Research Article

A General Scheme for Solving Systems of Linear First-Order Differential Equations Based on the Differential Transform Method

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In this study, we develop the differential transform method in a new scheme to solve systems of first-order differential equations. The differential transform method is a procedure to obtain the coefficients of the Taylor series of the solution of differential and integral equations. So, one can obtain the Taylor series of the solution of an arbitrary order, and hence, the solution of the given equation can be obtained with required accuracy. Here, we first give some basic definitions and properties of the differential transform method, and then, we prove some theorems for solving the linear systems of first order. Then, these theorems of our system are converted to a system of linear algebraic equations whose unknowns are the coefficients of the Taylor series of the solution. Finally, we give some examples to show the accuracy and efficiency of the presented method.

1. Introduction

The differential transform was first introduced by Zhou [1], and up to now, the DT method has been developed for solving various kinds of differential and integral equations in many literatures. For example, Ali [2] has developed the DT method for solving partial differential equations and Ayaz [3, 4] has applied this method to differential algebraic equations. Arikoglu and Ozkol [5] have solved the integrodifferential equations with boundary value conditions by the DT method. Odibat [6] has used the DT method for solving Volterra integral equations with separable kernels. Tari and Ziyade [7] have solved the system of two-dimensional nonlinear Volterra integrodifferential equations by the DT method. The systems of integral and integrodifferential equations, the multiorder fractional differential equations, the system of fractional differential equations, the singularly perturbed Volterra integral equations, and the time-fractional diffusion equation have been solved by the

DT method in [2, 6, 8–10]. Also, the DT method has been applied to nonlinear parabolic-hyperbolic partial differential equations, and a modified approach of DT has been developed to nonlinear partial differential equations in studies by Biazarand Abdul Halim-Haasan [8]. Patil and Kembayat [11] have solved the two-dimensional Fredholm integral equations. Abdewahid [12] introduced a new basic formula s for the one-dimensional differential transform. The main aim of this work is to introduce new useful algorithms depending on the DT method to solve systems of linear differential equations.

2. Analysis of Differential Transform

The basic definition and the fundamental theorems of the DTM and its applicability for various kinds of differential equations are given in [13–20]. For the convenience of the reader, we will present a review of the DTM. To do this, we assume that $f(x) \in C^\infty(I)$; then, for any point

$x_0 \in I$, the Taylor series of $f(x)$ about x_0 can be represented by

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k. \quad (1)$$

Definition 1. Let $f(x)$ be an analytic function about x_0 ; then, the k^{th} order differential transform of $f(x)$ is defined as

$$\text{DT}\{f(x)\} = F(k) = \frac{f^{(k)}(x_0)}{k!}. \quad (2)$$

From equations (1) and (2), we get

$$f(x) = \sum_{k=0}^{\infty} F(k) (x - x_0)^k, \quad (3)$$

which implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically. However, the relative derivatives are calculated by an iterative way which are described by the transformed equations of the original function. In real applications, the function $f(x)$ is expressed by a finite series, and equation (3) can be written as

$$f(x) = \sum_{k=0}^n \frac{d^k f(x_0)}{k! dx^k} (x - x_0)^k, \quad (4)$$

where n is decided by the convergence of natural frequency. The fundamental operations performed by differential transform can be readily obtained and are given.

Theorem 1. Let $U(k)$ and $V(k)$ be the differential transformations of the functions $u(x)$ and $v(x)$, respectively; then, we have the following properties.

Let $U(k)$ and $V(k)$ be the differential transformations of the functions $u(x)$ and $v(x)$, respectively; then, we have for the constants α and β the following properties:

- (1) If $f(x) = \alpha u(x) \pm \beta v(x)$, then $F(k) = \alpha U(k) \pm \beta V(k)$
- (2) If $f(x) = u(x) \cdot v(x)$, then $F(k) = \sum_{l=0}^k U(l) V(k-l)$
- (3) If $f(x) = (du(x)/dx)$, then $F(k) = (k+1)U(k+1)$
- (4) If $f(x) = (d^m u(x)/dx^m)$, then $F(k) = (k+1)(k+2)\dots(k+m)U(k+m)$
- (5) If $f(x) = \int_{x_0}^x u(t)dt$, then $F(k) = (U(k-1)/k)$, $k \geq 1$, $f(0) = 0$
- (6) If $f(x) = x^m$, then $F(k) = \delta(k-m)$
- (7) If $f(x) = e^{\alpha x}$, then $F(k) = (\alpha^k/k!)$
- (8) If $f(x) = \sin(\omega x + \alpha)$, then $F(k) = (\omega^k/k!) \sin((k\pi/2) + \alpha)$
- (9) If $f(x) = \cos(\omega x + \alpha)$, then $F(k) = (\omega^k/k!) \cos((k\pi/2) + \alpha)$
- (10) If $f(x) = x^m f^{(n)}(x)$, then $F(k) = \sum_{i=0}^k \delta_{im} ((k+n-i)!/(k-i)!) F(k+n-i)$
- (11) If $f(x) = x^n f^{(n)}(x)$, then $F(k) = \prod_{i=0}^{n-1} (k-i) F(k)$

