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Research Article

An Averaging Principle for Mckean–Vlasov-Type Caputo Fractional Stochastic Differential Equations

Weifeng Wang,[1] **Lei Yan**,[2] **Junhao Hu**,[1] **and Zhongkai Guo**[1]

[1]*School of Mathematics and Statistics, South-Central University for Nationalities, Wuhan 430074, China*
[2]*School of General Quality Education, Wuchang University of Technology, Wuhan 430223, China*

Correspondence should be addressed to Zhongkai Guo; zkguo@scuec.edu.cn

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In this paper, we want to establish an averaging principle for Mckean–Vlasov-type Caputo fractional stochastic differential equations with Brownian motion. Compared with the classic averaging condition for stochastic differential equation, we propose a new averaging condition and obtain the averaging convergence results for Mckean–Vlasov-type Caputo fractional stochastic differential equations.

1. Introduction

For complex systems, we usually want to locate an effective simplified model to approximate the original complex system or extract the main dynamical behavior of the original system. Based on these ideas, a lot of effective methods have been generated in dynamical systems, such as invariant manifolds, averaging principle, and homogenization principle. These effective methods have now been extended to deal with stochastic systems, such as stochastic invariant manifolds see [1, 2] and stochastic averaging principle, see [3–9].

Currently, the problem of averaging for stochastic differential equations have received a lot of attention and various types of stochastic differential equations have been studied, see [4, 6, 7, 10–12]. However, there are no relevant results of averaging principle for distribution dependent-type stochastic differential equations which we will consider in this paper.

On the contrary, the problem of averaging for stochastic fractional order differential equations have received a lot of attention in recent years, and some results [13] have been obtained under averaging condition consistence with the classic case (see [4, 5, 14]). Noting that the fractional order derivative is a nonlocal operator, therefore, the fractional order differential equation is more effective for describing certain phenomena in the real world (see [15–17]). Current research studies on stochastic fractional order differential equations mainly focused on the existence and uniqueness of the solutions, with fewer results from the dynamical system perspective.

Based on the above discussion, we shall study the averaging principle for the following Mckean–Vlasov-type Caputo fractional stochastic differential equations:

$$\begin{cases} D_t^{\alpha} X_t = f(t, X_t, \mu_t)\mathrm{d}t + g(t, X_t, \mu_t)\mathrm{d}B_t, \quad t \geq 0, \\ \mu(t) = \text{probability distribution of } X_t, \\ X_0 = x_0 \in L^2(\Omega, H), \end{cases} \tag{1}$$

where $\alpha \in (1/2, 1]$ and B_t is a scalar Brownian motion. Nonlinear terms f and g are H-valued functions defined on $R^+ \times H \times M_{\gamma^2}(H)$, and $M_{\gamma^2}(H)$ denotes a proper subset of probability measure on H. If the terms f and g do not depend on the probability distribution $\mu(t)$ of the process X at time t, such equations have been studied by [13] and other authors. If $\alpha = 1$, the equation becomes a classical Mckean–Vlasov-type stochastic differential equations which have been considered by many authors with different approaches (see [18–20]). In this paper, we just focused on $\alpha \in (1/2, 1)$, and more details can be seen in Section 2.

The paper is structured as follows. We introduce some notation and assumptions in Section 2. The existence and unique solution for distribution dependent fractional stochastic differential equations will be discussed in Section 3. An averaging principle for the above equation is established in Section 4.

2. Preliminaries

First, we introduce some notation. Let $C(H)$ be the space of continuous functions on H. Let $\mathscr{B}(H)$ be the Borel σ-algebra of subsets of H. $M(H)$ is the space of probability measures on $\mathscr{B}(H)$ and carries the usual topology of weak convergence. (μ, ϕ) denotes $\int_H \phi(x)\mu(dx)$. Let $\gamma(x) = 1 + |x|$, $\forall x \in H$, and then, define the Banach space

$$C_\rho(H) = \left\{ \phi \in C(H) \colon \|\phi\|_{C_\rho} \equiv \sup_{x\in H} \frac{|\phi(x)|}{\gamma^2(x)} + \sup_{x\neq y} \frac{|\phi(x)-\phi(y)|}{|x-y|} < \infty \right\}. \tag{2}$$

For any $p \geq 1$, let $M^s_{\gamma^p}(H)$ denote the Banach space of signed measures m on H, and $\|\mu\|_{\gamma^p} \equiv \int_H \gamma^p |m|(dx) < \infty$. $|m| = m^+ + m^-$ and $m = m^+ - m^-$ are the Jordan decomposition of m. $M_{\gamma^2}(H) = M^s_{\gamma^p}(H) \cap M(H)$ is the set of probability measures on $\mathscr{B}(H)$, and there exists second moments. Define the following metric:

$$\rho(\mu,\nu) = \sup\left\{ (\phi, \mu-\nu) \colon \|\phi\|_\rho = \sup_{x\in H} \frac{|\phi(x)|}{\gamma^2(x)} + \sup_{x\neq y} \frac{|\phi(x)-\phi(y)|}{|x-y|} \leq 1 \right\}. \tag{3}$$

Then, $(M_{\gamma^2}(H), \rho)$ is a complete metric space. Let $C([0,T], (M_{\gamma^2}(H),\rho))$ be the complete metric space of continuous functions from $[0,T]$ to $(M_{\gamma^2}(H),\rho)$ with the following metric:

$$D_T(\mu,\nu) = \sup_{t\in[0,T]} \rho(\mu(t),\nu(t)), \quad \text{for } \nu,\mu \in C\big([0,T], M_{\gamma^2}(H), \rho\big). \tag{4}$$

More details can be seen in [18].

