Research Article

Psi-Caputo Logistic Population Growth Model

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This article studies modeling of a population growth by logistic equation when the population carrying capacity K tends to infinity. Results are obtained using fractional calculus theories. A fractional derivative known as psi-Caputo plays a substantial role in the study. We proved existence and uniqueness of the solution to the problem using the psi-Caputo fractional derivative. The Chinese population, whose carrying capacity, K , tends to infinity, is used as evidence to prove that the proposed approach is appropriate and performs better than the usual logistic growth equation for a population with a large carrying capacity. A psi-Caputo logistic model with the kernel function $\sqrt{x+1}$ performed the best as it minimized the error rate to 3.20% with a fractional order of derivative $\alpha = 1.6455$.

1. Introduction

Real-life phenomena are often modeled by the mean of differential equations. Differential equations are built using two main tools which are derivatives and integrals. In the early stage of differential equations, only integer numbers were used as order of a derivative or integral. However, three centuries ago, a new type of derivative called fractional derivative was introduced [1]. This is a new form of derivative and integral tries to generalize the use of any real number or fraction as order of derivative or integral which was the genesis of fractional calculus. In the earlier days following its introduction, fractional calculus was mainly investigated theoretically. Hence, early research works were focused designing fractional models and then proving existence and uniqueness of their solutions. Samples such theoretical studies are found in [2–6] and the references therein. In all what follows, statements as integer order of derivation, classical approach, and classical differential equations are often used to refer to the model in which order of derivatives or integrals are all integers.

In recent decades, fractional calculus has taken a new direction. Indeed, after researchers have well established the theory of fractional differential equations, many are now

focusing on their application to solve real-life problems. It has since been proven in many works that the fractional differential equations can be successfully used for modeling some natural phenomena even when the classical differential equations fail to do so. Moreover, in the case where both classical and fractional approaches are applicable to solve a problem, it is common to observe that the fractional approach will minimize the fitting error compared to the classical approach. Fractional differential equations have been successfully used for modeling in finance [7] and epidemiology [8, 9], a few examples just to mention. More generally, samples of works in which fractional calculus has successfully been used for modeling are found in [10–17] and the references therein.

This work contributes in showing the power of fractional differential equations over the classical differential equations in modeling. A fractional model is built, its existence and uniqueness are proven, and the numerical study is carried out with the idea to support theoretical results. More generally, we referred to the model built in this work as the fractional logistic equation. Recent studies on finding solution of fractional logistic equation are found in [18, 19]. The population growth modeling is considered when the carrying capacity is very large. The Chinese population

which has such property was considered in simulation. It appeared that the ψ -Caputo model introduced by Almeida [20], with recent development found in [21], is a good tool for solving this problem. Moreover, square root function used as kernel better fits the Chinese population more than the classical logistic population growth model. The work is organized as follows. Preliminaries definitions of fractional calculus are given in Section 2. The logistic growth model is defined in Section 3. The ψ -Caputo is defined in Section 4. The approximation of population with large carrying capacity is given in Section 5, and finally, a simulation with Chinese population is done.

2. Preliminaries

The theory of fractional calculus is built upon definitions, theorems, and special notations. Moreover, fractional integrals and fractional derivatives are useful tools for designing and solving fractional differential equations. In this

section, selected definitions, lemma, and theorems which are useful for the remaining of the work are given.

Definition 1 (see [22]). The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $g: [a, b] \rightarrow R$ is defined as

$$({}_{RL}I_{0^+}^\alpha g)(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^\tau (t-s)^{\alpha-1} g(s) ds. \quad (1)$$

Equation (1) provided that the right-hand side of the integral is pointwise defined on $(0, +\infty)$; moreover, Γ is the well-known gamma function defined by $\Gamma(v) = \int_0^\infty e^{-t} t^{v-1} dt$, $\forall v > 0$.

Definition 2 (see [22]). The Caputo derivative of order $\alpha > 0$ of a function $g: [a, b] \rightarrow R$ is defined as

$$({}_CD_{0^+}^\alpha g)(t) = \begin{cases} \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} g^{(n)}(s) ds, & n-1 < \alpha < n, \alpha \in R, \\ g^{(n)}(t), & \alpha \in N, \end{cases} \quad (2)$$

where $n = [\alpha] + 1$, $[\alpha]$ is the integer part of α .

