Research Article

Some Fixed Point Theorems of Contractive Mappings in $\mathcal{P}_b^r$ -Cone Metric Spaces over Banach Algebras

Abba Auwalu ^{1,2} and Evren Hinçal²

¹Department of Mathematics and Computer Science, Sule Lamido University Kafin Hausa, P. M. B. 048, Kafin Hausa, Jigawa, Nigeria

²Department of Mathematics, Near East University, Nicosia, North Cyprus, Via Mersin 10, Turkey

Correspondence should be addressed to Abba Auwalu; abba.auwalu@slu.edu.ng

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In this paper, we introduce the concept of a $\mathcal{P}_b^r$ -cone metric space over Banach algebras and prove some fixed point results under various contractive mappings in such a space. Some examples are given to elucidate the results. Our results extend and generalize many existing results in the literature.

1. Introduction

In 2017, George et al. [1] introduced the concept of a rectangular cone b -metric space over Banach algebras as a generalization of metric space and many of its generalizations. They proved some fixed point results in such a space. Very recently, Fernandez et al. [2] introduced $\mathcal{P}_b$ -cone metric space over Banach algebras as a generalization of partial metric space and many of its generalizations. Motivated and inspired by these papers [1, 2], we introduce the concept of a $\mathcal{P}_b^r$ -cone metric space over Banach algebras which generalized both rectangular cone b -metric space over Banach algebras and $\mathcal{P}_b$ -cone metric space over Banach algebras. Furthermore, we prove some fixed point results under various contractive mappings in such a space. Examples are also given to elucidate our results. Our results extend and generalize many results in [1, 3–10].

2. Preliminaries

We start with definitions and some basic facts needed in the sequel.

Definition 1 (see [11]). Let $\mathcal{A}$ be a real Banach algebra, i.e., $\mathcal{A}$ is a real Banach space in which an operation of

multiplication is defined, for all $y, z, x \in \mathcal{A}$ and $k \in \mathbb{R}$, and the following are satisfied:

- (i) $y(zx) = (yz)x$
- (ii) $y(z + x) = yz + yx$ and $(y + z)x = yx + zx$
- (iii) $k(yz) = (ky)z = y(kz)$
- (iv) $\|yz\| \leq \|y\|\|z\|$

A Banach algebra $\mathcal{A}$ is called unital if there exists a unit $e \in \mathcal{A}$ such that $ey = ye = y$, for any $y \in \mathcal{A}$.

Definition 2 (see [5]). A subset $\mathcal{K}$ of Banach algebra $\mathcal{A}$ is called a cone if

- (i) $\mathcal{K}$ is nonempty, closed, and $\{\theta, e\} \subset \mathcal{A}$, where θ is the zero of $\mathcal{A}$
- (ii) $\alpha\mathcal{K} + \beta\mathcal{K} \subset \mathcal{K}$ for all nonnegative real numbers α, β
- (iii) $\mathcal{K}^2 = \mathcal{K}\mathcal{K} \subset \mathcal{K}$
- (iv) $\mathcal{K} \cap (-\mathcal{K}) = \{\theta\}$

For a given cone $\mathcal{K} \subset \mathcal{A}$, we define a partial ordering $\preceq$ with respect to $\mathcal{K}$ by $y \preceq z$ if and only if $z - y \in \mathcal{K}$. The notation $y \ll z$ will stand for $z - y \in \mathcal{K}^\circ$, where $\mathcal{K}^\circ$ denotes the interior of $\mathcal{K}$. $\mathcal{K}$ is called a solid cone if $\mathcal{K}^\circ \neq \emptyset$.

Definition 3 (see [6]). Let $\mathcal{K}$ be a solid cone in a Banach algebra $\mathcal{A}$. A sequence $\{y_n\} \subset \mathcal{K}$ is said to be a c -sequence if, for every $c \in \mathcal{K}^\circ$, there exists $N \in \mathbb{N}$ such that $y_n \ll c$ for all $n > N$.

Lemma 1 (see [8]). Let $\mathcal{K}$ be a solid cone in a Banach algebra $\mathcal{A}$:

- (1) If $\alpha, \beta \in \mathcal{A}$, $\gamma \in \mathcal{K}$, and $\alpha \leq \beta$, then $\gamma\alpha \leq \gamma\beta$
- (2) If $\alpha \leq \beta\alpha$, where $\alpha, \beta \in \mathcal{K}$ and $\rho(\beta) < 1$, then $\alpha = \theta$
- (3) If $\alpha \in \mathcal{K}$ and $\rho(\alpha) < 1$, then $\rho(\alpha^q) < 1$ for any fixed $q \in \mathbb{N}$

Lemma 2 (see [11, 12]). Let $\mathcal{A}$ be a unital Banach algebra and $\alpha \in \mathcal{A}$; then, $\lim_{m \rightarrow \infty} \|\alpha^m\|^{(1/m)}$ exists and the spectral radius $\rho(\alpha)$ satisfies

$$\rho(\alpha) = \lim_{m \rightarrow \infty} \|\alpha^m\|^{(1/m)} = \inf_{m \geq 1} \|\alpha^m\|^{(1/m)}. \quad (1)$$

If $\rho(\alpha) < |\beta|$, then $(\beta e - \alpha)$ is invertible in $\mathcal{A}$. Moreover,

$$(\beta e - \alpha)^{-1} = \sum_{j=0}^{\infty} \frac{\alpha^j}{\beta^{j+1}}, \quad (2)$$

$$\rho[(\beta e - \alpha)^{-1}] \leq \frac{1}{|\beta| - \rho(\alpha)},$$

where β is a complex constant.

Lemma 3 (see [11]). Let $\mathcal{A}$ be a unital Banach algebra and $\alpha, \beta \in \mathcal{A}$ such that α commutes with β . Then,

$$\begin{aligned} \rho(\alpha + \beta) &\leq \rho(\alpha) + \rho(\beta), \\ \rho(\alpha\beta) &\leq \rho(\alpha)\rho(\beta). \end{aligned} \quad (3)$$

Lemma 4 (see [13]). Let $\mathcal{K}$ be a solid cone in a Banach algebra $\mathcal{A}$, $\{y_n\}$ and $\{z_n\}$ be two c -sequences in $\mathcal{K}$. If $\alpha, \beta \in \mathcal{K}$ are two given vectors, then $\{\alpha y_n + \beta z_n\}$ is also a c -sequence in $\mathcal{K}$.

