Research Article

An Approximate Solution for a Class of Ill-Posed Nonhomogeneous Cauchy Problems

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In this paper, we consider a nonhomogeneous differential operator equation of first order $u'(t) + Au(t) = f(t)$. The coefficient operator A is linear unbounded and self-adjoint in a Hilbert space. We assume that the operator does not have a fixed sign. We associate to this equation the initial or final conditions $u(0) = \Phi$ or $u(T) = \Phi$. We note that the Cauchy problem is severely ill-posed in the sense that the solution if it exists does not depend continuously on the given data. Using a quasi-boundary value method, we obtain an approximate nonlocal problem depending on a small parameter. We show that regularized problem is well-posed and has a strongly solution. Finally, some convergence results are provided.

1. Introduction

The terms “inverse problems” and “ill-posed problems” have been steadily and surely gaining popularity in modern science since the middle of the 20th century. A little more than fifty years of studying problems of this kind have shown that a great number of problems from various branches of classical mathematics (computational algebra, differential and integral equations, partial differential equations, and functional analysis) can be classified as inverse or ill-posed, and they are among the most complicated ones (since they are unstable and usually nonlinear). At the same time, inverse and ill-posed problems began to be studied and applied systematically in physics, geophysics, medicine, astronomy, and all other areas of knowledge where mathematical methods are used. The reason is that solutions to inverse problems describe important properties of media under study, such as density and velocity of wave propagation, elasticity parameters, conductivity, dielectric permittivity and magnetic permeability, and properties and location of inhomogeneities in inaccessible areas.

Throughout this paper, H will denote a Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. A is a linear, unbounded self-adjoint operator which has a

continuous spectrum with nonfixed sign on H . We assume that A admits a bounded inverse A^{-1} within H and f is a given function in $C([0, T], H)$. Let T be a positive real number and Φ is an element in H . We consider the problem of finding a function $u: [0, T] \rightarrow H$, such that

$$\begin{cases} u'(t) + Au(t) = f(t), & t \in [0, T], \\ u(0) = \Phi \text{ or } u(T) = \Phi. \end{cases} \quad (1)$$

As is known, the nonhomogeneous problem is severely ill-posed in the sense of Hadamard, i.e., solutions do not always exist, and in the case of existence, these do not depend continuously on the given data. In fact, from small noise contaminated physical measurements, the corresponding solutions have large errors. It becomes difficult to do numerical calculations. Hence, a regularization is in order.

There have been many studies on the homogeneous case with the initial condition $u(0) = \Phi$ or Cauchy problems with final condition $u(T) = \Phi$ corresponding to A being of constant sign, using different approaches such as [1–6]. The unhomogeneous has been treated by many authors (see [7]).

We mention that the same problem for the homogeneous equation is treated by Yurchuk and Ababneh [8] and by Bessila [9] by introducing different nonlocal conditions.

In this paper, we introduce into unhomogeneous differential operator equation a nonlocal boundary condition depending on a small parameter $\varepsilon \in]0, 1[$ as follows:

$$\begin{cases} u'(t) + Au(t) = f(t), & t \in [0, T], \\ \varepsilon u(0) + (1 - \varepsilon)u(T) = \Phi, & \Phi \in H. \end{cases} \quad (2)$$

Throughout this work, we denote by $\{v_n\}_{n \geq 1}$ the eigenvectors corresponding to positive eigenvalues $\{\lambda_n\}_{n \geq 1}$ and by $\{w_n\}_{n \geq 1}$ the eigenvectors corresponding to negative eigenvalues $\{\mu_n\}_{n \geq 1}$ of the operator A . The eigenvectors of A $\{\{v_n\}_{n \geq 1}, \{w_n\}_{n \geq 1}\}$ form an orthogonal system in H with $\|v_n\| = 1$ and $\|w_n\| = 1, \forall n \geq 1$.

2. The Approximate Problem

We approximate problem (1) by the following problem:

$$\begin{cases} u'(t) + Au(t) = f(t), & 0 \leq t \leq T, \\ \varepsilon u(0) + (1 - \varepsilon)u(T) = \Phi, & \Phi \in H, \end{cases} \quad (3)$$

for $\Phi \in H, \varepsilon \in]0, 1[, t \in [0, T], f \in C([0, T], H)$.