Definition 3 (see [20]). Let $\alpha > 0$ and $g \in L^1[a, b]$ and $\psi \in C^1[a, b]$ be an increasing function with $\psi'(x) \neq 0$, $\forall x \in [a, b]$; then, $I_{0^+}^{\alpha, \psi} g(t)$ denotes the fractional integral of g with respect to ψ , and it is given by

$$I_{0^+}^{\alpha, \psi} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} g(s) ds. \quad (3)$$

This definition is known as ψ -Caputo integral if the fractional integral is taken in the Caputo sense. It is used in sequel for formulating the solution to the ψ -Caputo fractional model.

Definition 4 (see [20]). Let $\alpha > 0$ and $g, \psi \in C^n[a, b]$ with ψ be an increasing function and $\psi'(x) \neq 0$, $\forall x \in [a, b]$; then, ${}_CD_{0^+}^{\alpha, \psi} g(t)$ denotes the fractional derivative of g with respect to ψ (ψ -Caputo if the fractional derivative is taken in the Caputo sense), and it is given by

$${}_CD_{0^+}^{\alpha, \psi} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{n-\alpha-1} \left(\frac{1}{\psi'(s)} \frac{d}{dt} \right)^n g(s) ds. \quad (4)$$

Lemma 1 (see [20]). Let $\alpha > 0$, a natural number, n , such that $\alpha \in (n-1, n)$ if $g, \psi \in C^n[a, b]$; then,

$$I_{0^+}^{\alpha, \psi} ({}_CD_{0^+}^{\alpha, \psi} g)(t) = g(t) - \sum_{i=0}^{n-1} \frac{(1/\psi'(s) d/dt)^i g(0)}{i!} (\psi(t) - \psi(0))^i. \quad (5)$$

Lemma 2 (see [23]). Error rate of estimate.

Let us consider an experimental data set represented by a vector of size n , obtained from a time- or space-dependent process such that the i^{th} observation is denoted by $y_i = y(t_i)$. If the experimental is fitted by a function $\hat{y}$, in a way that the i^{th} fitted data value is denoted by $\hat{y}_i = \hat{y}(t_i)$,

the root-mean-squared deviation between the original data set and the fitted data set is computed by

$$\text{RMSD} = \sqrt{\frac{\sum_{t=0}^n (y(t) - \hat{y}(t))^2}{n}}. \quad (6)$$

On the contrary, the norm of the original data vector is obtained by averaging the sum of squared elements and then computing the square root of the results. This is expressed by the following formula:

$$\sqrt{\frac{\sum_{t=0}^n (y(t))^2}{n}}. \quad (7)$$

Dividing equation (6) by equation (7) produces an interesting tool to evaluate the fitting error which is represented as

$$ER = \frac{\sqrt{\sum_{t=0}^n (y(t) - \hat{y}(t))^2/n}}{\sqrt{\sum_{t=0}^n (y(t))^2/n}}. \quad (8)$$

The rate ER evaluates the magnitude of the error that occurs when the estimated values are used in place of the original values. It has an advantage over commonly used root square error and mean square error because it standardizes the magnitude of the data set used.

3. Logistic Population Growth Model

The theory of the population growth modeling using logistic equation was introduced by an economist named Malthus [24–26]. In fact, given an initial population with growth and reproductivity capacity, the theoretical expectation would be that the population size will approach infinity as the time increases indefinitely. However, Malthus has proven in his theory that a population growth tends to stabilize at some point when the time approaches infinity. He then proposed a model of population growth, called the logistic growth model, which is defined as

$$\frac{dN(t)}{dt} = rN(t) \left(1 - \frac{N(t)}{K} \right), \quad (9)$$

where r is the growth rate and K is the carrying capacity, which represents the maximum value that the population size may reach. Hence, the population size stabilizes when the carrying capacity is reached. A general solution of the classical logistic model defined by equation (9) is

$$N_c(t) = \frac{K}{1 + ((K - N_0)/N_0)e^{-rt}}, \quad (10)$$

where $N_0 = N(0)$ is the initial size of the population at $t = 0$.

Since we aim to prove that the fractional differential equation is better than the classical differential equation, the fractional differential equation equivalent to problem defined by equation (9) is built as follows:

$${}_c D^{\alpha, \psi} N(t) = rN(t) \left(1 - \frac{N(t)}{K} \right), \quad \alpha \in (0, 1), N(0) = 0, \quad (11)$$

or simply

$${}_c D^{\alpha, \psi} N(t) = f(t, N(t)), \quad \alpha \in (0, 1), N(0) = 0. \quad (12)$$

The following lemma introduces the general formula of the solution to the fractional logistic equation with initial condition, defined by equation (11).