Lemma 5 (see [13]). Let $\mathcal{A}$ be a unital Banach algebra. Let $\alpha \in \mathcal{A}$ and $\rho(\alpha) < 1$. Then, $\{\alpha^n\}$ is a c -sequence in $\mathcal{A}$.

Lemma 6 (see [6]). Let $\mathcal{K}$ be a solid cone in a Banach algebra $\mathcal{A}$:

- (1) If $\alpha, \beta, \gamma \in \mathcal{K}$ and $\alpha \leq \beta \ll \gamma$, then $\alpha \ll \gamma$
- (2) If $\alpha \in \mathcal{A}$ and $\theta \leq \alpha \ll \beta$ for each $\beta \in \mathcal{K}^\circ$, then $\alpha = \theta$
- (3) $\{y_n\} \subset \mathcal{K}$ is a c -sequence provided that $\{y_n\} \longrightarrow \theta$ as $n \longrightarrow \infty$

Definition 4 (see [2]). Let Y be a nonempty set and $\mathcal{A}$ a Banach algebra. Suppose that, for all $y, z, x \in Y$, a mapping $\mathcal{P}_b: Y \times Y \longrightarrow \mathcal{A}$ satisfies

- (1) $y = z \Leftrightarrow \mathcal{P}_b(y, y) = \mathcal{P}_b(y, z) = \mathcal{P}_b(z, z)$
- (2) $\theta \leq \mathcal{P}_b(y, y) \leq \mathcal{P}_b(y, z)$

$$(3) \mathcal{P}_b(y, z) = \mathcal{P}_b(z, y)$$

$$(4) \mathcal{P}_b(y, z) \leq s[\mathcal{P}_b(y, x) + \mathcal{P}_b(x, z)] - \mathcal{P}_b(x, x)$$

Then, $(Y, \mathcal{P}_b)$ is called a partial cone b -metric space over $\mathcal{A}$ with coefficient $s \geq 1$.

Definition 5 (see [1]). Let Y be a nonempty set and $\mathcal{K}$ a solid cone in a Banach algebra $\mathcal{A}$. Suppose that, for all $y, z \in Y$ and all distinct points $x_1, x_2 \in Y \setminus \{y, z\}$, a mapping $\mathcal{P}_{\text{rcb}}: Y \times Y \longrightarrow \mathcal{A}$ satisfies

$$(1) \theta \leq \mathcal{P}_{\text{rcb}}(y, z) \text{ and } \mathcal{P}_{\text{rcb}}(y, z) = \theta \Leftrightarrow y = z.$$

$$(2) \mathcal{P}_{\text{rcb}}(y, z) = \mathcal{P}_{\text{rcb}}(z, y).$$

$$(3) \text{ There exists } s \in \mathcal{K} \text{ with } e \leq s \text{ such that}$$

$$\mathcal{P}_{\text{rcb}}(y, z) \leq s[\mathcal{P}_{\text{rcb}}(y, x_1) + \mathcal{P}_{\text{rcb}}(x_1, x_2) + \mathcal{P}_{\text{rcb}}(x_2, z)]. \quad (4)$$

Then, $\mathcal{P}_{\text{rcb}}$ is called a rectangular cone b -metric on Y , and $(Y, \mathcal{P}_{\text{rcb}})$ is called a rectangular cone b -metric space over $\mathcal{A}$ with coefficient s .

3. Main Results

In this section, we introduce the concept of a partial rectangular cone b -metric space ($\mathcal{P}_b^r$ -cone metric space) over Banach algebras and give some of its topological property. Furthermore, the notions of convergent sequence, θ -Cauchy sequence, and θ -completeness in the setting of this new space are defined. Moreover, some fixed point theorems under various contractive mappings are proved in such a space.

Definition 6. Let Y be a nonempty set and $\mathcal{K}$ be a solid cone in a unital Banach algebra $\mathcal{A}$. Suppose that, for all $y, z \in Y$ and all distinct points $x_1, x_2 \in Y \setminus \{y, z\}$, a mapping $\mathcal{P}_b^r: Y \times Y \longrightarrow \mathcal{A}$ satisfies

$$(P1) y = z \Leftrightarrow \mathcal{P}_b^r(y, y) = \mathcal{P}_b^r(y, z) = \mathcal{P}_b^r(z, z).$$

$$(P2) \theta \leq \mathcal{P}_b^r(y, y) \leq \mathcal{P}_b^r(y, z).$$

$$(P3) \mathcal{P}_b^r(y, z) = \mathcal{P}_b^r(z, y).$$

$$(P4) \text{ There exists } s \in \mathcal{K} \text{ with } e \leq s \text{ such that}$$

$$\begin{aligned} \mathcal{P}_b^r(y, z) &\leq s[\mathcal{P}_b^r(y, x_1) + \mathcal{P}_b^r(x_1, x_2) + \mathcal{P}_b^r(x_2, z)] \\ &\quad - \mathcal{P}_b^r(x_1, x_1) - \mathcal{P}_b^r(x_2, x_2). \end{aligned} \quad (5)$$

Then, $\mathcal{P}_b^r$ is called a partial rectangular cone b -metric on Y , and $(Y, \mathcal{P}_b^r, \mathcal{A})$ is called a partial rectangular cone b -metric space over $\mathcal{A}$ with coefficient s (in short PRCbMS-BA).

Remark 1. In any PRCbMS-BA $(Y, \mathcal{P}_b^r, \mathcal{A})$, if $\mathcal{P}_b^r(y, z) = \theta$ for all $y, z \in Y$, then $y = z$, but the converse may not be true. Also, every rectangular cone b -metric space over $\mathcal{A}$ is a $\mathcal{P}_b^r$ -cone metric space over $\mathcal{A}$ with zero (θ) self distance, but there are $\mathcal{P}_b^r$ -cone metric spaces over $\mathcal{A}$ which are not a rectangular cone b -metric space over $\mathcal{A}$.

