Research Article

Inequalities Involving Essential Norm Estimates of Product-Type Operators

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Consider an open unit disk $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ in the complex plane $\mathbb{C}$, ξ a holomorphic function on $\mathbb{D}$, and ψ a holomorphic self-map of $\mathbb{D}$. For an analytic function f , the weighted composition operator is denoted and defined as follows: $(\mathcal{W}_{\xi, \psi} f)(z) = \xi(z)f(\psi(z))$. We estimate the essential norm of this operator from Dirichlet-type spaces to Bers-type spaces and Bloch-type spaces.

1. Introduction and Preliminaries

Consider an open unit disk $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ denote the class of all analytic functions on $\mathbb{D}$, $S(\mathbb{D})$ be the class of all holomorphic self-maps of $\mathbb{D}$, and H^∞ be the space of all bounded holomorphic functions on $\mathbb{D}$. Let $\xi \in H(\mathbb{D})$ and ψ be a holomorphic self-map of $\mathbb{D}$. For $z \in \mathbb{D}$, the composition operator and multiplication operator are, respectively, defined by

$$\begin{aligned} (C_\psi f)(z) &= f(\psi(z)), \\ (M_\xi f)(z) &= \xi(z)f(z), \end{aligned} \quad (1)$$

$f \in H(\mathbb{D})$.

The weighted composition operator is denoted and defined as

$$(\mathcal{W}_{\xi, \psi} f)(z) = \xi(z)f(\psi(z)), \quad f \in H(\mathbb{D}), \quad (2)$$

where $\mathcal{W}_{\xi, \psi}$ is a product-type operator as $\mathcal{W}_{\xi, \psi} = M_\xi C_\psi$. Clearly, this operator can be seen as a generalization of the composition operator and multiplication operator. It can be easily seen that, for $\xi \equiv 1$, the operator reduced to C_ψ . If $\psi(z) = z$, the operator gets reduced to M_ξ . This operator is basically a linear transformation of $H(\mathbb{D})$ defined by $(\mathcal{W}_{\xi, \psi} f)(z) = \xi(z)f(\psi(z)) = (M_\xi C_\psi f)(z)$, for f in $H(\mathbb{D})$

and z in $\mathbb{D}$. The basic aim is to give the operator-theoretic characterization of these operators in terms of function-theoretic characterization of their including functions. Various holomorphic function spaces on various domains have been studied for the boundedness and compactness of weighted composition operators acting on them. Moreover, a number of papers have been studied on these operators acting on different spaces of holomorphic functions on various domains. For more details, see [1–14] and the references therein. We say that a linear operator is bounded if the image of a bounded set is a bounded set. Moreover, a linear operator is said to be compact if it maps the bounded sets to those sets whose closure is compact. For each $\alpha > 0$, the weighted Bloch space $\mathfrak{B}$ is defined as follows:

$$\mathfrak{B} = \left\{ f \in H(\mathbb{D}): \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty \right\}. \quad (3)$$

In this expression, seminormed is defined. This space forms a Banach space with the natural norm defined by

$$\|f\|_{\mathfrak{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)|. \quad (4)$$

For $\alpha = 1$, this space gets reduced to classical Bloch space. A function $\omega: \mathbb{D} \rightarrow (0, \infty)$ is said to be a weight if it is continuous. For $z \in \mathbb{D}$, the weight ω is said to be radial if $\omega(z) = \omega(|z|)$. A weight ω is said to be a standard weight if

$\lim_{|z| \rightarrow 1^-} \omega(z) = 0$. For a weight function ω , the Bloch-type space $\mathfrak{B}_\omega$ is defined by

$$\mathfrak{B}_\omega = \left\{ f \in H(\mathbb{D}) : \sup_{z \in \mathbb{D}} \omega(z) |f'(z)| < \infty \right\}. \quad (5)$$

The little Bloch-type space $\mathfrak{B}_{\omega,0}$ is the closure of the set of polynomials in $\mathfrak{B}_\omega$ and is defined as follows:

$$\mathfrak{B}_{\omega,0} = \left\{ f \in \mathfrak{B}_\omega : \lim_{|z| \rightarrow 1^-} \omega(z) |f'(z)| = 0 \right\}. \quad (6)$$

Both $\mathfrak{B}_\omega$ and $\mathfrak{B}_{\omega,0}$ form a Banach space with the following norm:

$$\|f\|_{\mathfrak{B}_\omega} = |f(0)| + \sup_{z \in \mathbb{D}} \omega(z) |f'(z)|. \quad (7)$$

For more information about these spaces, one may refer [1–3, 5, 6, 15, 16] and the references therein. Likewise, for weight ω , the Bers-type space $\mathfrak{A}_\omega$ is defined as follows:

$$\mathfrak{A}_\omega = \left\{ f \in H(\mathbb{D}) : \|f\|_{\mathfrak{A}_\omega} := \sup_{z \in \mathbb{D}} \omega(z) |f(z)| < \infty \right\}. \quad (8)$$

It is a nonseparable Banach space with the norm $\|\cdot\|_{\mathfrak{A}_\omega}$. The closure of the set of polynomials in $\mathfrak{A}_\omega$ forms a separable Banach space. This set is denoted by $\mathfrak{A}_{\omega,0}$ and is defined as

$$\mathfrak{A}_{\omega,0} = \left\{ f \in \mathfrak{A}_\omega : \lim_{|z| \rightarrow 1^-} \omega(z) |f(z)| = 0 \right\}. \quad (9)$$

These spaces and their properties are discussed in many papers; some of these are [3, 15, 16] and the references therein. The Dirichlet space is defined as follows:

$$\mathfrak{D} = \left\{ f \in H(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty \right\}, \quad (10)$$

where $dA(z)$ denotes the normalized Lebesgue area measure on $\mathbb{D}$. With the following norm, it is a Hilbert space:

$$\|f\|_{\mathfrak{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z). \quad (11)$$

