Research Article

New Results on the Forgotten Topological Index and Coindex

Akbar Jahanbani¹,¹ Maryam Atapour²,² and Rana Khoeilar¹

¹Department of Mathematics, Azarbaijan Shahid Madani University Tabriz, Tabriz, Iran

²Department of Mathematics and Computer Science, Basic Science Faculty, University of Bonab, P.O. Box 55513-95133, Bonab, Iran

Correspondence should be addressed to Akbar Jahanbani; akbar.jahanbani92@gmail.com and Maryam Atapour; maryam.atapour@gmail.com

Received 7 September 2021; Accepted 22 October 2021; Published 2 November 2021

Academic Editor: Ali Ahmad

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The $\mathcal{F}$ -coindex (forgotten topological coindex) for a simple connected graph $\mathcal{G}$ is defined as the sum of the terms $\zeta_{\mathcal{G}}^2(y) + \zeta_{\mathcal{G}}^2(x)$ over all nonadjacent vertex pairs (x, y) of $\mathcal{G}$, where $\zeta_{\mathcal{G}}(y)$ and $\zeta_{\mathcal{G}}(x)$ are the degrees of the vertices y and x in $\mathcal{G}$, respectively. The $\mathcal{F}$ -index of a graph is defined as the sum of cubes of the vertex degrees of the graph. This was introduced in 1972 in the same paper where the first and second Zagreb indices were introduced to study the structure dependency of total π -electron energy. Therefore, considering the importance of the $\mathcal{F}$ -index and $\mathcal{F}$ -coindex, in this paper, we study these indices, and we present new bounds for the $\mathcal{F}$ -index and $\mathcal{F}$ -coindex.

1. Introduction

Suppose $\mathcal{G}$ be a simple graph with vertex set $V = V(\mathcal{G})$ and edge set $E(\mathcal{G})$. The integers $N = N(\mathcal{G}) = |V(\mathcal{G})|$ and $\epsilon = \epsilon(\mathcal{G}) = |E(\mathcal{G})|$ are the order and the size of the graph $\mathcal{G}$, respectively; we say $\mathcal{G}$ is a (N, ϵ) -graph. The open neighborhood of vertex v is $N_{\mathcal{G}}(x) = N(x) = \{y \in V(\mathcal{G}) \mid xy \in E(\mathcal{G})\}$, and the degree of v is $\zeta_{\mathcal{G}}(x) = |N(x)|$. We write Θ and δ for the maximum and minimum degrees of $\mathcal{G}$, respectively. A graph $\mathcal{G}$ is said to be t -regular if all of its vertices have degree t . An (r, s) -semiregular graph is a graph whose each vertex is of degree s or r , and a (k, s, t) -triangular graph is a graph whose each vertex is of degree k , s , or t . The complement $\bar{\mathcal{G}}$ of a graph $\mathcal{G}$ is a graph that has the same vertices as $\mathcal{G}$ and in which two vertices are adjacent if and only if they are not adjacent in $\mathcal{G}$. The number of vertex pairs (x_i, x_j) in $\mathcal{G}$ such that $x_i x_j \notin E(\mathcal{G})$ is $\bar{\epsilon} = N(N-1)/2 - \epsilon(\mathcal{G})$. A pendant vertex is a vertex of degree one. The number of pendant vertices in $\mathcal{G}$ is denoted by $r = r(\mathcal{G})$. We denote by π_1 the minimal nonpendant vertex degree.

In [1, 2], the first and second Zagreb indices are defined as the following:

$$\begin{aligned} M_1 = M_1(\mathcal{G}) &= \sum_{x \in V(\mathcal{G})} \zeta_{\mathcal{G}}^2(x) = \sum_{xy \in E(\mathcal{G})} (\zeta_{\mathcal{G}}(y) + \zeta_{\mathcal{G}}(x)), \\ M_2 = M_2(\mathcal{G}) &= \sum_{xy \in E(\mathcal{G})} \zeta_{\mathcal{G}}(y) \zeta_{\mathcal{G}}(x), \end{aligned} \quad (1)$$

respectively. The first and second Zagreb coindices are defined in [3, 4] as

$$\begin{aligned} \bar{M}_1(\mathcal{G}) &= \sum_{xy \notin E(\mathcal{G})} (\zeta_{\mathcal{G}}(y) + \zeta_{\mathcal{G}}(x)), \\ \bar{M}_2(\mathcal{G}) &= \sum_{xy \notin E(\mathcal{G})} \zeta_{\mathcal{G}}(y) \zeta_{\mathcal{G}}(x). \end{aligned} \quad (2)$$

Furtula and Gutman [5] defined the forgotten topological index ($\mathcal{F}$ -index) as the following:

$$\mathcal{F} = \mathcal{F}(\mathcal{G}) = \sum_{x \in V(\mathcal{G})} \zeta_{\mathcal{G}}^3(x) = \sum_{xy \in E(\mathcal{G})} (\zeta_{\mathcal{G}}^2(y) + \zeta_{\mathcal{G}}^2(x)). \quad (3)$$

For more information on the $\mathcal{F}$ -index, see [6–8]. The $\mathcal{F}$ -coindex introduced in [9] is as follows:

$$\overline{\mathcal{F}}(\mathcal{G}) = \sum_{xy \notin E(\mathcal{G})} (\zeta_{\mathcal{G}}^2(y) + \zeta_{\mathcal{G}}^2(x)). \quad (4)$$

Gutman in [10] introduced the hyper-Zagreb coindex as follows:

$$\overline{\text{HM}}(\mathcal{G}) = \sum_{xy \notin E(\mathcal{G})} (\zeta_{\mathcal{G}}(y) + \zeta_{\mathcal{G}}(x))^2. \quad (5)$$

Usha et al. [11] introduced the redefined first Zagreb indices as the following:

$$\text{ReZG}_1(\mathcal{G}) = \sum_{xy \in E(\mathcal{G})} \frac{\zeta_{\mathcal{G}}(y) + \zeta_{\mathcal{G}}(x)}{\zeta_{\mathcal{G}}(y)\zeta_{\mathcal{G}}(x)}. \quad (6)$$

Here, we introduce the redefined first Zagreb coindex as the following:

$$\overline{\text{ReZG}}_1(\mathcal{G}) = \sum_{xy \notin E(\mathcal{G})} \frac{\zeta_{\mathcal{G}}(y) + \zeta_{\mathcal{G}}(x)}{\zeta_{\mathcal{G}}(y)\zeta_{\mathcal{G}}(x)}. \quad (7)$$

