Research Article

Abel–Goncharov Type Multiquadric Quasi-Interpolation Operators with Higher Approximation Order

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Received 29 September 2020; Revised 31 May 2021; Accepted 12 June 2021; Published 5 July 2021

Academic Editor: Beny Neta

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A kind of Abel–Goncharov type operators is surveyed. The presented method is studied by combining the known multiquadric quasi-interpolant with univariate Abel–Goncharov interpolation polynomials. The construction of new quasi-interpolants $\mathcal{L}_m^{AG} f$ has the property of m ($m \in \mathbb{Z}, m > 0$) degree polynomial reproducing and converges up to a rate of $m + 1$. In this study, some error bounds and convergence rates of the combined operators are studied. Error estimates indicate that our operators could provide the desired precision by choosing the suitable shape-preserving parameter c and a nonnegative integer m . Several numerical comparisons are carried out to verify a higher degree of accuracy based on the obtained scheme. Furthermore, the advantage of our method is that the associated algorithm is very simple and easy to implement.

1. Introduction

Assume that f is a function defined on a domain $[a, b] \subset \mathbb{R}$ containing X and $x_i \in X$, $i = 0, \dots, N$ is some distinct points, where

$$a = x_0 < \dots < x_N = b, \quad (1)$$

has the form

$$\mathcal{L}[f; a, b](x) = \sum_{j=0}^N \lambda_j \mathcal{X}(x - x_j), \quad (2)$$

s.t.

$$\mathcal{L}[f; a, b](x_j) = f(x_j), \quad \text{for } j = 0, 1, \dots, N, \quad (3)$$

where $\mathcal{X}(\cdot)$ is an interpolation kernel. Radial basis functions have been employed to solve the above interpolation problems (2) and (3) by many research fellows. Multiquadrics first introduced by Hardy [1],

$$\phi_j(x) = \phi(x - x_j) = \sqrt{(x - x_j)^2 + c^2}, \quad j = 0, 1, \dots, N, \quad (4)$$

are particularly interesting, on account of their particular convergence property [2, 3]. In this study, we denote the multiquadrics and their shape-preserving parameter in (4) by notations $\phi_j(\cdot)$ and c , respectively. By means of accuracy, efficiency, and easy implementation, Franke [4] investigated that multiquadric interpolation is considered to be one of the most schemes in 29 interpolation methods. Based on distinct nodes $\{x_j\}_{j=0}^N$, Micchelli [5] proved that multiquadric interpolation is always solvable, but the resulting matrix in interpolation problems (2) and (3) quickly becomes ill-conditioned with the increase of the nodes. The well-known quasi-interpolation as a weaker form of (3) reproduces all polynomials of degree $\leq m$, that is,

$$\mathcal{L}[p; a, b](x) = p(x), \quad p \in \mathbb{P}_m, x \in \mathbb{R}, \quad (5)$$

where $\mathbb{P}_m = \{p: \deg(p) \leq m\}$. In this study, we will apply the quasi-interpolation scheme to overcome the ill-conditioning problem.

Beatson and Powell [6] first constructed a univariate quasi-interpolant $\mathcal{L}_B$ which reproduces constants. Wu and Schaback [7] introduced another quasi-interpolant $\mathcal{L}_D$ possessing shape-preserving and linear-reproducing properties. They proved that the error of the operator $\mathcal{L}_D$ is $\mathcal{O}(h^2 |\ln h|)$ when the shape parameter $c = \mathcal{O}(h)$ and $h = \max_{1 \leq j \leq n} \{ |x_j - x_{j-1}| \}$. Based on the operator $\mathcal{L}_D$ in [7], Ling [8] provided a multilevel quasi-interpolant and showed that its convergence rate is $\mathcal{O}(h^{2.5} |\ln h|)$ as $c = \mathcal{O}(h)$. To increase the degree of the multiquadric quasi-interpolation operator, Feng and Zhou [9] provided a kind of multiquadric quasi-interpolants, and the operators could have any degree of exactness. At the same time, by applying the operator $\mathcal{L}_B$ with Hermite interpolation polynomials, Wang et al. [10] proposed a kind of improved quasi-interpolation operators $\mathcal{L}_{H_{2m-1}}$ and gave the desired orders of convergence. By combining the operator $\mathcal{L}_B$ with Lidstone interpolating polynomials [11–13], Wu et al. [14] proposed a kind of Lidstone-type multiquadric quasi-interpolants $\tilde{\mathcal{L}}_{\Lambda_n}$ possessing any degree of polynomial reproducibility. The authors have given that the approximating capacity of the operator $\tilde{\mathcal{L}}_{\Lambda_n}$ is comparable with that of the operator $\mathcal{L}_{H_{2m-1}}$. Furthermore, many researchers applied multiquadric quasi-interpolants to solve differential equations [15–26]. Meanwhile, Ali et al. [27] constructed the SDI using Timmer triangular patches, which are used to visualize the energy data, i.e., spatial interpolation in visualizing rainfall data.

