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Research Article

Certain Properties of the Modified Degenerate Gamma Function

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In this paper, we prove some inequalities satisfied by the modified degenerate gamma function which was recently introduced. The tools employed include Holder's inequality, mean value theorem, Hermite–Hadamard's inequality, and Young's inequality. By some parameter variations, the established results reduce to the corresponding results for the classical gamma function.

1. Introduction

In recent times, degenerate special functions and polynomials have been a subject of intense discussion. See, for example, [1–5] and the related references therein.

In 2017, Kim and Kim [6] introduced the degenerate gamma function as

$$\Gamma_\lambda(x) = \int_0^\infty t^{x-1}(1+\lambda t)^{-1/\lambda}\mathrm{d}t, \tag{1}$$

where $\lambda \in (0,\infty)$ and $0 < \Re(x) < 1/\lambda$. This was motivated by the degenerate exponential function which is defined as [6]

$$e_\lambda^t = (1+\lambda t)^{1/\lambda}, \tag{2}$$

where $\lambda \in (0,\infty)$. It is clear that $\lim_{\lambda\longrightarrow 0} e_\lambda^t = e^t$ and $\lim_{\lambda\longrightarrow 0}\Gamma_\lambda(x) = \Gamma(x)$, where $\Gamma(x)$ is the classical gamma function.

In 2018, Kim et al. [7] introduced the modified degenerate gamma function which is defined as

$$\Gamma_\lambda(x) = \int_0^\infty t^{x-1}(1+\lambda)^{-1/\lambda}\mathrm{d}t, \tag{3}$$

where $\lambda \in (0,1)$ and $\Re(x) > 0$. This definition is equivalent to

$$\Gamma_k(x) = \int_0^\infty t^{x-1}\left(1+\frac{1}{k}\right)^{-kt}\mathrm{d}t, \tag{4}$$

where $1 < k < \infty$ and $\Re(x) > 0$. Here, $\lim_{k\longrightarrow\infty}\Gamma_k(x) = \Gamma(x)$. The modified degenerate gamma function (3) satisfies the following properties [7]:

$$\Gamma_\lambda(1) = \frac{\lambda}{\ln(1+\lambda)}, \tag{5}$$

$$\Gamma_\lambda(x+1) = \frac{\lambda x}{\ln(1+\lambda)}\Gamma_\lambda(x), \tag{6}$$

$$\Gamma_\lambda(m+1) = \frac{\lambda^{m+1} m!}{(\ln(1+\lambda))^{m+1}}, \quad m \in \mathbb{N}. \tag{7}$$

Derivatives of the modified degenerate gamma function are given as

$$\Gamma_\lambda^{(r)}(x) = \int_0^\infty (\ln t)^r t^{x-1}(1+\lambda)^{-t/\lambda}\mathrm{d}t, \tag{8}$$

where $r \in \mathbb{N}_0$.

In a recent work, He et al. [8] introduced the modified degenerate digamma function which is defined as

$$\psi_\lambda(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma_\lambda(x) = \frac{\Gamma_\lambda'(x)}{\Gamma_\lambda(x)} \tag{9}$$

and has the following representations among others:

$$\psi_\lambda(x) = -\gamma + \ln\left(\frac{\lambda}{\ln(1+\lambda)}\right) + \sum_{k=0}^{\infty}\frac{x-1}{(k+1)(k+x)} = -\gamma + \ln\left(\frac{\lambda}{\ln(1+\lambda)}\right) - \frac{1}{x} + \sum_{k=1}^{\infty}\frac{x}{k(k+x)}$$
$$= -\gamma + \ln\left(\frac{\lambda}{\ln(1+\lambda)}\right) + \int_0^1 \frac{1-t^{x-1}}{1-t}\mathrm{d}t, \tag{10}$$

where γ is the Euler–Mascheroni constant. It also satisfies the following basic properties:

$$\psi_\lambda(1) = -\gamma + \ln\left(\frac{\lambda}{\ln(1+\lambda)}\right),$$
$$\psi_\lambda(x+1) = \psi_\lambda(x) + \frac{1}{x}, \tag{11}$$

and similarly, it is clear that $\lim_{\lambda\longrightarrow 0}\psi_\lambda(x) = \psi(x)$, where $\psi(x)$ is the classical digamma function. For further properties of the function $\psi_\lambda(x)$, one may refer to [8].

In this paper, we continue to investigate the modified degenerate gamma function. Precisely, we prove some inequalities satisfied by this generalized function. The techniques we employed are analytical in nature.