Consider a function $K: [0, \infty) \rightarrow [0, \infty)$ which is right continuous and increasing. In this paper, we consider function K as a weight function. With a weight function K , the Dirichlet type space $\mathfrak{D}_K$ is given as follows:

$$\mathfrak{D}_K = \left\{ f \in H(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)|^2 K(1 - |z|^2) dA(z) < \infty \right\}. \quad (12)$$

Clearly, space $\mathfrak{D}_K$ forms a Hilbert space with the norm $\|\cdot\|_{\mathfrak{D}_K}$ defined by

$$\|f\|_{\mathfrak{D}_K}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 K(1 - |z|^2) dA(z). \quad (13)$$

Here, we have $K(t) = t^p$, $0 \leq p < \infty$, and $\mathfrak{D}_K$ gives $\mathfrak{D}_p$, that is, the usual Dirichlet-type space. This gives a classical Dirichlet space $\mathfrak{D}$ for a case when $p = 0$ and for $p = 1$, and we gain the Hardy space H^2 . These spaces have been studied widely in various papers. For example, Aleman, in [17], obtained that each element of $\mathfrak{D}_K$ can be written as a

quotient of two bounded functions in $\mathfrak{D}_K$. Kerman and Sawyer [18], by taking some conditions on weight function K , characterized Carleson measures and multipliers of $\mathfrak{D}_K$ in terms of maximal operator.

The Möbius invariant space generated by $\mathfrak{D}_K$ is denoted by $\mathcal{Q}_K$. The space $\mathcal{Q}_K$ contains those functions $f \in H(\mathbb{D})$ which satisfy the following:

$$\sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_{\mathfrak{D}_K} < \infty, \quad (14)$$

where $\sigma_a(z) = ((a - z)/(1 - \bar{a}z))$ is the Möbius transformation of $\mathbb{D}$. Wulan and Zhu, in [19], characterized Lacunary series in the $\mathcal{Q}_K$ space under some conditions on weight function K . Furthermore, Wulan and Zhou [20] characterized space $\mathcal{Q}_K$ in terms of fractional-order derivatives of function. They also established a relationship between Morrey type spaces and $\mathcal{Q}_K$ space in terms of fractional order derivatives. In the study of $\mathcal{Q}_K$ spaces, the following two conditions play a very important role:

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty, \quad (15)$$

$$\int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty, \quad (16)$$

where

$$\varphi_K(s) = \sup_{0 \leq t \leq 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty. \quad (17)$$

Let $M(\mathfrak{D}_K)$ be the class of multipliers of $\mathfrak{D}_K$, that is,

$$M(\mathfrak{D}_K) = \{f \in H(\mathbb{D}) : fg \in \mathfrak{D}_K, \forall g \in \mathfrak{D}_K\}. \quad (18)$$

Bao et al., in [21], characterized the interpolating sequences for $M(\mathfrak{D}_K)$ of space $\mathfrak{D}_K$, under certain conditions of weight function K . They also obtained corona theorem, $\bar{\partial}$ -equation, and corona-type decomposition theorem on $M(\mathfrak{D}_K)$. For more details, see [9, 15, 21–25] and the references therein.

From [26], one can see that if K satisfies (1), then

$$K_1(t) = \int_0^t K(s) \frac{ds}{s} \approx K(t), \quad 0 < t < 1. \quad (19)$$

If K satisfies (16), then

$$K_2(t) = t \int_t^\infty K(s) \frac{ds}{s^2} \approx K(t), \quad t > 0. \quad (20)$$

From condition (16), we get that $K(2t) \approx K(t)$ for $0 < t < 1$. Also, there exist $C > 0$ sufficiently small for which $t^{-C} K_1(t)$ is increasing and $K_2(t) t^{C-1}$ is decreasing. For more information about weight function K , one can refer [19–21].

The criterion of boundedness as well as compactness has been discussed in many papers. Recently, Gürbüz, in [27], studied the boundedness of generalized commutators of rough fractional maximal and integral operators on generalized weighted Morrey spaces, respectively, and, in [28], he investigated the generalized weighted Morrey estimates for the boundedness of Marcinkiewicz integrals with rough kernel associated with Schrödinger operator and their

commutators. Furthermore, in [29], Gürbüz studied the behavior of multi-sublinear fractional maximal operators and rough multilinear fractional integral both on product L^p and weighted L^p spaces and, in [30], he obtained the boundedness of the variation and oscillation operators for the family of multilinear integrals with Lipschitz functions on weighted Morrey spaces. Among others in [3], we obtained the following results about boundedness and compactness of $\mathcal{W}_{\xi,\psi}$ given as follows.

Theorem 1. Let ω and K be two weight functions, $\xi \in H(\mathbb{D})$, and ψ be a self-holomorphic map on $\mathbb{D}$. Then, the operator $\mathcal{W}_{\xi,\psi}: \mathfrak{D}_K \longrightarrow \mathfrak{B}_\omega$ is bounded if and only if the following conditions hold:

- (i) $M_1 = \sup_{z \in \mathbb{D}} ((\omega(z)|\xi'(z)|)/(\sqrt{K(1-|\psi(z)|^2)}(1-|\psi(z)|^2))) < \infty$
- (ii) $M_2 = \sup_{z \in \mathbb{D}} ((\omega(z)|\xi(z)\psi'(z)|)/(\sqrt{K(1-|\psi(z)|^2)}(1-|\psi(z)|^2)^2)) < \infty$

Furthermore, if the operator $\mathcal{W}_{\xi,\psi}: \mathfrak{D}_K \longrightarrow \mathfrak{B}_\omega$ is bounded, then

$$M_1 + M_2 \leq \|\mathcal{W}_{\xi,\psi}\|_{\mathfrak{D}_K \longrightarrow \mathfrak{B}_\omega} \leq 1 + M_1 + M_2. \quad (21)$$