By the means of construction idea in [10], we provide a kind of Abel–Goncharov type multiquadric quasi-interpolants by combining the operator $\mathcal{L}_B$ with Abel–Goncharov interpolating polynomials. The presented operators could reproduce polynomials of higher degree than $\mathcal{L}_B$. Under the suitable assumption of shape-preserving parameter c , we obtain the convergence rates of higher order. Therefore, we could derive the desired precision of the our operators with an optimal value of c .

The remaining organization of this study is arranged as follows. In Section 2, we give the definition of univariate Abel–Goncharov interpolation polynomials and derive three useful theorems for the error of approximation. Section 3 is devoted to construct Abel–Goncharov type multiquadric quasi-interpolants and study their approximation orders. In Section 4, numerical experiments are shown to compare the approximation capacity of our operators with that of Wang et al.’s quasi-interpolants. Finally, conclusion is given in Section 5.

2. Univariate Abel–Goncharov Interpolation Polynomial

We recall first Abel–Goncharov interpolation problem from [28–30]. Consider $m \in \mathbb{N}$, $a_0, a_1, \dots, a_m \in \mathbb{R}$, $a_0 < a_1 < \dots < a_{m-1} < a_m$, and suppose $f: [a_0, a_m] \rightarrow \mathbb{R}$ is a function with the first m derivatives $f^{(i)}$, $0 \leq i \leq m$. For the nodes $a_i \in [a_0, a_m]$, $0 \leq i \leq m$, and the values $f^{(i)}(a_i)$, $0 \leq i \leq m$, we introduce the Abel–Goncharov interpolation

problem of existing polynomial $P_m[f; a_0, a_1, \dots, a_m](x)$ of degree m , s.t.

$$(P_m[f; a_0, a_1, \dots, a_m])^{(i)}(a_i) = f^{(i)}(a_i), \quad 0 \leq i \leq m. \quad (6)$$

The determinant of the linear system,

$$D = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^m \\ 0 & 1! & 2a_1 & \cdots & ma_1^{m-1} \\ 0 & 0 & 2! & \cdots & m(m-1)a_2^{m-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & m! \end{vmatrix}, \quad (7)$$

is always nonzero, and problem (6) has a unique solution. The Abel–Goncharov interpolation polynomial $P_m[f; a_0, a_1, \dots, a_m](x)$ could be expressed as follows:

$$P_m[f; a_0, a_1, \dots, a_m](x) = \sum_{k=0}^m g_k(x) f^{(k)}(a_k), \quad (8)$$

where $g_k(x)$, $k = 0, 1, \dots, m$ are known as the Goncharov polynomials of degree k [31] with

$$\begin{cases} g_k^{(s)}(a_s) = 0, & \text{if } k \neq s, \\ g_k^{(k)}(a_k) = 1. \end{cases} \quad (9)$$

By means of [29–31], we have

$$g_0(x) = 1,$$

$$g_1(x) = x - a_0,$$

$$g_k(x) = \int_{a_0}^x dt_1 \int_{a_1}^{t_1} dt_2 \cdots \int_{a_{k-1}}^{t_{k-1}} dt_k \\ = \frac{1}{k!} \left[x^k - \sum_{j=0}^{k-1} g_j(x) \binom{k}{j} a_j^{k-j} \right], \quad k = 2, \dots, m. \quad (10)$$

The Abel–Goncharov interpolation polynomial $P_m[f; a_0, a_1, \dots, a_m](x)$ has the following properties:

$$\lim_{a_m \rightarrow a_0} P_m[f; a_0, a_1, \dots, a_m](x) = \sum_{k=0}^m \frac{(x - a_0)^k}{k!} f^{(k)}(a_0). \quad (11)$$

Remark 1. As the nodes $a_0 = a_1 = \dots = a_m$, the Abel–Goncharov interpolation polynomial $P_m[f; a_0, a_1, \dots, a_m](x)$ is the m^{th} Taylor polynomial of f about a_0 .