In order to obtain the existence and uniqueness of the solution of (1), we introduce the following conditions.

(i) H1 (Lipschitz condition): for all $x, y \in H$ and $t \in [0,T]$, $\mu \in C([0,T], M_{\gamma^2}(H), \rho)$, and there exists a bounded function $k_1(t) > 0$, such that

$$|f(t,x,\mu) - f(t,y,\nu)|^2 + |g(t,x,\mu) - g(t,y,\nu)|^2 \leq k_1(t)\big(|x-y|^2 + \rho^2(\mu,\nu)\big). \tag{5}$$

(ii) H2 (growth condition): for all $(x,t) \in H \times [0,T]$, there exists a bounded function $k_2(t) > 0$ such that

$$|f(t,x,\mu)|^2 + |g(t,x,\mu)|^2 \leq k_2(t)\big(1 + |x|^2\big). \tag{6}$$

In this paper, we assume there existence of a constant k such that $\max\{k_1(t), k_2(t)\} \leq k$.

First, we give an important lemma, which is a type of promotion form of Gronwall's inequality with singular kernels.

Lemma 1 (see [21, 22]). *Suppose $b \geq 0$, $\beta > 0$, and $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ (some $T \leq +\infty$), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + b\int_0^t (t-s)^{\beta-1} u(s) ds, \tag{7}$$

on this interval. Then,

$$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) \right] ds, \quad 0 \leq t < T. \tag{8}$$

3. Existence and Uniqueness

Consider the integral form of equation (1):

$$X_t = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, X_s, \mu_t) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(t, X_s, \mu_t) dB_t. \tag{9}$$

Under the assumptions of H1 and H2, we will prove the existence and uniqueness of solution for the above equation.

Definition 1. An $\mathscr{F}_t$-adapted stochastic process X_t with law $L(X_t) = \mu(t)$ is called a solution of (1) if X_t is continuous, and for $\forall t \in [0,T]$ with $X_0 = x_0$,

$$X_t = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, X_s, \mu_t) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(t, X_s, \mu_t) dB_t, \mathbb{P} - a.s. \tag{10}$$

Theorem 1. *Assume that H1 and H2 hold; then, for $\forall x_0 \in L^2(\Omega, H)$, equation (10) has a unique solution*

$X_t \in C([0,T]; L^2(\Omega, H))$ *with the associate probability distribution* $\mu_t = L(X_t)$, $t \in [0,T]$ *belonging to* $C([0,T], (M_\gamma^2(X), \rho))$, *such that*

$$\sup_{0\le t\le T} E|X_t|^2 < \infty. \tag{11}$$

We will proof the theorem by several steps.

(i) First, we prove that $X_t \in L^\infty([0,T], L^2(\Omega; H))$ for $\forall \mu \in C([0,T], (M_{\gamma^2}(H), \rho))$. Using the following inequality,

$$|a+b+c|^2 \le 3\left(|a|^2 + |b|^2 + |c|^2\right). \tag{12}$$

We see that

$$\begin{aligned} E|X(t)|^2 &\le 3E|x_0|^2 + 3E\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, X_s, \mu_s)\mathrm{d}s\right|^2 \\ &\quad + 3E\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(t, X_s, \mu_s)\mathrm{d}B_t\right|^2 \\ &\coloneqq 3I_1 + 3I_2 + 3I_3. \end{aligned} \tag{13}$$

For I_2, applying Cauchy–Schwarz's inequality and H2, it follows

$$\begin{aligned} I_2 &\le \frac{Tk}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2\alpha-2}\left(1 + E|X(s)|^2 + \|\mu_t\|_\gamma^2\right)\mathrm{d}s \\ &\le \frac{Tk}{\Gamma(\alpha)^2}\left[\frac{t^{2\alpha-1}}{2\alpha-1}\left(1 + \sup_{0\le t\le T}\|\mu_t\|_\gamma^2\right) + \int_0^t (t-s)^{2\alpha-2} E|X(s)|^2\mathrm{d}s\right] \\ &\le \frac{kT^{2\alpha}}{\Gamma(\alpha)^2(2\alpha-1)}\left(1 + \sup_{0\le t\le T}\|\mu_t\|_\gamma^2\right) + \frac{Tk}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2\alpha-2} E|X(s)|^2\mathrm{d}s. \end{aligned} \tag{14}$$

For I_3, by It $\hat{o}$'s isometry formula and H2, we have

$$\begin{aligned} I_3 &\le \frac{k}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2\alpha-2}\left(1 + E|X(s)|^2 + \|\mu_t\|_\gamma^2\right)\mathrm{d}s \\ &\le \frac{kT^{2\alpha-1}}{\Gamma(\alpha)^2(2\alpha-1)}\left(1 + \sup_{0\le t\le T}\|\mu_t\|_\gamma^2\right) \\ &\quad + \frac{k}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2\alpha-2} E|X(s)|^2\mathrm{d}s. \end{aligned} \tag{15}$$

Combining the above estimate results, we finally obtain

$$E|X(t)|^2 \le r_1 + r_2\int_0^t (t-s)^{(2\alpha-1)-1} E|X(s)|^2\mathrm{d}s, \tag{16}$$

where

$$\begin{aligned} r_1 &\coloneqq 3E|x_0|^2 + 3\frac{\left(kT^{2\alpha-1}\right)(T+1)}{\Gamma(\alpha)^2(2\alpha-1)}\left(1 + \sup_{0\le t\le T}|\mu_t|_\gamma^2\right), \\ r_2 &\coloneqq 3\frac{k(T+1)}{\Gamma(\alpha)^2}. \end{aligned} \tag{17}$$