Theorem 2. Let ω be a standard weight, $\xi \in H(\mathbb{D})$, and ψ be a self-holomorphic map on $\mathbb{D}$. Let K be a weight function. Assume that $\mathcal{W}_{\xi,\psi}: \mathfrak{D}_K \longrightarrow \mathfrak{B}_\omega$ is bounded. Then, the operator $\mathcal{W}_{\xi,\psi}: \mathfrak{D}_K \longrightarrow \mathfrak{B}_\omega$ is compact if and only if the following conditions hold:

- (i) $\lim_{|\psi(z)| \rightarrow 1} ((\omega(z)|\xi'(z)|)/(\sqrt{K(1-|\psi(z)|^2)}(1-|\psi(z)|^2))) = 0$
- (ii) $\lim_{|\psi(z)| \rightarrow 1} ((\omega(z)|\xi(z)\psi'(z)|)/(\sqrt{K(1-|\psi(z)|^2)}(1-|\psi(z)|^2)^2)) = 0$

Theorem 3. Let ω be a weight and K be a weight function, $\xi \in H(\mathbb{D})$, and ψ be a self-holomorphic map on $\mathbb{D}$. Then, the operator $\mathcal{W}_{\xi,\psi}: \mathfrak{D}_K \longrightarrow \mathfrak{A}_\omega$ is bounded if and only if the following condition holds:

$$l_1 = \sup_{z \in \mathbb{D}} \frac{\omega(z)|\xi(z)|}{\sqrt{K(1-|\psi(z)|^2)}(1-|\psi(z)|^2)} < \infty. \quad (22)$$

Theorem 4. Let ω be a standard weight, $\xi \in H(\mathbb{D})$, and ψ be a self-analytic map on $\mathbb{D}$. Let K be a weight function. Assume that the operator $\mathcal{W}_{\xi,\psi}: \mathfrak{D}_K \longrightarrow \mathfrak{A}_\omega$ is bounded. Then, $\mathcal{W}_{\xi,\psi}: \mathfrak{D}_K \longrightarrow \mathfrak{A}_\omega$ is compact if and only if the following condition is satisfied:

$$\lim_{|\psi(z)| \rightarrow 1} \frac{\omega(z)|\xi(z)|}{\sqrt{K(1-|\psi(z)|^2)}(1-|\psi(z)|^2)} = 0. \quad (23)$$

The aim of this paper is to provide some estimates of essential norm of the operator $\mathcal{W}_{\xi,\psi}: \mathfrak{D}_K \longrightarrow \mathfrak{B}_\omega$ as well as of $\mathcal{W}_{\xi,\psi}: \mathfrak{D}_K \longrightarrow \mathfrak{A}_\omega$.

Assume that $T: X_1 \longrightarrow X_2$ is a bounded linear operator for Banach spaces X_1 and X_2 . The essential norm of operator T is denoted and defined as follows:

$$\|T\|_{e, X_1 \longrightarrow X_2} = \inf \{ \|T - E\|_{X_1 \longrightarrow X_2} : E \text{ mapping } X_1 \text{ into } X_2 \text{ is compact} \}, \quad (24)$$

where $\|\cdot\|_{X_1 \longrightarrow X_2}$ is the operator norm. In other words, the essential norm is the distance from compact operators E mapping X_1 into X_2 to the bounded linear operator $T: X_1 \longrightarrow X_2$. If $X_1 = X_2$, that is, the two Banach spaces are same, then the norm is simply denoted by $\|\cdot\|_e$. For unbounded linear operator $T: X_1 \longrightarrow X_2$, we have $\|T\|_{e, X_1 \longrightarrow X_2} = \infty$. As the class of all compact operators is contained in the class of all bounded operators, in fact, this subset is closed, which implies that the operator T is compact if and only if $\|T\|_{e, X_1 \longrightarrow X_2} = 0$. Thus, the estimate of essential norm leads to the compactness of the operator. Various results on the essential norm of different operators such as multiplication, composition, differentiation, weighted composition, generalized weighted composition, and their different combinations are studied in numerous research papers, and some of the references are [31–37].

This study is formulated in a systematic way. Introduction and literature part is kept in Section 1. In Section 2, we estimated the essential norm of operator $\mathcal{W}_{\xi,\psi}: \mathfrak{D}_K \longrightarrow \mathfrak{B}_\omega$. Finally, in Section 3, we estimated the essential norm of operator $\mathcal{W}_{\xi,\psi}: \mathfrak{D}_K \longrightarrow \mathfrak{A}_\omega$. Throughout the paper, the notation $a \leq b$, for any two positive quantities a and b , which means that $a \leq Cb$, where C is some positive

constant. The value of constant C may change from one place to the other. We write $a \approx b$ if $a \leq b$ and $b \leq a$.

2. Essential Norm of Weighted Composition Operator from Dirichlet-Type Space to Bloch-Type Space

Theorem 5. Let ω be a standard weight, $\xi \in H(\mathbb{D})$, and ψ be a self-analytic map on $\mathbb{D}$. Let K be a weight function. Assume that $\mathcal{W}_{\xi,\psi}: \mathfrak{D}_K \longrightarrow \mathfrak{B}_\omega$ is bounded. Then,

$$\|\mathcal{W}_{\xi,\psi}\|_{e, \mathfrak{D}_K \longrightarrow \mathfrak{B}_\omega} \approx \max\{R, S\}, \quad (25)$$

where

$$R := \limsup_{|\psi(z)| \rightarrow 1} \frac{\omega(z)|\xi'(z)|}{\sqrt{K(1-|\psi(z)|^2)}(1-|\psi(z)|^2)}, \quad (26)$$

$$S := \limsup_{|\psi(z)| \rightarrow 1} \frac{\omega(z)|\xi(z)\psi'(z)|}{\sqrt{K(1-|\psi(z)|^2)}(1-|\psi(z)|^2)^2}.$$

Proof. At first, we show that

$$\|\mathcal{W}_{\xi,\psi}\|_{e,\mathfrak{D}_K \rightarrow \mathfrak{B}_\omega} \geq \max\{R, S\}. \quad (27)$$