The Abel–Goncharov interpolation polynomial $P_m[f; a_0, a_1, \dots, a_m](x)$ has the polynomial reproduction property as follows.

Theorem 1. The Abel–Goncharov interpolation polynomial $P_m[f; a_0, a_1, \dots, a_m](x)$ reproduces all polynomials of degree no more m .

Proof. Let us verify that $P_m[f; a_0, a_1, \dots, a_m](x) = f(x)$, for $f(x) = x^i, i = 0, 1, \dots, m, x \in [a_0, a_m]$. The Abel–Goncharov interpolation polynomial $P_m[f; a_0, a_1, \dots, a_m](x)$ can reproduce all polynomials of degree no more m . It is easily given that

$$P_m[x^i; a_0, a_1, \dots, a_m](x) = x^i, \quad i = 0, 1, \dots, m, \quad (12)$$

while

$$\begin{aligned} P_m[x^{m+1}; a_0, a_1, \dots, a_m](x) \\ = g_0(x)a_0^{m+1} + g_1(x)(m+1)a_1^m + \dots \\ + g_m(x)(m+1)m, \dots, 2 \cdot a_m \neq x^{m+1}. \end{aligned} \quad (13)$$

□

For function $f \in C^m[a_0, a_m]$, the Abel–Goncharov interpolation formula is obtained as follows:

$$f(x) = P_m[f; a_0, a_1, \dots, a_m](x) + R_m[f; a_0, a_1, \dots, a_m](x). \quad (14)$$

For bounds of the above remainder $R_m[f; a_0, a_1, \dots, a_m]$ even in points outside the interval $[c, d]$, we use the operator

$$f \longrightarrow P_m[f; a_0, a_1, \dots, a_m], \quad (15)$$

as acting on the space $C^m[c, d]$, where $c < a_0, a_m < d$. Based on Peano's kernel theorem [32], we provide the following integral expression for the remainder (14).

Theorem 2. Given $f \in C^m[c, d]$ and $x \in [c, d]$, for the remainder

$$R_m[f; a_0, a_1, \dots, a_m](x) = f(x) - P_m[f; a_0, a_1, \dots, a_m](x), \quad (16)$$

we consider the following integral representations:

$$\begin{aligned} R_m[f; a_0, a_1, \dots, a_m](x) \\ = \begin{cases} \frac{1}{(m-1)!} \int_x^{a_{m-1}} f^{(m)}(t) K_{a_0, a_1, \dots, a_m}(x, t) dt, & c \leq x \leq a_0, \\ \frac{1}{(m-1)!} \int_{a_0}^{a_{m-1}} f^{(m)}(t) K_{a_0, a_1, \dots, a_m}(x, t) dt, & a_0 \leq x \leq a_{m-1}, \\ \frac{1}{(m-1)!} \int_{a_0}^x f^{(m)}(t) K_{a_0, a_1, \dots, a_m}(x, t) dt, & a_{m-1} \leq x \leq d, \end{cases} \end{aligned} \quad (17)$$

where

$$K_{a_0, a_1, \dots, a_m}(x, t) = (x-t)_+^{m-1} - \sum_{k=0}^{m-1} \frac{(m-1)!}{(m-k-1)!} g_k(x) (a_k - t)_+^{m-k-1}, \quad (18)$$

and $(\cdot)_+^k$ denotes the positive part of the k^{th} power of the argument, such that

$$(s)_+^k = \begin{cases} s^k, & s > 0, \\ 0, & s \leq 0. \end{cases} \quad (19)$$

Proof. First say that in the interpolation polynomial (8), there are evaluations of derivatives of function f up to the order $m-1$ in the interval $[c, d]$. Finally, the approximant (8) has the degree of exactness m . By using Peano's kernel theorem, we then obtain

$$R_m[f; a_0, a_1, \dots, a_m](x) = \frac{1}{(m-1)!} \int_c^d f^{(m)}(t) K_{a_0, a_1, \dots, a_m}(x, t) dt, \quad (20)$$

where (18) is provided by using the linear functional $f \longrightarrow R_m[f; a_0, a_1, \dots, a_m]$ to $(x-t)_+^{m-1}$ viewed as a function of x .