The authors introduced the first general Zagreb coindex in [12], and it is defined as follows:

$$\overline{M}_1^{\alpha}(\mathcal{G}) = \sum_{xy \notin E(\mathcal{G})} (\zeta_{\mathcal{G}}^{\alpha-1}(y) + \zeta_{\mathcal{G}}^{\alpha-1}(x)), \quad (8)$$

where $\alpha \in \mathbb{R}$. The second general Zagreb coindex was introduced in [13] and defined as follows:

$$\overline{M}_2^{\alpha}(\mathcal{G}) = \sum_{xy \notin E(\mathcal{G})} (\zeta_{\mathcal{G}}(y)\zeta_{\mathcal{G}}(x))^{\alpha}. \quad (9)$$

Topological indices are numerical quantity derived from a molecular graph which correlate the physico-chemical properties of the molecular graph. Recently, topological indices have been studied by many researchers due to their applications in various sciences such as chemistry, physics, and electricity; see [14–16]. Among the topological indices, the first Zagreb index is one of the oldest and most applied topological indices, and for this reason, it is of great importance and has been considered by many researchers today. Furtula and Gutman in [5] recently investigated this index and named this index as “forgotten topological index” or “ F -index” and showed that the predictive ability of this index is almost similar to that of the first Zagreb index and for the entropy and acetic factor; both of them yield correlation coefficients greater than 0.95. Therefore, due to the importance of the F -index in this paper, we have decided to study this index.

In [17, 18], some of bounds for the general Zagreb coindices were obtained. Ranjini et al. [19] presented some of the bounds for Zagreb indices and the Zagreb coindices. In [20], some bounds were presented for the $\mathcal{F}$ -index and $\mathcal{F}$ -coindex. For more other bounds, see [21–23].

Given the importance of the forgotten topological index and the fact that it has recently attracted the attention of researchers and the interest of many readers, in this paper, we intend to discuss new bounds for this index.

2. Preliminaries

Here, we recall several published results that we will need for proof.

The following result obtained the relationship between the first Zagreb index and the maximum and minimum degrees.

Theorem 1 (see [24]). *Let $\mathcal{G}$ be an $(\aleph, \epsilon)$ -connected graph and $\aleph \geq 2$. Then,*

$$M_1 \geq \frac{4\epsilon^2}{\aleph} + \frac{1}{2}(\Theta - \delta)^2, \quad (10)$$

with equality if and only if $\mathcal{G}$ is isomorphic with a regular graph.

The following result comes from [18].

Theorem 2. *Suppose $\mathcal{G}$ be a $(\aleph, \epsilon)$ -graph and $\aleph \geq 2$. Then,*

$$M_1 \leq \frac{4\epsilon^2}{\aleph} + \frac{\aleph}{4}(\Theta - \delta)^2, \quad (11)$$

with equality if and only if $\mathcal{G}$ is isomorphic with the t -regular graph, $1 \leq t \leq \aleph - 1$.

In [19], the authors gave the relation of the second Zagreb coindex, the maximum degree, and the first Zagreb index as follows.

Theorem 3. *Let $\mathcal{G}$ be an $(\aleph, \epsilon)$ -graph and maximal degree be Θ . Then,*

$$\overline{M}_2(\mathcal{G}) \leq 2\Theta\epsilon(\aleph - 1) - \Theta M_1(\mathcal{G}). \quad (12)$$

Zhou and Trinajstić [25] proved the following result.

Theorem 4. *Suppose $\mathcal{G}$ be a $(\aleph, \epsilon)$ -graph. Then,*

$$\mathcal{F}(\mathcal{G}) \geq \frac{16\epsilon^3}{\aleph^2} - 2M_2(\mathcal{G}), \quad (13)$$

with equality if and only if $\mathcal{G}$ is regular.

Furtula and Gutman [5] showed the following.

Theorem 5. *Suppose $\mathcal{G}$ be a graph of size ϵ . Then,*

$$\mathcal{F}(\mathcal{G}) \geq \frac{M_1(\mathcal{G})^2}{2\epsilon}, \quad (14)$$

with equality if and only if $\mathcal{G}$ is regular.

In [26], Elumalai et al. obtained the following two results.

Theorem 6. *Suppose $\mathcal{G}$ be a simple graph of order $\aleph \geq 2$. Then,*

$$\mathcal{F}(\mathcal{G}) \geq \Theta^3 + \Theta^2 + \Phi, \quad (15)$$

with equality if and only if $\mathcal{G}$ is regular, where

$$\Phi = \frac{(M_1(\mathcal{G}) - \Theta^2 + \Theta_2^2)^2 + (\aleph - 2)(M_1(\mathcal{G}) - \Theta^2 + \Theta_2^2)}{(2\epsilon - \Theta + \Theta_2)} - (2\epsilon - \Theta + \Theta_2). \quad (16)$$

Theorem 7. Suppose $\mathcal{G}$ be a simple graph of order $\aleph \geq 2$. Then,

$$\mathcal{F}(\mathcal{G}) \geq \Theta^3 + \Theta_2^3 + Y, \quad (17)$$

with equality if and only if $\mathcal{G}$ is regular, where

$$Y = \frac{(M_1(\mathcal{G}) - \Theta^2 - \delta^2)^2 + (2\epsilon - \Theta - \delta)(ID(\mathcal{G}) - 1/\Theta - 1/\delta) - (\aleph - 2)^2}{(2\epsilon - \Theta - \delta)}. \quad (18)$$

In [5], Furtula and Gutman mentioned the following result.

Theorem 8. Suppose $\mathcal{G}$ be a connected $(\aleph, \epsilon)$ -graph and second Zagreb index $M_2(\mathcal{G})$. Then,

$$\mathcal{F}(\mathcal{G}) \leq 2M_2(\mathcal{G}) + \epsilon(\aleph - 2), \quad (19)$$

with equality if and only if $\mathcal{G}$ is the star graph.

The following result was first proved in [27].

Theorem 9. Suppose $\mathcal{G}$ be a simple graph of order $\aleph$. Then,

$$\overline{\mathcal{F}}(\mathcal{G}) + \mathcal{F}(\mathcal{G}) = (\aleph - 1)M_1(\mathcal{G}). \quad (20)$$

3. New Bounds for the $\mathcal{F}$ -Coindex

In this section, we will obtain some bounds for the $\mathcal{F}$ -coindex in terms of the maximal degree, minimum degree, order, size, pendant vertex, and the first and the second Zagreb indices.