2. Results and Discussion

Theorem 1. *For $s \in (0,1]$ and $x > 0$, the inequality holds:*

$$\left(\frac{\lambda x}{\ln(1+\lambda)}\right)^{1-s} \leq \frac{\Gamma_\lambda(x+1)}{\Gamma_\lambda(x+s)} \leq \left(\frac{\lambda(x+s)}{\ln(1+\lambda)}\right)^{1-s}. \tag{12}$$

Proof. The case for $s = 1$ is obvious. So, let $s \in (0,1)$ and $x > 0$. Then, by applying Holder's inequality for integrals, we have

$$\Gamma_\lambda(x+s) = \int_0^\infty t^{x+s-1}(1+\lambda)^{-(t/\lambda)}\mathrm{d}t = \int_0^\infty t^{(1-s)(x-1)}(1+\lambda)^{-(t(1-s)/\lambda)}t^{sx}(1+\lambda)^{-(st/\lambda)}\mathrm{d}t$$
$$\leq \left(\int_0^\infty t^{x-1}(1+\lambda)^{-(t/\lambda)}\mathrm{d}t\right)^{1-s}\left(\int_0^\infty t^{x}(1+\lambda)^{-(t/\lambda)}\mathrm{d}t\right)^{s} = [\Gamma_\lambda(x)]^{1-s}[\Gamma_\lambda(x+1)]^{s}, \tag{13}$$

and by using (6), we obtain

$$\Gamma_\lambda(x+s) \leq \left(\frac{\lambda x}{\ln(1+\lambda)}\right)^{s}\Gamma_\lambda(x). \tag{14}$$

By replacing s with $1-s$ in (14), followed by substituting x by $x+s$, we obtain

$$\Gamma_\lambda(x+1) \leq \left(\frac{\lambda(x+s)}{\ln(1+\lambda)}\right)^{1-s}\Gamma_\lambda(x+s). \tag{15}$$

Now, combining (14) and (15), we obtain

$$\left(\frac{\ln(1+\lambda)}{\lambda(x+s)}\right)^{1-s}\Gamma_\lambda(x+1) \leq \Gamma_\lambda(x+s) \leq \left(\frac{\lambda x}{\ln(1+\lambda)}\right)^{s}\Gamma_\lambda(x), \tag{16}$$

and by using (6), we obtain the desired results (12). □

Remark 1. Inequality (16) can also be rearranged as

$$\left(\frac{x}{x+s}\right)^{1-s} \leq \frac{\Gamma_\lambda(x+s)}{(\lambda x/\ln(1+\lambda))^{s}\Gamma_\lambda(x)} \leq 1, \tag{17}$$

which is the degenerate form of Wendel's inequality (see (7) of [9]). Furthermore, by Squeezes theorem, (17) implies that

$$\lim_{x\longrightarrow\infty}\frac{\Gamma_\lambda(x+s)}{(\lambda x/\ln(1+\lambda))^{s}\Gamma_\lambda(x)} = 1, \tag{18}$$

which is the degenerate form of Wendel's asymptotic relation (see (1) of [9]). The limit (18) also implies that

$$\lim_{x\longrightarrow\infty}\left(\frac{\lambda x}{\ln(1+\lambda)}\right)^{r-s}\frac{\Gamma_\lambda(x+s)}{\Gamma_\lambda(x+r)} = 1. \tag{19}$$

Theorem 2. *For $0 < u \leq v$, the inequality holds:*

$$\exp\{(v-u)\psi_\lambda(u)\} \leq \frac{\Gamma_\lambda(v)}{\Gamma_\lambda(u)} \leq \exp\{(v-u)\psi_\lambda(v)\}. \tag{20}$$

Proof. The case for $u = v$ is trivial. So, consider the function $\ln\Gamma_\lambda(x)$ on the interval $0 < u < v$. Then, by the mean value theorem, there exist a $k \in (u,v)$ such that

$$\frac{\ln\Gamma_\lambda(v) - \ln\Gamma_\lambda(u)}{v-u} = \psi_\lambda(k). \tag{21}$$

Since $\psi_\lambda(x)$ is increasing, then

$$\psi_\lambda(u) < \psi_\lambda(k) < \psi_\lambda(v), \tag{22}$$

which yields

$$(v-u)\psi_\lambda(u) < \ln\frac{\Gamma_\lambda(v)}{\Gamma_\lambda(u)} < (v-u)\psi_\lambda(v), \tag{23}$$

and by exponentiation, we obtain the desired result (20). □

Corollary 1. *For $s \in (0,1]$ and $x > 0$, the inequality holds:*

$$\exp\{(1-s)\psi_\lambda(x+s)\} \leq \frac{\Gamma_\lambda(x+1)}{\Gamma_\lambda(x+s)} \leq \exp\{(1-s)\psi_\lambda(x+1)\}. \tag{24}$$