Lemma 3. Assume f is an integrable function defined on $[0, T]$; then, the general solution of the fractional differential equation given by equation (12) is equivalent to the following integral equation:

$$N(t) = N_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(x) (\psi(t) - \psi(x))^{\alpha-1} f(x, N(x)) dx. \quad (13)$$

It is important to highlight the use of the ψ -Caputo fractional integral, equation (3) in Lemma 3.

Proof. Applying the operator $I_{0+}^{\alpha, \psi}$ to both sides of equation (12) leads to $N(t) - N_0 = I_{0+}^{\alpha, \psi} f(t, N(t))$. $\square$

4. Main Result of Psi-Caputo Logistic Population Growth Model

In this section, the theoretical study is carried around equation (12) in order to prove existence and uniqueness of a solution.

Given $C[0, T] = \{N \in C[0, T]\}$, let $\Phi = (C[0, T], \|\cdot\|)$ denote the Banach space of all continuous functions from $[0, T]$ to $\mathbb{R}$, endowed with the norm defined by $\|N\| = \sup_{0 \leq t \leq T} |N(t)|$.

An operator $\mathcal{F}: \Phi \rightarrow \Phi$ associated with the problem defined by equation (12) can be defined as

$$(\mathcal{F}N)(t) = N_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(x) (\psi(t) - \psi(x))^{\alpha-1} f(x, N(x)) dx. \quad (14)$$

The existence and uniqueness of solution for the problem given by equation (13) is discussed in sequel. Prior to establish proof of the main results, the following statements are established. These are mathematical hypotheses that are used in the proof sections.

(A1) $f: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous

(A2) There exists $L_N > 0$ such that $|f(t, N_1) - f(t, N_2)| \leq L_N |N_1 - N_2|$, $\forall t \in [0, T]$

(A3) There exists a function $g \in C([0, T], \mathbb{R}^+)$ and a nondecreasing function $\chi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $|f(t, u)| \leq g(t)\chi(|u|)$, $\forall (t, u) \in [0, T] \times \mathbb{R}$

(A4) There exists a constant $W > 0$ such that $W/N_0 + \|g\|_{\chi}(\lambda)(1/\Gamma(\alpha+1)(\psi(T) - \psi(0))^\alpha) > 1$

The following theorem establishes and proves existence of at least one solution to equation (12).

Theorem 1. Assume that (A1), (A3), and (A4) hold. Then, equation (12) has at least one solution in the interval $[0, T]$.

Proof. The proof will be split into several steps. In the first step, we show that the operator $\mathcal{F}: \Phi \rightarrow \Phi$ maps bounded

sets into bounded sets of Φ . Let $\mathfrak{B}_\lambda = \{N \in \Phi: \|N\| \leq \lambda\}$ be a bounded set in Φ ; then,

$$\begin{aligned} |(\mathcal{F}N)(t)| &\leq N_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(x) (\psi(t) - \psi(x))^{\alpha-1} |f(x, N(x))| dx \\ &\leq N_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(x) (\psi(t) - \psi(x))^{\alpha-1} g(x) \chi(\|N\|) dx. \end{aligned} \quad (15)$$

Applying the norm, $\sup_{0 \leq t \leq T}$ on both sides of equation (15) leads to

$$\begin{aligned} \|\mathcal{F}N\| &\leq N_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(x) (\psi(x) - \psi(x))^{\alpha-1} |f(x, N(x))| dx \\ &\leq N_0 + \frac{1}{\Gamma(\alpha+1)} \|g\| \chi(\lambda) (\psi(T) - \psi(0))^\alpha. \end{aligned} \quad (16)$$

The next step in the proof is to show that the operator $\mathcal{F}: \Phi \longrightarrow \Phi$ maps bounded sets into equi-continuous sets of Φ .

Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ and $N \in \mathfrak{B}_\lambda$. The following relation holds because of the above assumptions:

$$\begin{aligned} |(\mathcal{F}N)(t_2) - (\mathcal{F}N)(t_1)| &\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(x) [(\psi(t_2) - \psi(x))^{\alpha-1} - (\psi(t_1) - \psi(x))^{\alpha-1}] g(x) \chi(\|N\|) dx \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi'(x) [(\psi(t_2) - \psi(x))^{\alpha-1}] g(x) \chi(\|N\|) dx \right|, \end{aligned} \quad (17)$$

where the right-hand side of equation (17) tends to zero as $t_1 \longrightarrow t_2$. That is, $|(\mathcal{F}N)(t_2) - (\mathcal{F}N)(t_1)| \longrightarrow 0$ as $t_1 \longrightarrow t_2$.