In the base $\{\{v_n\}, \{w_n\}\}$, it can be written that

$$\begin{aligned} u_\varepsilon(t) = & \sum_{n \geq 1} \left[\frac{e^{-\lambda_n t}}{\varepsilon + (1 - \varepsilon)e^{-\lambda_n T}} \left(\xi_n - (1 - \varepsilon) \int_0^T g_n(s) e^{-\lambda_n(T-s)} ds \right) \right. \\ & + \left. \int_0^t g_n(s) e^{-\lambda_n(t-s)} ds \right] v_n \\ & + \sum_{n \geq 1} \left[\frac{e^{\mu_n(T-t)}}{\varepsilon e^{\mu_n T} + (1 - \varepsilon)} \left(\delta_n - (1 - \varepsilon) \int_0^T h_n(s) e^{-\mu_n(T-s)} ds \right) \right. \\ & + \left. \int_0^t h_n(s) e^{-\mu_n(t-s)} ds \right] w_n. \end{aligned} \quad (8)$$

2.1. Notation. If we denote

$$\begin{aligned} M_{n,\varepsilon} &= (\varepsilon + (1 - \varepsilon)e^{-\lambda_n T})^{-1}, \\ N_{n,\varepsilon} &= (\varepsilon e^{\mu_n T} + (1 - \varepsilon))^{-1}, \\ I_t(g_n) &= \int_0^t g_n(s) e^{-\lambda_n(t-s)} ds, \\ I_t(h_n) &= \int_0^t h_n(s) e^{-\mu_n(t-s)} ds, \end{aligned} \quad (9)$$

then the function $u_\varepsilon(t)$ can be written as follows:

where A is as above and $f \in C([0, T], H)$ is represented by

$$f(t) = \sum_{n \geq 1} g_n(t) v_n + \sum_{n \geq 1} h_n(t) w_n, \quad (4)$$

where

$$\begin{aligned} g_n(t) &= \langle f(t), v_n \rangle, \\ h_n(t) &= \langle f(t), w_n \rangle, \quad \forall n \geq 1. \end{aligned} \quad (5)$$

The vector $\Phi \in H$ can be represented as follows:

$$\Phi = \sum_{n \geq 1} \xi_n(t) v_n + \sum_{n \geq 1} \delta_n(t) w_n, \quad (6)$$

where $\xi_n = \langle v_n, \Phi \rangle$ and $\delta_n = \langle w_n, \Phi \rangle$.

The classical solution of problem (2) is the function as follows:

$$u_\varepsilon(t) = \left[\Phi - (1 - \varepsilon) \int_0^T f(s) e^{-A(T-s)} ds \right] \frac{e^{-AT}}{\varepsilon + (1 - \varepsilon)e^{-AT}} + \int_0^t f(s) e^{-A(t-s)} ds, \quad (7)$$

$$\begin{aligned} u_\varepsilon(t) = & \sum_{n \geq 1} \left[e^{-\lambda_n t} M_{n,\varepsilon} (\xi_n - (1 - \varepsilon) I_T(g_n)) + I_t(g_n) \right] v_n \\ & + \sum_{n \geq 1} \left[e^{\mu_n(T-t)} N_{n,\varepsilon} (\delta_n - (1 - \varepsilon) I_T(h_n)) + I_t(h_n) \right] w_n. \end{aligned} \quad (10)$$

Theorem 1.

Problem (3) is well-posed, and its unique solution is $u_\varepsilon(t)$ given by (8). Furthermore,

$$\sup_{0 \leq t \leq T} \|u_\varepsilon(t)\|^2 \leq \max \left\{ \frac{2}{\varepsilon^2}, \frac{2}{(1 - \varepsilon)^2} \right\} \|\Phi\|^2 + 2T^2 \|f\|^2. \quad (11)$$