$$(12) \text{ If } f(x) = e^{\alpha x} f^{(n)}(x), \text{ then } F(k) = \sum_{i=0}^k (\alpha^i (k+n-i)!/i! (k-i)!) F(k+n-i)$$

$$(13) \text{ If } f(x) = \{ (x f^{(n)}(x))' \}, \text{ then } F(k) = ((k+1)(k+n)!/k!) F(k+n)$$

$$(14) \text{ If } f(x) = \{ (x^m f^{(n)}(x))' \}, \text{ then } F(k) = ((k+1)(k+n-m+i)!/(k-m+i)!) F(k+n-m+i)$$

The proof of this theorem can be found in [2-5, 8, 10, 12-15, 20].

3. Description of the Method

In the first part of this section, we introduce a general algorithm depending on the differential transform method to solve systems of n linear differential equations with constant coefficients. The systems are assumed to be autonomous, which means that the independent variable t is not present explicitly. Such a system has the appearance

$$\begin{aligned} X' &= AX, \\ X(0) &= X_0, \end{aligned} \quad (5)$$

where A is an $n \times n$ matrix given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad (6)$$

$$X' = (x'_1 \ x'_2 \ \cdots \ x'_n)^T,$$

$$X = (x_1 \ x_2 \ \cdots \ x_n).$$

Theorem 2. Let $X(k)$ be the k^{th} differential transform of $X(t)$ and A^k be the k^{th} power of the matrix A . Then, the solution of the system given by (5) can be expressed as

$$X(t) = \sum_{k=0}^{\infty} X(k) t^k = \sum_{k=0}^{\infty} \frac{A^k}{k!} X(0) t^k. \quad (7)$$

Proof. By using the differential transform property (3) of Theorem 1, we get

$$(k+1)X(k+1) = AX(k), \quad k = 0, 1, 2, \dots \quad (8)$$

From this equation, we have the following recurrence relation:

$$X(k+1) = \frac{A}{(k+1)} X(k). \quad (9)$$

Then, for $k = 0, 1, 2, \dots$, we get

$$\begin{aligned} X(1) &= \frac{A}{1!} X(0), \quad X(2) = \frac{A}{2} X(1) = \frac{A^2}{2!} X(0), \\ X(3) &= \frac{A}{3} X(2) = \frac{A^3}{3!} X(0), \quad X(4) = \frac{A}{4} X(3) = \frac{A^4}{4!} X(0), \end{aligned} \quad (10)$$

for $k = j$, we get

$$X(j) = \frac{A^j}{j!} X(0), \quad (11)$$

where A^j is the power the matrix A ; then, the solution of the system is given by

$$X(t) = \sum_{k=0}^{\infty} X(k)t^k = \sum_{k=0}^{\infty} \frac{A^k}{k!} X(0)t^k. \quad (12)$$

Now, the proof is completed. $\square$

The principles that have been established can now be applied to the nonhomogeneous problem

$$X' = AX + W, X(0), \text{prescribed}, \quad (13)$$

where W can be a vector function of t .

Theorem 3. Let $X(k)$ and $W(k)$ be the k^{th} differential transforms of $X(t)$ and $W(t)$, respectively, and A^k be the k^{th}

power of the matrix A . Then, the solution of the system given by (13) can be expressed as

$$X(t) = \sum_{k=0}^{\infty} \left\{ \frac{1}{k!} A^k X(0) \right\} t^k + \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{k-1} \frac{(j)!}{(k)!} A^{k-1-j} W(j) \right\} t^k. \quad (14)$$