Note that the right-hand side of equation (17) is independent of $N \in \mathfrak{B}_\lambda$; therefore, by Arzela–Ascoli theorem, we conclude that $\mathcal{F}$ is completely continuous.

The last step to complete the assumptions of Leray–Schauder nonlinear alternative theorem is to show the boundedness of the set of all solution to equation $N = \delta \mathcal{F}N$. Assume that N is a solution equation (12); then, it follows from equation (13) that

$$\begin{aligned} |N(t)| &= |\delta(\mathcal{F}N)(t)| \\ &\leq \delta \left(N_0 + \frac{1}{\Gamma(\alpha+1)} \|g\| \chi(\lambda) (\psi(T) - \psi(0))^\alpha \right) \\ &\leq N_0 + \frac{1}{\Gamma(\alpha+1)} \|g\| \chi(\lambda) (\psi(T) - \psi(0))^\alpha. \end{aligned} \quad (18)$$

Inverting both sides of equation (18) and dividing them by $|N(t)|$ leads to the following relation:

$$\frac{\|N\|}{N_0 + (1/\Gamma(\alpha+1)) \|g\| \chi(\lambda) (\psi(T) - \psi(0))^\alpha} \leq 1. \quad (19)$$

Recalling (A4), there exists a constant $W > 0$, which is indeed such that $W \neq N$. Moreover, let us construct the set $\Omega = \{N \in \Phi: N < W\}$. It is obvious that the operator $\mathcal{F}: \Omega \longrightarrow \Phi$ is continuous and completely continuous. Based on the constructed Ω , $\nexists N \in \partial\Omega$ such that $N = \delta \mathcal{F}N$ for some $\delta \in (0, 1)$. Consequently, by the nonlinear alternative of Leray–Schauder type, we deduce that $\mathcal{F}$ has a fixed point $N \in \overline{\Omega}$ which is a solution to the problem defined by equation (13). $\square$

Theorem 2. Let us assume that (A1) and (A2) hold. If $L_N/\Gamma(\alpha+1)(\psi(T) - \psi(0))^\alpha < 1$, then the problem defined in equation (12) has a unique solution on $[0, T]$.

Proof. Consider the operator $\mathcal{F}$ defined in equation (14) and define a ball

$$\begin{aligned} \mathfrak{B}_\varepsilon &= \{N \in C[0, T]: \|N\| \leq \varepsilon\} \text{ with} \\ \varepsilon &\geq \frac{N_0 + M_N (1/\Gamma(\alpha+1) (\psi(T) - \psi(0))^\alpha)}{1 - L_N 1/\Gamma(\alpha+1) (\psi(T) - \psi(0))^\alpha}, \end{aligned} \quad (20)$$

where $M_N = \sup_{0 \leq t \leq T} |f(t, 0)|$.

First, let us show that $\mathcal{F}B_\varepsilon \subset B_\varepsilon$. It is done as follows. For any $N \in B_\varepsilon, t \in [0, T]$ and using equation (14), we have the following relation:

$$|(\mathcal{F}N)(t)| \leq N_0 + \frac{1}{\Gamma(\alpha)} \int_0^t |\psi'(x)| |\psi(t) - \psi(x)^{\alpha-1}| |f(x, N(0))| dx. \quad (21)$$

On the contrary, triangular inequality is applied on $|f(x, N(x))|$ and produced in the next relation:

$$\begin{aligned} |f(x, N(x))| &= |f(x, N(x)) - f(x, 0) + f(x, 0)| \\ &\leq |f(x, N(x)) - f(x, 0)| + |f(x, 0)| \\ &\leq L_N \|N\| + M_N \\ &\leq L_N \varepsilon + M_N. \end{aligned} \quad (22)$$

Substituting the appropriate fragment of equation in equation (21) by equation (22) leads to a new relation defined below:

$$\begin{aligned} \|\mathcal{F}N\| &\leq N_0 + \left(\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(x) (\psi(t) - \psi(x))^{\alpha-1} dx \right) (L_N \varepsilon + M_N) \\ &\leq N_0 + \left(\frac{1}{\Gamma(\alpha)} (\psi(T) - \psi(0))^\alpha \right) (L_N \varepsilon + M_N) \\ &\leq \varepsilon. \end{aligned} \quad (23)$$

Equation (23) is enough to conclude that $\mathcal{F}B_\varepsilon \subset B_\varepsilon$.