Proof. For each $0 \leq t \leq T$, we have

$$\begin{aligned} I_t(g_n) &\leq T \|g_n\|, \\ I_t(h_n) &\leq T \|h_n\|, \end{aligned} \quad (12)$$

$$\begin{aligned} \frac{e^{-\lambda_n t}}{\varepsilon + (1 - \varepsilon)e^{-\lambda_n T}} &\leq \frac{1}{\varepsilon}, \\ \frac{e^{\mu_n(T-t)}}{\varepsilon e^{\mu_n T} + (1 - \varepsilon)} &\leq \frac{1}{1 - \varepsilon}, \end{aligned} \quad (13)$$

and by using relations (12) and (13), we obtain

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \left\| \sum_{n=k}^{k+p} \left[\frac{e^{-\lambda_n t}}{\varepsilon + (1-\varepsilon)e^{-\lambda_n T}} \left((v_n, \Phi) - (1-\varepsilon) \int_0^T g_n(s) e^{-\lambda_n(T-s)} ds \right) + \int_0^t g_n(s) e^{-\lambda_n(t-s)} ds \right] v_n \right\|^2 \\
& + \sup_{0 \leq t \leq T} \left\| \sum_{n=k}^{k+p} \left[\frac{e^{\mu_n(T-t)}}{\varepsilon e^{\mu_n T} + (1-\varepsilon)} \left(\langle w_n, \Phi \rangle - (1-\varepsilon) \int_0^T h_n(s) e^{-\mu_n(T-s)} ds \right) + \int_0^t h_n(s) e^{-\mu_n(t-s)} ds \right] w_n \right\|^2 \\
& \leq 2 \left[\frac{1}{\varepsilon^2} \sum_{n=k}^{k+p} (v_n, \Phi)^2 + \frac{1}{(1-\varepsilon)^2} \sum_{n=k}^{k+p} \langle w_n, \Phi \rangle^2 + T^2 \sum_{n=k}^{k+p} \|g_n\|^2 + T^2 \sum_{n=k}^{k+p} \|h_n\|^2 \right].
\end{aligned} \tag{14}$$

Then,

$$\sup_{0 \leq t \leq T} \|u_\varepsilon(t)\|^2 \leq \max \left\{ \frac{2}{\varepsilon^2}, \frac{2}{(1-\varepsilon)^2} \right\} \|\Phi\|^2 + 2T^2 \|f\|^2, \tag{15}$$

and we deduce the convergence of series (10) which is the strong solution of problem (3). $\square$

3. The Convergence Results

Lemma 1. For all $\varepsilon \in]0, 1[$, we have the following estimates:

$$0 \leq \frac{\varepsilon(1 - e^{-\lambda_n T})}{\varepsilon + (1-\varepsilon)e^{-\lambda_n T}} \leq 1 - e^{-\lambda_n T} \leq 1, \tag{16}$$

$$\frac{\varepsilon}{\varepsilon + (1-\varepsilon)e^{-\lambda_n T}} \leq 1, \tag{17}$$

$$-1 \leq \frac{e^{\mu_n T} - 1}{e^{\mu_n T} + 1} \leq \frac{\varepsilon(e^{\mu_n T} - 1)}{\varepsilon e^{\mu_n T} + (1-\varepsilon)} \leq 0, \tag{18}$$

$$\frac{\varepsilon e^{\mu_n T}}{\varepsilon e^{\mu_n T} + (1-\varepsilon)} \leq 1, \tag{19}$$

$$\frac{\varepsilon^2(1 - e^{-\lambda_n T})^2}{(\varepsilon + (1-\varepsilon)e^{-\lambda_n T})^2} \leq \frac{\varepsilon^2 e^{2\lambda_n T}}{(1-\varepsilon)^2}, \tag{20}$$

$$\frac{\varepsilon^2(e^{\mu_n T} - 1)^2}{(\varepsilon e^{\mu_n T} + (1-\varepsilon))^2} \leq \frac{\varepsilon^2}{(1-\varepsilon)^2}, \tag{21}$$

$$\frac{\varepsilon^2 e^{2\mu_n T}}{(\varepsilon e^{\mu_n T} + (1-\varepsilon))^2} \leq \frac{\varepsilon^2}{(1-\varepsilon)^2}, \tag{22}$$

where $\lambda_N = \max_{n \in \mathbb{N}} \lambda_n$.