If $c \leq x \leq a_0$, then

$$\begin{aligned} R_m[f; a_0, a_1, \dots, a_m](x) \\ = \frac{1}{(m-1)!} \int_c^x f^{(m)}(t) K_{a_0, a_1, \dots, a_m}(x, t) dt \\ + \frac{1}{(m-1)!} \int_x^{a_0} f^{(m)}(t) K_{a_0, a_1, \dots, a_m}(x, t) dt \\ + \frac{1}{(m-1)!} \int_{a_0}^{a_1} f^{(m)}(t) K_{a_0, a_1, \dots, a_m}(x, t) dt \\ + \frac{1}{(m-1)!} \int_{a_1}^{a_2} f^{(m)}(t) K_{a_0, a_1, \dots, a_m}(x, t) dt \\ + \dots + \frac{1}{(m-1)!} \int_{a_{m-2}}^{a_{m-1}} f^{(m)}(t) K_{a_0, a_1, \dots, a_m}(x, t) dt \\ + \frac{1}{(m-1)!} \int_{a_{m-1}}^d f^{(m)}(t) K_{a_0, a_1, \dots, a_m}(x, t) dt. \end{aligned} \quad (21)$$

Let $t \in [c, x]$, then

$$\begin{aligned} K_{a_0, a_1, \dots, a_m}(x, t) &= (x-t)^{m-1} - \sum_{k=0}^{m-1} \frac{(m-1)!}{(m-k-1)!} g_k(x) (a_k - t)^{m-k-1} \\ &= (x-t)^{m-1} - (x-t)^{m-1} = 0, \end{aligned} \quad (22)$$

where $(x-t)^{m-1}$ is expressed as a polynomial in x of degree $m-1$.

Let $t \in [a_{m-1}, d]$, then

$$K_{a_0, a_1, \dots, a_m}(x, t) = 0. \quad (23)$$

Thus, the first case of (17) is proved by the above process. The rest of the expressions may be obtained by an analogous manner. $\square$

The following theorem provides the desired bounds.

Theorem 3. Given $f \in C^m[c, d]$ and $x \in [c, d]$, for the remainder (16), we have

$$|R_m[f; a_0, a_1, \dots, a_m](x)| \leq \begin{cases} C(m) \|f^{(m)}\|_{\infty} (a_{m-1} - x)^m, & c \leq x \leq a_0, \\ C(m) \|f^{(m)}\|_{\infty} (a_{m-1} - a_0)^m, & a_0 \leq x \leq a_{m-1}, \\ C(m) \|f^{(m)}\|_{\infty} (x - a_0)^m, & a_{m-1} \leq x \leq d, \end{cases} \quad (24)$$

where $\|\cdot\|_{\infty}$ denotes the sup-norm on $[c, d]$, and

$$C(m) = 1 + \left(\sum_{k=0}^{m-1} + \sum_{k=1}^{m-1} + \dots + \sum_{k=m-2}^{m-1} \right) \frac{1}{(m-k)!}. \quad (25)$$

Proof. If $c \leq x \leq a_0$, then we have from (17),

$$\begin{aligned} & R_m[f; a_0, a_1, \dots, a_m](x) \\ &= \frac{1}{(m-1)!} \int_x^{a_0} f^{(m)}(t) K_{a_0, a_1, \dots, a_m}(x, t) dt \\ &+ \frac{1}{(m-1)!} \int_{a_0}^{a_1} f^{(m)}(t) K_{a_0, a_1, \dots, a_m}(x, t) dt \\ &+ \frac{1}{(m-1)!} \int_{a_1}^{a_2} f^{(m)}(t) K_{a_0, a_1, \dots, a_m}(x, t) dt \\ &+ \dots + \frac{1}{(m-1)!} \int_{a_{m-2}}^{a_{m-1}} f^{(m)}(t) K_{a_0, a_1, \dots, a_m}(x, t) dt. \end{aligned} \quad (26)$$