Proof. By taking the k^{th} differential transform of both sides of (13), we get

$$DT\{X'\} = DT\{AX\} + DT\{W\}. \quad (15)$$

According to Theorem 1 and the operations of differential transform, we have the following recurrence relation:

$$X(k+1) = \frac{1}{k+1} \{AX(k) + W(k)\}, \quad k = 0, 1, 2, \dots \quad (16)$$

Consequently,

$$\begin{aligned} X(1) &= \frac{1}{1!} \{AX(0) + W(0)\}, W(0) = W(t)_{t=0}, \\ X(2) &= \frac{1}{2!} \{AX(1) + W(1)\} = \frac{1}{2!} A^2 X(0) + \frac{1}{2!} AW(0) + \frac{1}{2!} A^0 W(1), W(1) = \frac{1}{1!} \left\{ \frac{dW(t)}{dt} \right\}_{t=0}, \\ X(3) &= \frac{1}{3!} \{AX(2) + W(2)\} = \frac{1}{3!} A^3 X(0) + \frac{1}{3!} A^2 W(0) + \frac{1}{3!} AW(1) \\ &\quad + \frac{1}{3!} A^0 W(2), W(3) = \frac{1}{3!} \left\{ \frac{d^3 W(t)}{dt^3} \right\}_{t=0}, \\ X(4) &= \frac{1}{4!} \{AX(3) + W(3)\} = \frac{1}{4!} A^4 X(0) + \frac{1}{4!} A^3 W(0) + \frac{1}{4!} A^2 W(1) + \frac{2}{4!} AW(2) + \frac{3!}{4!} A^0 W(3), \\ X(5) &= \frac{1}{5!} \{AX(4) + W(4)\} = \frac{1}{5!} A^5 X(0) + \frac{1}{5!} A^4 W(0) \\ &\quad + \frac{1}{5!} A^3 W(1) + \frac{2}{5!} A^2 W(2) + \frac{3!}{5!} AW(3) + \frac{4!}{5!} A^0 W(4), \\ W(4) &= \frac{1}{4!} \left\{ \frac{d^4 W(t)}{dt^4} \right\}_{t=0}. \end{aligned} \quad (17)$$

Therefore, this leads to the general form of $X(k)$, where

$$X(k) = \frac{1}{(k)!} A^k X(0) + \sum_{j=0}^{k-1} \frac{(j)!}{(k)!} A^{k-1-j} W(j). \quad (18)$$

Then, the general solution of the system is given by

$$X(t) = \sum_{k=0}^{\infty} \left\{ \frac{1}{k!} A^k X(0) \right\} t^k + \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{k-1} \frac{(j)!}{(k)!} A^{k-1-j} W(j) \right\} t^k. \quad (19)$$

Now, the proof is completed. $\square$

4. Applications and Numerical Results

In order to illustrate the advantages and the accuracy of the results given by Theorems 2 and 3 for solving homogeneous and nonhomogeneous problems, we have applied the method to the following examples.

Example 1. Consider the following system of homogeneous differential equations.

$$\begin{aligned}x_1' &= x_1 + x_2, & x_1(0) &= 5, \\x_2' &= 0, & x_2(0) &= 7, \\x_3' &= -x_3, & x_3(0) &= 6.\end{aligned}\quad (20)$$

This system can be written in the matrix form as

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad (21)$$

or

$$\begin{aligned}X' &= AX, \\A &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\X(0) &= \begin{bmatrix} 5 \\ 7 \\ 6 \end{bmatrix}.\end{aligned}\quad (22)$$

Applying the result of Theorem 2, we get

$$\begin{aligned}X(t) &= \sum_{k=0}^{\infty} \frac{A^k X(0)}{k!} t^k = \begin{bmatrix} 5 \\ 7 \\ 6 \end{bmatrix} + \begin{bmatrix} 11 \\ 0 \\ -6 \end{bmatrix} t \\&+ \frac{1}{2!} \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix} t^2 + \frac{1}{3!} \begin{bmatrix} 11 \\ 0 \\ -6 \end{bmatrix} t^3 + \frac{1}{4!} \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix} t^4 \\&+ s + \frac{1}{5!} \begin{bmatrix} 11 \\ 0 \\ -6 \end{bmatrix} t^5 + \frac{1}{6!} \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix} t^6 + \frac{1}{7!} \begin{bmatrix} 11 \\ 0 \\ -6 \end{bmatrix} t^7 + \dots\end{aligned}\quad (23)$$

Then, the solutions are given by

$$\begin{aligned}x_1(t) &= 5 + 11t + \frac{5}{2}t^2 + \frac{11}{3!}t^3 + \frac{5}{4!}t^4 + \frac{11}{5!}t^5 + \frac{5}{6!}t^6 + \frac{11}{7!}t^7 + \dots \\&= 8e^t - 3e^{-t}, \\x_2(t) &= 7, \\x_3(t) &= 6(1-t) + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4 - \frac{1}{5!}t^5 + \frac{1}{6!}t^6 - \frac{1}{7!}t^7 + \dots \\&= 6e^{-t},\end{aligned}\quad (24)$$

which is the exact solution of the given problem.

