Since $|R_r| = (3m/2) + 1 < |\bar{R}_u|$, then $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 3m - 2 \geq |R_r|$.

- (d) The RNs of $\{a_r^2, a_{r+s}^2\}$, $\{a_r^3, a_{r+s}^3\}$, $\{a_r^1, a_{r+p}^1\}$, and $\{a_r^1, a_{r+p}^1\}$ are $\bar{R}_{15} = \{a_r^2, a_{r+s}^2\} = V(\mathbb{G}) - \{a_h^2 | r \equiv r + (s/2), r + ((s+m-2)/2) \pmod{m}\}$,
 $\bar{R}_{16} = \{a_r^3, a_{r+s}^3\} = V(\mathbb{G}) - \{a_h^3 | r \equiv r + (s/2), r + ((s+m-2)/2) \pmod{m}\}$,
 $\bar{R}_{17} = \{a_r^1, a_{r+p}^1\} = V(\mathbb{G}) - \{a_h^1 | r \equiv r + ((s+2)/2), r + ((s+m+2)/2) \pmod{m}\}$,
 $\bar{R}_{18} = \{a_r^1, a_{r+p}^1\} = V(\mathbb{G}) - \{a_h^1 | r \equiv r + ((p+1)/2) \pmod{m}\} \cup \{a_h^1 | r \equiv r + ((p+m+1)/2) \pmod{m}\}$,
 Clearly, $|\bar{R}_u| = 3m - 2$. Since $|R_r| = (3m/2) + 1 < |\bar{R}_u|$, then $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 3m - 2 \geq |R_r|$.

- (e) The RNs of $\{a_r^1, a_{r+1}^1\}$ and $\{a_r^1, a_{r-1}^1\}$ are $\bar{R}_{19} = R\{a_r^1, a_{r+1}^1\} = V(\mathbb{P}_m^3) - \{a_h^1 | h \equiv r - 1, r - 2, \dots, r - (m/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r - 2 \pmod{m}\} \cup \{a_h^1 | h \equiv r - 3 \pmod{m}\}$.
 Clearly, $|R\{a_r^1, a_{r+1}^1\}| = 2m - 1 < |R\{a_r^1, a_{r-1}^1\}|$ and $|R\{a_r^1, a_{r-1}^1\} \cap \cup_{r=1}^m R_r| = ((5m-4)/2) \geq |R_r|$.

- (f) The RN of $\{a_r^2, a_{r-1}^2\}$ is given by $\bar{R}_{20} = R\{a_r^2, a_{r-1}^2\} = V(\mathbb{P}_m^3) - \{a_h^2 | h \equiv r - 1, r - 2, \dots, r - (m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r - 1 \pmod{m}\}$.
 Clearly, $|R\{a_r^2, a_{r-1}^2\}| = 2m - 1 < |R\{a_r^2, a_{r-1}^2\}|$ and $|R\{a_r^2, a_{r-1}^2\} \cap \cup_{r=1}^m R_r| = ((5m-2)/2) \geq |R_r|$.

- (g) The RN of $\{a_r^2, a_r^3\}$ is given by $\bar{R}_{21} = R\{a_r^2, a_r^3\} = V(\mathbb{P}_m^3) - \{a_h^2 | h \equiv r - 1, r - 2, \dots, r - (m/2) + 1 \pmod{m}\}$.
 Clearly, $|R\{a_r^2, a_r^3\}| = 2m - 1 < |R\{a_r^2, a_r^3\}|$ and $|R\{a_r^2, a_r^3\} \cap \cup_{r=1}^m R_r| = ((5m+2)/2) \geq |R_r|$. $\square$

Theorem 6. If $\mathbb{N} \cong \mathbb{P}_m^3$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{P}_m^3) \leq (3m/(2m-1))$.

Proof

Case 1. When $m = 6$

The RNs are given as follows.

We note that the cardinalities of RNs as shown in Tables 24–26 are 16, 14, and 13, respectively. Similarly, the cardinality of RNs given in Table 27 is 11 which is less than the least among all the RNs. Moreover, $\bigcup_{r=1}^6 R_r = V(\mathbb{P}_6^3)$ which implies $|\bigcup_{r=1}^6 R_r| = 18$ and $|\bar{R}_r \cap \bigcup_{r=1}^6 R_r| \geq |R_r|$.

Now, we define a function $\kappa: V(\mathbb{P}_6^3) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = \kappa(a_r^3) = (1/11)$. Furthermore, R_r for $1 \leq t \leq 6$ of $\mathbb{P}_6^3$ are pairwise overlapping; hence, $\exists$ another minimal resolving function $\bar{\kappa}$ of $\mathbb{P}_6^3$ such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(\mathbb{P}_6^3) \leq \sum_{r=1}^{18} (1/11) = (18/11)$.

Case 2. $m \geq 8$

We have seen from Lemma 6 $|R\{a_r^1, a_r^2\}| = 2m - 1 \leq |R\{a, b\}|$ for all $\{a, b\} \subset V(\mathbb{P}_m^3)$ and $|R\{a, b\} \cap \bigcup_{r=1}^m R_r| \geq |R_r|$.

Let $\eta = |\bigcup_{r=1}^m R_r| = 3m$ and $\lambda = |R_r| = 2m - 1$. Then, we define a function $\kappa: V(\mathbb{P}_m^3) \rightarrow [0, 1]$ such that $\kappa(c) = \begin{cases} 1/\lambda & \text{for } c \in \bigcup_{r=1}^m R_r, \\ 0 & \text{for } c \in V(\mathbb{P}_m^3) - \bigcup_{r=1}^m R_r. \end{cases}$ We see that κ is a resolving function for $\mathbb{P}_m^3$ because $\kappa(R\{a, b\}) \geq 1 \forall uv \in E(\mathbb{P}_m^3)$. Now, suppose on contrary that there is another resolving function ρ , such that $\rho(u) \leq \kappa(u)$, for at least one $u \in V(\mathbb{P}_m^3)$ $\rho(u) \neq \kappa(u)$. As a result, $\rho(R\{a, b\}) < 1$, where $R\{a, b\}$ is an RN of $\mathbb{P}_m^3$ having the minimum cardinality of λ . This shows that ρ is not a resolving function, a contradiction. Thus, κ is a minimal resolving function for $\mathbb{P}_m^3$ that achieves minimum $|\kappa|$. Moreover, R_r for $1 \leq r \leq m$ of $\mathbb{P}_m^3$ are pairwise overlapping; hence, $\exists$ another minimal resolving function $\bar{\kappa}$ of $\mathbb{P}_m^3$ such that $|\bar{\kappa}| \leq |\kappa|$. Therefore, assigning $(1/\lambda)$ to all the vertices of $\mathbb{P}_m^3$ and calculating their sum, we get $\dim_f(\mathbb{P}_m^3) \leq \sum_{r=1}^{\eta} (1/\lambda) = (3m/(2m-1)) = (3m/(2m-1))$. $\square$

Corollary 2. If $\mathbb{N} \cong \mathbb{P}_m^4$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{P}_m^4) \leq (3m/(2m-1))$.

Proof. The above assertion is valid as $\mathbb{P}_m^4 \cong \mathbb{P}_m^3$. $\square$

Lemma 7. Let $\mathbb{N} \cong \mathbb{P}_m^5$ be the 4-faced QCL with $m \geq 6$ and $m \equiv 0 \pmod{2}$. For $1 \leq l, r \leq m$, $p \geq 3$, $s \geq 2$ $p \equiv 1 \pmod{2}$ and $s \equiv 0 \pmod{2}$. Then,

- $|R_l| = |R\{a_l^1, a_l^2\}| = |R_r| = |R\{a_r^2, a_{r-1}^3\}| = (3m/2) + 1$ with $a_0^3 = a_m^3$, $a_0^2 = a_m^2$, $|\bigcup_{l=1}^m R_l| = 3m$, $|\bigcup_{r=1}^m R_r| = 3m$ and $|\bigcup_{l=1}^m R_l \cup \bigcup_{r=1}^m R_r| = 3m$,
- $|R_r| < |\bar{R}_1| = |R\{a_r^1, a_r^3\}| = |\bar{R}_2| = |R\{a_r^2, a_{r+1}^3\}| = |\bar{R}_3| = |R\{a_r^2, a_{r-1}^3\}|$ and $|\bar{R}_u \cap \bigcup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_4| = |R\{a_r^1, a_{r+s}^1\}| = |\bar{R}_5| = |R\{a_r^2, a_{r+1}^2\}| = |\bar{R}_6| = |R\{a_r^2, a_{r+p}^2\}| = |\bar{R}_7| = |R\{a_r^1, a_{r+s}^3\}|$ with $a_{m+1}^1 = a_1^1$ and $|\bar{R}_u \cap \bigcup_{r=1}^m R_r| \geq |R_r|$.