Proof. (i) The value $(\varepsilon(1 - e^{-\lambda_n T})/(\varepsilon + (1-\varepsilon)e^{-\lambda_n T}))$ is positive and increasing with ε , and its derivative with respect to ε is as follows:

$$\frac{e^{-\lambda_n T}(1 - e^{-\lambda_n T})}{(\varepsilon + (1-\varepsilon)e^{-\lambda_n T})^2} \geq 0. \tag{23}$$

It takes its greatest value when $\varepsilon \rightarrow 1$, i.e.,

$$0 \leq \frac{\varepsilon(1 - e^{-\lambda_n T})}{\varepsilon + (1-\varepsilon)e^{-\lambda_n T}} \leq 1 - e^{-\lambda_n T} \leq 1. \tag{24}$$

(ii) We know that for all $\varepsilon \in]0, 1[$, we have

$$0 \leq (1-\varepsilon)e^{-\lambda_n T} \leq 1, \tag{25}$$

and then

$$\frac{\varepsilon}{\varepsilon + (1-\varepsilon)e^{-\lambda_n T}} \leq 1. \tag{26}$$

(iii) On the other hand, the following value:

$$\frac{\varepsilon(e^{\mu_n T} - 1)}{\varepsilon e^{\mu_n T} + (1-\varepsilon)}, \tag{27}$$

is negative and decreasing with ε as long as its derivative with respect to ε is as follows:

$$\frac{e^{\mu_n T} - 1}{(\varepsilon e^{\mu_n T} + (1-\varepsilon))^2} \leq 0, \tag{28}$$

and it takes its lowest value at $\varepsilon = (1/2)$, i.e.,

$$-1 \leq \frac{e^{\mu_n T} - 1}{e^{\mu_n T} + 1} \leq \frac{\varepsilon(e^{\mu_n T} - 1)}{\varepsilon e^{\mu_n T} + (1-\varepsilon)} \leq 0. \tag{29}$$

(iv) We have also

$$0 \leq \varepsilon e^{\mu_n T} \leq \varepsilon e^{\mu_n T} + (1-\varepsilon), \tag{30}$$

and then we obtain relation (19).

(v) For estimate (20), we have

$$\begin{aligned}
\varepsilon^2(1 - e^{-\lambda_n T})^2 & \leq \varepsilon^2, \\
(1-\varepsilon)e^{-\lambda_n T} & \leq \varepsilon + (1-\varepsilon)e^{-\lambda_n T},
\end{aligned} \tag{31}$$

and then

$$\frac{\varepsilon^2(1 - e^{-\lambda_n T})^2}{(\varepsilon + (1-\varepsilon)e^{-\lambda_n T})^2} \leq \frac{\varepsilon^2 e^{2\lambda_n T}}{(1-\varepsilon)^2} \leq \frac{\varepsilon^2 e^{2\lambda_N T}}{(1-\varepsilon)^2}. \tag{32}$$

(vi) Furthermore, we have

$$\begin{aligned} \varepsilon^2(e^{\mu_n T} - 1)^2 &\leq \varepsilon^2, \\ \frac{1}{(\varepsilon e^{\mu_n T} + (1 - \varepsilon))^2} &\leq \frac{1}{(1 - \varepsilon)^2}, \\ \frac{\varepsilon e^{\mu_n T}}{\varepsilon e^{\mu_n T} + (1 - \varepsilon)} &\leq \frac{\varepsilon e^{\mu_n T}}{1 - \varepsilon} \leq \frac{\varepsilon}{1 - \varepsilon}, \end{aligned} \quad (33)$$

and then we deduce (21) and (22). $\square$

Theorem 2.