Example 1. Let $\mathcal{A} = C_{\mathbb{R}}^1[0, 1]$ with the norm

$$\|y\| = \|y\|_\infty + \|y'\|_\infty, \quad \text{for all } y \in \mathcal{A}. \quad (6)$$

Define multiplication pointwisely on $\mathcal{A}$. Then, $\mathcal{A}$ is a Banach algebra with unit $e(t) = 1, \forall t \in [0, 1]$. Let $\mathcal{K} = \{y \in \mathcal{A} : y = y(t) \geq 0, t \in [0, 1]\}$. Then, $\mathcal{K}$ is a solid cone in $\mathcal{A}$. Let $Y = \{y_1, y_2, y_3, y_4\}$ and, for all $y, z \in Y$, define a mapping $\mathcal{P}_b^r : Y \times Y \longrightarrow \mathcal{K}$ by

$$\mathcal{P}_b^r(y, z)(t) = \begin{cases} \theta, & \text{if } y = z = y_1, \\ 2t, & \text{if } y, z \in \{y_1, y_2\}, y \neq z, \\ t, & \text{otherwise.} \end{cases} \quad (7)$$

Then, $(Y, \mathcal{P}_b^r, \mathcal{A})$ is a PRCbMS-BA with coefficient $s = 4/3$ which is not a rectangular cone b -metric space over $\mathcal{A}$ because $\mathcal{P}_b^r(y_2, y_2)(t) \neq \theta$ and $\mathcal{P}_b^r(y_1, y_2)(t) = 2t > t = \mathcal{P}_b^r(y_1, y_3)(t) + \mathcal{P}_b^r(y_3, y_4)(t) + \mathcal{P}_b^r(y_4, y_2)(t) - \mathcal{P}_b^r(y_3, y_3)(t) - \mathcal{P}_b^r(y_4, y_4)(t)$.

Definition 7. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a PRCbMS-BA and $\mathcal{K}$ be a solid cone in $\mathcal{A}$. For each $y \in Y$ and each $c \in \mathcal{K}^\circ$, let

$$B_{\mathcal{P}_b^r}(y, c) = \{z \in Y : \mathcal{P}_b^r(y, z) \ll c + \mathcal{P}_b^r(y, y)\}, \quad (8)$$

$$\mathcal{B} = \{B_{\mathcal{P}_b^r}(y, c) : y \in Y \text{ and } c \in \mathcal{K}^\circ\}.$$

Then,

$$\tau_{\mathcal{P}} = \left\{ \mathcal{U} \subset Y : \text{for all } y \in \mathcal{U} \text{ there exists } B_{\mathcal{P}_b^r} \in \mathcal{B} \text{ and } y \in B_{\mathcal{P}_b^r} \subset \mathcal{U} \right\} \cup \emptyset, \quad (9)$$

is a topology on Y , $B_{\mathcal{P}_b^r}(y, c)$ is a $\mathcal{P}_b^r$ -ball in $(Y, \mathcal{P}_b^r, \mathcal{A})$, $\mathcal{B}$ is a subbase for the topology $\tau_{\mathcal{P}}$ on Y , and $\mathcal{U}$ is a base generated by the subbase $\mathcal{B}$.

Definition 8. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a PRCbMS-BA, $\mathcal{K}$ be a solid cone in $\mathcal{A}$, $y^* \in Y$, and $\{y_n\}$ be a sequence in Y . If, for every $c \in \mathcal{K}^\circ$, there exists $N \in \mathbb{N}$ such that $\mathcal{P}_b^r(y_n, y^*) \ll c + \mathcal{P}_b^r(y^*, y^*)$ for all $n > N$, then $\{y_n\}$ is said to be convergent in Y and converges to y^* . This fact is denoted by $y_n \longrightarrow y^*$ as $n \longrightarrow \infty$ or $\lim_{n \longrightarrow \infty} y_n = y^*$.

Definition 9. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a PCbMS-BA, $\mathcal{K}$ be a solid cone in $\mathcal{A}$, and $\{y_n\}$ be a sequence in Y . Then, $\{y_n\}$ is called a θ -Cauchy sequence if $\{\mathcal{P}_b^r(y_n, y_m)\}$ is a c -sequence in $\mathcal{A}$. That is, if, for every $c \in \mathcal{K}^\circ$, there exists $N \in \mathbb{N}$ such that $\mathcal{P}_b^r(y_n, y_m) \ll c$ for all $n, m > N$.

Definition 10. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a PRCbMS-BA, $\mathcal{K}$ be a solid cone in $\mathcal{A}$, $y^* \in Y$, and $\{y_n\}$ be a sequence in Y . Then, $(Y, \mathcal{P}_b^r, \mathcal{A})$ is called θ -complete if every θ -Cauchy sequence $\{y_n\}$ in Y converges to a point $y^* \in Y$. That is,

$$\lim_{n, m \longrightarrow \infty} \mathcal{P}_b^r(y_n, y_m) = \lim_{n \longrightarrow \infty} \mathcal{P}_b^r(y_n, y^*) = \mathcal{P}_b^r(y^*, y^*) = \theta. \quad (10)$$

Lemma 7. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a PRCbMS-BA and $\{y_n\}$ be a sequence in Y . If $\{y_n\}$ converges to $y^* \in Y$, then

- (1) $\{\mathcal{P}_b^r(y_n, y^*)\}$ is a c -sequence.
- (2) For any $m \in \mathbb{N}$, $\{\mathcal{P}_b^r(y_n, y_{n+m})\}$ is a c -sequence.

Proof. It follows from Definitions 3, 6, and 8.

Firstly, we present a variant of the Banach contraction principle on $\mathcal{P}_b^r$ -cone metric space over Banach algebra $\mathcal{A}$. $\square$

Theorem 1. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a θ -complete PRCbMS-BA with $s \in \mathcal{K}$ such that $e \leq s$. Suppose $F : Y \longrightarrow Y$ is a function satisfying

$$\mathcal{P}_b^r(Fy, Fz) \leq \alpha \mathcal{P}_b^r(y, z), \quad \text{for all } y, z \in Y, \quad (11)$$

where $\alpha \in \mathcal{K}$ such that α commutes with s and $\rho(\alpha) < 1$. Then, F has a unique fixed point.