For $z \in \mathbb{D}$, define a function

$$\begin{aligned} h_z(w) = & a_0 \frac{(1-|z|^2)^{\varepsilon/2}}{\sqrt{K(1-|z|^2)}(1-\bar{z}w)^{1+(\varepsilon/2)}} \\ & + a_1 \frac{(1-|z|^2)^{1+(\varepsilon/2)}}{\sqrt{K(1-|z|^2)}(1-\bar{z}w)^{2+(\varepsilon/2)}}, \end{aligned} \quad (28)$$

where $a_0 = (2 + \varepsilon/2)$ and $a_1 = -(1 + \varepsilon/2)$. It can be easily checked that $h_z \in \mathbb{D}_K$ and, for all $z \in \mathbb{D}$, $\|h_z\|_{\mathfrak{D}_K} \leq 1$. On

calculation, we have $h'_z(z) = 0$ and $h_z(z) = 1/(\sqrt{K(1-|z|^2)}(1-|z|^2))$. Furthermore, on compact subsets of $\mathbb{D}$, h_z converges to zero as $|z| \rightarrow 1$. Hence, for any compact operator $E: \mathfrak{D}_K \rightarrow \mathfrak{B}_\omega$ and any $(\zeta_n)_{n \in \mathbb{N}}$ such that $|\psi(\zeta_n)| \rightarrow 1$, we obtain

$$\begin{aligned} \|\mathcal{W}_{\xi,\psi} - E\|_{\mathfrak{D}_K \rightarrow \mathfrak{B}_\omega} & \geq \|(\mathcal{W}_{\xi,\psi} - E)h_\psi(\zeta_n)\|_{\mathfrak{B}_\omega} \\ & \geq \|\mathcal{W}_{\xi,\psi}h_\psi(\zeta_n)\|_{\mathfrak{B}_\omega} - \|Eh_\psi(\zeta_n)\|_{\mathfrak{B}_\omega}. \end{aligned} \quad (29)$$

In the above inequality, take $\limsup_{|\psi(\zeta_n)| \rightarrow 1}$ on both sides, and we obtain

$$\|\mathcal{W}_{\xi,\psi} - E\|_{\mathfrak{D}_K \rightarrow \mathfrak{B}_\omega} \geq \limsup_{|\psi(\zeta_n)| \rightarrow 1} \|\mathcal{W}_{\xi,\psi}h_\psi(\zeta_n)\|_{\mathfrak{B}_\omega} \geq \limsup_{|\psi(\zeta_n)| \rightarrow 1} \frac{\omega(\zeta_n)|\xi'(\zeta_n)|}{\sqrt{K(1-|\psi(\zeta_n)|^2)}(1-|\psi(\zeta_n)|^2)} = R. \quad (30)$$

Again, for $z \in \mathbb{D}$, define another function:

$$\begin{aligned} k_z(w) = & b_0 \frac{(1-|z|^2)^{\varepsilon/2}}{\sqrt{K(1-|z|^2)}(1-\bar{z}w)^{1+\varepsilon/2}} \\ & + b_1 \frac{(1-|z|^2)^{1+\varepsilon/2}}{\sqrt{K(1-|z|^2)}(1-\bar{z}w)^{2+\varepsilon/2}}, \end{aligned} \quad (31)$$

where $b_0 = 1$ and $b_1 = -1$. In the similar manner, we can check that $k_z \in \mathbb{D}_K$ and, for all $z \in \mathbb{D}$, $\|k_z\|_{\mathfrak{D}_K} \leq 1$. On calculation, we have $k'_z(z) = 0$ and

$k'_z(z) = -\bar{z}/(\sqrt{K(1-|z|^2)}(1-|z|^2)^2)$. Furthermore, on compact subsets of $\mathbb{D}$, k_z converges to zero as $|z| \rightarrow 1$. Thus, for any compact operator $E: \mathfrak{D}_K \rightarrow \mathfrak{B}_\omega$ and any $(\zeta_n)_{n \in \mathbb{N}}$ such that $|\psi(\zeta_n)| \rightarrow 1$, we obtain

$$\begin{aligned} \|\mathcal{W}_{\xi,\psi} - E\|_{\mathfrak{D}_K \rightarrow \mathfrak{B}_\omega} & \geq \|(\mathcal{W}_{\xi,\psi} - E)k_\psi(\zeta_n)\|_{\mathfrak{B}_\omega} \geq \|\mathcal{W}_{\xi,\psi}k_\psi(\zeta_n)\|_{\mathfrak{B}_\omega} \\ & \quad - \|Ek_\psi(\zeta_n)\|_{\mathfrak{B}_\omega}. \end{aligned} \quad (32)$$

By taking $\limsup_{|\psi(\zeta_n)| \rightarrow 1}$ on both sides of the above inequality, we obtain

$$\|\mathcal{W}_{\xi,\psi} - E\|_{\mathfrak{D}_K \rightarrow \mathfrak{B}_\omega} \geq \limsup_{|\psi(\zeta_n)| \rightarrow 1} \|\mathcal{W}_{\xi,\psi}k_\psi(\zeta_n)\|_{\mathfrak{B}_\omega} \geq \limsup_{|\psi(\zeta_n)| \rightarrow 1} \frac{\omega(\zeta_n)|\xi(\zeta_n)\psi'(\zeta_n)|}{\sqrt{K(1-|\psi(\zeta_n)|^2)}(1-|\psi(\zeta_n)|^2)^2} = S. \quad (33)$$

On applying the definition of essential norm, we find that

$$\|\mathcal{W}_{\xi,\psi}\|_{e,\mathfrak{D}_K \rightarrow \mathfrak{B}_\omega} = \inf_E \|\mathcal{W}_{\xi,\psi} - E\|_{\mathfrak{D}_K \rightarrow \mathfrak{B}_\omega} \geq \max\{R, S\}. \quad (34)$$

Next, we prove that

$$\|\mathcal{W}_{\xi,\psi}\|_{e,\mathfrak{D}_K \rightarrow \mathfrak{B}_\omega} \leq \max\{R, S\}. \quad (35)$$