For all $\Phi \in H$, we have

$$\lim_{\varepsilon \rightarrow 0} \|\Phi - u_\varepsilon(T)\| = 0. \quad (34)$$

Proof. By definitions (6) and (8), we have

$$\begin{aligned} \|\Phi - u_\varepsilon(T)\|^2 &= \sum_{n \geq 1} \left[\frac{\varepsilon(1 - e^{-\lambda_n T})}{\varepsilon + (1 - \varepsilon)e^{-\lambda_n T}} \langle v_n, \Phi \rangle - \frac{\varepsilon}{\varepsilon + (1 - \varepsilon)e^{-\lambda_n T}} I_T(g_n) \right]^2 \\ &\quad + \sum_{n \geq 1} \left[\frac{\varepsilon(e^{\mu_n T} - 1)}{\varepsilon e^{\mu_n T} + (1 - \varepsilon)} \langle w_n, \Phi \rangle - \frac{\varepsilon e^{\mu_n T}}{\varepsilon e^{\mu_n T} + (1 - \varepsilon)} I_T(h_n) \right]^2. \end{aligned} \quad (35)$$

Using inequalities (12) and (16)–(19) (see Lemma 1), we obtain

$$\|\Phi - u_\varepsilon(T)\|^2 \leq 2(\|\Phi\|^2 + T^2\|f\|^2). \quad (36)$$

Now, we demonstrate that

$$\lim_{\varepsilon \rightarrow 0} \|\Phi - u_\varepsilon(T)\| = 0, \quad \forall \Phi \in M, \quad (37)$$

where M is some special set dense in the Hilbert space H . We take all Φ and f on M of the form as follows:

$$\begin{aligned} \Phi &= \sum_{n=1}^N \xi_n v_n + \sum_{n=1}^{+\infty} \delta_n w_n, \\ f &= \sum_{n=1}^N g_n v_n + \sum_{n=1}^{+\infty} h_n w_n, \end{aligned} \quad (38)$$

$\forall N < \infty$. Then, by virtue of the Banach–Steinhaus theorem, we obtain

$$\begin{aligned} \|\Phi - u_\varepsilon(T)\|^2 &\leq 2 \sum_{n=1}^N \frac{\varepsilon^2(1 - e^{-\lambda_n T})^2}{(\varepsilon + (1 - \varepsilon)e^{-\lambda_n T})^2} \langle v_n, \Phi \rangle^2 + 2 \sum_{n=1}^N \frac{\varepsilon^2}{(\varepsilon + (1 - \varepsilon)e^{-\lambda_n T})^2} I_T^2(g_n) \\ &\quad + 2 \sum_{n=1}^{+\infty} \frac{\varepsilon^2(e^{\mu_n T} - 1)^2}{(\varepsilon e^{\mu_n T} + (1 - \varepsilon))^2} \langle w_n, \Phi \rangle^2 + 2 \sum_{n=1}^{+\infty} \frac{\varepsilon^2 e^{2\mu_n T}}{(\varepsilon e^{\mu_n T} + (1 - \varepsilon))^2} I_T^2(h_n). \end{aligned} \quad (39)$$

Combining relations (12), (17), and (20)–(22) (see lemma 1), we obtain

$$\begin{aligned} \|\Phi - u_\varepsilon(T)\|^2 &\leq 2 \frac{\varepsilon^2 e^{2\lambda_N T}}{(1 - \varepsilon)^2} \left[\sum_{n=1}^N \langle v_n, \Phi \rangle^2 + T^2 \sum_{n=1}^N \|g_n\|^2 \right] \\ &\quad + 2 \frac{\varepsilon^2}{(1 - \varepsilon)^2} \left[\sum_{n=1}^{+\infty} \langle w_n, \Phi \rangle^2 + T^2 \sum_{n=1}^{+\infty} \|h_n\|^2 \right] \\ &\leq 2 \frac{\varepsilon^2 e^{2\lambda_N T}}{(1 - \varepsilon)^2} (\|\Phi\|^2 + T^2\|f\|^2), \end{aligned} \quad (40)$$

which gives that $\lim_{\varepsilon \rightarrow 0} \|\Phi - u_\varepsilon(T)\|^2 = 0$. $\square$

Lemma 2.