Example 2. Consider the system

$$X' = AX, \quad \text{where } A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad X(0) = \begin{bmatrix} 7 \\ 5 \\ 3 \\ 9 \end{bmatrix}. \quad (25)$$

Applying the result of Theorem 2, consequently, the solutions of the given system are given as

$$\begin{aligned}X(t) &= \begin{bmatrix} 7 \\ 5 \\ 3 \\ 9 \end{bmatrix} + \begin{bmatrix} 26 \\ 18 \\ 18 \\ 27 \end{bmatrix} t + \begin{bmatrix} 48 \\ 36 \\ \frac{81}{2} \\ \frac{81}{2} \end{bmatrix} t^2 + \begin{bmatrix} 60 \\ \frac{99}{2} \\ 54 \\ \frac{81}{2} \end{bmatrix} t^3 \\&+ \begin{bmatrix} \frac{459}{8} \\ \frac{405}{8} \\ \frac{405}{8} \\ \frac{243}{8} \end{bmatrix} t^4 + \begin{bmatrix} \frac{891}{20} \\ \frac{81}{2} \\ \frac{729}{20} \\ \frac{729}{40} \end{bmatrix} t^5 + \begin{bmatrix} \frac{1161}{40} \\ \frac{1053}{40} \\ \frac{1701}{80} \\ \frac{729}{80} \end{bmatrix} t^6 + \begin{bmatrix} \frac{81}{5} \\ \frac{8019}{560} \\ \frac{729}{70} \\ \frac{2187}{560} \end{bmatrix} t^7 + \dots.\end{aligned}\quad (26)$$

Then, the solutions of the given system are

$$\begin{aligned}
x_1(t) &= 7 + 26t + 48t^2 + 60t^3 + \frac{459}{8}t^4 + \frac{891}{20}t^5 + \frac{1161}{40}t^6 + \frac{81}{5}t^7 + \dots, \\
x_2(t) &= 5 + 18t + 36t^2 + \frac{99}{2}t^3 + \frac{405}{8}t^4 + \frac{81}{2}t^5 + \frac{1053}{40}t^6 + \frac{8019}{560}t^7 + \dots, \\
x_3(t) &= 3 + 18t + \frac{81}{2}t^2 + 54t^3 + \frac{405}{8}t^4 + \frac{729}{20}t^5 + \frac{1701}{80}t^6 + \frac{729}{70}t^7 + \dots, \\
x_4(t) &= 9 + 27t + \frac{81}{2}t^2 + \frac{81}{2}t^3 + \frac{243}{8}t^4 + \frac{729}{40}t^5 + \frac{729}{80}t^6 + \frac{2187}{560}t^7 + \dots.
\end{aligned} \tag{27}$$

The solutions can be written in matrix form as

$$\begin{aligned}
X(t) &= e^{3t} \begin{bmatrix} 7 \\ 5 \\ 3 \\ 9 \end{bmatrix} + te^{3t} \begin{bmatrix} 5 \\ 3 \\ 9 \\ 0 \end{bmatrix} + \frac{1}{2}t^2e^{3t} \begin{bmatrix} 3 \\ 9 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{6}t^3e^{3t} \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
X(t) &= \begin{bmatrix} 7 + 5t + 1.5t^2 + 1.5t^3 \\ 5 + 3t + 4.5t^2 \\ 3 + 9t \\ 9 \end{bmatrix} e^{3t},
\end{aligned} \tag{28}$$

which is the exact solution of the given problem.

Now, we will give some examples of nonhomogeneous initial value problems.