If $t \in [x, a_0]$, then

$$K_{a_0, a_1, \dots, a_m}(x, t) = - \sum_{k=0}^{m-1} \frac{(m-1)!}{(m-k-1)!} g_k(x) (a_k - t)^{m-k-1}, \quad (27)$$

such that

$$\begin{aligned} & \int_x^{a_0} K_{a_0, a_1, \dots, a_m}(x, t) f^{(m)}(t) dt \\ &= - \sum_{k=0}^{m-1} \frac{(m-1)!}{(m-k-1)!} g_k(x) \int_x^{a_0} (a_k - t)^{m-k-1} f^{(m)}(t) dt. \end{aligned} \quad (28)$$

We know that the integrands are of type $h(t)f^{(m)}(t)$ with $h(t)$ that does not change sign in $[x, a]$. By means of the first mean value theorem for integrals, we obtain for some $\xi_{0,k} \in [c, d]$, $k = 0, 1, \dots, m-1$,

$$\begin{aligned} & \int_x^{a_0} K_{a_0, a_1, \dots, a_m}(x, t) f^{(m)}(t) dt \\ &= - \sum_{k=0}^{m-1} \frac{(m-1)!}{(m-k-1)!} g_k(x) f^{(m)}(\xi_{0,k}) \int_x^{a_0} (a_k - t)^{m-k-1} dt. \end{aligned} \quad (29)$$

After some calculations, we obtain

$$\begin{aligned} & \int_x^{a_0} K_{a_0, a_1, \dots, a_m}(x, t) f^{(m)}(t) dt \\ &= - \sum_{k=0}^{m-1} \frac{(m-1)!}{(m-k-1)!} g_k(x) f^{(m)}(\xi_{0,k}) [(a_k - x)^{m-k} - (a_k - a_0)^{m-k}]. \end{aligned} \quad (30)$$

If $t \in [a_0, a_1]$, then

$$\begin{aligned} & K_{a_0, a_1, \dots, a_m}(x, t) = - \sum_{k=1}^{m-1} \frac{(m-1)!}{(m-k-1)!} g_k(x) (a_k - t)^{m-k-1} f^{(m)}(t), \\ & \int_{a_0}^{a_1} K_{a_0, a_1, \dots, a_m}(x, t) f^{(m)}(t) dt \\ &= - \sum_{k=1}^{m-1} \frac{(m-1)!}{(m-k-1)!} g_k(x) \int_{a_0}^{a_1} (a_k - t)^{m-k-1} f^{(m)}(t) dt. \end{aligned} \quad (31)$$

By applying the first mean theorem for the above integrals, we have for some $\xi_{1,k} \in [c, d]$, $k = 1, \dots, m-1$,

$$\begin{aligned}
& \int_{a_0}^{a_1} K_{a_0, a_1, \dots, a_m}(x, t) f^{(m)}(t) dt \\
&= - \sum_{k=1}^{m-1} \frac{(m-1)!}{(m-k-1)!} g_k(x) f^{(m)}(\xi_{1,k}) \int_{a_0}^{a_1} (a_k - t)^{m-k-1} dt.
\end{aligned} \quad (32)$$

After some calculations, we obtain

$$\begin{aligned}
& \int_{a_0}^{a_1} K_{a_0, a_1, \dots, a_m}(x, t) f^{(m)}(t) dt \\
&= - \sum_{k=1}^{m-1} \frac{(m-1)!}{(m-k-1)!} g_k(x) f^{(m)}(\xi_{1,k}) [(a_k - a_0)^{m-k} \\
&\quad - (a_k - a_1)^{m-k}].
\end{aligned} \quad (33)$$

If $t \in [a_1, a_2]$, then

$$\begin{aligned}
K_{a_0, a_1, \dots, a_m}(x, t) &= - \sum_{k=2}^{m-1} \frac{(m-1)!}{(m-k-1)!} g_k(x) (a_k - t)^{m-k-1} f^{(m)}(t), \\
& \int_{a_1}^{a_2} K_{a_0, a_1, \dots, a_m}(x, t) f^{(m)}(t) dt \\
&= - \sum_{k=2}^{m-1} \frac{(m-1)!}{(m-k)!} g_k(x) \int_{a_1}^{a_2} (a_k - t)^{m-k-1} f^{(m)}(t) dt.
\end{aligned} \quad (34)$$