For all $\varepsilon \in]0, 1[$, we have the following inequalities:

$$-1 \leq \frac{e^{-\lambda_n T} - 1}{e^{-\lambda_n T} + 1} \leq \frac{(\varepsilon - 1)(1 - e^{-\lambda_n T})}{\varepsilon + (1 - \varepsilon)e^{-\lambda_n T}} \leq 0, \quad (41)$$

$$-1 \leq \frac{1 - \varepsilon}{1 + \varepsilon} \leq \frac{1 - \varepsilon}{\varepsilon + (1 - \varepsilon)e^{-\lambda_n T}}, \quad (42)$$

$$0 \leq \frac{(1 - \varepsilon)(1 - e^{\mu_n T})}{\varepsilon e^{\mu_n T} + (1 - \varepsilon)} \leq 1 - e^{\mu_n T} \leq 1, \quad (43)$$

$$\frac{(1 - \varepsilon)e^{\mu_n T}}{\varepsilon e^{\mu_n T} + (1 - \varepsilon)} \leq 1, \quad (44)$$

$$\frac{1-\varepsilon}{\varepsilon+(1-\varepsilon)e^{-\lambda_n T}} \leq \frac{1-\varepsilon}{\varepsilon}, \quad (45)$$

$$\frac{(\varepsilon-1)^2(1-e^{-\lambda_n T})^2}{(\varepsilon+(1-\varepsilon)e^{-\lambda_n T})^2} \leq \frac{(1-\varepsilon)^2}{\varepsilon^2}, \quad (46)$$

$$\frac{(1-\varepsilon)(1-e^{\mu_n T})}{\varepsilon e^{\mu_n T} + (1-\varepsilon)} \leq \frac{1-\varepsilon}{\varepsilon e^{\mu_N T}}, \quad (47)$$

$$\frac{(1-\varepsilon)e^{\mu_n T}}{\varepsilon e^{\mu_n T} + (1-\varepsilon)} \leq \frac{1-\varepsilon}{\varepsilon e^{\mu_N T}}, \quad (48)$$

where $\mu_N = \min_{n \geq 1} \mu_n$.

Proof. (i) For all $\varepsilon \in]0, 1[$, the following value:

$$\frac{(\varepsilon-1)(1-e^{-\lambda_n T})}{\varepsilon+(1-\varepsilon)e^{-\lambda_n T}}, \quad (49)$$

is negative and increasing with ε since its derivative with respect to ε is as follows:

$$\frac{1-e^{-\lambda_n T}}{(\varepsilon+(1-\varepsilon)e^{-\lambda_n T})^2} \geq 0, \quad (50)$$

and it takes its lowest value when $\varepsilon = (1/2)$, and then we deduce (41).

(ii) We have also

$$\frac{1}{1+\varepsilon} \leq \frac{1}{\varepsilon+(1-\varepsilon)e^{-\lambda_n T}} \leq \frac{1}{\varepsilon}, \quad (51)$$

and then

$$-1 \leq \frac{1-\varepsilon}{1+\varepsilon} \leq \frac{1}{\varepsilon+(1-\varepsilon)e^{-\lambda_n T}}. \quad (52)$$

On the other hand, the following function:

$$\frac{(1-\varepsilon)(1-e^{\mu_n T})}{\varepsilon e^{\mu_n T} + (1-\varepsilon)}, \quad (53)$$

is positive and decreasing with ε since the derivative with respect to ε is as follows:

$$\frac{e^{\mu_n T}(e^{\mu_n T}-1)}{(\varepsilon e^{\mu_n T} + (1-\varepsilon))^2} \leq 0, \quad (54)$$

and it takes its greatest value for $\varepsilon = 0$, and then we obtain (43).

(iii) Furthermore, we know that for all $\varepsilon \in]0, 1[$,

$$\frac{1}{\varepsilon e^{\mu_n T} + (1-\varepsilon)} \leq \frac{1}{1-\varepsilon}, \quad (55)$$

and then relation (44) is obtained.

(iv) We have

$$\begin{aligned} 0 &\leq (1-\varepsilon)e^{-\lambda_n T} \leq 1-\varepsilon, \\ \varepsilon &\leq (1-\varepsilon)e^{-\lambda_n T} \leq 1+\varepsilon, \end{aligned} \quad (56)$$

and then we deduce (45) and (46).

(v) Finally, with simple increases, we can easily have inequalities (47) and (48). $\square$

Theorem 3. .