Theorem 10. Suppose $\mathcal{G}$ be a $(\aleph, \epsilon)$ -graph. Then,

$$\mathcal{F}(\overline{\mathcal{G}}) + \mathcal{F}(\mathcal{G}) = 2(\aleph - 1)^2(\overline{\epsilon} - 2\epsilon) + 3(\aleph - 1)M_1(\mathcal{G}). \quad (21)$$

Proof. By applying the definition of the $\mathcal{F}$ -index for complement graphs, we have

$$\begin{aligned} \mathcal{F}(\overline{\mathcal{G}}) &= \sum_{x \in V(\mathcal{G})} \zeta_{\overline{\mathcal{G}}}^3(x) = \sum_{x \in V(\mathcal{G})} (\aleph - 1 - \zeta_{\mathcal{G}}(x))^3, \\ &= \sum_{x \in V(\mathcal{G})} ((\aleph - 1)^3 - \zeta_{\mathcal{G}}^3(x) - 3(\aleph - 1)\zeta_{\mathcal{G}}^2(x) + 3(\aleph - 1)^2\zeta_{\mathcal{G}}(x)), \\ &= \sum_{x \in V(\mathcal{G})} (\aleph - 1)^3 - \sum_{x \in V(\mathcal{G})} \zeta_{\mathcal{G}}^3(x) + 3(\aleph - 1) \sum_{x \in V(\mathcal{G})} \zeta_{\mathcal{G}}^2(x) - 3 \sum_{x \in V(\mathcal{G})} (\aleph - 1)^2\zeta_{\mathcal{G}}(x), \\ &= \aleph(\aleph - 1)^3 - \mathcal{F}(\mathcal{G}) + 3(\aleph - 1)M_1(\mathcal{G}) - 6\epsilon(\aleph - 1)^2, \\ &= 2(\aleph - 1)^2(\overline{\epsilon} - 2\epsilon) - \mathcal{F}(\mathcal{G}) + 3(\aleph - 1)M_1(\mathcal{G}). \end{aligned} \quad (22)$$

Theorem 11. Suppose $\mathcal{G}$ be a $(\aleph, \epsilon)$ -graph. Then,

$$\overline{\mathcal{F}}(\overline{\mathcal{G}}) = 2\epsilon(\aleph - 1)^2 - 2(\aleph - 1)M_1(\mathcal{G}) + \mathcal{F}(\mathcal{G}). \quad (23)$$

Proof. For any vertex $y \in V(\mathcal{G})$, we have $\zeta_{\overline{\mathcal{G}}}(y) = \aleph - 1 - \zeta_{\mathcal{G}}(y)$, and by applying the definition of the $\mathcal{F}$ -coindex, we have

$$\begin{aligned} \overline{\mathcal{F}}(\overline{\mathcal{G}}) &= \sum_{xy \notin E(\mathcal{G})} (\zeta_{\overline{\mathcal{G}}}^2(y) + \zeta_{\overline{\mathcal{G}}}^2(x)), \\ &= \sum_{xy \in E(\mathcal{G})} ((\aleph - 1 - \zeta_{\mathcal{G}}(x))^2 + (\aleph - 1 - \zeta_{\mathcal{G}}(y))^2), \\ &= \sum_{xy \in E(\mathcal{G})} ((\aleph - 1)^2 + \zeta_{\mathcal{G}}^2(x) - 2(\aleph - 1)\zeta_{\mathcal{G}}(x) + (\aleph - 1)^2 + \zeta_{\mathcal{G}}^2(y) - 2(\aleph - 1)\zeta_{\mathcal{G}}(y)), \end{aligned}$$

$$\begin{aligned}
&= \sum_{xy \in E(\mathcal{G})} (2(\aleph - 1)^2 - 2(\aleph - 1)(\zeta_{\mathcal{G}}(x) + \zeta_{\mathcal{G}}(y)) + (\zeta_{\mathcal{G}}^2(x) + \zeta_{\mathcal{G}}^2(y))), \\
&= 2(\aleph - 1)^2 \epsilon - 2(\aleph - 1)M_1(\mathcal{G}) + \mathcal{F}(\mathcal{G}).
\end{aligned} \tag{24}$$

Now, we give a lower bound for the $\mathcal{F}$ -coindex in terms of pendant vertices. $\square$

Theorem 12. Suppose $\mathcal{G}$ be a connected graph of order $\aleph$ and r pendant vertices. Then,

$$\overline{\mathcal{F}}(\mathcal{G}) \geq 6\aleph r - 4r^2 - 7r. \tag{25}$$

Proof. It is easy to see that our result holds for $r = 0$. Now, we assume that $r \geq 1$. Here, we let that $\mathcal{G}$ has exactly one pendant vertex, called x , and y is the unique neighbor of x . Hence,

$$\begin{aligned}
\overline{\mathcal{F}}(\mathcal{G}) &\geq \sum_{z \in V(\mathcal{G}) \setminus \{y, x\}} (\zeta_{\mathcal{G}}^2(z) + 1) \geq \sum_{w \in V(\mathcal{G}) \setminus \{y, x\}} 5, \\
&= 5(\aleph - 2).
\end{aligned} \tag{26}$$

Here, we can let that $r \geq 2$. Each pair of pendant vertices contributes 2 to $\overline{\mathcal{F}}(\mathcal{G})$. The total contribution of pendant vertex pairs to $\overline{\mathcal{F}}(\mathcal{G})$ is $2\binom{r}{2}$. Assume that x is a pendant vertex in $\mathcal{G}$ and y is its unique neighbor. Then, for any nonpendant vertex z such that $w \in V(\mathcal{G}) \setminus \{y, x\}$, the contribution of vertex pairs $\{x, z\}$ to $\overline{\mathcal{F}}(\mathcal{G})$ is $1 + \zeta_{\mathcal{G}}^2(z)$. The total contribution of such vertex pairs $\{x, z\}$ to $\overline{\mathcal{F}}(\mathcal{G})$ is $(\aleph - r - 1)\binom{r}{1}(1 + \zeta_{\mathcal{G}}^2(z))$. Note that $\zeta_{\mathcal{G}}(z) \geq 2$ for any nonpendant vertex z in $\mathcal{G}$; therefore, we get $\overline{\mathcal{F}}(\mathcal{G}) \geq \binom{r}{2}(1 + 1) + (\aleph - r - 1)\binom{r}{2}(1 + 5) = 6\aleph r - 4r^2 - 7r$, as desired. $\square$ $\square$

Theorem 13. If $\mathcal{G}$ is a t -regular graph of order $\aleph$, then

$$\begin{aligned}
\mathcal{F}(\overline{\mathcal{G}}) &= (\aleph - 1)^2 (\aleph(\aleph - 1) - 3t\aleph) - \aleph t^3 + 3\aleph(\aleph - 1)t^2, \\
\overline{\mathcal{F}}(\mathcal{G}) &= \aleph t^2 (\aleph - t - 1), \\
\overline{\mathcal{F}}(\overline{\mathcal{G}}) &= \aleph t ((\aleph - 1)^2 - 2t(\aleph - 1) + t^2).
\end{aligned} \tag{27}$$

Proof. We know that any t -regular graph of order $\aleph$ has $\aleph t/2$ edges.