For $\delta \in [0, 1)$, consider $E_\delta: H(\mathbb{D}) \rightarrow H(\mathbb{D})$, defined as follows:

$$(E_\delta f)(z) = f_\delta(z) = f(\delta z), \quad f \in H(\mathbb{D}). \quad (36)$$

Clearly, E_δ is compact on $\mathfrak{D}_K$ and $\|E_\delta\|_{\mathfrak{D}_K \rightarrow \mathfrak{D}_K} \leq 1$. Consider a sequence $\{\delta_n\} \subset (0, 1)$ satisfying $\delta_n \rightarrow 1$ as $n \rightarrow \infty$. Then, for all $n \in \mathbb{N}$, operator $\mathcal{W}_{\xi,\psi}E_{\delta_n}: \mathfrak{D}_K \rightarrow \mathfrak{B}_\omega$ is compact. By using the definition of essential norm, we obtain

$$\|\mathcal{W}_{\xi,\psi}\|_{e,\mathfrak{D}_K \rightarrow \mathfrak{B}_\omega} \leq \limsup_{n \rightarrow \infty} \|\mathcal{W}_{\xi,\psi} - \mathcal{W}_{\xi,\psi}E_{\delta_n}\|_{\mathfrak{D}_K \rightarrow \mathfrak{B}_\omega}. \quad (37)$$

Therefore, we only have to prove that

$$\limsup_{n \rightarrow \infty} \|\mathcal{W}_{\xi,\psi} - \mathcal{W}_{\xi,\psi}E_{\delta_n}\|_{\mathfrak{D}_K \rightarrow \mathfrak{B}_\omega} \leq \max\{R, S\}. \quad (38)$$

Let f be a function in $\mathfrak{D}_K$ satisfying $\|f\|_{\mathfrak{D}_K} \leq 1$; then, we have

$$\begin{aligned} \|\mathcal{W}_{\xi,\psi} - \mathcal{W}_{\xi,\psi} E_{\delta_n}\|_{\mathfrak{B}_K \rightarrow \mathfrak{B}_\omega} &= |\xi(0)f(\psi(0)) - \xi(0)f(\delta_n\psi(0))| \\ &\quad + \|\xi(f - f_{\delta_n}) \circ \psi\|_{\mathfrak{B}_\omega}. \end{aligned} \quad (39)$$

Clearly, $\lim_{n \rightarrow \infty} |\xi(0)f(\psi(0)) - \xi(0)f(\delta_n\psi(0))| = 0$. Furthermore, consider a large enough $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $\delta_n \geq 1/2$. Thus, we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|\xi(f - f_{\delta_n}) \circ \psi\|_{\mathfrak{B}_\omega} &\leq l \limsup_{n \rightarrow \infty} \sup_{|\psi(z)| \leq \delta_N} \omega(z) |\xi(z)| \|(f - f_{\delta_n})'(\psi(z))\| |\psi'(z)| \\ &\quad + \limsup_{n \rightarrow \infty} \sup_{|\psi(z)| \leq \delta_N} \omega(z) |\xi(z)| \|(f - f_{\delta_n})'(\psi(z))\| |\psi'(z)| \\ &\quad + \limsup_{n \rightarrow \infty} \sup_{|\psi(z)| \leq \delta_N} \omega(z) |\xi'(z)| \|(f - f_{\delta_n})(\psi(z))\| \\ &\quad + \limsup_{n \rightarrow \infty} \sup_{|\psi(z)| \leq \delta_N} \omega(z) |\xi'(z)| \|(f - f_{\delta_n})(\psi(z))\| \\ &= S_1 + S_2 + S_3 + S_4, \end{aligned} \quad (40)$$

where

$$\begin{aligned} S_1 &= \limsup_{n \rightarrow \infty} \sup_{|\psi(z)| \leq \delta_N} \omega(z) |\xi(z)| \|(f - f_{\delta_n})'(\psi(z))\| |\psi'(z)|, \\ S_2 &= \limsup_{n \rightarrow \infty} \sup_{|\psi(z)| \leq \delta_N} \omega(z) |\xi(z)| \|(f - f_{\delta_n})'(\psi(z))\| |\psi'(z)|, \\ S_3 &= \limsup_{n \rightarrow \infty} \sup_{|\psi(z)| \leq \delta_N} \omega(z) |\xi'(z)| \|(f - f_{\delta_n})(\psi(z))\|, \\ S_4 &= \limsup_{n \rightarrow \infty} \sup_{|\psi(z)| \leq \delta_N} \omega(z) |\xi'(z)| \|(f - f_{\delta_n})(\psi(z))\|. \end{aligned} \quad (41)$$

Taking the operator $\mathcal{W}_{\xi,\psi}$ to 1 and z and using its boundedness, it easily follows that $\xi \in \mathbb{B}_\omega$ and

$$\tilde{E} = \sup_{z \in \mathbb{D}} \omega(z) |\psi'(z) \xi(z)| < \infty. \quad (42)$$

Also, on compact subsets of $\mathbb{D}$, $\delta_n f_{\delta_n}'$ uniformly converges to f' as $n \rightarrow \infty$; thus, we have

$$S_1 \leq \tilde{E} \limsup_{n \rightarrow \infty} \sup_{|w| \leq \delta_N} |f'(w) - \delta_n f'(\delta_n w)| = 0. \quad (43)$$

Similarly, for $\xi \in \mathbb{B}_\omega$ and the fact that f_{δ_n} converges uniformly to f on compact subsets of $\mathbb{D}$ as $n \rightarrow \infty$, we obtain

$$S_3 \leq \|\xi\|_{\mathfrak{B}_\omega} \limsup_{n \rightarrow \infty} \sup_{w \leq \delta_N} |f(w) - f(\delta_n w)| = 0. \quad (44)$$