With the help of Lemma 1, it follows

$$\begin{aligned} E|X(t)|^2 &\le r_1\left(1 + \int_0^t \sum_{n=1}^{\infty}\frac{\left(r_2\Gamma(2\alpha-1)\right)^n}{\Gamma((2\alpha-1)n)}(t-s)^{(2\alpha-1)n-1}\mathrm{d}s\right) \\ &\le r_1\left(1 + \sum_{n=1}^{\infty}\frac{\left(r_2\Gamma(2\alpha-1)T^{2\alpha-1}\right)^n}{\Gamma((2\alpha-1)n+1)}\right) \\ &= r_1\left(1 + E_{2\alpha-1,1}\left(r_2\Gamma(2\alpha-1)T^{2\alpha-1}\right)\right) < \infty, \end{aligned} \tag{18}$$

for $\forall t \in [0,T]$, and $E_{2\alpha-1,1}(\cdot)$ is a two-parameter function of the Mittag-Leffler type [21].

Then,

$$\sup_{0\le t\le T} E|X(t)|^2 < \infty, \tag{19}$$

and $X_t \in L^\infty([0,T], L^2(\Omega; H))$.

(ii) Now, we show that $X_t \in C([0,T], L^2(\Omega; H))$ for $\forall \mu \in C([0,T], (M_{\gamma^2}(H), \rho))$:

$$\begin{aligned}E\left\|X_t - X_{t_0}\right\|^2 &\le 2E\frac{1}{\Gamma(\alpha)^2}\left\|\int_0^t (t-s)^{\alpha-1} f(s, X_s, \mu_s)\mathrm{d}s - \int_0^{t_0} (t_0-s)^{\alpha-1} f(s, X_s, \mu_s)\mathrm{d}s\right\|^2 \\ &\quad + 2E\frac{1}{\Gamma(\alpha)^2}\left\|\int_0^t (t-s)^{\alpha-1} f(s, X_s, \mu_s)\mathrm{d}B_s - \int_0^{t_0} (t_0-s)^{\alpha-1} f(s, X_s, \mu_s)\mathrm{d}B_s\right\|^2 \\ &=: 2(J_1 + J_2).\end{aligned} \tag{20}$$

For J_1, we have

$$\begin{aligned}J_1 &\le 2E\frac{1}{\Gamma(\alpha)^2}\left\|\int_{t_0}^t (t-s)^{\alpha-1} f(s, X_s, \mu_s)\mathrm{d}s\right\|^2 \\ &\quad + 2E\frac{1}{\Gamma(\alpha)^2}\left\|\int_0^{t_0}\left((t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}\right) f(s, X_s, \mu_s)\mathrm{d}s\right\|^2 =: 2J_{11} + 2J_{12}.\end{aligned} \tag{21}$$

By the Cauchy–Schwartz inequality,

$$\begin{aligned}J_{11} &\le \frac{1}{\Gamma(\alpha)^2} k\int_{t_0}^t (t-s)^{2\alpha-2}\mathrm{d}s\int_{t_0}^t E\left|f(s, X_s, \mu_s)\right|^2\mathrm{d}s \\ &\le \frac{k}{\Gamma(\alpha)^2(2\alpha-1)}(t-t_0)^{2\alpha-1}\int_{t_0}^t\left(1 + E\left|X_s\right|^2 + \left\|\mu_s\right\|^2\right)\mathrm{d}s \\ &\le \frac{Ck}{\Gamma(\alpha)^2(2\alpha-1)}(t-t_0)^{2\alpha}.\end{aligned} \tag{22}$$

For J_{12}, we have

$$\begin{aligned}J_{12} &= E\frac{1}{\Gamma(\alpha)^2}\left|\int_0^{t_0}\left((t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}\right) f(s, X_s, \mu_s)\mathrm{d}s\right|^2 \\ &\le \frac{k}{\Gamma(\alpha)^2}\int_0^{t_0}\left((t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}\right)^2\mathrm{d}s\int_0^{t_0}\left(1 + E\left|X_s\right|^2 + \left\|\mu_s\right\|^2\right)\mathrm{d}s \\ &\le \frac{CTk}{\Gamma(\alpha)^2}\int_0^{t_0}\left((t_0-s)^{2(\alpha-1)} - (t-s)^{2(\alpha-1)}\right)\mathrm{d}s \\ &\le \frac{CTk}{\Gamma(\alpha)^2}\left[\frac{(t-t_0)^{2\alpha-1}}{2\alpha-1} + \frac{t_0^{2\alpha-1}}{2\alpha-1} - \frac{t^{2\alpha-1}}{2\alpha-1}\right] \le \frac{CTk}{\Gamma(\alpha)^2}\frac{(t-t_0)^{2\alpha-1}}{2\alpha-1}.\end{aligned} \tag{23}$$

For J_2, using the It$\hat{o}$ isometry formula, in the similar way as J_1, we can prove that

$$J_2 \le \frac{Ck}{\Gamma(\alpha)^2}\frac{(t-t_0)^{2\alpha-1}}{2\alpha-1}. \tag{24}$$

Results of J_1 and J_2 combined together show that

$$\begin{aligned}E\left\|X_t - X_{t_0}\right\|^2 &\le \frac{Ck(1+T)}{\Gamma(\alpha)^2}\frac{(t-t_0)^{2\alpha-1}}{2\alpha-1} \\ &\quad + \frac{Ck}{\Gamma(\alpha)^2(2\alpha-1)}(t-t_0)^{2\alpha},\end{aligned} \tag{25}$$

which implied $X_t \in C([0,T], L^2(\Omega; H))$ for each fixed $\mu \in C([0,T], (M_{\gamma^2}, \rho))$.