TABLE 24: $\bar{R}_u$ for $1 \leq u \leq 30$.

RNs	Elements	Equality
$R\{a_1^2, a_2^2\}$	$V(\mathbb{P}_6^3) - \{a_2^2, a_5^2\}$	$R\{a_4^2, a_6^2\}$
$R\{a_2^2, a_4^2\}$	$V(\mathbb{P}_6^3) - \{a_3^2, a_6^2\}$	$R\{a_5^1, a_6^1\}, R\{a_1^1, a_5^1\}$
$R\{a_3^2, a_5^2\}$	$V(\mathbb{P}_6^3) - \{a_1^2, a_4^2\}$	$R\{a_2^2, a_6^2\}$
$R\{a_1^3, a_3^3\}$	$V(\mathbb{P}_6^3) - \{a_2^3, a_5^3\}$	$R\{a_4^3, a_6^3\}$
$R\{a_2^3, a_5^3\}$	$V(\mathbb{P}_6^3) - \{a_3^3, a_6^3\}$	$R\{a_5^3, a_6^3\}, R\{a_1^3, a_5^3\}$
$R\{a_3^3, a_6^3\}$	$V(\mathbb{P}_6^3) - \{a_1^3, a_4^3\}$	$R\{a_2^3, a_6^3\}$
$R\{a_1^4, a_4^4\}$	$V(\mathbb{P}_6^3) - \{a_3^4, a_6^4\}$	$R\{a_5^4, a_6^4\}$
$R\{a_2^4, a_6^4\}$	$V(\mathbb{P}_6^3) - \{a_1^4, a_5^4\}$	$R\{a_1^5, a_5^5\}$
$R\{a_3^4, a_5^4\}$	$V(\mathbb{P}_6^3) - \{a_1^5, a_5^5\}$	$R\{a_6^5, a_2^5\}$
$R\{a_1^5, a_5^5\}$	$V(\mathbb{P}_6^3) - \{a_1^1\} \cup \{a_3^2\}$	
$R\{a_2^5, a_6^5\}$	$V(\mathbb{P}_6^3) - \{a_1^6\} \cup \{a_4^2\}$	
$R\{a_3^5, a_6^5\}$	$V(\mathbb{P}_6^3) - \{a_1^7\} \cup \{a_5^2\}$	
$R\{a_2^6, a_6^6\}$	$V(\mathbb{P}_6^3) - \{a_4^2, a_5^2\}$	
$R\{a_3^6, a_6^6\}$	$V(\mathbb{P}_6^3) - \{a_5^2, a_6^2\}$	
$R\{a_4^6, a_6^6\}$	$V(\mathbb{P}_6^3) - \{a_2^2, a_6^2\}$	
$R\{a_5^6, a_6^6\}$	$V(\mathbb{P}_6^3) - \{a_1^2, a_5^2\}$	
$R\{a_6^6, a_6^6\}$	$V(\mathbb{P}_6^3) - \{a_2^2, a_3^2\}$	
$R\{a_1^7, a_1^7\}$	$V(\mathbb{P}_6^3) - \{a_3^2, a_4^2\}$	

- $|R_r| < |\bar{R}_8| = |R\{a_r^1, a_{r+1}^1\}| = |\bar{R}_9| = |R\{a_r^1, a_{r+p}^1\}| = \bar{R}_{10} = |R\{a_r^2, a_{r+p}^3\}| = \bar{R}_{11} = |R\{a_r^2, a_{r-p}^3\}|$ and $|\bar{R}_u \cap \bigcup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_{12}| = |R\{a_r^2, a_{r+s}^2\}| = |\bar{R}_{13}| = |R\{a_r^1, a_{r+p}^2\}| = |\bar{R}_{14}| = |R\{a_r^1, a_{r+s}^2\}| = |\bar{R}_{15}| = |R\{a_r^1, a_{r+p}^3\}|$ and $|\bar{R}_u \cap \bigcup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_{16}| = |R\{a_r^2, a_r^3\}|$ and $|R\{a_r^2, a_r^3 \cap \bigcup_{r=1}^m R_r\}| \geq |R_r|$.

Proof

- The RNs of $\{a_l^1, a_l^2\}$ and $\{a_r^2, a_{r-1}^3\}$ are $R\{a_l^1, a_l^2\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv l+1, l+2, \dots, l+(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv l-1, l-2, \dots, l-(m/2) \pmod{m}\} \cup \{a_h^3 | h \equiv l-1, l-2, \dots, l-(m/2)+1 \pmod{m}\}$ and $R\{a_r^2, a_{r-1}^3\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r-1, r-2, \dots, r-(m/2)+1 \pmod{m}\}$, respectively. We note that $\bigcup_{r=1}^m R_r = V(\mathbb{P}_m^5)$ and $|\bigcup_{r=1}^m R_r| = 3m$.
- The RNs of $\{a_r^1, a_r^3\}$, $\{a_r^2, a_{r+1}^3\}$ and $\{a_r^2, a_{r-1}^3\}$ are $\bar{R}_1 = R\{a_r^1, a_r^3\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv r+2, r+3, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r \pmod{m}\} \cup \{a_h^3 | h \equiv r-1 \pmod{m}\}$, $\bar{R}_2 = R\{a_r^2, a_{r+1}^3\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv r+2, r+3, \dots, r+(m+2)/2 \pmod{m}\} \cup \{a_h^2 | h \equiv r \pmod{m}\} \cup \{a_h^3 | h \equiv r+(m/2) \pmod{m}\}$ and $\bar{R}_3 = R\{a_r^2, a_{r-1}^3\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv r-2, r-3, \dots, r-(m+2)/2 \pmod{m}\} \cup \{a_h^2 | h \equiv r \pmod{m}\} \cup \{a_h^3 | h \equiv r-(m/2) \pmod{m}\}$, respectively. We can see that $|\bar{R}_u| = (5m/2) + 2$. Since $|R\{a_r^1, a_r^2\}| = |R\{a_r^2, a_{r-1}^3\}| = (3m/2) + 1 < |\bar{R}_u|$, then $|\bar{R}_u \cap \bigcup_{r=1}^m R_r| = 3m - 4 \geq |R_r|$, where $1 \leq u \leq 3$.
- The RNs of $\{a_r^1, a_{r+s}^1\}$, $\{a_r^2, a_{r+p}^2\}$ and $\{a_r^1, a_{r+s}^3\}$ are $\bar{R}_4 = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv r+(s/2), r+(m+s)/2 \pmod{m}\} \cup \{a_h^3 | h \equiv r+(s-2)/2, r+$

TABLE 25: $\bar{R}_u$ for $31 \leq u \leq 78$.

