Research Article

Linear Barycentric Rational Method for Two-Point Boundary Value Equations

Qian Ge ^{1,2} and Xiaoping Zhang ²

¹School of Mathematics and Statistics, Shandong Normal University, Jinan 250358, China

²School of Science, Shandong Jianzhu University, Jinan 250101, China

Correspondence should be addressed to Xiaoping Zhang; geqian@sdjzu.edu.cn

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Linear barycentric rational method for solving two-point boundary value equations is presented. The matrix form of the collocation method is also obtained. With the help of the convergence rate of the interpolation, the convergence rate of linear barycentric rational collocation method for solving two-point boundary value problems is proved. Several numerical examples are provided to validate the theoretical analysis.

1. Introduction

The analysis of many physical phenomena and engineering problems can be reduced to solving the boundary value problem of differential equation, most of which need to be solved by the numerical method. The barycentric interpolation method is a high precision calculation method, and a strong form of collocation that relies on differential equation, which has been studied extensively by many scholars. The linear barycentric rational method (LBRM) [1–3] has been used to solve certain problems such as delay Volterra integro-differential equations [4], Volterra integral equations [5–7], biharmonic equation [8], beam force vibration equation [9], boundary value problems [10], heat conduction problems [11], plane elastic problems [12], incompressible plane elastic problems [13], nonlinear problems [14], and so on [1, 15].

In this article, we pay our attention to the numerical solution of two-point boundary value problems:

$$(Tu)(x) := u''(x) + qu(x) = f(x), \quad x \in (a, b), \quad (1)$$

$$u(a) = u_\ell, u(b) = u_r. \quad (2)$$

Let the interval $[a, b]$ be partitioned into n uniform part with $h = (b - a)/n$ and $x_0, x_1, \dots, x_n$ with its related function $f(x_i)$, $i = 0, 1, \dots, n$. For any $0 \leq d \leq n$, with

$P(x_i)$, $i = 0, 1, \dots, n - d$, to be the interpolation function at the point $x_i, x_{i+1}, \dots, x_{i+d}$, then we have $P_i(x_k) = f(x_k)$, $k = i, i + 1, \dots, i + d$, and

$$r(x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x) P_i(x)}{\sum_{i=0}^{n-d} \lambda_i(x)}, \quad (3)$$

where

$$\lambda_i(x) = \frac{(-1)^i}{(x - x_i) \cdots (x - x_{i+d})}. \quad (4)$$

Change the polynomial $P_i(x)$ into the Lagrange interpolation form as

$$P_i(x) = \sum_{k=i}^{i+d} \prod_{j=i, j \neq k}^{i+d} \frac{x - x_j}{x_k - x_j} f_k. \quad (5)$$

Combining (7) and (5) together, we get

$$\sum_{i=0}^{n-d} \lambda_i(x) P_i(x) = \sum_{i=0}^{n-d} (-1)^i \sum_{k=i}^{i+d} \frac{1}{x - x_k} \prod_{j=i, j \neq k}^{i+d} \frac{1}{x_k - x_j} f_k = \sum_{k=0}^n \frac{w_k}{x - x_k} f_k, \quad (6)$$

where $w_k = \sum_{i \in J_k} (-1)^i \prod_{j=i, j \neq k}^{i+d} 1/(x_k - x_j)$ and $J_k = \{i \in I; k - d \leq i \leq k\}$.

Then we get

$$r(x) = \frac{\sum_{j=0}^n [w_j/(x-x_j)] f_j}{\sum_{j=0}^n w_j/(x-x_j)}, \quad (7)$$

where its basis function is

$$L_j(x) = \frac{w_j/(x-x_j)}{\sum_{k=0}^n w_k/(x-x_k)}. \quad (8)$$

For the equidistant point, its weight function is

$$w_j = (-1)^{n-j} C_n^j. \quad (9)$$

The Chebyshev point of the second kind is

$$x_j = \cos \frac{j\pi}{n}, \quad j = 0, 1, \dots, n, \quad (10)$$

and its weight function is

$$w_j = (-1)^j \delta_j, \quad \delta_j = \begin{cases} \frac{1}{2}, & j = 0, n, \\ 1, & \text{otherwise.} \end{cases} \quad (11)$$

Consider the barycentric interpolation function as

$$u_n(x) = \sum_{j=0}^n L_j(x) u_j, \quad (12)$$

and the numerical scheme is given as

$$\sum_{j=0}^n u_j L_j''(x) + q \sum_{j=0}^n u_j L_j(x) = f(x). \quad (13)$$

By using the notation of the differential matrix, equation (13) is denoted as matrices in the form of

$$\sum_{j=0}^n D_{ij}^{(2)} u_j + q \sum_{j=0}^n \delta_{ij} u_j = f(x_i), \quad (14)$$

where $i = 1, 2, \dots, n$.