Example 3. Consider the following system:

$$\begin{aligned}
x_1' &= t^2, & x_1(0) &= 5, \\
x_2' &= x_2 + t, & x_2(0) &= 7, \\
x_3' &= 2x_3 + \sin t, & x_3(0) &= 9.
\end{aligned} \tag{29}$$

The exact solutions are obtained by [17] as follows

$$\begin{aligned}
x_1(t) &= 5 + \frac{1}{3}t^3, \\
x_2(t) &= \frac{1}{8}e^t - t - 1, \\
x_3(t) &= \frac{48}{5}e^{2t} - \frac{2}{5}\sin t - \frac{1}{5}\cos t.
\end{aligned} \tag{30}$$

To find the solutions of the given system applying the results in Theorem 3, the approximate solutions up to $k=6$ are given by

$$\begin{aligned}
X(t) &= X(0) + [AX(0) + W(0)]t + \frac{1}{2!} [A^2X(0) + AW(0) + W(1)]t^2 \\
&+ \frac{1}{3!} [A^3X(0) + A^2W(0) + AW(1) + 2W(2)]t^3 + \\
&+ \frac{1}{4!} [A^4X(0) + A^3W(0) + A^2W(1) + 2AW(2) + 6A^0W(3)]t^4 \\
&+ \frac{1}{5!} [A^5X(0) + A^4W(0) + A^3W(1) + 2A^2W(2) + 6AW(3) + 24A^0W(4)]t^5 \\
&+ \frac{1}{6!} [A^6X(0) + A^5W(0) + A^4W(1) + 2A^3W(2) + 6A^2W(3) + 24AW(4) + 120A^0W(5)]t^6.
\end{aligned} \tag{31}$$

Therefore,

$$\begin{aligned}
 X(t) &= \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} + \begin{bmatrix} 0 \\ 7 \\ 18 \end{bmatrix} t + \frac{1}{2!} \left[\begin{bmatrix} 0 \\ 7 \\ 36 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right] t^2 \\
 &\quad + \frac{1}{3!} \left[\begin{bmatrix} 0 \\ 7 \\ 72 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right] t^3 \\
 &\quad + \frac{1}{4!} \left[\begin{bmatrix} 0 \\ 7 \\ 144 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right] t^4 \\
 &\quad + \frac{1}{5!} \left[\begin{bmatrix} 0 \\ 7 \\ 288 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right] t^5 \\
 &\quad + \frac{1}{6!} \left[\begin{bmatrix} 0 \\ 7 \\ 576 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 16 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right] t^6 \\
 X(t) &= \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} + \begin{bmatrix} 0 \\ 7 \\ 18 \end{bmatrix} t + \frac{1}{2!} \begin{bmatrix} 0 \\ 8 \\ 37 \end{bmatrix} t^2 + \frac{1}{3!} \begin{bmatrix} 2 \\ 8 \\ 74 \end{bmatrix} t^3 \\
 &\quad + \frac{1}{4!} \begin{bmatrix} 0 \\ 8 \\ 147 \end{bmatrix} t^4 + \frac{1}{5!} \begin{bmatrix} 0 \\ 8 \\ 295 \end{bmatrix} t^5 + \frac{1}{6!} \begin{bmatrix} 0 \\ 8 \\ \frac{2945}{5} \end{bmatrix} t^6 + \dots
 \end{aligned} \tag{32}$$

Then, the solutions are

$$x_1(t) = 5 + \frac{1}{3}t^3,$$

$$\begin{aligned}
 x_2(t) &= 7 + 7t + 8 \left(\frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5 + \frac{1}{6!}t^6 + \dots \right) \\
 &= 8e^t - t - 1,
 \end{aligned}$$

$$\begin{aligned}
 x_3(t) &= 9 + 18t + \frac{37}{2!}t^2 + \frac{74}{3!}t^3 + \frac{147}{4!}t^4 + \frac{294}{5!}t^5 + \frac{589}{6!}t^6 \\
 &= \frac{46}{5}e^{2t} - \frac{2}{3}\sin t - \frac{1}{5}\cos t,
 \end{aligned}$$

(33)

which is the exact solution of the given problem.

Example 4. Consider the following system:

$$\begin{aligned}
 x_1' &= 5x_1 + x_2 + t^2, & x_1(0) &= 5, \\
 x_2' &= 5x_2 + x_3 + t, & x_2(0) &= 7, \\
 x_3' &= 5x_3 + \sin t, & x_3(0) &= 9.
 \end{aligned} \tag{34}$$

The exact solution of this system is given by [17] as follows:

$$\begin{aligned}
 x_1 &= \left(5 + \frac{74}{26^3} \right) e^{5t} + \left(7 + \frac{1}{25} + \frac{10}{26^2} \right) t e^{5t} \\
 &\quad + \frac{1}{2} \left(9 + \frac{1}{26} \right) t^2 e^{5t} - \frac{2}{26^3} (55 \sin t + 37 \cos t) - \frac{1}{5}t^2 - \frac{1}{25}t, \\
 x_2 &= \left(7 + \frac{1}{25} - \frac{10}{26^2} \right) e^{5t} + \frac{1}{2} \left(9 + \frac{1}{26} \right) t e^{5t} \\
 &\quad + \frac{2}{26^2} (12 \sin t + t \cos t) - \frac{1}{5}t - \frac{1}{25}, \\
 x_3 &= \left(9 + \frac{1}{26} \right) e^{5t} - \frac{5}{26} \sin t - \frac{1}{26} \cos t.
 \end{aligned} \tag{35}$$