The next step of the proof is to show that the operator is a contraction. $\forall N_1, N_2 \in \Phi$, and the following relation holds:

$$\begin{aligned} |(\mathcal{F}N_1)(t) - (\mathcal{F}N_2)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(x) (\psi(t) - \psi(x))^{\alpha-1} |f(x, N_1(x)) - f(x, N_2(x))| dx \\ &\leq \left(\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(x) (\psi(t) - \psi(x))^{\alpha-1} dx \right) L_N \|N_1 - N_2\| \\ &\leq L_N \left(\frac{1}{\Gamma(\alpha+1)} (\psi(T) - \psi(0))^\alpha \right) \|N_1 - N_2\| \\ &\leq \|N_1 - N_2\|. \end{aligned} \quad (24)$$

From equation (24), $\mathcal{F}N$ is said to be a contraction. By the Banach contraction mapping theorem, the fractional differential equation, equation (12), has a unique solution on $[0, T]$. $\square$

5. Population Growth with Carrying Capacity K Approaching Infinity

The carrying capacity of a population is computed with accuracy using nonlinear optimization routine on the existing sample data over a given time interval. From

equation (9), it is obvious that the carrying capacity has a high impact on the model. Some researchers focused in the past on the population growth behavior based on the carrying capacity dynamic (see [24, 27, 28]). Considering the human population growth modeling, we will observe that there exist some countries that try to control their population growth. China is a country with large population for which the concept of “large carrying capacity,” and $K \rightarrow \infty$ is applicable. Birth control in China has reduced the population growth rate. In this regard, the true population size is usually smaller than what is theoretically

FIGURE 1: Classical logistic growth.

expected using forecasting tools. This leads to a population size which is far away smaller than the carrying capacity. Mathematically, this is denoted as

$$N(t) \ll K. \quad (25)$$

That is,

$$\frac{N(t)}{K} \longrightarrow 0 \text{ as } t \longrightarrow +\infty. \quad (26)$$

Under the conditions defined by equation (25) and equation (26), the initial logistic model equation (9) becomes an exponential model:

$$\frac{dN(t)}{dt} = rN(t). \quad (27)$$

A general solution of the exponential model defined by equation (27) is

$$N(t) = N_0 e^{rt}. \quad (28)$$

The solution to the exponential model defined by equation (28) shows that the population size will increase infinitely as the time approaches infinity. Such result means that the Chinese population, for instance, will grow infinitely. This coincides with neither the Malthusian theory nor what is practically known about the Chinese population. In what follows, an alternative approach is proposed for the modeling of large size population with carrying capacity $K \longrightarrow +\infty$. Recalling the ψ -derivative from Definition 4, the model defined by equation (27) can be written in fractional sense as

$${}_C D_{0+}^{\alpha, \psi} N(t) = rN(t), N(0) = N_0. \quad (29)$$

Applying the fractional integral operator $I_{0+}^{\alpha, \psi}$ to both sides of equation (29) and using Theorem 7.2 from [29] is enough to find the solution of the model defined by equation (29) as

$$N(t) = N_0 E_\alpha[r(\psi(t) - \psi(0))^\alpha], \quad (30)$$

where $E_\alpha(t) = \sum_{i=0}^{\infty} t^i / \Gamma(\alpha i + 1)$, $t \in \mathbb{R}$ is the Mittag-Leffler function.

In simulations, a choice of the Kernels is made based on their shape in a way to best fit the data.

6. Simulation Results

In this section, we proved based on the numerical study with the Chinese population that the proposed method in Section 5 is the most appropriate for fitting a population with large carrying capacity than the usual logistic and exponential models. Historical data of the Chinese population was retrieved from the World Bank website [30]. A nonlinear optimization routine “lsqcurvefit” from Matlab was used to estimate the parameter that would best fit the classical logistic model equation (9), the exponential model equation (27), and the psi-Caputo fractional model equation (30). Recalling Lemma 2, in which error metric is proposed, the following results were obtained. Physical meaning of parameters used in this section is given prior to presenting all the results’ scenarios.