RNs	Elements	Equality
$R\{a_2^2, a_1^3\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_2^2\}$	
$R\{a_3^2, a_3^3\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_2^1, a_6^1\} \cup \{a_2^2\}$	
$R\{a_4^2, a_3^3\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_2^2\}$	
$R\{a_5^2, a_4^3\}$	$V(\mathbb{P}_6^3) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_2^2\}$	
$R\{a_6^2, a_5^3\}$	$V(\mathbb{P}_6^3) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_2^2\}$	
$R\{a_2^2, a_6^3\}$	$V(\mathbb{P}_6^3) - \{a_4^1, a_5^1, a_6^1\} \cup \{a_5^2\}$	
$R\{a_1^2, a_2^3\}$	$V(\mathbb{P}_8^3) - \{a_2^2, a_4^2\} \cup \{a_3^3, a_4^3\}$	$R\{a_4^1, a_5^1\}, R\{a_3^1, a_6^1\}, R\{a_3^2, a_6^2\}$
$R\{a_5^2, a_1^3\}$	$V(\mathbb{P}_8^3) - \{a_5^2, a_7^2\} \cup \{a_5^3, a_7^3\}$	$R\{a_5^1, a_6^1\}, R\{a_1^1, a_4^1\}, R\{a_1^2, a_4^2\}, R\{a_5^1, a_8^1\}, R\{a_5^2, a_8^2\}, R\{a_3^1, a_3^3\}$
$R\{a_1^2, a_4^3\}$	$V(\mathbb{P}_8^3) - \{a_2^2, a_6^2\} \cup \{a_3^3, a_3^3\}$	$R\{a_1^1, a_6^1\}, R\{a_1^2, a_5^1\}, R\{a_1^1, a_6^1\}$
$R\{a_1^2, a_3^3\}$	$V(\mathbb{P}_8^3) - \{a_1^2, a_5^1\} \cup \{a_3^3, a_3^3\}$	$R\{a_4^1, a_6^1\}, R\{a_1^2, a_2^2\}, R\{a_3^1, a_6^1\}, R\{a_4^2, a_5^2\}, R\{a_3^2, a_6^2\}$
$R\{a_2^2, a_4^3\}$	$V(\mathbb{P}_8^3) - \{a_1^3, a_6^1\} \cup \{a_3^3, a_5^3\}$	$R\{a_1^1, a_5^1\}, R\{a_2^2, a_3^2\}, R\{a_1^1, a_4^1\}, R\{a_1^2, a_8^1\}, R\{a_5^2, a_6^2\}, R\{a_2^1, a_4^2\}$
$R\{a_1^2, a_5^3\}$	$V(\mathbb{P}_8^3) - \{a_1^1, a_4^1\} \cup \{a_3^3, a_6^3\}$	$R\{a_2^1, a_5^1\}, R\{a_1^1, a_6^1\}, R\{a_2^3, a_4^1\}, R\{a_2^1, a_6^2\}, R\{a_2^2, a_5^2\}$
$R\{a_1^2, a_2^3\}$	$V(\mathbb{P}_8^3) - \{a_2^2, a_5^1\} \cup \{a_3^3, a_2^3\}$	$R\{a_4^3, a_5^3\}, R\{a_1^3, a_4^3\}$
$R\{a_3^2, a_3^3\}$	$V(\mathbb{P}_8^3) - \{a_3^1, a_6^1\} \cup \{a_2^2, a_6^2\}$	$R\{a_3^3, a_6^3\}, R\{a_2^3, a_5^3\}$
$R\{a_3^2, a_4^3\}$	$V(\mathbb{P}_8^3) - \{a_1^1, a_4^1\} \cup \{a_2^2, a_2^2\}$	$R\{a_1^3, a_6^3\}, R\{a_3^3, a_6^3\}$

TABLE 26: $\overline{R}_u$ for $79 \leq u \leq 84$.

RNs	Elements
$R\{a_1^1 a_1^2\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_6^2\} \cup \{a_5^3\}$
$R\{a_1^1 a_2^2\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_2^1, a_6^1\} \cup \{a_2^2\} \cup \{a_6^3\}$
$R\{a_1^1 a_3^2\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_3^1, a_6^1\} \cup \{a_3^2\} \cup \{a_6^3\}$
$R\{a_2^1 a_2^2\}$	$V(\mathbb{P}_6^3) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_5^2\} \cup \{a_3^3\}$
$R\{a_1^1 a_6^2\}$	$V(\mathbb{P}_6^3) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_4^2\} \cup \{a_3^3\}$
$R\{a_1^1 a_6^2\}$	$V(\mathbb{P}_6^3) - \{a_4^1, a_5^1, a_6^1\} \cup \{a_5^2\} \cup \{a_4^3\}$

TABLE 27: R_r with minimum cardinalities for $1 \leq r \leq 6$.

RNs	Elements
$R_1 = R\{a_1^1 a_1^2\}$	$V(\mathbb{P}_6^3) - \{a_1^2, a_3^1, a_4^1\} \cup \{a_2^2, a_3^2\} \cup \{a_3^3, a_3^3\}$
$R_2 = R\{a_2^1 a_2^2\}$	$V(\mathbb{P}_6^3) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_2^2, a_4^2\} \cup \{a_3^3, a_4^3\}$
$R_3 = R\{a_3^1 a_3^2\}$	$V(\mathbb{P}_6^3) - \{a_4^1, a_5^1, a_6^1\} \cup \{a_2^2, a_5^2\} \cup \{a_3^3, a_5^3\}$
$R_4 = R\{a_4^1 a_4^2\}$	$V(\mathbb{P}_6^3) - \{a_1^2, a_5^2, a_6^2\} \cup \{a_5^2, a_6^2\} \cup \{a_3^3, a_6^3\}$
$R_5 = R\{a_5^1 a_5^2\}$	$V(\mathbb{P}_6^3) - \{a_1^2, a_2^2, a_6^2\} \cup \{a_2^2, a_6^2\} \cup \{a_3^3, a_6^3\}$
$R_6 = R\{a_6^1 a_6^2\}$	$V(\mathbb{P}_6^3) - \{a_1^2, a_2^2, a_3^2\} \cup \{a_2^2, a_2^2\} \cup \{a_3^3, a_3^3\}$

$((s+m-2)/2)(\text{mod } m)\}$, $\overline{R}_5 = R\{a_r^2, a_{r+1}^2\} = R\{a_r^1, a_{r+2}^1\}$, for $s \geq 4$, $\overline{R}_6 = R\{a_r^2, a_{r+p}^2\} = R\{a_r^1, a_{r+s}^1\}$, $\overline{R}_7 = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv r + ((s+2)/2), r + ((m+s)/2)(\text{mod } m)\} \cup \{a_h^2 | h \equiv r + (s/2), r + ((m+s)/2)(\text{mod } m)\}$. Clearly, $|\overline{R}_u| = 3m - 4$. Since $|R\{a_r^1, a_r^2\}| = |R\{a_r^2, a_{r-1}^1\}| = (3m/2) + 1 < |\overline{R}_u|$, then $|\overline{R}_u \cap \cup_{r=1}^m R_r| = 3m - 4 \geq |R_r|$.

- (d) The RNs of $\{a_r^1, a_{r+p}^1\}$, $\{a_r^2, a_{r+p}^3\}$ and $\{a_r^2, a_{r-p}^3\}$ are $\bar{R}_8 = R\{a_r^1, a_{r+1}^1\} = V(\mathbb{P}_m^5) - \{a_h^1|h \equiv r, \quad r + (m/2) \pmod{m}\} \cup \{a_h^3|h \equiv r \pmod{m}\}$, $\bar{R}_9 = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{P}_m^5) - \{a_h^1|h \equiv r + ((p-1)/2), r + ((p+m-1)/2) \pmod{m}\} \cup \{a_h^3|h \equiv r + ((p-1)/2) \pmod{m}\}$, $\bar{R}_{10} = R\{a_r^2, a_{r+p}^3\} = V(\mathbb{P}_m^5) - \{a_h^1|h \equiv r-1 \pmod{m}\} \cup \{a_h^2|h \equiv r + ((p+1)/2) \pmod{m}\} \cup \{a_h^3|h \equiv r + ((p+m-1)/2) \pmod{m}\}$ and $\bar{R}_{11} = R\{a_r^2, a_{r-p}^3\} = V(\mathbb{P}_m^5) - \{a_h^1|h \equiv r-1 \pmod{m}\} \cup \{a_h^2|h \equiv r - ((p+1)/2) \pmod{m}\} \cup \{a_h^3|h \equiv r - ((p+m+1)/2) \pmod{m}\}$.

(mod m)), respectively. We can see from these sets that $|\overline{R}_u| = 3(m-1) > |R_r|$ and $|\overline{R}_u \cap \cup_{r=1}^m R_r| = 3m-4 \geq |R_r|$.