To find the approximate solutions of the given system, applying the results of Theorem 3, we have

$$\begin{aligned}
X(t) &= X(0) + [AX(0) + W(0)]t + \frac{1}{2!} [A^2X(0) + AW(0) + W(1)]t^2 \\
&+ \frac{1}{3!} [A^3X(0) + A^2W(0) + AW(1) + 2W(2)]t^3 + \\
&+ \frac{1}{4!} [A^4X(0) + A^3W(0) + A^2W(1) + 2AW(2) + 6A^0W(3)]t^4 \\
&+ \frac{1}{5!} [A^5X(0) + A^4W(0) + A^3W(1) + 2A^2W(2) + 6AW(3) + 24A^0W(4)]t^5 \\
&+ \frac{1}{6!} [A^6X(0) + A^5W(0) + A^4W(1) + 2A^3W(2) + 6A^2W(3) + 24AW(4) + 120A^0W(5)]t^6.
\end{aligned} \tag{36}$$

Therefore,

$$\begin{aligned}
X(t) &= \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} + \begin{bmatrix} 32 \\ 44 \\ 45 \end{bmatrix} t + \frac{1}{2!} \begin{bmatrix} 204 \\ 265 \\ 225 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t^2 \\
&+ \frac{1}{3!} \begin{bmatrix} 1285 \\ 1550 \\ 1125 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} t^3 \\
&\frac{1}{4!} \begin{bmatrix} 7975 \\ 8875 \\ 5625 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 11 \\ 35 \\ 25 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} t^4 \\
&+ \frac{1}{5!} \begin{bmatrix} 48750 \\ 50000 \\ 28125 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 90 \\ 200 \\ 125 \end{bmatrix} + \begin{bmatrix} 50 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ -5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} t^5 \\
&+ \frac{1}{6!} \begin{bmatrix} 293750 \\ 278125 \\ 140625 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 650 \\ 1125 \\ 625 \end{bmatrix} + \begin{bmatrix} 250 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -10 \\ -25 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^6, \\
X(t) &= \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} + \begin{bmatrix} 32 \\ 44 \\ 45 \end{bmatrix} t + \frac{1}{2!} \begin{bmatrix} 204 \\ 266 \\ 226 \end{bmatrix} t^2
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{3!} \begin{bmatrix} 1288 \\ 1556 \\ 1130 \end{bmatrix} t^3 + \frac{1}{4!} \begin{bmatrix} 7996 \\ 8910 \\ 5649 \end{bmatrix} t^4 + \frac{1}{5!} \begin{bmatrix} 48890 \\ 50199 \\ 28245 \end{bmatrix} t^5 \\
& + \frac{1}{6!} \begin{bmatrix} 293750 \\ 278125 \\ 140625 \end{bmatrix} t^6 + \dots.
\end{aligned} \tag{37}$$

Then, the solution of the given system is

$$\begin{aligned}
x_1(t) &= 5 + 32t + \frac{204}{2!}t^2 + \frac{1288}{3!}t^3 + \frac{7996}{4!}t^4 + \frac{48890}{5!}t^5 + \frac{293750}{6!}t^6 + \dots, \\
x_2(t) &= 7 + 44t + \frac{266}{2!}t^2 + \frac{1556}{3!}t^3 + \frac{8910}{4!}t^4 + \frac{50199}{5!}t^5 + \frac{278125}{6!}t^6 + \dots, \\
x_3(t) &= 9 + 45t + \frac{226}{2!}t^2 + \frac{1127}{3!}t^3 + \frac{5649}{4!}t^4 + \frac{28245}{5!}t^5 + \frac{140625}{6!}t^6 + \dots,
\end{aligned} \tag{38}$$

which are the exact solutions of the given system.

5. Conclusions

In this work, we successfully apply the differential transform method to find useful formulas for solving homogeneous or nonhomogeneous systems of n first-order differential equations. The present methods reduce the computational difficulties of the traditional methods, and all the calculations can be made simple manipulations. Several examples were tested by applying the results of Theorems 2 and 3, and the results have shown a remarkable performance.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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