(iii) By virtue of the fixed point theorem for contraction mappings, we can show that, for each fixed $\mu \in C([0,T],(M_{\gamma^2},\rho))$, equation (10) has a unique solution in $C([0,T],L^2(\Omega;H))$. Similar arguments are also discussed in [18]. Now, we define an operator $\Phi_\mu(\cdot)$ on $C([0,T],L^2(\Omega;H))$:

$$\left(\Phi_\mu X\right)(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,X_s,\mu_s)\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(t,X_s,\mu_s)\mathrm{d}B_t. \tag{26}$$

It is easy to verify that Φ_μ is from $C([0,T],L^2(\Omega;H))$ into itself.

For $X_t, Y_t \in C([0,T],L^2(\Omega;H))$ with $x_0 = y_0$, let $|\cdot|_\zeta$ denote the norm of $C([0,T],L^2(\Omega;H))$ and $\beta = 2\alpha - 1 > 0$; then, we obtain

$$\begin{aligned}\left|\left(\Phi_\mu X\right)(t) - \left(\Phi_\mu Y\right)(t)\right|_\varsigma^2 &\le 2\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(f(s,X_s,\mu_s) - f(s,Y_s,\mu_s)\right)\mathrm{d}s\right|_\varsigma^2 \\ &+ 2\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(g(t,X_s,\mu_s) - g(t,Y_s,\mu_s)\right)\mathrm{d}B_t\right|_\varsigma^2.\end{aligned} \tag{27}$$

Using the Cauchy–Schwartz inequality and It $\hat{o}$ isometry formula, it is readily seen that

$$\left|\left(\Phi_\mu X\right)(t) - \left(\Phi_\mu Y\right)(t)\right|_\varsigma^2 \le \frac{2Tk + 2k}{\Gamma(\alpha)^2}\int_0^t (t-s)^{\beta-1}\left|X_s - Y_s\right|_\varsigma^2 \mathrm{d}s. \tag{28}$$

Our goal is now to prove the following inequality:

$$\left|\left(\Phi_\mu^n X\right)(t) - \left(\Phi_\mu^n X\right)(t)\right|_\varsigma^2 \le \frac{1}{\beta}\left(\frac{2Tk + 2k}{\Gamma(\alpha)^2}\right)^n \frac{\Gamma(\beta)^n}{\Gamma(n\beta)} t^{n\beta}\left|X_t - Y_t\right|_\varsigma^2. \tag{29}$$

The proof is based on mathematical induction over n. For $n = 1$,

$$\left|\left(\Phi_\mu X\right)(t) - \left(\Phi_\mu Y\right)(t)\right|_\varsigma^2 \le \frac{2Tk + 2k}{\Gamma(\alpha)^2}\left|X_t - Y_t\right|_\varsigma^2 \frac{t^\beta}{\beta}, \tag{30}$$

which is fulfilled.

For the induction step from $n = l$ to $n = l + 1$, we assume that, for $n = l$, equation (29) is satisfied; then,

$$\begin{aligned}\left|\left(\Phi_\mu^{l+1} X\right)(t) - \left(\Phi_\mu^{l+1} Y\right)(t)\right|_\varsigma^2 &\le \frac{2Tk + 2k}{\Gamma(\alpha)^2}\int_0^t (t-s)^{\beta-1}\left|\left(\Phi_\mu^l X\right)(s) - \left(\Phi_\mu^l Y\right)(s)\right|_\varsigma^2 \mathrm{d}s \\ &\le \frac{2Tk + 2k}{\Gamma(\alpha)^2}\int_0^t (t-s)^{\beta-1}\frac{1}{\beta}\left(\frac{2Tk + 2k}{\Gamma(\alpha)^2}\right)^l \frac{\Gamma(\beta)^l}{\Gamma(k\beta)} s^{l\beta}\left|X_s - Y_s\right|_\varsigma^2 \mathrm{d}s \\ &\le \left(\frac{2Tk + 2k}{\Gamma(\alpha)^2}\right)^{l+1}\frac{1}{\beta}\frac{\Gamma(\beta)^k}{\Gamma(l\beta)}\left|X_t - Y_t\right|_\varsigma^2 \int_0^t (t-s)^{\beta-1} s^{l\beta}\mathrm{d}s.\end{aligned} \tag{31}$$

Thus, we only need to discuss the following integral:

$$\int_0^t (t-s)^{\beta-1} s^{l\beta}\mathrm{d}s. \tag{32}$$

Let $s = tz$; then,

$$\begin{aligned}\int_0^t (t-s)^{\beta-1} s^{l\beta}\mathrm{d}s &= \int_0^1 (1-z)^{\beta-1} t^{\beta-1} t^{l\beta} z^{l\beta} t\mathrm{d}z \\ &= t^{(l+1)\beta}\int_0^1 (1-z)^{\beta-1} z^{l\beta}\mathrm{d}z \\ &= t^{(l+1)\beta} B(l\beta + 1, \beta) = t^{(l+1)\beta}\frac{\Gamma(\beta)\Gamma(l\beta + 1)}{\Gamma((l+1)\beta + 1)},\end{aligned} \tag{33}$$

where $B(\cdot,\cdot)$ is the Beta function. Substitute the above equality into (31), and we derive that