Proof. Let y_0 be a point in Y . We define a sequence $\{y_n\}$ in Y by

$$y_n = Fy_{n-1} = F^n y_0, \quad \text{for all } n \geq 1. \quad (12)$$

If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$, then $y^* = y_n = Fy_n$ is a fixed point of F , and the result is proved. Hence, we assume that $y_n \neq y_{n+1}$ for all $n \geq 0$. We will show that $y_n \neq y_{n+q}$ for all $n \geq 0$ and $q \geq 1$. Suppose that $y_n = y_{n+q}$ for some $n \geq 0, q \geq 1$; then, $y_{n+1} = y_{n+q+1}$ and $Fy_n = Fy_{n+q}$. Then, (11) implies that

$$\begin{aligned} \mathcal{P}_b^r(y_n, y_{n+1}) &= \mathcal{P}_b^r(y_{n+q}, y_{n+q+1}) \leq \alpha \mathcal{P}_b^r(y_{n+q-1}, y_{n+q}) \\ &\leq \cdots \leq \alpha^q \mathcal{P}_b^r(y_n, y_{n+1}). \end{aligned} \quad (13)$$

Using Lemma 1, we obtain that $\mathcal{P}_b^r(y_n, y_{n+1}) = \theta$, that is, $y_n = y_{n+1}$, which is a contradiction. Therefore, $y_n \neq y_m$ for all distinct $n, m \in \mathbb{N}$. Hence, from (11) and (12), we have that

$$\begin{aligned} \mathcal{P}_b^r(y_n, y_{n+1}) &= \mathcal{P}_b^r(Fy_{n-1}, Fy_n) \leq \alpha \mathcal{P}_b^r(y_{n-1}, y_n) \\ &\leq \alpha^2 \mathcal{P}_b^r(y_{n-2}, y_{n-1}) \leq \cdots \\ \therefore \mathcal{P}_b^r(y_n, y_{n+1}) &\leq \alpha^n \mathcal{P}_b^r(y_0, y_1), \quad \text{for all } n \in \mathbb{N}. \end{aligned} \quad (14)$$

Similarly, for all $n, m, q \in \mathbb{N}$, we obtain that

$$\begin{aligned} \mathcal{P}_b^r(y_{n+q}, y_{m+q}) &= \mathcal{P}_b^r(Fy_{n+q-1}, Fy_{m+q-1}) \\ &\leq \alpha \mathcal{P}_b^r(y_{n+q-1}, y_{m+q-1}) \\ &\leq \alpha^2 \mathcal{P}_b^r(y_{n+q-2}, y_{m+q-2}) \leq \cdots \\ \therefore \mathcal{P}_b^r(y_{n+q}, y_{m+q}) &\leq \alpha^q \mathcal{P}_b^r(y_n, y_m), \\ &\text{for all } n, m, q \in \mathbb{N}. \end{aligned} \quad (15)$$

Observe that $\rho(s)$ exists because of Lemma 2, and since $\rho(\alpha) < 1$, there exists $q_1 \in \mathbb{N}$ such that $\rho(s)\rho(\alpha)^{q_1} < 1$ holds. Since α commutes with s , by Lemmas 2 and 3, we have that

$$\rho(s\alpha^{q_1}) \leq \rho(s)\rho(\alpha)^{q_1} < 1 \text{ and } (e - s\alpha^{q_1}) \text{ is invertible in } \mathcal{A}. \quad (16)$$

Hence, by condition (P4), for all $y, z, x_1, x_2 \in Y$, we have

$$\begin{aligned} \mathcal{P}_b^r(y, z) &\leq s[\mathcal{P}_b^r(y, x_1) + \mathcal{P}_b^r(x_1, x_2) + \mathcal{P}_b^r(x_2, z)] - \mathcal{P}_b^r(x_1, x_1) - \mathcal{P}_b^r(x_2, x_2) \\ \therefore \mathcal{P}_b^r(y, z) &\leq s[\mathcal{P}_b^r(y, x_1) + \mathcal{P}_b^r(x_1, x_2) + \mathcal{P}_b^r(x_2, z)], \quad \text{for all } y, z, x_1, x_2 \in Y. \end{aligned} \quad (17)$$

This, using (15) and (16), implies that

$$\begin{aligned} \mathcal{P}_b^r(y_n, y_m) &\leq s[\mathcal{P}_b^r(y_n, y_{n+q_1}) + \mathcal{P}_b^r(y_{n+q_1}, y_{m+q_1}) + \mathcal{P}_b^r(y_{m+q_1}, y_m)] \\ &\leq s[\alpha^n \mathcal{P}_b^r(y_0, y_{q_1}) + \alpha^{q_1} \mathcal{P}_b^r(y_n, y_m) + \alpha^m \mathcal{P}_b^r(y_{q_1}, y_0)] \\ &\quad \cdot (e - s\alpha^{q_1}) \mathcal{P}_b^r(y_n, y_m) \leq s[\alpha^n \mathcal{P}_b^r(y_0, y_{q_1}) + \alpha^m \mathcal{P}_b^r(y_{q_1}, y_0)] \\ &\quad \cdot (e - s\alpha^{q_1}) \mathcal{P}_b^r(y_n, y_m) \leq (e - s\alpha^{q_1})^{-1} s[\alpha^n \mathcal{P}_b^r(y_0, y_{q_1}) + \alpha^m \mathcal{P}_b^r(y_{q_1}, y_0)]. \end{aligned} \quad (18)$$

Using Lemmas 4 and 5, we deduce that $\{\mathcal{P}_b^r(y_n, y_m)\}$ is a c -sequence in $\mathcal{A}$. Therefore, $\{y_n\}$ is a θ -Cauchy sequence in Y . From the hypothesis, $(Y, \mathcal{P}_b^r, \mathcal{A})$ is θ -complete; hence, there exists a point $y^* \in Y$ such that $\{y_n\}$ converges to y^* . That is,

$$\lim_{n \rightarrow \infty} \mathcal{P}_b^r(y_n, y^*) = \lim_{n, m \rightarrow \infty} \mathcal{P}_b^r(y_n, y_m) = \mathcal{P}_b^r(y^*, y^*) = \theta. \quad (19)$$

Next, we will show that y^* is the unique fixed point of F :

$$\begin{aligned} \mathcal{P}_b^r(y^*, Fy^*) &\leq s[\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1}) + \mathcal{P}_b^r(y_{n+1}, Fy^*)] \\ &\quad - \mathcal{P}_b^r(y_n, y_n) - \mathcal{P}_b^r(y_{n+1}, y_{n+1}) \\ &\leq s[\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1}) + \mathcal{P}_b^r(Fy_n, Fy^*)] \\ &\leq s[\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1}) + \alpha \mathcal{P}_b^r(y_n, y^*)] \\ &\leq s[(e + \alpha) \mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1})]. \end{aligned} \quad (20)$$

By Lemmas 6 and 7, we have $\mathcal{P}_b^r(y^*, y_n) \rightarrow \theta$ as $n \rightarrow \infty$ and $\mathcal{P}_b^r(y_n, y_{n+1}) \rightarrow \theta$ as $n \rightarrow \infty$. Hence, we deduce that $\mathcal{P}_b^r(y^*, Fy^*) = \theta$. That is, $y^* = Fy^*$. So, y^* is a fixed point of F . For uniqueness, we let z^* be another fixed point of F . Then, it follows from (11) that

$$\mathcal{P}_b^r(y^*, z^*) = \mathcal{P}_b^r(Fy^*, Fz^*) \leq \alpha \mathcal{P}_b^r(y^*, z^*). \quad (21)$$

By Lemma 1, we get that $\mathcal{P}_b^r(y^*, z^*) = \theta$, and hence, $y^* = z^*$.