For all $\Phi \in H$, we have

$$\lim_{\varepsilon \rightarrow 1} \|\Phi - u_\varepsilon(0)\| = 0. \quad (57)$$

Proof. We have

$$\begin{aligned} \Phi - u_\varepsilon(0) &= \sum_{n \geq 1} \left[\frac{(\varepsilon-1)(1-e^{-\lambda_n T})}{\varepsilon+(1-\varepsilon)e^{-\lambda_n T}} \langle v_n, \Phi \rangle + \frac{1-\varepsilon}{\varepsilon+(1-\varepsilon)e^{-\lambda_n T}} I_T(g_n) \right] v_n \\ &\quad + \sum_{n \geq 1} \left[\frac{(1-\varepsilon)(1-e^{\mu_n T})}{\varepsilon e^{\mu_n T} + (1-\varepsilon)} \langle w_n, \Phi \rangle + \frac{(1-\varepsilon)e^{\mu_n T}}{\varepsilon e^{\mu_n T} + (1-\varepsilon)} I_T(h_n) \right] w_n, \end{aligned} \quad (58)$$

and then by using (12) and (41)–(44), we obtain

$$\|\Phi - u_\varepsilon(0)\|^2 \leq 2(\|\Phi\|^2 + T^2\|f\|^2). \quad (59)$$

Now, we prove that

$$\lim_{\varepsilon \rightarrow 1} \|\Phi - u_\varepsilon(0)\| = 0, \quad \forall \Phi \in M, \quad (60)$$

where M is a particular set dense in H . We choose Φ and f on M having the form:

$$\begin{aligned} \Phi &= \sum_{n=1}^{+\infty} \xi_n v_n + \sum_{n=1}^N \delta_n w_n, \\ f &= \sum_{n=1}^{+\infty} g_n v_n + \sum_{n=1}^N h_n w_n, \end{aligned} \quad (61)$$

$\forall N < \infty$. Then,

$$\begin{aligned} \|\Phi - u_\varepsilon(0)\|^2 &= \sum_{n=1}^{+\infty} \left[\frac{(\varepsilon - 1)(1 - e^{-\lambda_n T})}{\varepsilon + (1 - \varepsilon)e^{-\lambda_n T}} \langle v_n, \Phi \rangle + \frac{1 - \varepsilon}{\varepsilon + (1 - \varepsilon)e^{-\lambda_n T}} I_T(g_n) \right]^2 \\ &\quad + \sum_{n=1}^N \left[\frac{(1 - \varepsilon)(1 - e^{\mu_n T})}{\varepsilon e^{\mu_n T} + (1 - \varepsilon)} \langle w_n, \Phi \rangle + \frac{(1 - \varepsilon)e^{\mu_n T}}{\varepsilon e^{\mu_n T} + (1 - \varepsilon)} I_T(h_n) \right]^2. \end{aligned} \quad (62)$$

Using inequalities (45)–(48) in Lemma 2 and relation (12), we obtain that

$$\begin{aligned} \|\Phi - u_\varepsilon(0)\|^2 &\leq 2 \frac{(1 - \varepsilon)^2}{\varepsilon^2} \left[\sum_{n=1}^{+\infty} \langle v_n, \Phi \rangle^2 + T^2 \sum_{n=1}^{+\infty} \|g_n\|^2 \right] \\ &\quad + 2 \frac{(1 - \varepsilon)^2}{\varepsilon^2} e^{-2\mu_n T} \left[\sum_{n=1}^N \langle w_n, \Phi \rangle^2 + T^2 \sum_{n=1}^N \|h_n\|^2 \right] \\ &\leq 2 \frac{(1 - \varepsilon)^2}{\varepsilon^2} e^{-2\mu_n T} (\|\Phi\|^2 + T^2 \|f\|^2). \end{aligned} \quad (63)$$

Finally, according to the Banach–Steinhaus theorem, (57) results from (59) and (63). $\square$

Conclusion 1. Note that, in this work, using a quasi-boundary value method, we study the nonhomogeneous differential-operator equation introducing nonlocal conditions. The results given in this paper generalized the results of the work given by Yurchuk and Ababneh [8] where they considered the homogeneous case. [10–16]

Data Availability

The content of our article is devoid of any particular data and therefore does not require any provision in this direction.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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