$$
\begin{aligned}
&\left|\left(\Phi_\mu^{l+1} X\right)(t)-\left(\Phi_\mu^{l+1} Y\right)(t)\right|_\varsigma^2\\
&\leq\left(\frac{2Tk+2k}{\Gamma^2(\alpha)}\right)^{l+1}\frac{1}{\beta}\frac{\Gamma(\beta)^l}{\Gamma(l\beta)}\left|X_t-Y_t\right|_\varsigma^2 t^{(l+1)\beta}\frac{\Gamma(\beta)\Gamma(l\beta+1)}{\Gamma((l+1)\beta+1)}\\
&=\left(\frac{2Tk+2k}{\Gamma^2(\alpha)}\right)^{l+1}\frac{1}{\beta}\Gamma(\beta)^{l+1}\frac{\Gamma(l\beta+1)}{\Gamma((l+1)\beta+1)\Gamma(l\beta)}t^{(l+1)\beta}\left|X_t-Y_t\right|_\varsigma^2\\
&=\left(\frac{2Tk+2k}{\Gamma^2(\alpha)}\right)^{l+1}\frac{1}{\beta}\Gamma(\beta)^{l+1}\frac{l\beta\Gamma(l\beta)}{(l+1)\beta\Gamma((l+1)\beta)\Gamma(l\beta)}t^{(l+1)\beta}\left|X_t-Y_t\right|_\varsigma^2\\
&\leq\left(\frac{2Tk+2k}{\Gamma^2(\alpha)}\right)^{l+1}\frac{1}{\beta}\Gamma(\beta)^{l+1}\frac{t^{(l+1)\beta}}{\Gamma((l+1)\beta)}\left|X_t-Y_t\right|_\varsigma^2.
\end{aligned}
\tag{34}
$$

By the above discussion, we finally obtain

$$
\begin{aligned}
&\left|\left(\Phi_\mu^{n} X\right)(t)-\left(\Phi_\mu^{n} Y\right)(t)\right|_\varsigma^2\\
&\leq\left(\frac{2Tk+2k}{\Gamma^2(\alpha)}\right)^{n}\frac{1}{\beta}\Gamma(\beta)^n\frac{T^{n\beta}}{\Gamma(n\beta)}\left|X_t-Y_t\right|_\varsigma^2.
\end{aligned}
\tag{35}
$$

Note that $(Tk+k/\Gamma(\alpha))^n(1/\beta)\Gamma(\beta)^n(T^{n\beta}/\ \Gamma(n\beta))\longrightarrow 0$ as $n\longrightarrow+\infty$. Then, we have $(Tk+k/\Gamma(\alpha))^n(1/\beta)\Gamma(\beta)^n(T^{n\beta}/\Gamma(n\beta))<1$ for any sufficiently large n. And, this shows that $\Phi_\mu(\cdot)$ is a contraction map on $C([0,T],L^2(\Omega;H))$. So, it has a unique fixed point for $\mu\in C([0,T],(M_{\gamma^2}(H),\rho))$.

(iv) $L(X_\mu)=\{L(X_\mu(t))\colon t\in[0,T]\}$ is the probability law of X_μ. Now, we prove that $L(X_\mu)\in C([0,T],(M_{\gamma^2},\rho))$. Notice that $X_\mu\in C([0,T],L^2(\Omega;H)),L(X_\mu)\in M_{\gamma^2}(H)$ for $\forall t\in[0,T]$. So, we only need to prove $t\longrightarrow L(X_\mu(t))$ is continuous. In step (ii), we have

$$E\left|X_\mu(t)-X_\mu(s)\right|^2\longrightarrow 0, \tag{36}$$

as $t\longrightarrow s$.

By the definition of ρ, we have

$$
\begin{aligned}
\left|\left(\phi,L\left(X_\mu(t)\right)-L\left(X_\mu(s)\right)\right)\right|&=E\left|\phi\left(X_\mu(t)\right)-\phi\left(X_\mu(s)\right)\right|\\
&\leq\|\phi\|_\rho E\left|X_\mu(t)-X_\mu(s)\right|\leq\|\phi\|_\rho\left(E\left|X_\mu(t)-X_\mu(s)\right|^2\right)^{1/2},
\end{aligned}
\tag{37}
$$

which implies

$$\lim_{t\longrightarrow s}\rho\left(L\left(X_\mu(t)\right),L\left(X_\mu(s)\right)\right)=0. \tag{38}$$

Hence, we verify that $L(X_\mu)\in C([0,T],(M_{\gamma^2},\rho))$.

(v) Define Ψ on $C([0,T],(M_{\gamma^2},\rho))$ as follows:

$$\Psi\colon \mu\longrightarrow L\left(X_\mu\right). \tag{39}$$

In the following, we will show that the operator Ψ has a unique fixed point in $C([0,T],(M_{\gamma^2},\rho))$. Take $\mu,\nu\in C([0,T],(M_{\gamma^2},\rho))$, and let $X_\mu(t)$ and $X_\nu(t)$ be the corresponding solutions of the following equations:

$$
\begin{aligned}
X_\mu(t)&=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,X_s,\mu_s)\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(t,X_s,\mu_s)\mathrm{d}B_t,\\
X_\nu(t)&=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,X_s,\nu_s)\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(t,X_s,\nu_s)\mathrm{d}B_t.
\end{aligned}
\tag{40}
$$