Kindly, observe that Theorem 1 extends and generalizes Theorem 3.5 in [1], Theorem 3.1 in [3], Theorem 2.1 in [4], Theorem 2.1 in [5], and Theorem 3.1 in [6]. $\square$

Example 2. Let $\mathcal{A} = C_{\mathbb{R}}^1[0, 1]$ with the norm

$$\|y\| = \|y\|_{\infty} + \|y'\|_{\infty}, \quad \text{for all } y \in \mathcal{A}. \quad (22)$$

Define multiplication pointwisely on $\mathcal{A}$. Then, $\mathcal{A}$ is a Banach algebra with unit $e(t) = 1, \forall t \in [0, 1]$. Let $\mathcal{K} = \{y \in \mathcal{A} : y = y(t) \geq 0, t \in [0, 1]\}$. Then, $\mathcal{K}$ is a solid cone in $\mathcal{A}$. Let $Y = \{0, 1, 2, 3\}$ and, for all $y, z \in Y$, define a mapping $\mathcal{P}_b^r : Y \times Y \rightarrow \mathcal{K}$ by

$$\mathcal{P}_b^r(y, z)(t) = \begin{cases} y^2 t, & \text{if } y = z \neq 0, \\ 2(y^2 + z^2)t, & \text{if } y, z \notin \{2, 3\}, y \neq z, \\ (y^2 + z^2)t & \text{if } y, z \in \{2, 3\}, y \neq z, \\ \frac{1}{2}t, & \text{if } y = z = 0. \end{cases} \quad (23)$$

Then, $(Y, \mathcal{P}_b^r, \mathcal{A})$ is a θ -complete PRCbMS-BA with coefficient $s = 2$. Define a mapping $F : Y \rightarrow Y$ as follows:

$$Fy = \begin{cases} 0, & \text{if } y \in \{0, 1\}, \\ 1, & \text{if } y \in \{2, 3\}. \end{cases} \quad (24)$$

Hence, the mapping F satisfies all the conditions of Theorem 1 and $y^* = 0 \in Y$ is the unique fixed point of F .

Secondly, we present a variant of the Reich contraction principle on $\mathcal{P}_b^r$ -cone metric space over Banach algebra $\mathcal{A}$.

Theorem 2. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a θ -complete PRCbMS-BA with $s \in \mathcal{K}$ such that $e \leq s$. Suppose $F : Y \rightarrow Y$ is a function satisfying

$$\mathcal{P}_b^r(Fy, Fz) \leq \alpha \mathcal{P}_b^r(y, z) + \beta \mathcal{P}_b^r(y, Fy) + \gamma \mathcal{P}_b^r(z, Fz), \quad (25)$$

for all $y, z \in Y$, where $\alpha, \beta, \gamma \in \mathcal{K}$ commutes, $\rho(\alpha) + \rho(\beta + \gamma) < 1$, and $\min\{\rho(\beta), \rho(\gamma)\} < (1/\rho(s))$. Then, F has a unique fixed point.

Proof. Let y_0 be a point in Y . We define a sequence $\{y_n\}$ in Y by

$$y_{n+1} = Fy_n = F^{n+1}y_0, \quad \text{for all } n \geq 0. \quad (26)$$

From (25) and (26), we have

$$\begin{aligned} \mathcal{P}_b^r(y_{n+1}, y_n) &= \mathcal{P}_b^r(Fy_n, Fy_{n-1}) \\ &\leq \alpha \mathcal{P}_b^r(y_n, y_{n-1}) + \beta \mathcal{P}_b^r(y_n, Fy_n) \\ &\quad + \gamma \mathcal{P}_b^r(y_{n-1}, Fy_{n-1}) \\ \therefore (e - \beta) \mathcal{P}_b^r(y_{n+1}, y_n) &\leq (\alpha + \gamma) \mathcal{P}_b^r(y_n, y_{n-1}). \end{aligned} \quad (27)$$

On the contrary, we have

$$\begin{aligned} \mathcal{P}_b^r(y_{n+1}, y_n) &= \mathcal{P}_b^r(Fy_n, Fy_{n-1}) = \mathcal{P}_b^r(Fy_{n-1}, Fy_n) \\ &\leq \alpha \mathcal{P}_b^r(y_{n-1}, y_n) + \beta \mathcal{P}_b^r(y_{n-1}, Fy_{n-1}) + \gamma \mathcal{P}_b^r(y_n, Fy_n) \\ \therefore (e - \gamma) \mathcal{P}_b^r(y_{n+1}, y_n) &\leq (\alpha + \beta) \mathcal{P}_b^r(y_n, y_{n-1}). \end{aligned} \quad (28)$$

Adding up (27) and (28), we have

$$(2e - \lambda) \mathcal{P}_b^r(y_{n+1}, y_n) \leq (2\alpha + \lambda) \mathcal{P}_b^r(y_n, y_{n-1}), \quad (29)$$

where $\lambda = (\beta + \gamma) \in \mathcal{K}$. Now, we observe that

$$2\rho(\lambda) \leq 2\rho(\alpha) + 2\rho(\lambda) = 2[\rho(\alpha) + \rho(\beta + \gamma)] < 2. \quad (30)$$