- (e) The RNs of $\{a_r^2, a_{r+s}^2\}, \{a_r^1, a_{r+p}^2\}, \{a_r^1, a_{r+s}^2\}, \{a_r^1, a_{r+p}^3\}$ and $R\{a_r^1, a_{r+s}^3\}$ are $\overline{R}_{12} = R\{a_r^2, a_{r+s}^2\} = V(\mathbb{P}_m^5) - \{a_h^2|h \equiv r + (s/2), r + ((s+m)/2) \pmod{m}\}$, $\overline{R}_{13} = R\{a_r^1, a_{r+p}^2\} = V(\mathbb{P}_m^5) - \{a_h^2|h \equiv r + ((p-1)/2) \pmod{m}\} \cup \{a_h^3|h \equiv r + ((p+m-1)/2) \pmod{m}\}$, $\overline{R}_{14} = R\{a_r^1, a_{r+s}^2\} = V(\mathbb{P}_m^5) - \{a_h^1|h \equiv r + ((m+s)/2) \pmod{m}\} \cup \{a_h^3|h \equiv r + ((s-2)/2) \pmod{m}\}$, $\overline{R}_{15} = R\{a_r^1, a_{r+p}^3\} = V(\mathbb{P}_m^5) - \{a_h^1|h \equiv r + ((p+1)/2) \pmod{m}\} \cup \{a_h^3|h \equiv r + ((p-1)/2) \pmod{m}\}$, respectively. We note that $|\overline{R}_u| = 3m - 2 > |R_r|$ and $|\overline{R}_u \cap \bigcup_{r=1}^m R_r| = 3m - 2 \geq |R_r|$.
- (f) The RN of a_r^2, a_r^3 is $|\overline{R}_{16}| = R\{a_r^2, a_r^3\} = V(\mathbb{P}_m^5) - \{a_h^2|h \equiv r+1, r+2, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^3|h \equiv r+1, r+2, \dots, r+(m/2)-1 \pmod{m}\}$.

Clearly, $|R\{a_r^1 a_r^2\}| = |R\{a_r^2 a_{r-1}^3\}| = (3m/2) + 1 < |R\{a_r^2 a_r^3\}|$ and $|R\{a_r^2 a_r^3\} \cap \bigcup_{r=1}^m R_r| = 2m + 1 \geq |R_r|$. $\square$

Theorem 7. *If $\mathbb{N} \cong \mathbb{P}_m^5$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{P}_m^5) \leq (6m/(3m+2))$.*

Proof

Case 1. When $m = 6$.

The RNs are given as follows.

In Tables 28–31, the cardinalities of RNs are 16, 15, 14, and 13, respectively. In the same manner, Table 32 is having RNs withholding the cardinality of 10. Moreover, $\bigcup_{r=1}^{12} R_r = V(\mathbb{P}_6^5)$ which implies $|\bigcup_{r=1}^{12} R_r| = 18$ and $|\overline{R}_r \cap \bigcup_{r=1}^{12} R_r| \geq |R_r|$.

Now, we define a function $\kappa: V(\mathbb{P}_6^5) \longrightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = \kappa(a_r^3) = (1/10)$. We have seen that R_r for $1 \leq t \leq 12$ of $\mathbb{P}_6^5$ are pairwise overlapping; hence, $\exists$ another minimal resolving function $\bar{\kappa}$ of $\mathbb{P}_6^5$ such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(\mathbb{P}_6^5) \leq \sum_{r=1}^{18^6} (1/10) = (9/5)$.

TABLE 28: $\bar{R}_u$ for $1 \leq u \leq 31$.

RNs	Elements	Equality
$R\{a_1^2, a_2^2\}$	$V(\mathbb{P}_6^5) - \{a_2^2, a_5^2\}$	$R\{a_4^2, a_6^2\}$
$R\{a_2^2, a_4^2\}$	$V(\mathbb{P}_6^5) - \{a_3^2, a_6^2\}$	$R\{a_1^2, a_5^2\}$
$R\{a_3^2, a_6^2\}$	$V(\mathbb{P}_6^5) - \{a_4^2, a_6^2\}$	$R\{a_2^2, a_6^2\}$
$R\{a_1^2, a_2^2\}$	$V(\mathbb{P}_6^5) - \{a_2^2\} \cup \{a_4^3\}$	$R\{a_2^2, a_6^2\}$
$R\{a_2^2, a_3^2\}$	$V(\mathbb{P}_6^5) - \{a_2^2\} \cup \{a_5^3\}$	$R\{a_1^2, a_4^3\}$
$R\{a_3^2, a_4^2\}$	$V(\mathbb{P}_6^5) - \{a_3^2\} \cup \{a_6^3\}$	$R\{a_2^2, a_5^3\}$
$R\{a_4^2, a_5^2\}$	$V(\mathbb{P}_6^5) - \{a_4^2\} \cup \{a_3^3\}$	$R\{a_3^2, a_6^3\}$
$R\{a_5^2, a_6^2\}$	$V(\mathbb{P}_6^5) - \{a_5^2\} \cup \{a_3^3\}$	
$R\{a_6^2, a_1^2\}$	$V(\mathbb{P}_6^5) - \{a_6^2\} \cup \{a_3^3\}$	
$R\{a_1^2, a_3^2\}$	$V(\mathbb{P}_6^5) - \{a_5^2\} \cup \{a_3^3\}$	
$R\{a_2^2, a_4^2\}$	$V(\mathbb{P}_6^5) - \{a_6^2\} \cup \{a_3^3\}$	
$R\{a_3^2, a_5^2\}$	$V(\mathbb{P}_6^5) - \{a_1^2\} \cup \{a_3^3\}$	
$R\{a_4^2, a_6^2\}$	$V(\mathbb{P}_6^5) - \{a_2^2\} \cup \{a_4^3\}$	
$R\{a_1^2, a_5^2\}$	$V(\mathbb{P}_6^5) - \{a_3^2\} \cup \{a_2^3\}$	
$R\{a_2^2, a_6^2\}$	$V(\mathbb{P}_6^5) - \{a_4^2\} \cup \{a_6^3\}$	
$R\{a_1^2, a_3^2\}$	$V(\mathbb{P}_6^5) - \{a_3^2\} \cup \{a_3^3\}$	
$R\{a_2^2, a_3^2\}$	$V(\mathbb{P}_6^5) - \{a_4^2\} \cup \{a_3^3\}$	
$R\{a_3^2, a_4^2\}$	$V(\mathbb{P}_6^5) - \{a_5^2\} \cup \{a_3^3\}$	
$R\{a_4^2, a_5^2\}$	$V(\mathbb{P}_6^5) - \{a_6^2\} \cup \{a_3^3\}$	
$R\{a_5^2, a_6^2\}$	$V(\mathbb{P}_6^5) - \{a_1^2\} \cup \{a_4^3\}$	$R\{a_1^2, a_3^3\}$
$R\{a_1^2, a_3^2\}$	$V(\mathbb{P}_6^5) - \{a_5^2\} \cup \{a_3^3\}$	$R\{a_2^2, a_5^3\}$
$R\{a_2^2, a_4^2\}$	$V(\mathbb{P}_6^5) - \{a_6^2\} \cup \{a_3^3\}$	$R\{a_3^2, a_6^3\}$
$R\{a_3^2, a_5^2\}$	$V(\mathbb{P}_6^5) - \{a_1^2\} \cup \{a_4^3\}$	
$R\{a_4^2, a_6^2\}$	$V(\mathbb{P}_6^5) - \{a_2^2\} \cup \{a_6^3\}$	
$R\{a_5^2, a_1^2\}$	$V(\mathbb{P}_6^5) - \{a_3^2\} \cup \{a_2^3\}$	
$R\{a_6^2, a_1^2\}$	$V(\mathbb{P}_6^5) - \{a_4^2\} \cup \{a_6^3\}$	

TABLE 29: $\overline{R}_u$ for $32 \leq u \leq 47$.

RNs	Elements	Equality
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_4^2\} \cup \{a_3^3\}$	$R\{a_4^1, a_5^1\}, R\{a_3^1, a_6^1\}, R\{a_3^2, a_6^2\}$
$R\{a_1^1, a_3^1\}$	$V(\mathbb{P}_6^5) - \{a_2^2, a_5^2\} \cup \{a_3^3\}$	$R\{a_1^1, a_4^1\}, R\{a_5^1, a_6^1\}, R\{a_2^1, a_4^1\}, R\{a_1^1, a_8^1\}, R\{a_5^2, a_8^2\}, R\{a_1^3, a_3^3\}$
$R\{a_1^1, a_4^1\}$	$V(\mathbb{P}_6^5) - \{a_3^2, a_6^2\} \cup \{a_3^3\}$	$R\{a_2^1, a_5^1\}, R\{a_1^1, a_6^1\}, R\{a_1^1, a_6^1\}$
$R\{a_2^1, a_3^1\}$	$V(\mathbb{P}_6^5) - \{a_6^1\} \cup \{a_2^2\} \cup \{a_3^3\}$	
$R\{a_2^1, a_5^1\}$	$V(\mathbb{P}_6^5) - \{a_1^1\} \cup \{a_2^2\} \cup \{a_6^3\}$	
$R\{a_3^1, a_6^1\}$	$V(\mathbb{P}_6^5) - \{a_2^1\} \cup \{a_5^2\} \cup \{a_3^3\}$	

TABLE 30: $\bar{R}_u$ for $48 \leq u \leq 62$.