Thus,

$$\begin{aligned}E\left|X_\mu(t)-X_\nu(t)\right|^2 &\le \frac{2T}{\Gamma(\alpha)^2}E\int_0^t (t-s)^{2\alpha-2}\left|f\left(s,X_\mu(s),\mu_s\right)-f\left(s,X_\nu(s),\nu_s\right)\right|^2\mathrm{d}s\\&+\frac{2}{\Gamma(\alpha)^2}E\int_0^t (t-s)^{2\alpha-2}\left|g\left(s,X_\mu(s),\mu_s\right)-g\left(s,X_\nu(s),\nu_s\right)\right|^2\mathrm{d}s\\&\le\frac{2kT}{\Gamma(\alpha)^2}E\int_0^t (t-s)^{2\alpha-2}\left[\left|X_\mu(s)-X_\nu(s)\right|^2+\rho^2(\mu_s,\nu_s)\right]\mathrm{d}s\\&+\frac{2k}{\Gamma(\alpha)^2}E\int_0^t (t-s)^{2\alpha-2}\left[\left|X_\mu(s)-X_\nu(s)\right|^2+\rho^2(\mu_s,\nu_s)\right]\mathrm{d}s.\end{aligned}\tag{41}$$

After simple calculation, we have that

$$\sup_{0\le t\le T}E\left|X_\mu(t)-X_\nu(t)\right|^2\le\frac{2k(T+1)}{\Gamma(\alpha)^2}\frac{T^{2\alpha-1}}{2\alpha-1}\left[\sup_{0\le t\le T}E\left|X_\mu(t)-X_\nu(t)\right|^2+D_T^2(\mu,\nu)\right].\tag{42}$$

Select the appropriate $T=T_0>0$, such that

$$\frac{2k(T_0+1)}{\Gamma(\alpha)^2}\frac{T_0^{2\alpha-1}}{2\alpha-1}<\frac{1}{3}.\tag{43}$$

Then, it follows

$$sup_{0\le t\le T_0}E\left|X_\mu(t)-X_\nu(t)\right|^2<\frac{1}{2}D_{T_0}^2(\mu,\nu).\tag{44}$$

By the definition of $\rho(\mu,\nu)$ and $D_T^2(\mu,\nu)$, we can obtain

$$\rho^2(\mu,\nu)\le E\left|X_\mu(t)-X_\nu(t)\right|^2.\tag{45}$$

Taking sup-norm on both sides, we obtain

$$D_T^2(\Psi(\mu),\Psi(\nu))\le\sup_{0\le t\le T_0}E\left|X_\mu(t)-X_\nu(t)\right|^2.\tag{46}$$

Combine this result with equation (44), and we finally derive

$$D_{T_0}^2(\Psi(\mu),\Psi(\nu))<\frac{1}{2}D_{T_0}^2(\mu,\nu).\tag{47}$$

Since Ψ is a contraction in $C([0,T_0],(M_{\gamma^2}(H),\rho))$, it has a unique fixed point. Thus, equation (10) has a unique solution X_t with $\mu=L(X_t)$ on $[0,T_0]$. Because X_t belongs to $C([0,T],L^2(\Omega;H))$, we can extend the solution to $[0,T]$ by considering $[0,T_0]$, $[T_0,2T_0]$, and so on. This completes the proof.

4. An Averaging Principle

In this section, we study an averaging principle for the following distribution dependent fractional stochastic differential equations in H:

$$\begin{aligned}X_\epsilon(t)&=x_0+\frac{\epsilon}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,X_\epsilon(s),\mu_\epsilon(s))\mathrm{d}s\\&+\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s,X_\epsilon(s),\mu_\epsilon(s))\mathrm{d}B_t,\end{aligned}\tag{48}$$

where $x_0\in L^2(\Omega;H)$. We will show that the solution of (48) will be approximated by the following simpler or averaged process under certain conditions:

$$\begin{aligned}Z_\epsilon(t)&=x_0+\frac{\epsilon}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\overline{f}(Z_\epsilon(s),\nu_\epsilon(s))\mathrm{d}s\\&+\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\overline{g}(Z_\epsilon(s),\nu_\epsilon(s))\mathrm{d}B_t.\end{aligned}\tag{49}$$

Equation (49) is called the averaged equation for (48). Now, we prove that the solution of (49) converges to the solution of the original equation (48) under the following additional conditions.

H3:

$$\begin{aligned}&\frac{1}{t^{2\alpha-1}}\int_0^t(t-s)^{2(\alpha-1)}|f(s,x,\mu)-\overline{f}(x,\mu)|^2\mathrm{d}s\\&\le\varphi_1(t)\left(1+|x|^2+\|\mu\|^2\right).\end{aligned}\tag{50}$$

H4:

$$\begin{aligned}&\frac{1}{t^{2\alpha-1}}\int_0^t(t-s)^{2(\alpha-1)}\left|g(s,x,\mu)-\overline{g}(x,\mu)^2\right|\mathrm{d}s\\&\le\varphi_2(t)\left(1+|x|^2+\|\mu\|^2\right),\end{aligned}\tag{51}$$

where $\varphi_i(t)$ are positive and bounded with $\lim_{t\longrightarrow+\infty}\varphi_i(t)=0$ for $i=1,2$.

Remark 1. Note that when we take $\alpha=1$, then this condition is consistence with the classic case, see [4].