Equation (13) is written as matrices in the form of

$$[\mathbf{D}^{(2)} + \mathbf{qI}] \mathbf{u} = \mathbf{f}, \quad (15)$$

where $\mathbf{L} := \mathbf{D}^{(2)} + \mathbf{qI}$, $\mathbf{u} = [u_0, u_1, u_2, \dots, u_n]^T$, $\mathbf{D}^{(k)} = [D_{ij}^{(k)}]_{(n+1) \times (n+1)}$,

$$D_{ij}^{(1)} = \begin{cases} \frac{\omega_j/\omega_i}{x_i - x_j}, & i \neq j, \\ -\sum_{k \neq i} D_{ik}^{(1)}, & i = j, \end{cases} \quad (16)$$

$$D_{ij}^{(2)} = \begin{cases} 2D_{ij}^{(1)} \left(D_{ii}^{(1)} - \frac{1}{x_i - x_j} \right), & i \neq j, \\ -\sum_{k \neq i} D_{ik}^{(2)}, & i = j, \end{cases}$$

and $\mathbf{q} = \text{diag}[q]$, $f = [f(x_0), f(x_1), f(x_2), \dots, f(x_n)]^T$. Using interpolation formulas, boundary conditions can be discretized into

$$\sum_{j=0}^n D_{1j}^{(1)} u_j = a, \quad \sum_{j=0}^n D_{nj}^{(1)} u_j = b. \quad (17)$$

2. Convergence and Error Analysis

With the error function of difference formula

$$e(x) := u(x) - r(x) = (x - x_i) \cdots (x - x_{i+d}) [x_i, x_{i+1}, \dots, x_{i+d}, x] f, \quad (18)$$

and

$$e(x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x) (u(x) - P_i(x))}{\sum_{i=0}^{n-d} \lambda_i(x)} = \frac{A(x)}{B(x)} = O(h^{d+1}), \quad (19)$$

where $A(x) := \sum_{i=0}^{n-d} (-1)^i [x_i, \dots, x_{i+d}, x] f$, $B(x) := \sum_{i=0}^{n-d} \lambda_i(x)$. Taking the numerical scheme

$$\sum_{j=0}^n y_j L_j''(x) + q \sum_{j=0}^n y_j L_j(x) = f(x). \quad (20)$$

Combining (20) and (1), we have

$$Te(x) := e''(x) + qe(x) = R_f(x), \quad (21)$$

where $R_f(x) = f(x) - f(x_k)$, $k = 0, 1, 2, \dots, n$.

The following Lemma has been proved by Jean-Paul Berrut in [13].

Lemma 1 (see [13]). For $e(x)$ defined in (18), we have

$$\begin{cases} |e(x)| \leq Ch^{d+1}, & u \in C^{d+2}[a, b], \\ |e'(x)| \leq Ch^d, & u \in C^{d+3}[a, b], \\ |e''(x)| \leq Ch^{d-1}, & u \in C^{d+4}[a, b], d \geq 1. \end{cases} \quad (22)$$

Let $u(x)$ be the solution of (1) and $u_n(x)$ is the numerical solution, then we have

$$Tu_n(x_k) = f(x_k), \quad k = 0, 1, 2, \dots, n, \quad (23)$$

and

$$\lim_{n \rightarrow \infty} u_n(x) = u(x). \quad (24)$$

The results can be obtained in the reference of [14].

Based on the above lemma, we derive the following theorem.