By applying the first mean theorem for the above integrals, we have for some $\xi_{2,k} \in [c, d], k = 2, \dots, m-1$,

$$\begin{aligned}
& \int_{a_1}^{a_2} K_{a_0, a_1, \dots, a_m}(x, t) f^{(m)}(t) dt \\
&= - \sum_{k=2}^{m-1} \frac{(m-1)!}{(m-k-1)!} g_k(x) f^{(m)}(\xi_{2,k}) \int_{a_1}^{a_2} (a_k - t)^{m-k-1} dt.
\end{aligned} \quad (35)$$

After some calculations, we obtain

$$\begin{aligned}
& \int_{a_1}^{a_2} K_{a_0, a_1, \dots, a_m}(x, t) f^{(m)}(t) dt \\
&= - \sum_{k=2}^{m-1} \frac{(m-1)!}{(m-k)!} g_k(x) f^{(m)}(\xi_{2,k}) [(a_k - a_1)^{m-k} - (a_k - a_2)^{m-k}].
\end{aligned} \quad (36)$$

In the same way, we have, if $t \in [a_2, a_3]$, then for $\xi_{3,k} \in [c, d], k = 3, \dots, m-1$,

$$\begin{aligned}
& \int_{a_2}^{a_3} K_{a_0, a_1, \dots, a_m}(x, t) f^{(m)}(t) dt \\
&= - \sum_{k=3}^{m-1} \frac{(m-1)!}{(m-k)!} g_k(x) f^{(m)}(\xi_{3,k}) [(a_k - a_2)^{m-k} - (a_k - a_3)^{m-k}].
\end{aligned} \quad (37)$$

If $t \in [a_{m-3}, a_{m-2}]$, then for $\xi_{m-2,k} \in [c, d], k = m-2, m-1$,

$$\begin{aligned}
& \int_{a_{m-3}}^{a_{m-2}} K_{a_0, a_1, \dots, a_m}(x, t) f^{(m)}(t) dt \\
&= - \sum_{k=m-2}^{m-1} \frac{(m-1)!}{(m-k)!} g_k(x) f^{(m)}(\xi_{m-2,k}) [(a_k - a_{m-3})^{m-k} \\
&\quad - (a_k - a_{m-2})^{m-k}].
\end{aligned} \quad (38)$$

If $t \in [a_{m-2}, a_{m-1}]$, then for $\xi_{m-1,k} \in [c, d], k = m-1$,

$$\begin{aligned}
& \int_{a_{m-2}}^{a_{m-1}} K_{a_0, a_1, \dots, a_m}(x, t) f^{(m)}(t) dt \\
&= - \sum_{k=m-1}^{m-1} \frac{(m-1)!}{(m-k)!} g_k(x) f^{(m)}(\xi_{m-1,k}) [(a_k - a_{m-2})^{m-k} \\
&\quad - (a_k - a_{m-1})^{m-k}].
\end{aligned} \quad (39)$$

By definition of (19), $K_{a_0, a_1, \dots, a_m}(x, t)$ of (18) is zero as $a_{m-1} \leq t \leq a_m$. Substituting into (26) the left-hand sides of (30), (33), (36), (37), (38), and (39) by their respective right-hand sides, we finally get the expression as follows:

$$\begin{aligned}
& R_m[f; a_0, a_1, \dots, a_m](x, t) \\
&= - \sum_{k=0}^{m-1} \frac{1}{(m-k)!} g_k(x) f^{(m)}(\xi_{0,k}) [(a_k - x)^{m-k} - (a_k - a_0)^{m-k}] \\
&\quad - \sum_{k=1}^{m-1} \frac{1}{(m-k)!} g_k(x) f^{(m)}(\xi_{1,k}) [(a_k - a_0)^{m-k} - (a_k - a_1)^{m-k}] \\
&\quad \dots \\
&\quad - \sum_{k=m-2}^{m-1} \frac{1}{(m-k)!} g_k(x) f^{(m)}(\xi_{m-2,k}) [(a_k - a_{m-3})^{m-k} \\
&\quad - (a_k - a_{m-2})^{m-k}] \\
&\quad - \sum_{k=m-1}^{m-1} \frac{1}{(m-k)!} g_k(x) f^{(m)}(\xi_{m-1,k}) [(a_k - a_{m-2})^{m-k} \\
&\quad - (a_k - a_{m-1})^{m-k}].
\end{aligned} \quad (40)$$