(1) By applying Theorem 10, we have

$$\begin{aligned}
\mathcal{F}(\overline{\mathcal{G}}) &= 2(\aleph - 1)^2 (\overline{\epsilon} - 2\epsilon) + 3(\aleph - 1)M_1(\mathcal{G}) - \mathcal{F}(\mathcal{G}), \\
&= 2(\aleph - 1)^2 \left(\frac{\aleph(\aleph - 1)}{2} - \frac{3\aleph t}{2} \right) + 3(\aleph - 1)\aleph t^2 - \aleph t^3, \\
&= (\aleph - 1)^2 (\aleph(\aleph - 1) - 3t\aleph) - \aleph t^3 + 3\aleph(\aleph - 1)t^2.
\end{aligned} \tag{28}$$

(2) By applying Theorem 9, we can write

$$\mathcal{F}(\overline{\mathcal{G}}) = (\aleph - 1)M_1(\mathcal{G}) - \mathcal{F}(\mathcal{G}) = \aleph(\aleph - 1)t^2 - \aleph t^3. \tag{29}$$

(3) Similarly, by applying Theorem 11, we obtain

$$\begin{aligned}
\overline{\mathcal{F}}(\overline{\mathcal{G}}) &= 2(\aleph - 1)^2 \epsilon - 2(\aleph - 1)M_1(\mathcal{G}) + \mathcal{F}(\mathcal{G}), \\
&= 2(\aleph - 1)^2 \frac{\aleph t}{2} - 2\aleph t^2 (\aleph - 1) + \aleph t^3, \\
&= \aleph t ((\aleph - 1)^2 - 2t(\aleph - 1) + t^2).
\end{aligned} \tag{30}$$

Now, we give lower and upper bounds for the $\mathcal{F}$ -coindex. $\square$

Theorem 14. Suppose $\mathcal{G}$ be a connected $(\aleph, \epsilon)$ -graph, maximum degree be Θ , and nonpendant minimum degree be π_1 and with s leaves. Then,

$$\begin{aligned}
\overline{\mathcal{F}}(\mathcal{G}) &\leq \left(s(\aleph - s - 1) \left(\Theta + \frac{1}{\Theta} \right) + \left(\frac{\Theta}{\pi_1} + \frac{\pi_1}{\Theta} \right) \left(\left(\frac{\aleph^2}{2} + \frac{s^2}{2} + \frac{3s}{2} - \frac{\aleph}{2} - \epsilon \right) + s(s - 1) - \aleph s \right) \right) (\aleph - 2)^2, \\
\overline{\mathcal{F}}(\mathcal{G}) &\geq \left(s(\aleph - s - 1) \left(\pi_1 + \frac{1}{\pi_1} \right) + \left(\frac{\pi_1}{\Theta} + \frac{\Theta}{\pi_1} \right) \left(\left(\frac{\aleph^2}{2} + \frac{s^2}{2} + \frac{3s}{2} - \frac{\aleph}{2} - \epsilon \right) + s(s - 1) - \aleph s \right) \right).
\end{aligned} \tag{31}$$

Proof. It can be easily seen that the number of vertices pairs (x_i, x_j) is as follows:

$$\begin{aligned}
 X_1 &= \{(x_i, x_j) | x_i x_j \notin E(\mathcal{G})\}, |X_1| = \frac{\aleph(\aleph-1)}{2} - \epsilon, \\
 X_2 &= \{(x_i, x_j) | x_i x_j \notin E(\mathcal{G}) \text{ and } \zeta_{\mathcal{G}}(x_i) = \zeta_{\mathcal{G}}(x_j) = 1\}, |X_2| = \frac{s(s-1)}{2}, \\
 X_3 &= \{(x_i, x_j) | x_i x_j \notin E(\mathcal{G}) \zeta_{\mathcal{G}}(x_i) = 1 \text{ or } \zeta_{\mathcal{G}}(x_j) = 1\}, |X_3| = s(\aleph-s-1), \\
 X_4 &= \{(x_i, x_j) | x_i x_j \notin E(\mathcal{G}) \zeta_{\mathcal{G}}(x_i) > 1, \text{ and } \zeta_{\mathcal{G}}(x_j) > 1\}, |X_4| = \frac{\aleph(\aleph-1)}{2} - \epsilon - \frac{s(s-1)}{2} - s(\aleph-s-1).
 \end{aligned} \tag{32}$$

For any vertex x_i , we have $\pi_1 \leq \zeta_{\mathcal{G}}(x_i) \leq \Theta$ and $\pi_1 \leq \zeta_{\mathcal{G}}(x_i) \zeta_{\mathcal{G}}(x_j) \leq \Theta^2$ for nonpendant vertices. Therefore, $x_i x_j \notin E(\mathcal{G})$.

We continue the proof with the following four cases.

Let $x_i x_j \notin E(\mathcal{G})$. $\square$

Case 1. If $\zeta_{\mathcal{G}}(x_i) > 1$ and $\zeta_{\mathcal{G}}(x_j) > 1$, we have

$$\frac{\zeta_{\mathcal{G}}(x_i)}{\zeta_{\mathcal{G}}(x_j)} + \frac{\zeta_{\mathcal{G}}(x_j)}{\zeta_{\mathcal{G}}(x_i)} \leq \left(\frac{\Theta}{\pi_1} + \frac{\pi_1}{\Theta} \right). \tag{33}$$

Case 2. If $\zeta_{\mathcal{G}}(x_i) > 1$ and $\zeta_{\mathcal{G}}(x_j) = 1$, hence, we get

$$\frac{\zeta_{\mathcal{G}}(x_i)}{\zeta_{\mathcal{G}}(x_j)} + \frac{\zeta_{\mathcal{G}}(x_j)}{\zeta_{\mathcal{G}}(x_i)} \leq \left(\Theta + \frac{1}{\Theta} \right). \tag{34}$$