Proof. Let $v = x+1$ and $u = x+s$ in Theorem 2. □

Corollary 2. *For $x \geq 0$, the inequality holds:*

$$\frac{\lambda}{\ln(1+\lambda)}\exp\{x\psi_\lambda(x+1)\} \leq \Gamma_\lambda(x+1) \leq \frac{\lambda}{\ln(1+\lambda)}\exp\{x\psi_\lambda(1)\}. \tag{25}$$

Proof. Let $v = x+1$ and $u = 1$ in Theorem 2. □

Remark 2. Inequality (24) is the degenerate form of inequality (3.4) of [10].

Theorem 3. *For $0 < u \leq v$, the inequality holds:*

$$\exp\left\{(v-u)\frac{\psi_\lambda(u)+\psi_\lambda(v)}{2}\right\} \leq \frac{\Gamma_\lambda(v)}{\Gamma_\lambda(u)} \leq \exp\left\{(v-u)\psi_\lambda\left(\frac{u+v}{2}\right)\right\}. \tag{26}$$

Proof. Let $0 < u \leq v$, and consider the function $\psi_\lambda(x)$ on the interval $[u, v]$. Since $\psi_\lambda(x)$ is concave, then by the classical Hermite–Hadamard inequality, we have

$$\frac{\psi_\lambda(u)+\psi_\lambda(v)}{2} \leq \frac{1}{v-u}\int_u^v \psi_\lambda(t)dt \leq \psi_\lambda\left(\frac{u+v}{2}\right), \tag{27}$$

which translates to

$$(v-u)\frac{\psi_\lambda(u)+\psi_\lambda(v)}{2} \leq \ln\frac{\Gamma_\lambda(v)}{\Gamma_\lambda(u)} \leq (v-u)\psi_\lambda\left(\frac{u+v}{2}\right), \tag{28}$$

and by exponentiation, we obtain the desired result (26). □

Corollary 3. *For $s \in (0,1]$ and $x > 0$, the inequality holds:*

$$\exp\left\{(1-s)\frac{\psi_\lambda(x+1)+\psi_\lambda(x+s)}{2}\right\} \leq \frac{\Gamma_\lambda(x+1)}{\Gamma_\lambda(x+s)} \leq \exp\left\{(1-s)\psi_\lambda\left(x+\frac{s+1}{2}\right)\right\}. \tag{29}$$

Proof. Let $v = x+1$ and $u = x+s$ in Theorem 3. □

Corollary 4. *For $x \geq 0$, the inequality holds:*

$$\frac{\lambda}{\ln(1+\lambda)}\exp\left\{\frac{x}{2}\left[\psi_\lambda(x+1)+\psi_\lambda(1)\right]\right\} \leq \Gamma_\lambda(x+1) \leq \frac{\lambda}{\ln(1+\lambda)}\exp\left\{x\psi_\lambda\left(\frac{x}{2}+1\right)\right\}. \tag{30}$$

Proof. Let $v = x+1$ and $u = 1$ in Theorem 3. □

Remark 3. Inequalities (26), (29), and (30) are, respectively, better than (20), (24), and (25)

Remark 4. By letting $\lambda \longrightarrow 0$, inequality (29) reduces to

$$\exp\left\{(1-s)\frac{\psi(x+1)+\psi(x+s)}{2}\right\} \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq \exp\left\{(1-s)\psi\left(x+\frac{s+1}{2}\right)\right\}. \tag{31}$$

The upper bound of (31) coincides with the upper bound of inequality (1.2) in the work [11] which was obtained by a different procedure. However, the lower bound of (31) is better than the lower bound of inequality (1.2) in [11] since

$$\frac{\psi(x+1)+\psi(x+s)}{2} \geq \sqrt{\psi(x+1)\psi(x+s)} \geq \psi(x+s) \geq \psi(x+\sqrt{s}). \tag{32}$$

This is by virtue of the arithmetic-geometric mean inequality and the monotonicity property of $\psi(x)$.