This implies that $\rho(\lambda) < 1 < 2$; then, by Lemma 2, it follows that $(2e - \lambda)$ is invertible and $(2e - \lambda)^{-1} = \sum_{j=0}^{\infty} (\lambda^j / 2^{j+1})$. From (29), we obtain

$$\begin{aligned} \mathcal{P}_b^r(y_{n+1}, y_n) &\leq (2e - \lambda)^{-1} (2\alpha + \lambda) \mathcal{P}_b^r(y_n, y_{n-1}) \\ &\leq k \mathcal{P}_b^r(y_n, y_{n-1}), \end{aligned} \quad (31)$$

where $k = (2e - \lambda)^{-1} (2\alpha + \lambda) \in \mathcal{K}$. Hence,

$$\begin{aligned} \mathcal{P}_b^r(y_{n+1}, y_n) &\leq k \mathcal{P}_b^r(y_n, y_{n-1}) \leq k^2 \mathcal{P}_b^r(y_{n-1}, y_{n-2}) \leq \dots \\ \therefore \mathcal{P}_b^r(y_{n+1}, y_n) &\leq k^n \mathcal{P}_b^r(y_1, y_0), \quad \text{for all } n \in \mathbb{N}. \end{aligned} \quad (32)$$

We claim that $\rho(k) < 1$. Indeed, since α commutes with $\lambda = \beta + \gamma$, it follows that

$$\begin{aligned} (2e - \lambda)^{-1} (2\alpha + \lambda) &= \left(\sum_{j=0}^{\infty} \frac{\lambda^j}{2^{j+1}} \right) (2\alpha + \lambda) \\ &= 2 \left(\sum_{j=0}^{\infty} \frac{\lambda^j}{2^{j+1}} \right) \alpha + \sum_{j=0}^{\infty} \frac{\lambda^{j+1}}{2^{j+1}} \\ &= 2\alpha \left(\sum_{j=0}^{\infty} \frac{\lambda^j}{2^{j+1}} \right) + \lambda \left(\sum_{j=0}^{\infty} \frac{\lambda^j}{2^{j+1}} \right) \\ &= (2\alpha + \lambda) \left(\sum_{j=0}^{\infty} \frac{\lambda^j}{2^{j+1}} \right) \\ &= (2\alpha + \lambda) (2e - \lambda)^{-1}. \end{aligned} \quad (33)$$

Therefore, $(2\alpha + \lambda)$ commutes with $(2e - \lambda)^{-1}$. Then, by Lemmas 2 and 3, we obtain

$$\begin{aligned} \rho(k) &= \rho((2e - \lambda)^{-1} (2\alpha + \lambda)) \leq \rho((2e - \lambda)^{-1}) \rho(2\alpha + \lambda) \\ &\leq \frac{1}{2 - \rho(\lambda)} [2\rho(\alpha) + \rho(\lambda)] < 1 \quad (\text{since } \rho(\alpha) + \rho(\lambda) < 1). \end{aligned} \quad (34)$$

If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$, then $y^* = y_n = Fy_n$ is a fixed point of F , and the result is proved. Hence, we assume that $y_n \neq y_{n+1}$, for all $n \geq 0$. We will show that $y_n \neq y_{n+q}$, for all $n \geq 0$ and $q \geq 1$. Suppose that $y_n = y_{n+q}$, for some $n \geq 0, q \geq 1$; then, $y_{n+1} = y_{n+q+1}$ and $Fy_n = Fy_{n+q}$. Then, (31) implies that

$$\mathcal{P}_b^r(y_{n+1}, y_n) = \mathcal{P}_b^r(y_{n+q+1}, y_{n+q}) \leq k^q \mathcal{P}_b^r(y_{n+1}, y_n). \quad (35)$$

Using Lemma 1, we obtain that $\mathcal{P}_b^r(y_{n+1}, y_n) = \theta$, that is, $y_{n+1} = y_n$, which is a contradiction. Therefore, $y_n \neq y_m$ for all distinct $n, m \in \mathbb{N}$. Next, from (25), (26), and (32), we have

$$\begin{aligned} \mathcal{P}_b^r(y_n, y_m) &= \mathcal{P}_b^r(Fy_{n-1}, Fy_{m-1}) \\ &\leq \alpha \mathcal{P}_b^r(y_{n-1}, y_{m-1}) + \beta \mathcal{P}_b^r(y_{n-1}, Fy_{n-1}) + \gamma \mathcal{P}_b^r(y_{m-1}, Fy_{m-1}) \\ &= \alpha \mathcal{P}_b^r(y_{n-1}, y_{m-1}) + \beta \mathcal{P}_b^r(y_{n-1}, y_n) + \gamma \mathcal{P}_b^r(y_{m-1}, y_m) \\ &\leq \alpha \mathcal{P}_b^r(y_{n-1}, y_{m-1}) + \beta k^{n-1} \mathcal{P}_b^r(y_0, y_1) + \gamma k^{m-1} \mathcal{P}_b^r(y_0, y_1) \\ \therefore \mathcal{P}_b^r(y_n, y_m) &\leq q \mathcal{P}_b^r(y_{n-1}, y_{m-1}) + (q^n + q^m) \mathcal{P}_b^r(y_0, y_1), \end{aligned} \quad (36)$$

where $q \in \{\alpha, \beta, \gamma, k\}$ such that $\rho(q) = \max\{\rho(\alpha), \rho(\beta), \rho(\gamma), \rho(k)\}$. Hence, from (36), we also obtain

$$\mathcal{P}_b^r(y_n, y_m) \leq q^p \mathcal{P}_b^r(y_{n-p}, y_{m-p}) + p(q^n + q^m) \mathcal{P}_b^r(y_0, y_1), \quad (37)$$

for all $p \in \{1, 2, \dots, \min\{n, m\}\}$. Observe that $\rho(s)$ exists because of Lemma 2, and since $\rho(q) < 1$, there exists $q_0 \in \mathbb{N}$

such that $\rho(s)\rho(q)^{q_0} < 1$ holds. Furthermore, since q commutes with s , by Lemmas 2 and 3, we have that

$$\rho(sq^{q_0}) \leq \rho(s)\rho(q)^{q_0} < 1 \text{ and } (e - sq^{q_0}) \text{ is invertible in } \mathcal{A}. \quad (38)$$

Therefore, from (37), we further obtain

$$\mathcal{P}_b^r(y_n, y_{n+q_0}) \leq q^n \mathcal{P}_b^r(y_0, y_{q_0}) + n(q^n + q^{n+q_0}) \mathcal{P}_b^r(y_0, y_1), \quad (39)$$

$$\mathcal{P}_b^r(y_{m+q_0}, y_m) \leq q^m \mathcal{P}_b^r(y_{q_0}, y_0) + m(q^{m+q_0} + q^m) \mathcal{P}_b^r(y_0, y_1), \quad (40)$$

$$\mathcal{P}_b^r(y_{n+q_0}, y_{m+q_0}) \leq q^{q_0} \mathcal{P}_b^r(y_n, y_m) + q_0(q^{n+q_0} + q^{m+q_0}) \mathcal{P}_b^r(y_0, y_1). \quad (41)$$