Now, consider S_2 . We have $S_2 \leq \limsup_{n \rightarrow \infty} (P_1 + P_2)$, where

$$\begin{aligned} P_1 &= \sup_{|\psi(z)| > \delta_N} \omega(z) |\xi(z)| |\psi'(z)| |f'(\psi(z))|, \\ P_2 &= \sup_{|\psi(z)| > \delta_N} \delta_n \omega(z) |\xi(z)| |\psi'(z)| |f'(\delta_n \psi(z))|. \end{aligned} \quad (45)$$

First, we consider P_1 . As $\|f\|_{\mathfrak{B}_K} \leq 1$, we obtain

$$\begin{aligned} P_1 &= \sup_{|\psi(z)| > \delta_N} \omega(z) |\xi(z)| |\psi'(z)| |f'(\psi(z))| \\ &\leq \frac{1}{\delta_N} \|f\|_{\mathfrak{B}_K} \sup_{|\psi(z)| > \delta_N} \omega(z) |\xi(z)| |\psi'(z)| \frac{|\psi(z)|}{\sqrt{K(1-|\psi(z)|^2)(1-|\psi(z)|^2)^2}}. \end{aligned} \quad (46)$$

On taking limit as $N \rightarrow \infty$, we obtain

$$\limsup_{n \rightarrow \infty} P_1 \leq \limsup_{|\psi(z)| \rightarrow 1} \frac{\omega(z) |\xi(z)| |\psi'(z)|}{\sqrt{K(1-|\psi(z)|^2)(1-|\psi(z)|^2)^2}} = S. \quad (47)$$

In the similar manner, we obtain

$$\limsup_{n \rightarrow \infty} P_2 \leq \limsup_{|\psi(z)| \rightarrow 1} \frac{\omega(z) |\xi(z)| |\psi'(z)|}{\sqrt{K(1-|\psi(z)|^2)(1-|\psi(z)|^2)^2}} = S. \quad (48)$$

On combining the above two inequalities, we obtain

$$S_2 \leq \limsup_{n \rightarrow \infty} (P_1 + P_2) \leq S. \quad (49)$$

Next, consider S_4 . We have $S_4 \leq \limsup_{n \rightarrow \infty} (P_3 + P_4)$, where

$$\begin{aligned} P_3 &= \sup_{|\psi(z)| > \delta_N} \omega(z) |\xi'(z)| |f(\psi(z))|, \\ P_4 &= \sup_{|\psi(z)| > \delta_N} \omega(z) |\xi'(z)| |f(\delta_n \psi(z))|. \end{aligned} \quad (50)$$

By similar calculation, we obtain

$$\begin{aligned} P_3 &= \sup_{|\psi(z)| > \delta_N} \omega(z) |\xi'(z)| |f(\psi(z))| \\ &\leq \frac{1}{\delta_N} \|f\|_{\mathfrak{B}_K} \sup_{|\psi(z)| > \delta_N} \omega(z) |\xi'(z)| \frac{1}{\sqrt{K(1-|\psi(z)|^2)(1-|\psi(z)|^2)}}. \end{aligned} \quad (51)$$

On taking limit $N \rightarrow \infty$, we obtain

$$\limsup_{n \rightarrow \infty} P_3 \leq \limsup_{|\psi(z)| \rightarrow 1} \frac{\omega(z)|\xi'(z)|}{\sqrt{K(1-|\psi(z)|^2)(1-|\psi(z)|^2)}} = R. \quad (52)$$

In the similar manner, we obtain

$$\limsup_{n \rightarrow \infty} P_4 \leq \limsup_{|\psi(z)| \rightarrow 1} \frac{\omega(z)|\xi'(z)|}{\sqrt{K(1-|\psi(z)|^2)(1-|\psi(z)|^2)}} = R. \quad (53)$$

Combining the above two inequalities, we obtain

$$S_4 \leq \limsup_{n \rightarrow \infty} (P_3 + P_4) \leq R. \quad (54)$$

On combining (40), (43), (44), (49), and (54), we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|\mathcal{W}_{\xi, \psi} - \mathcal{W}_{\xi, \psi} E_{\delta_n}\|_{\mathfrak{D}_K \rightarrow \mathfrak{B}_\omega} &= \limsup_{n \rightarrow \infty} \sup_{\|f\|_{\mathfrak{D}_K} \leq 1} \|\mathcal{W}_{\xi, \psi} - \mathcal{W}_{\xi, \psi} E_{\delta_n}\|_{\mathfrak{B}_\omega} \\ &= \limsup_{n \rightarrow \infty} \sup_{\|f\|_{\mathfrak{D}_K} \leq 1} \|\xi(f - f_{\delta_n}) \circ \psi\|_{\mathfrak{B}_\omega} \\ &\leq R + S. \end{aligned} \quad (55)$$

Thus, inequalities (37) and (55) imply that

$$\|\mathcal{W}_{\xi, \psi}\|_{e, \mathfrak{D}_K \rightarrow \mathfrak{B}_\omega} \leq R + S \leq \max\{R, S\}. \quad (56)$$

Hence, inequalities (34) and (56) complete the theorem.

The following corollary can be easily obtained from Theorem 5. $\square$

Corollary 1. Let ω be a standard weight and ψ be a self-analytic map on $\mathbb{D}$. Let K be a weight function. Assume that $C_\psi: \mathfrak{D}_K \rightarrow \mathfrak{B}_\omega$ is bounded. Then,

$$\|C_\psi\|_{e, \mathfrak{D}_K \rightarrow \mathfrak{B}_\omega} \approx \limsup_{|\psi(z)| \rightarrow 1} \frac{\omega(z)|\psi'(z)|}{\sqrt{K(1-|\psi(z)|^2)(1-|\psi(z)|^2)}}. \quad (57)$$

3. Essential Norm of Weighted Composition Operator from Dirichlet-Type Space to Bers-Type Space

In this section, we consider the Bers-type spaces and estimated the essential norm of weighted composition operator from $\mathfrak{D}_K$ to $\mathfrak{B}_\omega$.