Theorem 4. *For $x > 0$ and $s \in (0,1]$, the inequality holds:*

$$\left(\frac{\ln(\lambda+1)}{\lambda}\right)^{1-s}(x+s)^{s-1} \leq \frac{\Gamma_\lambda(x+s)}{\Gamma_\lambda(x+1)} \leq \frac{\ln(\lambda+1)}{\lambda}s^{1-s}\Gamma_\lambda(s)(x+s)^{s-1}. \tag{33}$$

Proof. Let $\Delta(x) = (x+s)^{1-s}\Gamma_\lambda(x+s)/\Gamma_\lambda(x+1)$ for $x>0$ and $s\in(0,1]$. Then,

$$\lim_{x\longrightarrow 0}\Delta(x) = \frac{\ln(\lambda+1)}{\lambda}s^{1-s}\Gamma_\lambda(s). \tag{34}$$

Also, inequality (17) implies that

$$\lim_{x\longrightarrow\infty}\Delta(x) = \left(\frac{\ln(\lambda+1)}{\lambda}\right)^{1-s}. \tag{35}$$

Furthermore,

$$\frac{\Delta'(x)}{\Delta(x)} = \frac{1-s}{x+s} + \psi_\lambda(x+s) - \psi_\lambda(x+1) \le 0, \tag{36}$$

which shows that $\Delta(x)$ is decreasing. Hence, $\Delta(\infty)\le\Delta(x)\le\Delta(0)$ which yields (33) □

Remark 5. As a particular case, by letting $s=1/2$ and $\lambda\longrightarrow 0$, we obtain

$$\sqrt{\frac{2}{\pi}\left(x+\frac{1}{2}\right)} < \frac{\Gamma(x+1)}{\Gamma(x+(1/2))} < \sqrt{\left(x+\frac{1}{2}\right)}. \tag{37}$$

The upper bound of (37) agrees with the upper bound of (3) in [12]. Comparing the lower bounds of (37) and (3) in [12] reveals that the lower bound of (37) is stronger if $0<x<1/\pi-2$ and is weaker if $x>1/\pi-2$.

Theorem 5. *Let $u>1$ and $(1/u)+(1/v)=1$. Then,*

$$\Gamma_\lambda(x+y) \le [\Gamma_\lambda(ux)]^{1/u}[\Gamma_\lambda(vy)]^{1/v}, \tag{38}$$

holds for $x>0$ and $y>0$.

Proof. By Holder's inequality for integrals, we have

$$\begin{aligned}\Gamma_\lambda(x+y) &= \int_0^\infty t^{x+y-1}(1+\lambda)^{-(t/\lambda)}dt = \int_0^\infty t^{x-(1/u)}(1+\lambda)^{-(t/\lambda u)}t^{y-(1/v)}(1+\lambda)^{-(t/\lambda v)}dt\\ &\le\left(\int_0^\infty t^{ux-1}(1+\lambda)^{-(t/\lambda)}dt\right)^{1/u}\left(\int_0^\infty t^{vy-1}(1+\lambda)^{-(t/\lambda)}dt\right)^{1/v} = [\Gamma_\lambda(ux)]^{1/u}[\Gamma_\lambda(vy)]^{1/u},\end{aligned} \tag{39}$$

which concludes the proof. □

Remark 6. Applying Young's inequality on the right-hand side of (38) reveals that

$$\Gamma_\lambda(x+y) \le \frac{\Gamma_\lambda(ux)}{u} + \frac{\Gamma_\lambda(vy)}{v}. \tag{40}$$

Theorem 6. *Let $r_1, r_2\in\{2n: n\in\mathbb{N}_0\}$, $k_1>1$, $(1/k_1)+(1/k_2)=1$, and $(r_1/k_1)+(r_2/k_2)\in\mathbb{N}_0$. Then, inequality holds for $x>0$ and $y>0$:*

$$\Gamma_\lambda^{((r_1/k_1)+(r_2/k_2))}\left(\frac{x}{k_1}+\frac{y}{k_2}\right) \le \left[\Gamma_\lambda^{(r_1)}(x)\right]^{1/k_1}\left[\Gamma_\lambda^{(r_2)}(y)\right]^{1/k_2}, \tag{41}$$