Hence, from (P4), (38), (39), (40) and (41), we have

$$\begin{aligned} \mathcal{P}_b^r(y_n, y_m) &\leq s[\mathcal{P}_b^r(y_n, y_{n+q_0}) + \mathcal{P}_b^r(y_{n+q_0}, y_{m+q_0}) + \mathcal{P}_b^r(y_{m+q_0}, y_m)] \\ &\quad - \mathcal{P}_b^r(y_{n+q_0}, y_{n+q_0}) - \mathcal{P}_b^r(y_{m+q_0}, y_{m+q_0}) \\ &\leq s \left[\begin{aligned} &q^n \mathcal{P}_b^r(y_0, y_{q_0}) + n(q^n + q^{n+q_0}) \mathcal{P}_b^r(y_0, y_1) + q^{q_0} \mathcal{P}_b^r(y_n, y_m) + q_0(q^{n+q_0} + q^{m+q_0}) \mathcal{P}_b^r(y_0, y_1) \\ &+ q^m \mathcal{P}_b^r(y_{q_0}, y_0) + m(q^{m+q_0} + q^m) \mathcal{P}_b^r(y_0, y_1) \end{aligned} \right] \\ (e - sq^{q_0}) \mathcal{P}_b^r(y_n, y_m) &\leq s[(q^n + q^m) \mathcal{P}_b^r(y_0, y_{q_0}) + [q^n(n + (n + q_0)q^{q_0}) \\ &\quad + q^m(m + (m + q_0)q^{q_0})] \mathcal{P}_b^r(y_0, y_1)] \\ \therefore \mathcal{P}_b^r(y_n, y_m) &\leq (e - sq^{q_0})^{-1} s[(q^n + q^m) \mathcal{P}_b^r(y_0, y_{q_0}) + [q^n(n + (n + q_0)q^{q_0}) \\ &\quad + q^m(m + (m + q_0)q^{q_0})] \mathcal{P}_b^r(y_0, y_1)]. \end{aligned} \quad (42)$$

Using Lemmas 4 and 5, we deduce that $\{\mathcal{P}_b^r(y_n, y_m)\}$ is a c -sequence in $\mathcal{A}$. Therefore, $\{y_n\}$ is a θ -Cauchy sequence in Y . From the hypothesis, $(Y, \mathcal{P}_b^r, \mathcal{A})$ is θ -complete; hence, there exists a point $y^* \in Y$ such that $\{y_n\}$ converges to y^* . That is,

$$\lim_{n \rightarrow \infty} \mathcal{P}_b^r(y_n, y^*) = \lim_{n, m \rightarrow \infty} \mathcal{P}_b^r(y_n, y_m) = \mathcal{P}_b^r(y^*, y^*) = \theta. \quad (43)$$

Next, we will show that y^* is the unique fixed point of F :

$$\begin{aligned} \mathcal{P}_b^r(y^*, Fy^*) &\leq s[\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1}) + \mathcal{P}_b^r(y_{n+1}, Fy^*)] \\ &\quad - \mathcal{P}_b^r(y_n, y_n) - \mathcal{P}_b^r(y_{n+1}, y_{n+1}) \\ &\leq s[\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1}) + \mathcal{P}_b^r(Fy_n, Fy^*)] \\ &\leq s[\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1}) + \alpha \mathcal{P}_b^r(y_n, y^*) + \beta \mathcal{P}_b^r(y_n, Fy_n) + \gamma \mathcal{P}_b^r(y^*, Fy^*)] \\ \therefore \mathcal{P}_b^r(y^*, Fy^*) &\leq s[(e + \alpha) \mathcal{P}_b^r(y^*, y_n) + (e + \beta) \mathcal{P}_b^r(y_n, y_{n+1}) + \gamma \mathcal{P}_b^r(y^*, Fy^*)]. \end{aligned} \quad (44)$$

On the contrary, we have

$$\begin{aligned}
 \mathcal{P}_b^r(Fy^*, y^*) &\leq s[\mathcal{P}_b^r(Fy^*, y_{n+1}) + \mathcal{P}_b^r(y_{n+1}, y_n) + \mathcal{P}_b^r(y_n, y^*)] \\
 &\quad - \mathcal{P}_b^r(y_{n+1}, y_{n+1}) - \mathcal{P}_b^r(y_n, y_n) \\
 &\leq s[\mathcal{P}_b^r(Fy^*, Fy_n) + \mathcal{P}_b^r(y_{n+1}, y_n) + \mathcal{P}_b^r(y_n, y^*)] \\
 &\leq s[\alpha\mathcal{P}_b^r(y^*, y_n) + \beta\mathcal{P}_b^r(y^*, Fy^*) + \gamma\mathcal{P}_b^r(y_n, Fy_n) + \mathcal{P}_b^r(y_{n+1}, y_n) + \mathcal{P}_b^r(y_n, y^*)] \\
 \therefore \mathcal{P}_b^r(y^*, Fy^*) &\leq s[(e + \alpha)\mathcal{P}_b^r(y^*, y_n) + (e + \gamma)\mathcal{P}_b^r(y_n, y_{n+1}) + \beta\mathcal{P}_b^r(y^*, Fy^*)].
 \end{aligned} \tag{45}$$