Theorem 6. Let ω be a standard weight, $\xi \in H(\mathbb{D})$, and ψ be a self-analytic map on $\mathbb{D}$. Let K be a weight function. Assume that the operator $\mathcal{W}_{\xi, \psi}: \mathfrak{D}_K \rightarrow \mathfrak{B}_\omega$ is bounded. Then,

$$\|\mathcal{W}_{\xi, \psi}\|_{e, \mathfrak{D}_K \rightarrow \mathfrak{B}_\omega} \approx \limsup_{|\psi(z)| \rightarrow 1^-} \frac{\omega(z)|\xi(z)|}{\sqrt{K(1-|\psi(z)|^2)(1-|\psi(z)|^2)}}. \quad (58)$$

Proof. Firstly, we prove that

$$\|\mathcal{W}_{\xi, \psi}\|_{e, \mathfrak{D}_K \rightarrow \mathfrak{B}_\omega} \geq \limsup_{|\psi(z)| \rightarrow 1^-} \frac{\omega(z)|\xi(z)|}{\sqrt{K(1-|\psi(z)|^2)(1-|\psi(z)|^2)}}. \quad (59)$$

Consider a function $f_z \in \mathfrak{D}_K$ such that $\|f_z\|_{\mathfrak{D}_K} \leq 1$, and on compact subsets of $\mathbb{D}$, f_z converges to zero as $|z| \rightarrow 1$. Thus, for any compact operator $E: \mathfrak{D}_K \rightarrow \mathfrak{B}_\omega$ and any $(\zeta_n)_{n \in \mathbb{D}}$ such that $|\psi(\zeta_n)| \rightarrow 1^-$, we obtain

$$\begin{aligned} \|\mathcal{W}_{\xi, \psi} - E\|_{\mathfrak{D}_K \rightarrow \mathfrak{B}_\omega} &\geq \|\mathcal{W}_{\xi, \psi} - E\|_{\mathfrak{D}_K \rightarrow \mathfrak{B}_\omega} f_{\psi(\zeta_n)} \|_{\mathfrak{B}_\omega} \\ &\geq \|\mathcal{W}_{\xi, \psi} f_{\psi(\zeta_n)}\|_{\mathfrak{B}_\omega} - \|E f_{\psi(\zeta_n)}\|_{\mathfrak{B}_\omega}. \end{aligned} \quad (60)$$

Taking $\limsup_{|\psi(\zeta_n)| \rightarrow 1^-}$ on both sides, we obtain

$$\begin{aligned} \|\mathcal{W}_{\xi, \psi} - E\|_{\mathfrak{D}_K \rightarrow \mathfrak{B}_\omega} &\geq \limsup_{|\psi(\zeta_n)| \rightarrow 1^-} \|\mathcal{W}_{\xi, \psi} f_{\psi(\zeta_n)}\|_{\mathfrak{B}_\omega} \\ &\geq \limsup_{|\psi(\zeta_n)| \rightarrow 1^-} \frac{\omega(\zeta_n)|\xi(\zeta_n)|}{\sqrt{K(1-|\psi(\zeta_n)|^2)(1-|\psi(\zeta_n)|^2)}}. \end{aligned} \quad (61)$$

On applying the definition of essential norm, we find that

$$\|\mathcal{W}_{\xi,\psi}\|_{e,\mathfrak{D}_K \rightarrow \mathfrak{A}_\omega} = \inf_E \|\mathcal{W}_{\xi,\psi} - E\|_{\mathfrak{D}_K \rightarrow \mathfrak{A}_\omega} \geq \limsup_{|\psi(\zeta_n)| \rightarrow 1^-} \frac{\omega(\zeta_n)|\xi(\zeta_n)|}{\sqrt{K(1-|\psi(\zeta_n)|^2)}(1-|\psi(\zeta_n)|^2)}. \quad (62)$$

Finally, we prove that

$$\|\mathcal{W}_{\xi,\psi}\|_{e,\mathfrak{D}_K \rightarrow \mathfrak{A}_\omega} \leq \limsup_{|\psi(\zeta_n)| \rightarrow 1^-} \frac{\omega(\zeta_n)|\xi(\zeta_n)|}{\sqrt{K(1-|\psi(\zeta_n)|^2)}(1-|\psi(\zeta_n)|^2)}. \quad (63)$$

For this, consider $E_\delta: H(\mathbb{D}) \rightarrow H(\mathbb{D})$ with $\delta \in [0, 1)$ and a sequence $\{\delta_n\} \subset (0, 1)$ satisfying $\delta_n \rightarrow 1$ as $n \rightarrow \infty$ defined in Theorem 5. Then, for all $n \in \mathbb{N}$, the operator $\mathcal{W}_{\xi,\psi}E_{\delta_n}: \mathfrak{D}_K \rightarrow \mathfrak{A}_\omega$ is compact. By using the definition of essential norm, we obtain

$$\|\mathcal{W}_{\xi,\psi}\|_{e,\mathfrak{D}_K \rightarrow \mathfrak{A}_\omega} \leq \limsup_{n \rightarrow \infty} \|\mathcal{W}_{\xi,\psi} - \mathcal{W}_{\xi,\psi}E_{\delta_n}\|_{\mathfrak{D}_K \rightarrow \mathfrak{A}_\omega}. \quad (64)$$

So, we only have to prove that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|\mathcal{W}_{\xi,\psi} - \mathcal{W}_{\xi,\psi}E_{\delta_n}\|_{\mathfrak{D}_K \rightarrow \mathfrak{A}_\omega} \\ \leq \limsup_{|\psi(\zeta_n)| \rightarrow 1^-} \frac{\omega(\zeta_n)|\xi(\zeta_n)|}{\sqrt{K(1-|\psi(\zeta_n)|^2)}(1-|\psi(\zeta_n)|^2)}. \end{aligned} \quad (65)$$

Let f be a function in $\mathfrak{D}_K$ satisfying $\|f\|_{\mathfrak{D}_K} \leq 1$; then, we have

$$\|(\mathcal{W}_{\xi,\psi} - \mathcal{W}_{\xi,\psi}E_{\delta_n})f\|_{\mathfrak{D}_K \rightarrow \mathfrak{A}_\omega} = \|\xi(f - f_{\delta_n}) \circ \psi\|_{\mathfrak{A}_\omega}. \quad (66)$$