Theorem 2. *Let $H1-H4$ hold. Then, for $\forall\delta_1>0$, there exist constants $L>0$, $\epsilon_1\in(0,\epsilon_0]$ and $\beta\in(0,1)$ such that, for any $\epsilon\in(0,\epsilon_1]$, $1/2<\alpha<1$, we have*

$$\sup_{0\le t\le L\varepsilon^{(-\beta/2\alpha-1)}} E\left|X_\varepsilon(t)-Z_\varepsilon(t)\right|^2\le\delta_1. \tag{52}$$

Let us consider

$$\begin{aligned}X_\epsilon(t)-Z_\epsilon(t)&=\frac{\epsilon}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left[f\left(s,X_\epsilon(s),\mu_\epsilon(s)\right)-\overline{f}\left(Z_\epsilon(s),\nu_\epsilon(s)\right)\right]\mathrm{d}s\\&+\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left[g\left(s,X_\epsilon(s),\mu_\epsilon(s)\right)-\overline{g}\left(Z_\epsilon(s),\nu_\epsilon(s)\right)\right]\mathrm{d}B_t.\end{aligned} \tag{53}$$

By the arithmetic inequality, it follows that

$$\begin{aligned}&E\left|X_\epsilon(t)-Z_\epsilon(t)\right|^2\le\\&2\left\{\begin{aligned}&E\left|\frac{\epsilon}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left[f\left(s,X_\epsilon(s),\mu_\epsilon(s)\right)-\overline{f}\left(Z_\epsilon(s),\nu_\epsilon(s)\right)\right]\mathrm{d}s\right|^2\\&+E\left|\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left[g\left(s,X_\epsilon(s),\mu_\epsilon(s)\right)-\overline{g}\left(Z_\epsilon(s),\nu_\epsilon(s)\right)\right]\mathrm{d}B_t\right|^2\end{aligned}\right\}\\&=:2\left(Q_1+Q_2\right).\end{aligned} \tag{54}$$

For Q_1, we have

$$\begin{aligned}Q_1&=E\left|\frac{\epsilon}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left[f\left(s,X_\epsilon(s),\mu_\epsilon(s)\right)-\overline{f}\left(Z_\epsilon(s),\nu_\epsilon(s)\right)\right]\mathrm{d}s\right|^2\\&\le\frac{2\epsilon^2}{\Gamma(\alpha)^2}\left\{\begin{aligned}&E\left|\int_0^t(t-s)^{\alpha-1}\left[f\left(s,X_\epsilon(s),\mu_\epsilon(s)\right)-f\left(s,Z_\epsilon(s),\nu_\epsilon(s)\right)\right]\mathrm{d}s\right|^2\\&+E\left|\int_0^t(t-s)^{\alpha-1}\left[f\left(s,Z_\epsilon(s),\nu_\epsilon(s)\right)-\overline{f}\left(Z_\epsilon(s),\nu_\epsilon(s)\right)\right]\mathrm{d}s\right|^2\end{aligned}\right\}.\end{aligned} \tag{55}$$

Applying the Cauchy–Schwarz inequality, H1, H3, and the definition of the metric ρ, we obtain

$$Q_1 \leq \frac{2tk\epsilon^2}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2\alpha-2}E\left[\left|X_\epsilon(s)-Z_\epsilon(s)\right|^2+\rho^2\left(\mu_\epsilon(s),\nu_\epsilon(s)\right)\right]ds + \frac{2\epsilon^2}{\Gamma(\alpha)^2}t^{2\alpha}\left\{\begin{array}{l} \frac{1}{t^{2\alpha-1}}E\int_0^t (t-s)^{2\alpha-2}\left|f\left(s,Z_\epsilon(s),\nu_\epsilon(s)\right)-\overline{f}\left(Z_\epsilon(s),\nu_\epsilon(s)\right)\right|^2 ds \\ \leq \frac{4tk\epsilon^2}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2\alpha-2}E\left|X_\epsilon(s)-Z_\epsilon(s)\right|^2 ds \\ +\frac{2\epsilon^2}{\Gamma(\alpha)^2}t^{2\alpha}\varphi_1(t)\left(1+E\left|Z_\epsilon(s)\right|^2+\left\|\nu_\epsilon(s)\right\|\right)^2 \end{array}\right\}. \tag{56}$$

For Q_2, using the Itô isometry formula, we obtain

$$\begin{aligned} Q_2 &= \frac{\epsilon}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2\alpha-2}E\left|g\left(s,X_\epsilon(s),\mu_\epsilon(s)\right)-\overline{g}\left(Z_\epsilon(s),\nu_\epsilon(s)\right)\right|^2 ds \\ &\leq \frac{2\epsilon}{\Gamma(\alpha)^2}\left\{\begin{array}{l} \int_0^t (t-s)^{2\alpha-2}E\left|g\left(s,X_\epsilon(s),\mu_\epsilon(s)\right)-g\left(Z_\epsilon(s),\nu_\epsilon(s)\right)\right|^2 ds \\ +\int_0^t (t-s)^{2\alpha-2}E\left|g\left(Z_\epsilon(s),\nu_\epsilon(s)\right)-\overline{g}\left(Z_\epsilon(s),\nu_\epsilon(s)\right)\right|^2 ds \end{array}\right\}. \end{aligned} \tag{57}$$

Applying conditions H1 and H4, we derive

$$\begin{aligned} Q_2 &\leq \frac{2k\epsilon}{\Gamma(\alpha)^2}\left\{\begin{array}{l} \int_0^t (t-s)^{2(\alpha-1)}\left[E\left|X_\epsilon(s)-Z_\epsilon(s)\right|^2+\rho^2\left(\mu_\epsilon(s),\nu_\epsilon(s)\right)\right]ds \\ +\int_0^t (t-s)^{2(\alpha-1)}E\left|g\left(Z_\epsilon(s),\nu_\epsilon(s)\right)-\overline{g}\left(Z_\epsilon(s),\nu_\epsilon(s)\right)\right|^2 ds \end{array}\right\} \\ &\leq \frac{4k\epsilon}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2(\alpha-1)}E\left|X_\epsilon(s)-Z_\epsilon(s)\right|^2 ds \\ &\quad+\frac{2\epsilon}{\Gamma(\alpha)^2}t^{2\alpha-1}\varphi_2(t)\left(1+E\left|Z_\epsilon(s)\right|^2+\left\|\nu_\epsilon(s)\right\|^2\right). \end{aligned} \tag{58}$$