RNs	Elements	Equality
$R\{a_1^1, a_3^3\}$	$V(\mathbb{P}_6^5) - \{a_1^1, a_5^1\} \cup \{a_2^2, a_5^2\}$	
$R\{a_1^1, a_4^3\}$	$V(\mathbb{P}_6^5) - \{a_1^1, a_6^1\} \cup \{a_2^2, a_6^2\}$	
$R\{a_1^1, a_5^3\}$	$V(\mathbb{P}_6^5) - \{a_1^1, a_5^1\} \cup \{a_1^2, a_2^2\}$	
$R\{a_1^1, a_6^3\}$	$V(\mathbb{P}_6^5) - \{a_1^1, a_6^1\} \cup \{a_2^2, a_5^2\}$	
$R\{a_1^1, a_5^3\}$	$V(\mathbb{P}_6^5) - \{a_1^1, a_5^1\} \cup \{a_1^2, a_6^2\}$	
$R\{a_1^1, a_6^3\}$	$V(\mathbb{P}_6^5) - \{a_1^1, a_6^1\} \cup \{a_2^2, a_5^2\}$	
$R\{a_1^1, a_2^3\}$	$V(\mathbb{P}_6^5) - \{a_2^1, a_5^1\} \cup \{a_2^2, a_3^2\}$	$R\{a_4^3, a_5^3\}, R\{a_1^3, a_4^3\}$
$R\{a_1^1, a_3^3\}$	$V(\mathbb{P}_6^5) - \{a_1^1, a_6^1\} \cup \{a_2^2, a_6^2\}$	$R\{a_5^3, a_6^3\}, R\{a_2^3, a_5^3\}$
$R\{a_1^1, a_4^3\}$	$V(\mathbb{P}_6^5) - \{a_1^1, a_4^1\} \cup \{a_1^2, a_4^2\}$	$R\{a_1^3, a_6^3\}, R\{a_3^3, a_6^3\}$
$R\{a_1^1, a_5^3\}$	$V(\mathbb{P}_6^5) - \{a_2^1, a_5^1\} \cup \{a_1^2, a_3^2\}$	$R\{a_4^3, a_6^3\}, R\{a_1^3, a_6^3\}, R\{a_2^3, a_4^3\}, R\{a_2^3, a_5^3\}, R\{a_1^2, a_2^2\}, R\{a_1^2, a_3^2\}, R\{a_1^2, a_4^2\}, R\{a_1^2, a_5^2\}, R\{a_1^2, a_6^2\}, R\{a_2^2, a_3^2\}, R\{a_2^2, a_4^2\}, R\{a_2^2, a_5^2\}, R\{a_2^2, a_6^2\}, R\{a_3^2, a_4^2\}, R\{a_3^2, a_5^2\}, R\{a_3^2, a_6^2\}, R\{a_4^2, a_5^2\}, R\{a_4^2, a_6^2\}, R\{a_5^2, a_6^2\}$
$R\{a_1^1, a_6^3\}$	$V(\mathbb{P}_6^5) - \{a_2^1, a_6^1\} \cup \{a_3^2, a_4^2\}$	
$R\{a_2^1, a_1^1\}$	$V(\mathbb{P}_6^5) - \{a_2^1, a_6^1\} \cup \{a_3^2, a_4^2\}$	
$R\{a_2^1, a_1^1\}$	$V(\mathbb{P}_6^5) - \{a_2^1, a_6^1\} \cup \{a_3^2, a_4^2\}$	
$R\{a_2^1, a_1^1\}$	$V(\mathbb{P}_6^5) - \{a_1^1, a_4^1\} \cup \{a_3^2, a_4^2\}$	

Case 2. $m \geq 8$.

It can be seen from Lemma 7 that among all the RNs of $\mathbb{P}_m^5$, $|R\{a_r^1 a_r^2\}| = |R\{a_r^2 a_r^3\}| = (3m/2) + 1 \leq |R\{a, b\}|$ for all $\{a, b\} \subset V(\mathbb{P}_m^5)$ and $|R\{a, b\} \cap \cup_{r=1}^m R_r| = (3m/2) + 1 \leq |R_r|$.

Let $\eta = |\cup_{r=1}^m R_r| = 3m$ and $\lambda = |R_r| = (3m/2) + 1$. Then, we define a function $\kappa: V(\mathbb{P}_m^5) \longrightarrow [0, 1]$ such that $\kappa(c) = \begin{cases} 1/\lambda & \text{for } c \in \cup_{r=1}^m R_r, \\ 0 & \text{for } c \in V(\mathbb{P}_m^5) - \cup_{r=1}^m R_r. \end{cases}$ It is

found that κ is a resolving function for $\mathbb{P}_m^5$ because $\kappa(R\{a, b\}) \geq 1 \forall \{a, b\} \subset V(\mathbb{P}_m^5)$. On contrary, assume that there is another resolving function ρ , such that $\rho(u) \leq \kappa(u)$, for at least one $u \in V(\mathbb{P}_m^5)$ $\rho(u) \neq \kappa(u)$. As a result, $\rho(R\{a, b\}) < 1$, where $R\{a, b\}$ is an RN of $\mathbb{P}_m^5$ having the minimum cardinality of λ . This shows that ρ is not a resolving function, a contradiction. Thus, κ is a minimal resolving function for $\mathbb{P}_m^5$ that achieves minimum $|\kappa|$. Moreover, R_t for $1 \leq t \leq 2m$ of $\mathbb{P}_m^5$ are

TABLE 31: $\bar{R}_u$ for $63 \leq u \leq 68$.

RNs	Elements
$R\{a_1^2, a_3^3\}$	$V(\mathbb{P}_6^5) - \{a_2^2, a_3^2, a_4^2\} \cup \{a_2^3, a_3^3\}$
$R\{a_2^2, a_3^3\}$	$V(\mathbb{P}_6^5) - \{a_3^2, a_4^2, a_5^2\} \cup \{a_3^3, a_4^3\}$
$R\{a_3^2, a_3^3\}$	$V(\mathbb{P}_6^5) - \{a_4^2, a_5^2, a_6^2\} \cup \{a_4^3, a_5^3\}$
$R\{a_4^2, a_4^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_2^2, a_6^2\} \cup \{a_1^3, a_2^3\}$
$R\{a_5^2, a_5^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_2^2, a_6^2\} \cup \{a_1^3, a_2^3\}$
$R\{a_6^2, a_6^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_2^2, a_3^2\} \cup \{a_1^3, a_2^3\}$

TABLE 32: R_r with minimum cardinalities for $1 \leq r \leq 12$.