It is clear that, for each $x_i x_j \notin E(\mathcal{G})$, $\zeta_{\mathcal{G}}(x_i), \zeta_{\mathcal{G}}(x_j) \leq \aleph - 2$. Hence, by applying the definition of the $\mathcal{F}$ -coindex and above facts, we can write

$$\begin{aligned}
 \frac{\overline{\mathcal{F}}}{(\mathcal{G})}(\aleph-2)^2 &= \frac{\sum_{x_i x_j \notin E(\mathcal{G})} (\zeta_{\mathcal{G}}^2(x_j) + \zeta_{\mathcal{G}}^2(x_i))}{(n-2)^2} \leq \sum_{x_i x_j \notin E(\mathcal{G})} \frac{\zeta_{\mathcal{G}}^2(x_j) + \zeta_{\mathcal{G}}^2(x_i)}{\zeta_{\mathcal{G}}(x_i) \zeta_{\mathcal{G}}(x_j)}, \\
 &= \sum_{x_i x_j \notin E(\mathcal{G})} \left(\frac{\zeta_{\mathcal{G}}(x_i)}{\zeta_{\mathcal{G}}(x_j)} + \frac{\zeta_{\mathcal{G}}(x_j)}{\zeta_{\mathcal{G}}(x_i)} \right), \\
 &= \sum_{\substack{x_i x_j \notin E(\mathcal{G}), \\ \zeta_{\mathcal{G}}(x_i)=1, \\ \zeta_{\mathcal{G}}(x_j)=1}} \left(\frac{\zeta_{\mathcal{G}}(x_i)}{\zeta_{\mathcal{G}}(x_j)} + \frac{\zeta_{\mathcal{G}}(x_j)}{\zeta_{\mathcal{G}}(x_i)} \right) + \sum_{\substack{x_i x_j \notin E(\mathcal{G}), \\ \zeta_{\mathcal{G}}(x_i) > 1, \\ \zeta_{\mathcal{G}}(x_j)=1}} \left(\frac{\zeta_{\mathcal{G}}(x_i)}{\zeta_{\mathcal{G}}(x_j)} + \frac{\zeta_{\mathcal{G}}(x_j)}{\zeta_{\mathcal{G}}(x_i)} \right) + \sum_{\substack{x_i x_j \notin E(\mathcal{G}), \\ \zeta_{\mathcal{G}}(x_i) > 1, \\ \zeta_{\mathcal{G}}(x_j) > 1}} \left(\frac{\zeta_{\mathcal{G}}(x_i)}{\zeta_{\mathcal{G}}(x_j)} + \frac{\zeta_{\mathcal{G}}(x_j)}{\zeta_{\mathcal{G}}(x_i)} \right) \\
 &\leq \sum_{\substack{x_i x_j \notin E(\mathcal{G}), \\ \zeta_{\mathcal{G}}(x_i)=1, \\ \zeta_{\mathcal{G}}(x_j)=1}} 2 + \sum_{\substack{x_i x_j \notin E(\mathcal{G}), \\ \zeta_{\mathcal{G}}(x_i) > 1, \\ \zeta_{\mathcal{G}}(x_j)=1}} \left(\Theta + \frac{1}{\Theta} \right) + \sum_{\substack{x_i x_j \notin E(\mathcal{G}), \\ \zeta_{\mathcal{G}}(x_i) > 1, \\ \zeta_{\mathcal{G}}(x_j) > 1}} \left(\frac{\Theta}{\pi_1} + \frac{\pi_1}{\Theta} \right), \\
 &= s(s-1) + \left(\Theta + \frac{1}{\Theta} \right) s(\aleph-s-1) + \left(\frac{\Theta}{\pi_1} + \frac{\pi_1}{\Theta} \right) \left(\frac{\aleph(\aleph-1)}{2} - \epsilon - \frac{s(s-1)}{2} - s(\aleph-s-1) \right), \\
 &= s(\aleph-s-1) \left(\Theta + \frac{1}{\Theta} \right) + \left(\frac{\Theta}{\pi_1} + \frac{\pi_1}{\Theta} \right) \left(\left(\frac{\aleph^2}{2} + \frac{s^2}{2} + \frac{3s}{2} - \frac{n}{2} - \epsilon \right) + s(s-1) - \aleph s \right),
 \end{aligned} \tag{35}$$

as desired.

Case 4. If $\zeta_{\mathcal{G}}(x_i) > 1$ and $\zeta_{\mathcal{G}}(x_j) = 1$, we get that

Case 3. If $\zeta_{\mathcal{G}}(x_i) > 1$ and $\zeta_{\mathcal{G}}(x_j) > 1$, we get

$$\frac{\zeta_{\mathcal{G}}(x_i)}{\zeta_{\mathcal{G}}(x_j)} + \frac{\zeta_{\mathcal{G}}(x_j)}{\zeta_{\mathcal{G}}(x_i)} \geq \left(\frac{\pi_1}{\Theta} + \frac{\Theta}{\pi_1} \right). \quad (36)$$

Thus,

$$\frac{\zeta_{\mathcal{G}}(x_i)}{\zeta_{\mathcal{G}}(x_j)} + \frac{\zeta_{\mathcal{G}}(x_j)}{\zeta_{\mathcal{G}}(x_i)} \geq \left(\pi_1 + \frac{1}{\pi_1} \right). \quad (37)$$