Furthermore, consider a large enough $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $\delta_n \geq 1/2$. Thus, we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|\xi(f - f_{\delta_n}) \circ \psi\|_{\mathfrak{A}_\omega} &\leq \limsup_{n \rightarrow \infty} \sup_{|\psi(z)| \leq \delta_N} \omega(z)|\xi(z)| |(f - f_{\delta_n})(\psi(z))| \\ &+ \limsup_{n \rightarrow \infty} \sup_{|\psi(z)| \leq \delta_N} \omega(z)|\xi(z)| |(f - f_{\delta_n})(\psi(z))| = A_1 + A_2, \end{aligned} \quad (67)$$

where

$$A_1 = \limsup_{n \rightarrow \infty} \sup_{|\psi(z)| \leq \delta_N} \omega(z)|\xi(z)| |(f - f_{\delta_n})(\psi(z))|, \quad (68)$$

$$A_2 = \limsup_{n \rightarrow \infty} \sup_{|\psi(z)| \leq \delta_N} \omega(z)|\xi(z)| |(f - f_{\delta_n})(\psi(z))|.$$

Similar to Theorem 5, for $\xi \in \mathfrak{A}_\omega$ and the fact that f_{δ_n} converges uniformly to f on compact subsets of $\mathbb{D}$ as $n \rightarrow \infty$, we obtain

$$A_1 \leq \|\xi\|_{\mathfrak{A}_\omega} \limsup_{n \rightarrow \infty} \sup_{w \leq \delta_N} |f(w) - f(\delta_n w)| = 0. \quad (69)$$

Next, we consider A_2 . We have $A_2 \leq \limsup_{n \rightarrow \infty} (R_1 + R_2)$, where

$$\begin{aligned} R_1 &= \sup_{|\psi(z)| > \delta_N} \omega(z)|\xi(z)| |f(\psi(z))|, \\ R_2 &= \sup_{|\psi(z)| > \delta_N} \omega(z)|\xi(z)| |f(\delta_n \psi(z))|. \end{aligned} \quad (70)$$

On calculation, we obtain

$$\begin{aligned} R_1 &= \sup_{|\psi(z)| > \delta_N} \omega(z)|\xi(z)| |f(\psi(z))| \\ &\leq \frac{1}{\delta_N} \|f\|_{\mathfrak{D}_K} \sup_{|\psi(z)| > \delta_N} \omega(z)|\xi(z)| \frac{1}{\sqrt{K(1-|\psi(z)|^2)}(1-|\psi(z)|^2)}. \end{aligned} \quad (71)$$

Taking $N \rightarrow \infty$, we obtain

$$\limsup_{n \rightarrow \infty} R_1 \leq \limsup_{|\psi(z)| \rightarrow 1^-} \frac{\omega(z)|\xi(z)|}{\sqrt{K(1-|\psi(z)|^2)}(1-|\psi(z)|^2)}. \quad (72)$$

Similarly,

$$\limsup_{n \rightarrow \infty} R_2 \leq \limsup_{|\psi(z)| \rightarrow 1^-} \frac{\omega(z)|\xi(z)|}{\sqrt{K(1-|\psi(z)|^2)}(1-|\psi(z)|^2)}. \quad (73)$$

On combining the above two inequalities, we obtain

$$A_2 \leq \limsup_{n \rightarrow \infty} (R_1 + R_2) \leq \limsup_{|\psi(z)| \rightarrow 1^-} \frac{\omega(z)|\xi(z)|}{\sqrt{K(1-|\psi(z)|^2)}(1-|\psi(z)|^2)}. \quad (74)$$

From inequalities (67), (69), and (74), we obtain

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \|\mathcal{W}_{\xi, \psi} - \mathcal{W}_{\xi, \psi} E_{\delta_n}\|_{\mathfrak{D}_K \rightarrow \mathfrak{A}_\omega} &= \limsup_{n \rightarrow \infty} \sup_{\|f\|_{\mathfrak{D}_K} \leq 1} \|(\mathcal{W}_{\xi, \psi} - \mathcal{W}_{\xi, \psi} E_{\delta_n})f\|_{\mathfrak{A}_\omega} \\
&= \limsup_{n \rightarrow \infty} \sup_{\|f\|_{\mathfrak{D}_K} \leq 1} \|\xi(f - f_{\delta_n}) \circ \psi\|_{\mathfrak{A}_\omega} \\
&\leq \limsup_{|\psi(z)| \rightarrow 1^-} \frac{\omega(z)|\xi(z)|}{\sqrt{K(1-|\psi(z)|^2)(1-|\psi(z)|^2)}}.
\end{aligned} \tag{75}$$

Thus, inequalities (64) and (75) imply that

$$\|\mathcal{W}_{\xi, \psi}\|_{e, \mathfrak{D}_K \rightarrow \mathfrak{A}_\omega} \leq \limsup_{|\psi(z)| \rightarrow 1^-} \frac{\omega(z)|\xi(z)|}{\sqrt{K(1-|\psi(z)|^2)(1-|\psi(z)|^2)}}. \tag{76}$$

Hence, inequalities (62) and (76) complete the theorem.

The following corollary can be easily obtained from Theorem 6. $\square$

Corollary 2. Let ω be a standard weight and ψ be a self-analytic map on $\mathbb{D}$. Let K be a weight function. Assume that the operator $C_\psi: \mathfrak{D}_K \rightarrow \mathfrak{A}_\omega$ is bounded. Then,

$$\|C_\psi\|_{e, \mathfrak{D}_K \rightarrow \mathfrak{A}_\omega} \approx \limsup_{|\psi(z)| \rightarrow 1^-} \frac{\omega(z)}{\sqrt{K(1-|\psi(z)|^2)(1-|\psi(z)|^2)}}. \tag{77}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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