Therefore, from the above discussion, (56)–(58), and Theorem 1, we have

$$\begin{aligned} E\left|X_\epsilon(t)-Z_\epsilon(t)\right|^2 &\le \frac{4tk\epsilon^2}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2(\alpha-1)}E\left|X_\epsilon(s)-Z_\epsilon(s)\right|^2 ds \\ &+ \frac{2\epsilon^2}{\Gamma(\alpha)^2}t^{2\alpha}\varphi_1\left((t)\left(1+E\left|Z_\epsilon(s)\right|^2+\left\|\nu_\epsilon(s)\right\|^2\right)\right) \\ &+ \frac{4k\epsilon}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2(\alpha-1)}E\left|X_\epsilon(s)-Z_\epsilon(s)\right|^2 ds + \frac{2\epsilon}{\Gamma(\alpha)^2}t^{2\alpha-1}\varphi_2\Gamma(\alpha)^2\left(1+E\left|Z_\epsilon(s)\right|^2+\left\|\nu_\epsilon(s)\right\|^2\right) \\ &= 2\epsilon T^{2\alpha-1}\left(\frac{c_1\epsilon}{\Gamma(\alpha)^2}T+\frac{c_2}{\Gamma(\alpha)^2}\right)+4\epsilon\left(\frac{k\epsilon}{\Gamma(\alpha)^2}T+\frac{k}{\Gamma(\alpha)^2}\right)\int_0^t (t-s)^{(2\alpha-1)-1}E\left|X_\epsilon(s)-Z_\epsilon(s)\right|^2 ds. \end{aligned} \tag{59}$$

Denote $r_1 = 2((c_1\epsilon/\Gamma(\alpha)^2)T + (c_2/\Gamma(\alpha)^2))$ and $r_2 = 4((k\epsilon/\Gamma(\alpha)^2)T + (k/\Gamma(\alpha)^2))$; using Lemma 1, we have

$$\begin{aligned} E\left|X_\epsilon(t)-Z_\epsilon(t)\right|^2 &\le \epsilon T^{2\alpha-1}r_1\left(1+\int_0^t\sum_{n=1}^{\infty}\frac{\left(r_2\epsilon\Gamma(2\alpha-1)\right)^n}{\Gamma((2\alpha-1)n)}(t-s)^{(2\alpha-1)n-1}ds\right) \\ &\le \epsilon T^{2\alpha-1}r_1\left(1+\sum_{n=1}^{\infty}\frac{\left(r_2\epsilon\Gamma(2\alpha-1)T^{2\alpha-1}\right)^n}{\Gamma((2\alpha-1)n+1)}\right) \\ &\le \epsilon T^{2\alpha-1}r_1\left(1+E_{2\alpha-1,1}\left(r_2\epsilon\Gamma(2\alpha-1)T^{2\alpha-1}\right)\right). \end{aligned} \tag{60}$$

Select some $\beta\in(0,1)$, $L>0$, such that, for $\forall t\in(0,L\epsilon^{(-\beta/2\alpha-1)}]$, we obtain

$$\sup_{0\le t\le L\epsilon^{(-\beta/2\alpha-1)}} E\left|X_\epsilon(t)-Z_\epsilon(t)\right|^2 \le C\epsilon^{1-\beta}, \tag{61}$$

where $C = r_1(1+E_{2\alpha-1,1}(r_2L\epsilon^{(-\beta/2\alpha-1)}\Gamma(2\alpha-1)))$.

Consequently, for $\forall\delta_1>0$, one can select some $\epsilon_1\in(0,\epsilon_0]$ such that, for each $\epsilon\in(0,\epsilon_1]$, $\forall t\in(0,L\epsilon^{(-\beta/2\alpha-1)}]$, we have

$$\sup_{0\le t\le L\epsilon^{(-\beta/2\alpha-1)}} E\left|X_\epsilon(t)-Z_\epsilon(t)\right|^2 \le \delta_1. \tag{62}$$

This completes the proof.

Remark 2. Using the definition of ρ, we obtain

$$\rho\left(\mu_\epsilon(t),\nu_\epsilon(t)\right) \le E\left|X_\epsilon(t)-Z_\epsilon(t)\right| \le E\left|X_\epsilon(t)-Z_\epsilon(t)\right|^2. \tag{63}$$

From the above estimate, we actually obtain

$$\begin{aligned} D_T\left(\mu_\epsilon,\nu_\epsilon\right) &= \sup_{0\le t\le T}\rho\left(\mu_\epsilon(t),\nu_\epsilon(t)\right) \\ &\le s\sup_{0\le t\le T}E\left|X_\epsilon(t)-Z_\epsilon(t)\right|^2 \longrightarrow 0, \quad \epsilon\longrightarrow 0, \end{aligned} \tag{64}$$

which means that, as $\epsilon\longrightarrow 0$, $\mu_\epsilon(t)$ corresponding to $X_\epsilon(t)$ converges to $\nu_\epsilon(t)$ of $Z_\epsilon(t)$ in $C([0,T;(M_\gamma^2,\rho)])$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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