Proof. By using (8) and Holder's inequality, we have

$$\begin{aligned}\Gamma_\lambda^{((r_1/k_1)+(r_2/k_2))}\left(\frac{x}{k_1}+\frac{y}{k_2}\right) &= \int_0^\infty (\ln t)^{((r_1/k_1)+(r_2/k_2))}t^{(x/k_1)+(y/k_2)-1}(1+\lambda)^{-(t/\lambda)}dt\\ &= \int_0^\infty (\ln t)^{r_1/k_1}t^{x-1/k_1}(1+\lambda)^{-(t/\lambda k_1)}(\ln t)^{r_2/k_2}t^{y-1/k_2}(1+\lambda)^{-t/\lambda k_2}dt\\ &\le\left(\int_0^\infty (\ln t)^{r_1}t^{x-1}(1+\lambda)^{-(t/\lambda)}dt\right)^{1/k_1}\left(\int_0^\infty (\ln t)^{r_2}t^{y-1}(1+\lambda)^{-(t/\lambda)}dt\right)^{1/k_2} = \left[\Gamma_\lambda^{(r_1)}(x)\right]^{1/k_1}\left[\Gamma_\lambda^{(r_2)}(y)\right]^{1/k_2},\end{aligned} \tag{42}$$

which concludes the proof. □

Remark 7. If $r_1=r_2=r$, then (41) reduces to

$$\Gamma_\lambda^{(r)}\left(\frac{x}{k_1}+\frac{y}{k_2}\right) \le \left[\Gamma_\lambda^{(r)}(x)\right]^{1/k_1}\left[\Gamma_\lambda^{(r)}(y)\right]^{1/k_2}, \tag{43}$$

which implies that function (8) is log-convex for any even order derivative. Moreover, if $r=0$ in (43), we obtain

$$\Gamma_\lambda\left(\frac{x}{k_1}+\frac{y}{k_2}\right) \le [\Gamma_\lambda(x)]^{1/k_1}[\Gamma_\lambda(y)]^{1/k_2}, \tag{44}$$

which shows that the modified degenerate gamma function is log-convex.

Remark 8. If $r_1=r$, $r_2=r+2$, $k_1=k_2=2$, and $x=y$, then (41) reduces to the Turan-type inequality:

$$\left(\Gamma_\lambda^{(r+1)}(x)\right)^2 \le \Gamma_\lambda^{(r)}(x)\Gamma_\lambda^{(r+2)}(x). \tag{45}$$

Remark 9. If $r_1 = s-1$, $r_2 = s+1$, $k_1 = k_2 = 2$, and $x = y$, then (35) reduces to the Turan-type inequality:

$$\left(\Gamma_\lambda^{(s)}(x)\right)^2 \le \Gamma_\lambda^{(s-1)}(x)\Gamma_\lambda^{(s+1)}(x), \tag{46}$$

where $s \in \{2n+1\colon n \in \mathbb{N}_0\}$. This is the degenerate version of main result of [13].

Theorem 7. *For $x \ge 1$, the inequality holds:*

$$\Gamma_\lambda(x) \ge \frac{\lambda}{\ln(1+\lambda)} - 1 + \frac{1}{x}. \tag{47}$$

Proof. Let $\phi(x) = \Gamma_\lambda(x) - (1/x)$. Then,

$$\phi(x+1) - \phi(x) = \Gamma_\lambda(x+1) - \Gamma_\lambda(x) + \frac{1}{x} - \frac{1}{x+1} = \Gamma_\lambda(x)\left(\frac{\lambda x}{\ln(1+\lambda)} - 1\right) + \frac{1}{x} - \frac{1}{x+1} > 0. \tag{48}$$

Thus, $\phi(x)$ is increasing, and for $x \ge 1$, we have $\phi(x) \ge \phi(1)$ which completes the proof. □

Lemma 1. *The function $\beta(x) = x\Gamma_\lambda'(x)$ is increasing for all $x > 0$.*

Proof. By using (6) and (9) and the monotonicity property of $\psi_\lambda(x)$, we have

$$\beta(x) = x\Gamma_\lambda(x)\psi_\lambda(x) = \frac{\ln(\lambda+1)}{\lambda}\Gamma_\lambda(x+1)\psi_\lambda(x), \tag{49}$$

and consequently, we obtain

$$\frac{\lambda}{\ln(\lambda+1)}\frac{\beta'(x)}{\Gamma_\lambda(x+1)} = \psi_\lambda(x+1)\psi_\lambda(x) + \psi_\lambda'(x) > \psi_\lambda^2(x) + \psi_\lambda'(x) > 0, \tag{50}$$

which completes the proof. □

Theorem 8. *The inequalities hold for $x > 0$:*

$$\Gamma_\lambda(x)\Gamma_\lambda\left(\frac{1}{x}\right) \ge \left(\frac{\lambda}{\ln(\lambda+1)}\right)^2, \tag{51}$$

$$\Gamma_\lambda(x) + \Gamma_\lambda\left(\frac{1}{x}\right) \ge \frac{2\lambda}{\ln(\lambda+1)}. \tag{52}$$