Theorem 1. Let $u_n(x): Tu_n(x) = f(x)$, $u_n^*(x): Tu_n^*(x) = f^*(x)$, and $f(x) \in C[a, b]$, we have

$$|u_n(x) - u_n^*(x)| \leq Ch^{d-1}. \quad (25)$$

Proof. As $\mathbf{L} := \mathbf{D}^{(2)} + \mathbf{QI}$, where

$$\mathbf{D}^{(2)} = \begin{bmatrix} D_{00}^{(2)} & D_{01}^{(2)} & D_{02}^{(2)} & D_{03}^{(2)} & \cdots & D_{0n}^{(2)} \\ D_{10}^{(2)} & D_{11}^{(2)} & D_{12}^{(2)} & D_{13}^{(2)} & \cdots & D_{1n}^{(2)} \\ D_{20}^{(2)} & D_{21}^{(2)} & D_{22}^{(2)} & D_{23}^{(2)} & \cdots & D_{2n}^{(2)} \\ D_{30}^{(2)} & D_{31}^{(2)} & D_{32}^{(2)} & D_{33}^{(2)} & \cdots & D_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ D_{n-1,0}^{(2)} & D_{n-1,1}^{(2)} & D_{n-1,2}^{(2)} & D_{n-1,3}^{(2)} & \cdots & D_{n-1,n}^{(2)} \\ D_{n0}^{(2)} & D_{n1}^{(2)} & D_{n2}^{(2)} & D_{n3}^{(2)} & \cdots & D_{nn}^{(2)} \end{bmatrix}, \quad (26)$$

and

$$\mathbf{L} := \mathbf{D}^{(2)} + \mathbf{QI} = \begin{bmatrix} q + D_{00}^{(2)} & D_{01}^{(2)} & D_{02}^{(2)} & D_{03}^{(2)} & \cdots & D_{0n}^{(2)} \\ D_{10}^{(2)} & q + D_{11}^{(2)} & D_{12}^{(2)} & D_{13}^{(2)} & \cdots & D_{1n}^{(2)} \\ D_{20}^{(2)} & D_{21}^{(2)} & q + D_{22}^{(2)} & D_{23}^{(2)} & \cdots & D_{2n}^{(2)} \\ D_{30}^{(2)} & D_{31}^{(2)} & D_{32}^{(2)} & q + D_{33}^{(2)} & \cdots & D_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ D_{n-1,0}^{(2)} & D_{n-1,1}^{(2)} & D_{n-1,2}^{(2)} & D_{n-1,3}^{(2)} & \cdots & D_{n-1,n}^{(2)} \\ D_{n0}^{(2)} & D_{n1}^{(2)} & D_{n2}^{(2)} & D_{n3}^{(2)} & \cdots & q + D_{nn}^{(2)} \end{bmatrix}. \quad (27)$$

Putting column 2, column 3, column n added to column 1, we have

$$\mathbf{D}^{(2)} = \begin{bmatrix} 0 & D_{01}^{(2)} & D_{02}^{(2)} & D_{03}^{(2)} & \cdots & D_{0n}^{(2)} \\ 0 & D_{11}^{(2)} & D_{12}^{(2)} & D_{13}^{(2)} & \cdots & D_{1n}^{(2)} \\ 0 & D_{21}^{(2)} & D_{22}^{(2)} & D_{23}^{(2)} & \cdots & D_{2n}^{(2)} \\ 0 & D_{31}^{(2)} & D_{32}^{(2)} & D_{33}^{(2)} & \cdots & D_{3n}^{(2)} \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & D_{n-1,1}^{(2)} & D_{n-1,2}^{(2)} & D_{n-1,3}^{(2)} & \cdots & D_{n-1,n}^{(2)} \\ 0 & D_{n1}^{(2)} & D_{n2}^{(2)} & D_{n3}^{(2)} & \cdots & D_{nn}^{(2)} \end{bmatrix}, \quad (28)$$

which means the matrix $\mathbf{D}^{(2)}$ is the singular matrix.