By Lemmas 6 and 7, we have $\mathcal{P}_b^r(y^*, y_n) \rightarrow \theta$ as $n \rightarrow \infty$ and $\mathcal{P}_b^r(y_n, y_{n+1}) \rightarrow \theta$ as $n \rightarrow \infty$. Hence, from (44) and (45), we deduce that $\mathcal{P}_b^r(y^*, Fy^*) \leq s\gamma\mathcal{P}_b^r(y^*, Fy^*)$ and $\mathcal{P}_b^r(y^*, Fy^*) \leq s\beta\mathcal{P}_b^r(y^*, Fy^*)$. Since $\min\{\rho(\beta), \rho(\gamma)\} < (1/\rho(s))$, by Lemma 1, we have $\mathcal{P}_b^r(y^*, Fy^*) = \theta$ so that $y^* = Fy^*$. That is, y^* is a fixed point of F . For uniqueness, we let z^* be another fixed point of F . Then, it follows from (25) that

$$\begin{aligned}
 \mathcal{P}_b^r(y^*, z^*) &= \mathcal{P}_b^r(Fy^*, Fz^*) \\
 &\leq \alpha\mathcal{P}_b^r(y^*, z^*) + \beta\mathcal{P}_b^r(y^*, Fy^*) + \gamma\mathcal{P}_b^r(z^*, Fz^*) \\
 &= \alpha\mathcal{P}_b^r(z^*, y^*) + \beta\mathcal{P}_b^r(y^*, y^*) + \gamma\mathcal{P}_b^r(z^*, z^*) \\
 &\leq \alpha\mathcal{P}_b^r(z^*, y^*) + \beta\mathcal{P}_b^r(y^*, z^*) + \gamma\mathcal{P}_b^r(z^*, y^*) \\
 \therefore \mathcal{P}_b^r(y^*, z^*) &\leq (\alpha + \beta + \gamma)\mathcal{P}_b^r(y^*, z^*).
 \end{aligned} \tag{46}$$

By Lemmas 1 and 3, we have that $\mathcal{P}_b^r(y^*, z^*) = \theta$, and hence, $y^* = z^*$, i.e., the fixed point of F is unique.

Note that Theorem 2 extends and generalizes Theorem 3.1 in [7].

Thirdly, we present a variant of the quasi-contraction principle on $\mathcal{P}_b^r$ -cone metric space over Banach algebra $\mathcal{A}$. $\square$

Corollary 1. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a θ -complete PRCbMS-BA with $s \in \mathcal{K}$ such that $e \leq s$. Let $F: Y \rightarrow Y$ be a function satisfying

$$\mathcal{P}_b^r(Fy, Fz) \leq \alpha M(y, z), \quad \text{for all } y, z \in Y, \tag{47}$$

where $M(y, z) \in \{\mathcal{P}_b^r(y, z), \mathcal{P}_b^r(y, Fy), \mathcal{P}_b^r(z, Fz)\}$, $\alpha \in \mathcal{K}$ commutes with s , and $\rho(\alpha) < 1$. Then, F has a unique fixed point.

Proof. Since (47) implies (25), the proof follows from Theorem 2.

Note that Corollary 1 generalizes Theorem 3.1 in [8–10].

Finally, we present a variant of the Kannan contraction principle on $\mathcal{P}_b^r$ -cone metric space over Banach algebra $\mathcal{A}$. $\square$

Corollary 2. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a θ -complete PRCbMS-BA with $s \in \mathcal{K}$ such that $e \leq s$. Suppose $F: Y \rightarrow Y$ is a function satisfying

$$\mathcal{P}_b^r(Fy, Fz) \leq \beta[\mathcal{P}_b^r(y, Fy) + \mathcal{P}_b^r(z, Fz)], \quad \text{for all } y, z \in Y, \tag{48}$$

where $\alpha \in \mathcal{K}$ such that $\rho(\beta) < 1/2$ and $\rho(s\beta) < 1$. Then, F has a unique fixed point.

Proof. Put $\alpha = \theta$ and $\beta = \gamma$ in Theorem 2, and the result follows.

Note that Corollary 2 generalizes Theorem 2.4 in [4], Theorem 2.3 in [5], and Theorem 3.3 in [6]. $\square$

4. Conclusion

The concept of a partial rectangular cone b -metric space over Banach algebras was introduced, and some fixed point results under various contractive mappings were proved in such a space. Some examples were also given to elucidate the results. Our results extend and generalized many existing results in [1, 3–10].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors have contributed equally and significantly in writing this article.

References

- [1] R. George, H. A. Nabwey, R. Rajagopalan, S. Radenović, and K. P. Reshma, "Rectangular cone b -metric spaces over Banach algebra and contraction principle," *Fixed Point Theory and Applications*, vol. 14, pp. 1–15, 2017.
- [2] J. Fernandez, N. Malviya, D. Dolicanin, and D. Pucic, "The P_b -cone metric spaces over Banach algebra with applications," *Filomat*, vol. 34, no. 3, pp. 983–998, 2020.
- [3] S. Jain and P. Chaubey, "Partial cone rectangular metric spaces and fixed point theorems," *Journal of Advanced Mathematical Studies*, vol. 13, no. 3, pp. 294–301, 2020.
- [4] R. George, S. Radenović, K. P. Reshma, and S. Shukla, "Rectangular b -metric space and contraction principles,"

- Journal of Nonlinear Sciences and Applications*, vol. 8, no. 6, pp. 1005–1013, 2015.
- [5] H. Liu and S. Xu, “Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings,” *Fixed Point Theory and Applications*, vol. 320, pp. 1–10, 2013.
 - [6] S. Xu and S. Radenović, “Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality,” *Fixed Point Theory and Applications*, vol. 102, pp. 1–12, 2014.
 - [7] R. George and Z. D. Mitrović, “On Reich contraction principle in rectangular cone b -metric space over Banach algebra,” *Journal of Advanced Mathematical Studies*, vol. 11, no. 1, pp. 10–16, 2018.
 - [8] S. Shukla, S. Balasubramanian, and M. Pavlović, “A generalized Banach fixed point theorem,” *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 39, no. 4, pp. 1529–1539, 2016.
 - [9] J. Vujakovic, A. Auwalu, and V. Sesum-Cavic, “Some new results for Reich type mappings on cone b -metric spaces over Banach algebras,” *University Thought Publication in Natural Sciences*, vol. 8, no. 2, pp. 54–60, 2018.
 - [10] A. Auwalu, “A note on some fixed point theorems for generalized expansive mappings in cone metric spaces over Banach algebras,” *AIP Conference Proceedings*, vol. 1997, Article ID 020004, 2018.
 - [11] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, NY, USA, 2nd edition, 1991.
 - [12] H. Huang and S. Radenović, “Common fixed point theorems of generalized Lipschitz mappings in cone b -metric spaces over Banach algebras and applications,” *Journal of Nonlinear Sciences and Applications*, vol. 8, no. 5, pp. 787–799, 2015.
 - [13] H. Huang and S. Radenović, “Some fixed point results of generalized Lipschitz mappings in cone b -metric spaces over Banach algebras,” *Journal of Computational Analysis and Applications*, vol. 20, no. 3, pp. 566–583, 2016.