RNs	Elements
$R_1 = R\{a_2^2, a_3^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_5^2, a_6^2\} \cup \{a_2^2, a_3^2, a_4^2\} \cup \{a_2^3, a_3^3\}$
$R_2 = R\{a_3^2, a_3^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_2^2, a_6^2\} \cup \{a_2^2, a_3^2, a_4^2\} \cup \{a_2^3, a_3^3\}$
$R_3 = R\{a_4^2, a_4^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_2^2, a_6^2\} \cup \{a_2^2, a_3^2, a_4^2\} \cup \{a_2^3, a_3^3\}$
$R_4 = R\{a_5^2, a_5^3\}$	$V(\mathbb{P}_6^5) - \{a_2^2, a_3^2, a_4^2\} \cup \{a_2^2, a_3^2, a_4^2\} \cup \{a_2^3, a_3^3\}$
$R_5 = R\{a_6^2, a_6^3\}$	$V(\mathbb{P}_6^5) - \{a_3^2, a_4^2, a_5^2\} \cup \{a_2^2, a_3^2, a_4^2\} \cup \{a_2^3, a_3^3\}$
$R_6 = R\{a_1^2, a_1^3\}$	$V(\mathbb{P}_6^5) - \{a_4^2, a_5^2, a_6^2\} \cup \{a_2^2, a_3^2, a_4^2\} \cup \{a_2^3, a_3^3\}$
$R_7 = R\{a_1^2, a_1^3\}$	$V(\mathbb{P}_6^5) - \{a_2^2, a_3^2, a_4^2\} \cup \{a_2^2, a_3^2, a_4^2\} \cup \{a_2^3, a_3^3\}$
$R_8 = R\{a_2^2, a_2^3\}$	$V(\mathbb{P}_6^5) - \{a_3^2, a_4^2, a_5^2\} \cup \{a_2^2, a_3^2, a_4^2\} \cup \{a_2^3, a_3^3\}$
$R_9 = R\{a_3^2, a_3^3\}$	$V(\mathbb{P}_6^5) - \{a_4^2, a_5^2, a_6^2\} \cup \{a_2^2, a_3^2, a_4^2\} \cup \{a_2^3, a_3^3\}$
$R_{10} = R\{a_4^2, a_4^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_5^2, a_6^2\} \cup \{a_2^2, a_3^2, a_4^2\} \cup \{a_2^3, a_3^3\}$
$R_{11} = R\{a_5^2, a_5^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_2^2, a_6^2\} \cup \{a_2^2, a_3^2, a_4^2\} \cup \{a_2^3, a_3^3\}$
$R_{12} = R\{a_6^2, a_6^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_2^2, a_3^2\} \cup \{a_2^2, a_3^2, a_4^2\} \cup \{a_2^3, a_3^3\}$

pairwise overlapping; hence, $\exists$ another minimal resolving function $\bar{\kappa}$ of $\mathbb{P}_m^5$ such that $|\bar{\kappa}| \leq |\kappa|$. Therefore, assigning $(1/\lambda)$ to all the vertices of $\mathbb{P}_m^5$ and calculating their sum, we get $\dim_f(\mathbb{P}_m^5) \leq \sum_{r=1}^m (1/\lambda) = (3m/(3m+2)/2) = (6m/(3m+2))$. $\square$

Corollary 3. If $\aleph \cong \mathbb{P}_m^6$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{P}_m^6) \leq (6m/(3m+2))$.

Proof. This can be derived easily as $\mathbb{P}_m^6 \cong \mathbb{P}_m^5$. $\square$

Lemma 8. Let $\aleph \cong \mathbb{P}_m^7$ be the 4-faced QCL with $m \geq 6$ and $m \equiv 0 \pmod{2}$. For $1 \leq l, r \leq m$, $p \geq 3$, $s \geq 2$ $p \equiv 1 \pmod{2}$ and $s \equiv 0 \pmod{2}$. Then,

- $|R_l| = |R\{a_l^1, a_l^3\}| = |R_r| = |R\{a_r^1, a_r^3\}|$, $|\cup_{l=1}^m R_l| = 3m$, $|\cup_{r=1}^m R_r| = 3m$ and $|\cup_{l=1}^m R_l| \cup |\cup_{r=1}^m R_r| = 3m$.
- $|R_r| < |\bar{R}_1| = |R\{a_r^1, a_{r+1}^1\}| = |\bar{R}_2| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_3| = |R\{a_r^2, a_{r+1}^2\}| = |\bar{R}_4| = |R\{a_r^2, a_{r+p}^2\}| = |\bar{R}_5| = |R\{a_r^1, a_{r+s}^1\}| = |\bar{R}_6| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_7| = |R\{a_r^2, a_{r+s}^2\}|$ with $a_{m+1}^1 = a_1^1$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_8| = |R\{a_r^1, a_r^2\}| = |\bar{R}_9| = |R\{a_r^1, a_{r+s}^1\}| = |\bar{R}_{10}| = |R\{a_r^2, a_{r+s}^2\}| = |\bar{R}_{11}| = |R\{a_r^3, a_{r+1}^3\}| = |\bar{R}_{12}| = |R\{a_r^3, a_{r+p}^3\}|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_{13}| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_{14}| = |R\{a_r^1, a_{r+s}^1\}|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_{15}| = |R\{a_r^2, a_r^3\}|$ and $|R\{a_r^2, a_r^3 \cap \cup_{r=1}^m R_r\}| \geq |R_r|$.
- $|R_r| < |\bar{R}_{16}| = |R\{a_r^2, a_{r-1}^2\}|$ and $|R\{a_r^2, a_{r-1}^2 \cap \cup_{r=1}^m R_r\}| \geq |R_r|$.
- $|R_r| < |\bar{R}_{17}| = |R\{a_r^2, a_{r+p}^2\}|$ and $|\bar{R}_{17} \cap \cup_{r=1}^m R_r| \geq |R_r|$.

$$(h) |R_r| < |\bar{R}_{18}| = |R\{a_r^1, a_{r+1}^1\}| \text{ and } |\bar{R}_{18} \cap \cup_{r=1}^m R_r| \geq |R_r|.$$

Proof

- The RNs of $\{a_l^1, a_l^3\}$ and a_r^1, a_{r-1}^3 are $R_l = R\{a_l^1, a_l^3\} = V(\mathbb{P}_m^7) - \{a_h^1 | h \equiv l+1, l+2, \dots, l+(m/2) \pmod{m}\} \cup \{a_h^3 | h \equiv l, l+2, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^3 | t \equiv l+1, l+2, \dots, l+(m/2) \pmod{m}\}$ and $R_r = R\{a_r^1, a_{r-1}^3\} = V(\mathbb{P}_m^7) - \{a_h^1 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r, r-2, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^3 | t \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\}$, respectively. Clearly, $|R_r| = (3/2)m + 1$ for $1 \leq r \leq m$.

We note that $\cup_{r=1}^m R_r = V(\mathbb{P}_m^7)$ and $|\cup_{r=1}^m R_r| = 3m$.

- The RNs of $\{a_r^1, a_{r+1}^1\}$, $\{a_r^1, a_{r+p}^1\}$, $\{a_r^2, a_{r+p}^2\}$, $\{a_r^1, a_{r+s}^1\}$, $\{a_r^2, a_{r+s}^2\}$ and $\{a_r^2, a_{r+p}^2\}$ are $\bar{R}_1 = R\{a_r^1, a_{r+1}^1\} = V(\mathbb{G}) - \{a_h^3 | h \equiv r, r+(m/2) \pmod{m}\} = \bar{R}_3 = R\{a_r^2, a_{r+1}^2\}$, $\bar{R}_2 = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{G}) - \{a_h^3 | h \equiv r + ((p-1)/2), r + ((p+m-1)/2) \pmod{m}\} = \bar{R}_4 = R\{a_r^2, a_{r+p}^2\} = \bar{R}_5 = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{G}) - \{a_h^3 | h \equiv r + (s/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r + ((m+s-2)/2) \pmod{m}\}$, $\bar{R}_6 = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{G}) - \{a_h^3 | h \equiv r + ((p+1)/2), r + ((p+m+1)/2) \pmod{m}\}$ and $\bar{R}_7 = R\{a_r^2, a_{r+s}^2\} = V(\mathbb{G}) - \{a_h^3 | h \equiv r + (s/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r + ((m+s)/2) \pmod{m}\}$, respectively. We can see that $|R_r| < |\bar{R}_u| = 3m - 2$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- The RNs of $\{a_r^1, a_r^2\}$, $\{a_r^1, a_{r+s}^1\}$, $\{a_r^2, a_{r+s}^2\}$, $\{a_r^3, a_{r+1}^3\}$ and $\{a_r^3, a_{r+p}^3\}$ are $\bar{R}_8 = R\{a_r^1, a_r^2\} = V(\mathbb{P}_m^7) - \{a_h^2 | h \equiv r-1, r+1 \pmod{m}\} \cup \{a_h^3 | h \equiv r, r-1 \pmod{m}\}$, $\bar{R}_9 = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{P}_m^7) - \{a_h^1 | h \equiv r + ((p+1)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((p+1)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((s+m)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((s+m)/2) \pmod{m}\}$, $\bar{R}_{10} = \{a_r^2, a_{r+s}^2\}$, $\bar{R}_{11} = R\{a_r^3, a_{r+1}^3\} = R\{a_r^2, a_{r+2}^2\}$ and $\bar{R}_{12} = R\{a_r^3, a_{r+p}^3\} = R\{a_r^2, a_{r+s}^2\}$ where $s \geq 4$, respectively.