Similarly we have

$$\mathbf{L} = \begin{bmatrix} q & D_{01}^{(2)} & D_{02}^{(2)} & D_{03}^{(2)} & \cdots & D_{0n}^{(2)} \\ q & q + D_{11}^{(2)} & D_{12}^{(2)} & D_{13}^{(2)} & \cdots & D_{1n}^{(2)} \\ q & D_{21}^{(2)} & q + D_{22}^{(2)} & D_{23}^{(2)} & \cdots & D_{2n}^{(2)} \\ q & D_{31}^{(2)} & D_{32}^{(2)} & q + D_{33}^{(2)} & \cdots & D_{3n}^{(2)} \\ q & \cdots & \cdots & \cdots & \cdots & \cdots \\ q & D_{n-1,1}^{(2)} & D_{n-1,2}^{(2)} & D_{n-1,3}^{(2)} & \cdots & D_{n-1,n}^{(2)} \\ q & D_{n1}^{(2)} & D_{n2}^{(2)} & D_{n3}^{(2)} & \cdots & q + D_{nn}^{(2)} \end{bmatrix}, \quad (29)$$

and then we assume $|\mathbf{L}| \neq 0$ with $q \neq 0$,
 $u_n(x) = \sum_{j=0}^n L_j(x) f_j$, $u_n^*(x) = \sum_{j=0}^n L_j(x) f_j^*$,
 $U_n = (f(x_0), f(x_1), \dots, f(x_n))^T$
and $U_n^* = (f^*(x_0), f^*(x_1), \dots, f^*(x_n))^T$.

By

$$U_n - U_n^* = \mathbf{L}^{-1}(\mathbf{L}U_n - F_n^*), \quad (30)$$

which means

$$u_n(x) - u_n^*(x) = \sum_{j=0}^n M_j(x) Te(x), \quad (31)$$

where $M_j(x)$ is the element of matrix $\mathbf{L}^{-1}$.

Then we have

$$|u_n(x) - u_n^*(x)| \leq \left| \sum_{j=0}^n M_j(x) \right| |Te(x)| \leq Ch^{d-1}. \quad (32)$$

The proof is completed.

We know that the central difference method can achieve quadratic convergence and the convergence order is the same as that of $d=3$. When $d>3$, the convergence of the barycentric rational method is better than that of the central difference method. $\square$

3. Numerical Example

Example 1. Consider the two-point boundary value:

$$-y'' + 400y = -400 \cos^2 \pi x - 2\pi^2 \cos 2\pi x, \quad (33)$$

$$y(0) = y(1) = 0, \quad (34)$$

and its analysis solution is

$$y(x) = \frac{e^{-20}}{1 + e^{-20}} e^{20x} + \frac{1}{1 + e^{-20}} e^{-20x} - \cos^2 \pi x. \quad (35)$$

In this example, we consider the two-point boundary value equations with the boundary condition $y(0) = y(1) = 0$. In Table 1, the convergence rate of equidistant nodes with different d is $O(h^d)$; in Table 2, the convergence rate of the Chebyshev point of the second kind with different d is $O(h^{d+2})$, $d \geq 2$. From Theorem 1, the convergence rate is $O(h^{d-1})$, and there are no convergence rates as $d=1$. Here the convergence rate is $O(h)$ and $O(h^2)$ in Tables 1 and 2 for $d=1$, respectively, and we will give exact analysis in other paper.

Example 2. Consider the two-point boundary value.

$$y'' + y' \sin x + ye^x = -16\pi^2 \sin 4\pi x + 4\pi \sin x \cos 4\pi x + e^x (2 + \sin 4\pi x), \quad -1 < x < 1, \quad (36)$$

with the boundary condition

$$y(-1) + y'(-1) = 2 + 4\pi, \quad y(1) + y'(1) = 2 + 4\pi, \quad (37)$$

and its analysis solution is

$$y(x) = 2 + \sin 4\pi x. \quad (38)$$

In this example, we consider the variable coefficient of two-point boundary value equations with the boundary

TABLE 1: Errors of the equidistant nodes with different d .

n	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$
10	$5.6796e-02$	$4.7174e-02$	$3.6770e-02$	$2.9692e-02$	$2.5240e-02$
20	$3.1866e-02$	$2.0707e-02$	$1.2295e-02$	$7.5473e-03$	$4.6949e-03$
40	$1.1899e-02$	$4.8530e-03$	$1.8024e-03$	$6.8958e-04$	$2.6519e-04$
80	$4.0465e-03$	$8.2037e-04$	$1.7027e-04$	$3.6416e-05$	$7.8060e-06$
160	$1.4721e-03$	$1.1821e-04$	$1.2993e-05$	$1.4706e-06$	$1.6657e-07$
320	$4.9182e-04$	$1.5796e-05$	$8.9500e-07$	$5.2131e-08$	$3.0430e-09$
640	$1.5719e-04$	$2.0375e-06$	$5.8674e-08$	$1.7409e-09$	$5.1202e-11$