$$\begin{aligned} \overline{\mathcal{F}}(\mathcal{G}) &= \sum_{x_i, x_j \notin E(\mathcal{G})} \zeta_{\mathcal{G}}^2(x_i) + \zeta_{\mathcal{G}}^2(x_j) \geq \sum_{x_i, x_j \notin E(\mathcal{G})} \frac{\zeta_{\mathcal{G}}^2(x_i) + \zeta_{\mathcal{G}}^2(x_j)}{\zeta_{\mathcal{G}}(x_i)\zeta_{\mathcal{G}}(x_j)}, \\ &= \sum_{x_i, x_j \notin E(\mathcal{G})} \left(\frac{\zeta_{\mathcal{G}}(x_i)}{\zeta_{\mathcal{G}}(x_j)} + \frac{\zeta_{\mathcal{G}}(x_j)}{\zeta_{\mathcal{G}}(x_i)} \right), \\ &= \sum_{\substack{x_i, x_j \notin E(\mathcal{G}), \\ \zeta_{\mathcal{G}}(x_i)=1, \\ \zeta_{\mathcal{G}}(x_j)=1}} \left(\frac{\zeta_{\mathcal{G}}(x_i)}{\zeta_{\mathcal{G}}(x_j)} + \frac{\zeta_{\mathcal{G}}(x_j)}{\zeta_{\mathcal{G}}(x_i)} \right) + \sum_{\substack{x_i, x_j \notin E(\mathcal{G}), \\ \zeta_{\mathcal{G}}(x_i) > 1, \\ \zeta_{\mathcal{G}}(x_j)=1}} \left(\frac{\zeta_{\mathcal{G}}(x_i)}{\zeta_{\mathcal{G}}(x_j)} + \frac{\zeta_{\mathcal{G}}(x_j)}{\zeta_{\mathcal{G}}(x_i)} \right) + \sum_{\substack{x_i, x_j \notin E(\mathcal{G}), \\ \zeta_{\mathcal{G}}(x_i) > 1, \\ \zeta_{\mathcal{G}}(x_j) > 1}} \left(\frac{\zeta_{\mathcal{G}}(x_i)}{\zeta_{\mathcal{G}}(x_j)} + \frac{\zeta_{\mathcal{G}}(x_j)}{\zeta_{\mathcal{G}}(x_i)} \right) \\ &\geq \sum_{\substack{x_i, x_j \notin E(\mathcal{G}), \\ \zeta_{\mathcal{G}}(x_i)=1, \\ \zeta_{\mathcal{G}}(x_j)=1}} 2 + \sum_{\substack{x_i, x_j \notin E(\mathcal{G}), \\ \zeta_{\mathcal{G}}(x_i) > 1, \\ \zeta_{\mathcal{G}}(x_j)=1}} \left(\pi_1 + \frac{1}{\pi_1} \right) + \sum_{\substack{x_i, x_j \notin E(\mathcal{G}), \\ \zeta_{\mathcal{G}}(x_i) > 1, \\ \zeta_{\mathcal{G}}(x_j) > 1}} \left(\frac{\pi_1}{\Theta} + \frac{\Theta}{\pi_1} \right), \\ &= s(s-1) + \left(\pi_1 + \frac{1}{\pi_1} \right) s(\aleph - s - 1) + \left(\frac{\pi_1}{\Theta} + \frac{\Theta}{\pi_1} \right) \left(\frac{\aleph(\aleph-1)}{2} - \epsilon - \frac{s(s-1)}{2} - s(\aleph - s - 1) \right), \\ &= s(\aleph - s - 1) \left(\pi_1 + \frac{1}{\pi_1} \right) + \left(\frac{\pi_1}{\Theta} + \frac{\Theta}{\pi_1} \right) \left(\left(\frac{\aleph^2}{2} + \frac{s^2}{2} + \frac{3s}{2} - \frac{\aleph}{2} - \epsilon \right) + s(s-1) - \aleph s \right), \end{aligned} \quad (38)$$

and the proof is completed.

By setting $s = 0$ in Theorem 14, we can obtain the following results.

Corollary 1. Suppose $\mathcal{G}$ be a connected $(\aleph, \epsilon)$ -graph, maximum degree be Θ , and nonpendant minimum degree be π_1 . Then,

$$\begin{aligned} \overline{\mathcal{F}}(\mathcal{G}) &\leq \left(\frac{\Theta}{\pi_1} + \frac{\pi_1}{\Theta} \right) \left(\frac{\aleph^2}{2} - \frac{\aleph}{2} - \epsilon \right) (\aleph - 2)^2, \\ \overline{\mathcal{F}}(\mathcal{G}) &\geq \left(\frac{\pi_1}{\Theta} + \frac{\Theta}{\pi_1} \right) \left(\frac{\aleph^2}{2} - \frac{\aleph}{2} - \epsilon \right). \end{aligned} \quad (39)$$

Here, we give a lower bound for the $\mathcal{F}$ -coindex.

Proposition 1. Suppose $\mathcal{G}$ be a connected graph of order $\aleph$ and size ϵ . Then,

$$\overline{\mathcal{F}}(\mathcal{G}) \geq \aleph^2 - 2\epsilon - \aleph. \quad (40)$$

Proof. By applying the definition of the $\mathcal{F}$ -coindex, we have

$$\overline{\mathcal{F}}(\mathcal{G}) = \sum_{x_i, x_j \notin E(\mathcal{G})} (\zeta_{\mathcal{G}}^2(x_i) + \zeta_{\mathcal{G}}^2(x_j)) \geq \sum_{x_i, x_j \notin E(\mathcal{G})} 2 = 2 \left(\frac{\aleph(\aleph-1)}{2} - \epsilon \right) = \aleph^2 - 2\epsilon - \aleph. \quad (41)$$

In the following result, we obtain the lower bound for the $\mathcal{F}$ -coindex. $\square$

Proof. Clearly, for each $x_i, x_j \notin E(\mathcal{G})$, we have $\max\{\zeta_{\mathcal{G}}(x_i), \zeta_{\mathcal{G}}(x_j)\} \leq \aleph - 2$. It follows that

Theorem 15. Suppose $\mathcal{G}$ be a connected $(\aleph, \epsilon)$ -graph and $\aleph \geq 2$. Then,

$$\overline{\mathcal{F}}(\mathcal{G}) \geq \frac{\overline{M}_1^5(\mathcal{G}) + 2\overline{M}_2^2(\mathcal{G})}{2(\aleph - 2)^2}. \quad (42)$$

$$\begin{aligned} 2(n-2)^2 \overline{\mathcal{F}}(\mathcal{G}) &= 2(\aleph - 2)^2 \sum_{x_i, x_j \notin E(\mathcal{G})} (\zeta_{\mathcal{G}}^2(x_i) + \zeta_{\mathcal{G}}^2(x_j)) \geq \sum_{x_i, x_j \notin E(\mathcal{G})} (\zeta_{\mathcal{G}}^2(x_i) + \zeta_{\mathcal{G}}^2(x_j))^2, \\ &= \sum_{x_i, x_j \notin E(\mathcal{G})} (\zeta_{\mathcal{G}}^4(x_i) + \zeta_{\mathcal{G}}^4(x_j) + 2\zeta_{\mathcal{G}}^2(x_i)\zeta_{\mathcal{G}}^2(x_j)), \\ &= \sum_{x_i, x_j \notin E(\mathcal{G})} (\zeta_{\mathcal{G}}^4(x_i) + \zeta_{\mathcal{G}}^4(x_j)) + 2 \sum_{x_i, x_j \notin E(\mathcal{G})} (\zeta_{\mathcal{G}}(x_i)\zeta_{\mathcal{G}}(x_j))^2, \\ &= \overline{M}_1^5(\mathcal{G}) + 2\overline{M}_2^2(\mathcal{G}), \end{aligned} \quad (43)$$

and the proof is completed.