Clearly, $R_r = 3m - 4 < |R\{a_r^1, a_r^2\}|$ and $|R\{a_r^1, a_r^2\} \cap \cup_{r=1}^m R_r| = 3m - 4 \geq |R_r|$.

- The RNs of $\{a_r^1, a_{r+p}^1\}$ and $\{a_r^1, a_{r+s}^1\}$ are $\bar{R}_{13} = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{G}) - \{a_h^1 | h \equiv r + ((p+1)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((p+1)/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r + ((p+m-1)/2) \pmod{m}\}$ and $\bar{R}_{14} = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{G}) - \{a_h^1 | h \equiv r + ((m+s)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((m+s)/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r + (s/2) \pmod{m}\}$, respectively. Clearly, $R_r = 3(m-1) < |R\{a_r^1, a_r^2\}|$ and $|R\{a_r^1, a_r^2\} \cap \cup_{r=1}^m R_r| = 3m - 4 \geq |R_r|$.
- The RN of a_r^2, a_r^3 is $\bar{R}_{15} = R\{a_r^2, a_r^3\} = V(\mathbb{P}_m^7) - \{a_h^1 | h \equiv r, r-1, \dots, r-(m/2)+1 \pmod{m}\}$. Clearly, $|R_r| = (5m/2) < |R\{a_r^2, a_r^3\}|$ and $|R\{a_r^2, a_r^3\} \cap \cup_{r=1}^m R_r| = ((5m+2)/2) \geq |R_r|$.
- The RN of a_r^2, a_{r-1}^2 is $\bar{R}_{16} = R\{a_r^2, a_r^3\} = V(\mathbb{P}_m^7) - \{a_h^1 | h \equiv r, r+1, \dots, r+(m/2)-1 \pmod{m}\}$. Clearly, $|R_r| = (5m/2) < |R\{a_r^2, a_r^3\}|$ and $|R\{a_r^2, a_r^3\} \cap \cup_{r=1}^m R_r| = ((5m+2)/2) \geq |R_r|$.

TABLE 33: $\bar{R}_u$ for $1 \leq u \leq 39$.

RNs	Elements	Equality
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1, a_5^1\}$	$R\{a_4^1, a_5^1\}, R\{a_2^1, a_3^1\}, R\{a_4^1, a_5^1\}, R\{a_4^1, a_7^1\}, R\{a_4^1, a_7^1\}$
$R\{a_1^1, a_3^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1, a_5^1\}$	$R\{a_1^1, a_6^1\}, R\{a_2^1, a_3^1\}, R\{a_2^1, a_5^1\}, R\{a_1^1, a_4^1\}, R\{a_2^1, a_4^1\}, R\{a_5^1, a_8^1\}, R\{a_5^1, a_8^1\}, R\{a_5^1, a_8^1\}$
$R\{a_1^1, a_4^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1, a_5^1\}$	$R\{a_7^1, a_8^1\}, R\{a_3^1, a_4^1\}, R\{a_7^1, a_8^1\}, R\{a_2^1, a_5^1\}, R\{a_2^1, a_5^1\}, R\{a_1^1, a_6^1\}, R\{a_1^1, a_6^1\}$
$R\{a_1^1, a_5^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1, a_5^1\}$	
$R\{a_1^1, a_6^1\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_4^1\}$	
$R\{a_1^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_1^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_2^1, a_3^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_2^1, a_4^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_2^1, a_5^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_2^1, a_6^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_2^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_2^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_3^1, a_4^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_3^1, a_5^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_3^1, a_6^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_3^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_3^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_4^1, a_5^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_4^1, a_6^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_4^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_4^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_5^1, a_6^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_5^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_5^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_6^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_6^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	

- (g) The RN of $\{a_r^2, a_{r+p}^3\}$ is $\bar{R}_{17} = R\{a_r^2, a_{r+p}^3\} = V(\mathbb{G}) - \{a_h^1 | h \equiv r + ((p+m-1)/2) \pmod{m}\}$.
Clearly, $|R_r| < |\bar{R}_{17}| = 3m-1$ and $|R\{a_r^2, a_{r+p}^3\} \cap \cup_{r=1}^m R_r| = ((5m+2)/2) \geq |R_r|$.
- (h) The RN of $\{a_r^1, a_{r+1}^2\}$ is $\bar{R}_{18} = R\{a_r^1, a_{r+1}^2\} = V(\mathbb{G}) - \{a_h^1 | h \equiv r+1, r+2, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r+3 \pmod{m}\} \cup \{a_h^3 | h \equiv r, r+2 \pmod{m}\}$.
Clearly, $|R_r| < |\bar{R}_{18}| = 3m-1$ and $|R\{a_r^1, a_{r+1}^2\} \cap \cup_{r=1}^m R_r| = ((5m+2)/2) \geq |R_r|$. $\square$

Theorem 8. If $\mathbb{N} \cong \mathbb{P}_m^7$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{P}_m^7) \leq (6m/(3m+2))$.

Proof

Case 1. When $m = 6$.

The RNs are given as follows.

Tables 33–37 show the RNs of $\mathbb{P}_6^7$ with cardinalities 16, 15, 14, 13, and 10, respectively. Moreover, in Table 37, $\cup_{r=1}^{12} R_r = V(\mathbb{P}_6^7)$ which implies $|\cup_{r=1}^{12} R_r| = 18$ and $|\bar{R}_u \cap \cup_{r=1}^{12} R_r| \geq |R_r|$, where $1 \leq u \leq 81$.

Now, we define a function $\kappa: V(\mathbb{P}_6^7) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = \kappa(a_r^3) = (1/10)$. Furthermore, R_r for $1 \leq r \leq 12$ are pairwise overlapping; hence, $\exists$ another minimal resolving function $\bar{\kappa}$ of $\mathbb{P}_6^7$ such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(\mathbb{P}_6^7) \leq \sum_{r=1}^{18} (1/10) = (9/5)$.

Case 2. $m \geq 8$.

As Lemma 8 clears the fact that $|R\{a_r^2, a_r^3\}| = |R\{a_r^2, a_{r-1}^3\}| = (3/2)m+1 \leq |R\{a, b\}|$ for all $\{a, b\} \subset V(\mathbb{P}_m^7)$ and $|R\{a, b\} \cap \cup_{r=1}^m R_r| \leq |R_r|$.

Let $\eta = |\cup_{r=1}^m R_r| = 3m$ and $\lambda = |R_r| = (3/2)m+1$. Then, we define a function $\kappa: V(\mathbb{P}_m^7) \rightarrow [0, 1]$ such that $\kappa(c) = \begin{cases} 1/\lambda & \text{for } c \in \cup_{r=1}^m R_r, \\ 0 & \text{for } c \in V(\mathbb{P}_m^7) - \cup_{r=1}^m R_r. \end{cases}$ We find that κ is a resolving function for $\mathbb{P}_m^7$ because $\kappa(R\{a, b\}) \geq 1 \forall \{a, b\} \subset V(\mathbb{P}_m^7)$. Assume on contrary that

TABLE 34: $\bar{R}_u$ for $40 \leq u \leq 51$.

RNs	Elements
$R\{a_1^1, a_4^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_1^1, a_5^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_1^1, a_6^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_1^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_1^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_2^1, a_3^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_2^1, a_4^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_2^1, a_5^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_2^1, a_6^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_2^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_2^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_3^1, a_4^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_3^1, a_5^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_3^1, a_6^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_3^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_3^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_4^1, a_5^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_4^1, a_6^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_4^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_4^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$

there is another resolving function ρ , such that $\rho(u) \leq \kappa(u)$, for at least one $u \in V(\mathbb{P}_m^7)$. As a result, $\rho(R\{a, b\}) < 1$, where $R\{a, b\}$ is an RN of $\mathbb{P}_m^7$ having the minimum cardinality of λ . This shows that ρ is not a resolving function, a contradiction. Thus, κ is a minimal resolving function for $\mathbb{P}_m^7$ that achieves minimum $|\kappa|$. Also R_r for $1 \leq t \leq 2m$ of $\mathbb{P}_m^7$ are pairwise overlapping; hence, $\exists$ another minimal resolving function $\bar{\kappa}$ of $\mathbb{P}_m^7$ such that $|\bar{\kappa}| \leq |\kappa|$. Therefore, assigning $(1/\lambda)$ to all the vertices of $\mathbb{P}_m^7$ and calculating their sum, we get $\dim_f(\mathbb{P}_m^7) \leq \sum_{r=1}^{\eta} (1/\lambda) = (3m/((3/2)m+1)) = (6m/(3m+2))$. $\square$

6. Conclusion

In this article,

- we have found the upper bounds of FMD of symmetric networks called by TCL ($\mathbb{T}_m$), QCLs ($\mathbb{Q}_m^1$ and $\mathbb{Q}_m^2$), and PCLs ($\mathbb{P}_m^1, \mathbb{P}_m^2, \mathbb{P}_m^3, \mathbb{P}_m^5$, and $\mathbb{P}_m^7$).
- Table 38 shows the summary of main results, and Table 39 gives the values of FMDs as they tend to ∞ .