- (i) Carrying capacity, K is the maximum size that a population can reach without causing any degradation
- (ii) The growth rate r of a population is the percentage change in size of a given population in unit period (in this case per year)
- (iii) Constant α is the fractional order of derivative involved in the model design

Figure 1 depicts the original data alongside fitting line of the estimation using the classical logistic approach. The best parameters values for this model were found as the growth rate of $r = 0.0145$, which means over the period from 1960 to 2020 the logistic model revealed that the overall growth rate

FIGURE 2: Exponential growth.

FIGURE 3: ψ -Caputo with logarithm kernel.

of Chinese population was less than 2%. Moreover, the carrying capacity, $K = 3.3198 \times 10^{23}$, is theoretically the maximum size that Chinese population could reach. Using the classical logistic model, the original data set would be fitted while making the total error rate of $ER = 6.27\%$.

Figure 2 represents the solution obtained fitting original data set with classical exponential. Fitted model and original data are both shown in the figure. This approach has produced similar result as the classical logistic approach. In fact, the best estimated parameters are $r = 0.0145$ and

$K = 3.3198 \times 10^{23}$ for a minimum error rate of $ER = 6.27\%$. At this point, a partial result concerning large carrying capacity is obtained. In fact, for a large carrying capacity, the logistic growth model and exponential growth model coincide.

Figure 3 represents the result obtained when using the ψ -Caputo approach method with the kernel function being $\ln(x+1)$ to fit the original data. The best parameters of the model were estimated as growth rate $r = 0.0950$, $K = 3.3198 \times 10^{23}$ and fractional order of derivative $\alpha = 3.0096$. Fitting true data using those parameters

FIGURE 4: ψ -Caputo with square root kernel.

produced a total error of $ER = 1.57\%$. This model is non-realistic and irrelevant in this study since one expects the order of derivative, $\alpha \in (0, 1) \cup (1, 2)$.

Figure 4 shows true data alongside fitted data using the ψ -Caputo model with $\sqrt{x+1}$ as the kernel function. The best parameters estimated for this model are the growth rate $r = 0.0484$ and the fractional order of derivative $\alpha = 1.6455$. The use of those parameters produced a total error of $ER = 3.20\%$.

7. Conclusion

In this work, we studied population growth modeling using fractional calculus. We designed a fractional model and proved its existence and uniqueness, and then, we simulated the results using Chinese population data. At the end of the study, we observed the following. It can be cumbersome to give a practical or physical meaning to a fractional order of derivative associated to a model's solution. However, it is still important to provide an explanation for what is obtained in a modeling process. Let us, for instance, state that, in a more general physical sense, the first derivative of a trajectory equation is the speed or velocity. For the population growth model, the first derivative represents the speed by which the said population grows. It is also important to give a meaning to a fractional order of derivative α when used in fractional differential equations. In this regard, one can say that, given a modeling problem for which there exists a solution based on classical approach, if this solution involves a first-order derivative, then one would expect from its fractional counterpart to be optimal with a fractional derivative order α value such that $\alpha \in (0, 1) \cup (1, 2)$. In fact, if the fractional approach to a problem involving a first-order differential equation is optimal when $\alpha = 1$, this coincides with the classical solution. Hence, the fractional approach in such case does not improve anything compared

to the classical solution. However, if the solution is such that $\alpha \in (0, 1) \cup (1, 2)$, then one can proudly say that the fractional differential approach is better compared to the classical approach for this problem.

In this study, when the carrying capacity of population is very large and that the said population is subjected to a restricted growth, then the logistic approach and the exponential approach coincide. However, they might not be suitable for modeling the growth of such population. The ψ -Caputo method in both cases minimized the error. Moreover, the kernel function $\sqrt{x+1}$ is the most appropriate in this study because it best fits the data with an $\alpha \in (0, 1) \cup (1, 2)$, which is in line with the solution obtained using the classical differential equation approach, for which the solution is based on a first-order derivative.

An expansion of this work in future could consist of investigating the results using other types of fractional derivatives and integrals such as Hadamard and ψ -Hilfer and compare the results with those obtained so far.

Data Availability

The numerical data used to support the findings of this study have been deposited in the World Bank repository (<https://data.worldbank.org/indicator/SP.POP.TOTL?locations=CN>).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Each of the authors, M.A., Y.Y.Y., and K.A. contributed to each part of this work equally and read and approved the final version of the manuscript.

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