TABLE 2: Errors of the Chebyshev point with different d .

n	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$
10	$4.7235e-02$	$3.4143e-02$	$2.3198e-02$	$1.0658e-02$	$4.2299e-03$
20	$2.0966e-02$	$4.9675e-03$	$2.3057e-03$	$1.5490e-03$	$1.0185e-03$
40	$5.0812e-03$	$3.3376e-04$	$7.5530e-05$	$3.1018e-05$	$1.2517e-05$
80	$1.2482e-03$	$2.0099e-05$	$2.1222e-06$	$4.5138e-07$	$9.5181e-08$
160	$2.9995e-04$	$1.2091e-06$	$6.1512e-08$	$6.5988e-09$	$6.9630e-10$
320	$7.3360e-05$	$7.3634e-08$	$1.8414e-09$	$1.0610e-10$	$2.8788e-10$
640	$1.8092e-05$	$4.5328e-09$	$5.7970e-11$	$1.5463e-10$	$6.0430e-09$

TABLE 3: Errors of the equidistant nodes with different d .

n	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$
10	$3.1903e+00$	$5.8183e+00$	$4.2584e+00$	$3.3515e+00$	$3.8150e+00$
20	$9.0854e-01$	$1.8487e-01$	$4.7472e-02$	$5.6182e-02$	$2.1227e-02$
40	$1.9690e-01$	$3.8711e-02$	$5.9273e-04$	$1.9234e-04$	$4.9300e-05$
80	$4.4235e-02$	$3.9481e-03$	$4.9399e-06$	$2.3577e-06$	$2.4228e-07$
160	$9.6887e-03$	$3.4240e-04$	$1.9523e-07$	$1.9939e-08$	$2.8181e-09$
320	$2.0060e-03$	$2.7549e-05$	$5.5624e-09$	$1.6331e-09$	$7.9052e-08$
640	$3.7877e-04$	$2.1177e-06$	$1.0976e-09$	$2.1374e-08$	$1.2137e-06$

TABLE 4: Errors of the Chebyshev point with different d .

n	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$
10	$1.3687e+01$	$1.0866e+02$	$2.9315e+02$	$6.7980e+02$	$1.4700e+03$
20	$2.9375e+00$	$2.2412e+01$	$2.0846e+01$	$2.2744e+00$	$3.1001e+01$
40	$1.0678e+00$	$4.9833e+00$	$9.5860e-01$	$1.3524e+00$	$7.5788e-01$
80	$4.4216e-01$	$1.1571e+00$	$2.6706e-02$	$9.3346e-02$	$8.5273e-03$
160	$1.9821e-01$	$2.7788e-01$	$1.7104e-03$	$5.6811e-03$	$2.5039e-05$
320	$9.3181e-02$	$6.8032e-02$	$4.9655e-04$	$3.4460e-04$	$2.6914e-06$
640	$4.5032e-02$	$1.6827e-02$	$7.8365e-05$	$2.1132e-05$	$1.3480e-07$

condition $y(-1) + y'(-1) = 2 + 4\pi$, $y(1) + y'(1) = 2 + 4\pi$. In Table 3, the convergence rate of equidistant nodes with different d is $O(h^d)$; in Table 4, the convergence rate of the Chebyshev point of second kind with different d is $O(h^{d+2})$, $d \geq 2$.

4. Concluding Remarks

In this paper, the numerical approximation of linear bar-centric rational collocation method for solving two-point boundary value equations is presented. The matrix form of the algorithm is given for the simple calculation; with the help of Newton formula, the error function of the

convergence rate $O(h^{d-1})$ is also obtained. For the constant coefficient and variable coefficient of two-point boundary value equations, numerical results show that the convergence rate can reach $O(h^d)$ for the equidistant nodes and $O(h^{d+2})$ for the Chebyshev point of the second kind with $d \geq 2$. For the special case of $d = 1$, there are still convergence rates with $O(h)$, and the analysis of this phenomenon will be presented in other papers.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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