TABLE 35: $\bar{R}_u$ for $52 \leq u \leq 69$.

RNs	Elements	Equality
$R\{a_1^1, a_1^2\}$	$V(\mathbb{P}_6^7) - \{a_2^2, a_2^3\} \cup \{a_1^3, a_6^3\}$	
$R\{a_1^1, a_2^2\}$	$V(\mathbb{P}_6^7) - \{a_1^2, a_3^2\} \cup \{a_1^3, a_2^3\}$	
$R\{a_1^1, a_3^2\}$	$V(\mathbb{P}_6^7) - \{a_2^2, a_2^3\} \cup \{a_2^3, a_3^3\}$	
$R\{a_1^1, a_4^2\}$	$V(\mathbb{P}_6^7) - \{a_3^2, a_5^2\} \cup \{a_3^3, a_4^3\}$	
$R\{a_1^1, a_5^2\}$	$V(\mathbb{P}_6^7) - \{a_4^2, a_5^2\} \cup \{a_4^3, a_5^3\}$	
$R\{a_1^1, a_6^2\}$	$V(\mathbb{P}_6^7) - \{a_2^2, a_5^2\} \cup \{a_5^3, a_6^3\}$	
$R\{a_1^1, a_1^3\}$	$V(\mathbb{P}_6^7) - \{a_1^2, a_5^2\} \cup \{a_2^2, a_2^3\}$	$R\{a_1^1, a_6^1\}, R\{a_1^2, a_2^2\}, R\{a_4^2, a_6^2\}$
$R\{a_1^1, a_1^4\}$	$V(\mathbb{P}_6^7) - \{a_1^2, a_6^2\} \cup \{a_3^2, a_6^2\}$	$R\{a_1^1, a_5^1\}, R\{a_2^2, a_5^2\}, R\{a_1^2, a_5^2\}$
$R\{a_1^1, a_1^5\}$	$V(\mathbb{P}_6^7) - \{a_1^2, a_4^2\} \cup \{a_2^2, a_4^2\}$	$R\{a_2^1, a_6^1\}, R\{a_3^2, a_5^2\}, R\{a_2^2, a_6^2\}$

TABLE 36: $\bar{R}_u$ for $70 \leq u \leq 81$.

RNs	Elements
$R\{a_1^2, a_1^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_4^3, a_5^3\}$
$R\{a_1^2, a_2^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_2^1, a_6^1\} \cup \{a_3^3, a_6^3\}$
$R\{a_1^2, a_3^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_1^3, a_5^3\}$
$R\{a_1^2, a_4^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_3^1, a_4^1\} \cup \{a_1^3, a_2^3\}$
$R\{a_1^2, a_5^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_4^1, a_5^1\} \cup \{a_2^3, a_3^3\}$
$R\{a_1^2, a_6^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_3^3, a_4^3\}$
$R\{a_2^2, a_1^3\}$	$V(\mathbb{P}_6^7) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_3^3, a_5^3\}$
$R\{a_2^2, a_2^3\}$	$V(\mathbb{P}_6^7) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_4^3, a_5^3\}$
$R\{a_2^2, a_3^3\}$	$V(\mathbb{P}_6^7) - \{a_4^1, a_5^1, a_6^1\} \cup \{a_5^3, a_6^3\}$
$R\{a_2^2, a_4^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_5^3, a_6^3\}$
$R\{a_2^2, a_5^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_1^3, a_2^3\}$
$R\{a_2^2, a_6^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_2^3, a_3^3\}$

TABLE 37: R_r with minimum cardinalities for $1 \leq r \leq 12$.

RNs	Elements
$R_1 = R\{a_1^1, a_1^3\}$	$V(\mathbb{P}_6^7) - \{a_2^1, a_3^1, a_4^1\} - \{a_2^2, a_3^2, a_4^2\} - \{a_2^3, a_3^3\}$
$R_2 = R\{a_1^1, a_2^3\}$	$V(\mathbb{P}_6^7) - \{a_3^1, a_4^1, a_5^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_3^3, a_4^3\}$
$R_3 = R\{a_1^1, a_3^3\}$	$V(\mathbb{P}_6^7) - \{a_4^1, a_5^1, a_6^1\} - \{a_3^2, a_5^2, a_6^2\} - \{a_4^3, a_5^3\}$
$R_4 = R\{a_1^1, a_4^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_5^1, a_6^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_3^3, a_6^3\}$
$R_5 = R\{a_1^1, a_5^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_2^1, a_6^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_1^3, a_6^3\}$
$R_6 = R\{a_1^1, a_6^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_2^1, a_3^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_1^3, a_2^3\}$
$R_7 = R\{a_1^1, a_1^4\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_5^1, a_6^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_3^3, a_6^3\}$
$R_8 = R\{a_1^1, a_2^4\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_2^1, a_6^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_1^3, a_6^3\}$
$R_9 = R\{a_1^1, a_3^4\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_2^1, a_3^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_1^3, a_2^3\}$
$R_{10} = R\{a_1^1, a_4^4\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_2^1, a_3^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_1^3, a_2^3\}$
$R_{11} = R\{a_1^1, a_5^4\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_4^1, a_5^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_3^3, a_4^3\}$
$R_{12} = R\{a_1^1, a_6^4\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_4^1, a_5^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_3^3, a_4^3\}$

TABLE 38: FMD of rotationally symmetric networks having different FMD for $m \geq 8$.

Network	Upper bound of $\dim_f$	Comment
$\mathbb{T}_m$	$2m/(m+1)$	Bounded
$\mathbb{Q}_m^1$	2	Bounded and constant
$\mathbb{Q}_m^2$	2	Bounded and constant
$\mathbb{P}_m^1$	$6m/(5m-2)$	Bounded
$\mathbb{P}_m^2$	$6m/(3m+2)$	Bounded
$\mathbb{P}_m^3$	$3m/(2m-1)$	Bounded
$\mathbb{P}_m^5$	$6m/(3m+2)$	Bounded
$\mathbb{P}_m^7$	$6m/(3m+2)$	Bounded

TABLE 39: Values of FMD as they tends to ∞ , where $m \geq 8 \wedge m \equiv 0 \pmod{2}$.

Network	Values of FMD as they tends to ∞
$\mathbb{T}_m$	$\lim_{m \rightarrow \infty} (2m/(m+1))$
$\mathbb{P}_m^2$	$\lim_{m \rightarrow \infty} (6m/(5m-2))$
$\mathbb{P}_m^3$	$\lim_{m \rightarrow \infty} (6m/(3m+2))$
$\mathbb{P}_m^5$	$\lim_{m \rightarrow \infty} (3m/(2m-1))$
$\mathbb{P}_m^7$	$\lim_{m \rightarrow \infty} (6m/(3m+2))$

- (iii) the networks having the maximum FMD of 2 are $\mathbb{Q}_m^1$ and $\mathbb{Q}_m^2$.
- (iv) in contrast, $\mathbb{T}_m$ is the network bearing the minimum FMD of $(2m/(m+1))$.
- (v) the obtained results are the generalization of [23].
- (vi) to find the extremal values of FMD of asymmetric networks is still an open problem.
- (vii) moreover, the evaluation of the FMD and LFMD of connected networks such as convex polytopes is an open problem as well.

Data Availability

The data used to support the findings of this study are included within this paper and are available from the corresponding author upon request.

Conflicts of Interest

The authors have no conflicts of interest.

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