Now, we give an upper bound for the $\mathcal{F}$ -coindex in terms of the second Zagreb coindex and the redefined first Zagreb coindex. $\square$

$$\overline{\mathcal{F}}(\mathcal{G}) \leq 2(\aleph - 2)^3 \overline{\text{ReZG}}_1(\mathcal{G}) - 2(\aleph - 2)^2 \overline{M}_2(\mathcal{G}). \quad (44)$$

Proof. By applying the definition of the $\mathcal{F}$ -coindex, we have

Theorem 16. Suppose $\mathcal{G}$ be a graph of order $\aleph$. Then,

$$\begin{aligned} \frac{\overline{\mathcal{F}}(\mathcal{G})}{(n-2)^2} &= \frac{\sum_{x_i, x_j \notin E(\mathcal{G})} (\zeta_{\mathcal{G}}^2(x_i) + \zeta_{\mathcal{G}}^2(x_j))}{(n-2)^2} \leq \sum_{v_i, v_j \notin E(\mathcal{G})} \left(\frac{\zeta_{\mathcal{G}}(x_i)}{\zeta_{\mathcal{G}}(x_j)} + \frac{\zeta_{\mathcal{G}}(x_j)}{\zeta_{\mathcal{G}}(x_i)} \right), \\ &= \sum_{x_i, x_j \notin E(\mathcal{G})} \left(\left(\frac{1}{\zeta_{\mathcal{G}}(x_i)} + \frac{1}{\zeta_{\mathcal{G}}(x_j)} \right) (\zeta_{\mathcal{G}}(x_j) + \zeta_{\mathcal{G}}(x_i)) - 2\zeta_{\mathcal{G}}(x_i)\zeta_{\mathcal{G}}(x_j) \right), \\ &= \sum_{x_i, x_j \notin E(\mathcal{G})} \left(\left(\frac{1}{\zeta_{\mathcal{G}}(x_i)} + \frac{1}{\zeta_{\mathcal{G}}(x_j)} \right) (\zeta_{\mathcal{G}}(x_j) + \zeta_{\mathcal{G}}(x_i)) \right) - 2 \sum_{x_i, x_j \notin E(\mathcal{G})} \zeta_{\mathcal{G}}(x_i)\zeta_{\mathcal{G}}(x_j) \\ &\leq 2(n-2) \sum_{x_i, x_j \notin E(\mathcal{G})} \left(\frac{1}{\zeta_{\mathcal{G}}(x_i)} + \frac{1}{\zeta_{\mathcal{G}}(x_j)} \right) - 2 \sum_{x_i, x_j \notin E(\mathcal{G})} \zeta_{\mathcal{G}}(x_i)\zeta_{\mathcal{G}}(x_j), \\ &= 2(n-2) \sum_{x_i, x_j \notin E(\mathcal{G})} \left(\frac{\zeta_{\mathcal{G}}(x_i) + \zeta_{\mathcal{G}}(x_j)}{\zeta_{\mathcal{G}}(x_i)\zeta_{\mathcal{G}}(x_j)} \right) - 2\overline{M}_2(\mathcal{G}), \\ &= 2(\aleph - 2) \overline{\text{ReZG}}_1(\mathcal{G}) - 2\overline{M}_2(\mathcal{G}), \end{aligned} \quad (45)$$

which leads to the result.

In the following results, we obtain upper bounds for the $\mathcal{F}$ -coindex in terms of the first and second Zagreb indices and hyper-Zagreb coindex. $\square$

Theorem 17. Suppose $\mathcal{G}$ be a (N, ϵ) -graph. Then,

$$\overline{\mathcal{F}}(\mathcal{G}) + \overline{M}_2(\mathcal{G}) \leq 3\overline{\epsilon}(N-2)^2, \quad (46)$$

$$\overline{\mathcal{F}}(\mathcal{G}) + \overline{M}_1(\mathcal{G}) \leq 2\overline{\epsilon}(N^2 - 3N + 2), \quad (47)$$

$$\overline{\mathcal{F}}(\mathcal{G}) + \overline{HM}(\mathcal{G}) \leq 6\overline{\epsilon}(N-2)^2. \quad (48)$$

The equalities hold if and only if $\mathcal{G} \cong K_N$ or $\mathcal{G}$ is $(N-2)$ -regular or $(N-2, N-1)$ -semiregular.

Proof. By applying the definitions of the $\mathcal{F}$ -coindex and second Zagreb coindex, we can write

$$\begin{aligned} \overline{\mathcal{F}}(\mathcal{G}) + \overline{M}_2(\mathcal{G}) &= \sum_{x_i, x_j \notin E(\mathcal{G})} (\zeta_{\mathcal{G}}^2(x_i) + \zeta_{\mathcal{G}}^2(x_j) + \zeta_{\mathcal{G}}(x_i)\zeta_{\mathcal{G}}(x_j)) \leq \sum_{x_i, x_j \notin E(\mathcal{G})} ((N-2)^2 + (N-2)^2 + (N-2)^2), \\ &= 3\overline{\epsilon}(N-2)^2. \end{aligned} \quad (49)$$

Likewise, we have

$$\begin{aligned} \overline{\mathcal{F}}(\mathcal{G}) + \overline{M}_1(\mathcal{G}) &= \sum_{x_i, x_j \notin E(\mathcal{G})} (\zeta_{\mathcal{G}}^2(x_i) + \zeta_{\mathcal{G}}^2(x_j) + \zeta_{\mathcal{G}}(x_i) + \zeta_{\mathcal{G}}(x_j)) \leq \sum_{x_i, x_j \notin E(\mathcal{G})} ((N-2)^2 + (N-2)^2 + 2(N-2)), \\ &= 2\overline{\epsilon}(N^2 - 3N + 2), \end{aligned} \quad (50)$$

and

$$\begin{aligned} \overline{\mathcal{F}}(\mathcal{G}) + \overline{HM}(\mathcal{G}) &= \sum_{x_i, x_j \notin E(\mathcal{G})} (\zeta_{\mathcal{G}}^2(x_i) + \zeta_{\mathcal{G}}^2(x_j) + (\zeta_{\mathcal{G}}(x_i) + \zeta_{\mathcal{G}}(x_j))^2), \\ &= \sum_{x_i, x_j \notin E(\mathcal{G})} (\zeta_{\mathcal{G}}^2(x_i) + \zeta_{\mathcal{G}}^2(x_j) + \zeta_{\mathcal{G}}^2(x_i) + \zeta_{\mathcal{G}}^2(x_j) + 2\zeta_{\mathcal{G}}(x_i)\zeta_{\mathcal{G}}(x_j)), \\ &= \sum_{x_i, x_j \notin E(\mathcal{G})} (2\zeta_{\mathcal{G}}^2(x_i) + 2\zeta_{\mathcal{G}}^2(x_j) + 2\zeta_{\mathcal{G}}(x_i)\zeta_{\mathcal{G}}(x_j)) \leq \sum_{x_i, x_j \notin E(\mathcal{G})} (4(N-2)^2 + 2(N-2)^2), \\ &= 6\overline{\epsilon}(N-2)^2. \end{aligned} \quad (51)$$

The equalities hold in (46)–(48) if and only if $\zeta_{\mathcal{G}}(x_i) = \zeta_{\mathcal{G}}(x_j) = N-2$, for each $x_i, x_j \notin E(\mathcal{G})$. This implies that each vertex of $\mathcal{G}$ has degree $N-1$ or $N-2$; that is, $\mathcal{G} \cong K_N$ or $\mathcal{G}$ is $(N-2)$ -regular or $(N-2, N-1)$ -semiregular.