Proof. By letting $k_1 = k_2 = 2$ and replacing x and y with $1+x$ and $1+(1/x)$ in (44), we obtain

$$\ln\Gamma_\lambda\left(1 + \frac{x}{2} + \frac{1}{2x}\right) \le \frac{\ln\Gamma_\lambda(1+x)}{2} + \frac{\ln\Gamma_\lambda(1+(1/x))}{2}. \tag{53}$$

Also, since $x + (1/x) \ge 2$ for $x > 0$, then $1 + (x/2) + (1/2x) \ge 2$. Now, let $\phi(x) = \Gamma_\lambda(x)\Gamma_\lambda(1/x)$ for $x > 0$. Then, by using (6), we have

$$\phi(x) = \left(\frac{\ln(\lambda+1)}{\lambda}\right)^2 \Gamma_\lambda(1+x)\Gamma_\lambda\left(1+\frac{1}{x}\right). \tag{54}$$

Next, by using (5), (53), and (54), we obtain

$$\ln\phi(x) \ge 2\ln\left(\frac{\ln(\lambda+1)}{\lambda}\right) + 2\ln\Gamma_\lambda\left(1 + \frac{x}{2} + \frac{1}{2x}\right) \ge 2\ln\left(\frac{\ln(\lambda+1)}{\lambda}\right) + 2\ln\Gamma_\lambda(2) = \ln\left(\frac{\lambda}{\ln(\lambda+1)}\right)^2, \tag{55}$$

which gives (51). Next, let $\theta(x) = \Gamma_\lambda(x) + \Gamma_\lambda(1/x)$ for $x > 0$. Then,

$$x\theta'(x) = x\Gamma_\lambda'(x) - \frac{1}{x}\Gamma_\lambda'\left(\frac{1}{x}\right). \tag{56}$$

It follows from Lemma 1 that $\theta(x)$ is increasing if $x > 1$ and decreasing if $0 < x < 1$. For both cases, we have $\theta(x) > \theta(1) = 2\lambda/\ln(\lambda+1)$ which gives inequality (52). □

Remark 10. Inequality (52) can be obtained from inequality (51) by applying the arithmetic-geometric mean inequality.

Theorem 9. *Let $r, s \in \{2n\colon n \in \mathbb{N}_0\}$ and $r \ge s$. Then, inequality holds for $x > 0$:*

$$\left(\exp\Gamma_\lambda^{(r)}(x)\right)^2 \le \exp\Gamma_\lambda^{(r-s)}(x) \cdot \exp\Gamma_\lambda^{(r+s)}(x). \tag{57}$$

Proof. We adopt the technique of Mortici [14] to estimate the function

$$\frac{\Gamma_\lambda^{(r-s)}(x)+\Gamma_\lambda^{(r+s)}(x)}{2}-\Gamma_\lambda^{(r)}(x)=\frac{1}{2}\int_0^\infty (\ln t)^{r-s}t^{x-1}(1+\lambda)^{-(t/\lambda)}\mathrm{d}t+\frac{1}{2}\int_0^\infty (\ln t)^{r+s}t^{x-1}(1+\lambda)^{-(t/\lambda)}\mathrm{d}t$$
$$-\int_0^\infty (\ln t)^{s}t^{x-1}(1+\lambda)^{-(t/\lambda)}\mathrm{d}t=\frac{1}{2}\int_0^\infty\left[\frac{1}{(\ln t)^s}+(\ln t)^s-2\right](\ln t)^{r}t^{x-1}(1+\lambda)^{-(t/\lambda)}\mathrm{d}t \quad (58)$$
$$=\frac{1}{2}\int_0^\infty\left[1-(\ln t)^s\right]^2(\ln t)^{r-s}t^{x-1}(1+\lambda)^{-(t/\lambda)}\mathrm{d}t\geq 0.$$

Thus,

$$2\Gamma_\lambda^{(r)}(x)\leq\Gamma_\lambda^{(r-s)}(x)+\Gamma_\lambda^{(r+s)}(x), \quad (59)$$

and by exponentiation, we arrive at (57). □

3. Concluding Remarks

In this work, we have proved several inequalities satisfied by the modified degenerate gamma function which was recently introduced. When $\lambda\longrightarrow 0$, the established results reduce to the corresponding results for the classical gamma function. It is our fervent hope that the present results will inspire further research on the modified degenerate gamma function.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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