By using Theorems 1, 8, 10, and 9, we get the following result. $\square$

Theorem 18. Suppose $\mathcal{G}$ be a (N, ϵ) -graph. Then,

$$\begin{aligned} \mathcal{F}(\overline{\mathcal{G}}) &\geq 2(N-1)^2(\overline{\epsilon} - 2\epsilon) + 3(N-1)\left(\frac{4\epsilon^2}{N} + \frac{1}{2}(\Theta - \delta)^2\right) - (2M_2(\mathcal{G}) + \epsilon(N-2)), \\ \overline{\mathcal{F}}(\mathcal{G}) &\geq (N-1)\left(\frac{4\epsilon^2}{N} + \frac{1}{2}(\Theta - \delta)^2\right) - (2M_2(\mathcal{G}) + \epsilon(N-2)). \end{aligned} \quad (52)$$

The following results were obtained by combining Theorems 1, 4, 5, 6, and 11.

Theorem 19. Suppose $\mathcal{G}$ be a (N, ϵ) -graph. Then,

$$\begin{aligned}\overline{\mathcal{F}}(\overline{\mathcal{G}}) &\geq 2\epsilon(N-1)^2 - 2(N-1)\left(\left(\frac{4\epsilon^2}{N} + \frac{1}{2}(\Theta - \delta)^2\right)\right) + \left(\frac{16\epsilon^3}{N^2} - 2M_2(\mathcal{G})\right), \\ \overline{\mathcal{F}}(\overline{\mathcal{G}}) &\geq 2\epsilon(N-1)^2 - 2(N-1)\left(\left(\frac{4\epsilon^2}{N} + \frac{1}{2}(\Theta - \delta)^2\right)\right) + \left(\frac{M_1(\mathcal{G})^2}{2\epsilon}\right), \\ \overline{\mathcal{F}}(\overline{\mathcal{G}}) &\geq 2\epsilon(N-1)^2 - 2(N-1)\left(\left(\frac{4\epsilon^2}{N} + \frac{1}{2}(\Theta - \delta)^2\right)\right) + (\Theta^3 + \Theta_2^3 + \Phi).\end{aligned}\tag{53}$$

The following results were obtained by combining Theorems 2, 4, 10, and 9.

Theorem 20. Suppose $\mathcal{G}$ be a (N, ϵ) -graph. Then,

$$\begin{aligned}\mathcal{F}(\overline{\mathcal{G}}) &\leq 2(N-1)^2(\overline{\epsilon} - 2\epsilon) + 3(N-1)\left(\frac{4\epsilon^2}{N} + \frac{N}{4}(\Theta - \delta)^2\right) - \left(\frac{16\epsilon^3}{N^2} - 2M_2(\mathcal{G})\right), \\ \overline{\mathcal{F}}(\mathcal{G}) &\leq (N-1)\left(\frac{4\epsilon^2}{N} + \frac{N}{4}(\Theta - \delta)^2\right) - \left(\frac{16\epsilon^3}{N^2} - 2M_2(\mathcal{G})\right).\end{aligned}\tag{54}$$

By applying Theorems 2, 8, and 11, we can obtain the following theorem.

The following results were obtained by Theorems 2, 5, 10, and 9.

Theorem 21. Suppose $\mathcal{G}$ be a (N, ϵ) -graph. Then,

Theorem 22. Suppose $\mathcal{G}$ be a (N, ϵ) -graph. Then,

$$\begin{aligned}\overline{\mathcal{F}}(\overline{\mathcal{G}}) &\leq 2\epsilon(N-1)^2 - 2(N-1)\left(\left(\frac{4\epsilon^2}{N} + \frac{N}{4}(\Theta - \delta)^2\right)\right) \\ &\quad + (2M_2(\mathcal{G}) + \epsilon(N-2)).\end{aligned}\tag{55}$$

$$\begin{aligned}\mathcal{F}(\overline{\mathcal{G}}) &\leq 2(N-1)^2(\overline{\epsilon} - 2\epsilon) + 3(N-1)\left(\frac{4\epsilon^2}{N} + \frac{N}{4}(\Theta - \delta)^2\right) - \left(\frac{M_1(\mathcal{G})^2}{2\epsilon}\right), \\ \overline{\mathcal{F}}(\mathcal{G}) &\leq (N-1)\left(\frac{4\epsilon^2}{N} + \frac{N}{4}(\Theta - \delta)^2\right) - \left(\frac{M_1(\mathcal{G})^2}{2\epsilon}\right).\end{aligned}\tag{56}$$

The following results were obtained by combining Theorems 7, 10, and 9.

Theorem 23. Suppose $\mathcal{G}$ be a (N, ϵ) -graph. Then,

$$\begin{aligned}\mathcal{F}(\overline{\mathcal{G}}) &\leq 2(N-1)^2(\overline{\epsilon} - 2\epsilon) + 3(N-1)\left(\frac{4\epsilon^2}{N} + \frac{N}{4}(\Theta - \delta)^2\right) - (\Theta^3 + \Theta_2^3 + Y), \\ \overline{\mathcal{F}}(\mathcal{G}) &\leq (N-1)\left(\frac{4\epsilon^2}{N} + \frac{N}{4}(\Theta - \delta)^2\right) - (\Theta^3 + \Theta_2^3 + Y).\end{aligned}\tag{57}$$

4. Conclusion

In this paper, we investigate the relationship between the $\mathcal{F}$ -coindex and the other topological coindices, such as the first and second Zagreb coindices, the hyper-Zagreb coindex, and the redefined first Zagreb coindex. However, there are still open and challenging problems for researchers, for example, the problem on the relationship among the $\mathcal{F}$ -coindex and GA-coindex, harmonic coindex, Randić coindex, etc.

Data Availability

The data involved in the examples of our manuscript are included within the article.

Disclosure

The authors would like to declare that the work described was original research that has not been published previously [28].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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