In order to obtain the desired bounds (24), we have the following inequality for $k = 0, 1, m$:

$$\begin{aligned}
|g_k(x)| &= \left| \int_{a_0}^x dt_1 \int_{a_1}^{t_1} dt_2, \dots, \int_{a_{k-1}}^{t_{k-1}} dt_k \right| \\
&\leq (a_{m-1} - x)^k,
\end{aligned} \quad (41)$$

where the inequality follows from (10):

$$\begin{aligned}
|g_0(x)| &= 1 \leq (a_{m-1} - x)^0, \\
|g_1(x)| &= |x - x_0| \leq (a_{m-1} - x), \\
|g_k(x)| &= \left| \int_{a_0}^x dt_1 \int_{a_1}^{t_1} dt_2, \dots, \int_{a_{k-1}}^{t_{k-1}} dt_k \right| \\
&\leq \left| \int_{a_0}^x dt_1 \int_{a_1}^x dt_2, \dots, \int_{a_{k-1}}^x dt_k \right| \\
&\leq |(x - a_0)(x - a_1), \dots, (x - a_{k-1})| \\
&\leq (a_{m-1} - x)^k, \quad (k = 2, 3, \dots, m).
\end{aligned} \tag{42}$$

Finally, we have, after some calculations,

$$\begin{aligned}
&|R_m[f; a_0, a_1, \dots, a_m]| \\
&\leq \|f^{(m)}\|_{\infty} \left[(a_{m-1} - x)^m \left(\sum_{k=0}^{m-1} \frac{1}{(m-k)!} \right. \right. \\
&\quad \left. \left. + \sum_{k=1}^{m-1} \frac{1}{(m-k)!} + \dots + \sum_{k=m-2}^{m-1} \frac{1}{(m-k)!} + \sum_{k=m-1}^{m-1} \frac{1}{(m-k)!} \right) \right] \\
&\leq \left[1 + \left(\sum_{k=0}^{m-1} + \sum_{k=1}^{m-1} + \dots + \sum_{k=m-2}^{m-1} \right) \frac{1}{(m-k)!} \right] \|f^{(m)}\|_{\infty} (a_{m-1} - x)^m.
\end{aligned} \tag{43}$$

Similarly, the remaining expressions of (24) could be proved. $\square$

Since the algebraic degree of exactness of the operator $P_m[f; a_0, a_1, \dots, a_m]$ is equal to m , the following result can be given in an analogous manner.

Theorem 4. Given $f \in C^{m+1}[c, d]$ and $x \in [c, d]$, for the remainder (16), we have

$$\begin{aligned}
&|R_m[f; a_0, a_1, \dots, a_m](x)| \\
&\leq \begin{cases} C(m+1) \|f^{(m+1)}\|_{\infty} (a_m - x)^{m+1}, & c \leq x \leq a_0, \\ C(m+1) \|f^{(m+1)}\|_{\infty} (a_m - a_0)^{m+1}, & a_0 \leq x \leq a_m, \\ C(m+1) \|f^{(m+1)}\|_{\infty} (x - a_0)^{m+1}, & a_m \leq x \leq d, \end{cases}
\end{aligned} \tag{44}$$

where $C(m)$ is defined in (25).

3. The Abel–Goncharov Type Multiquadric Quasi-Interpolants

3.1. A Kind of Abel–Goncharov Type Multiquadric Quasi-Interpolants. For $N+1$ pairwise distinct points $x_i \in [a, b]$, $0 \leq i \leq N$, and the values $f^{(i)}(x_i)$, $i = 0, 1, \dots, N$, we attach to each node x_i , $i = 0, 1, \dots, N$, a set of nodes X_{im} , $m \in \mathbb{N}$, $m \in N$, $i = 0, 1, \dots, N$, defined by

$$X_{im} = \{x_i, x_{i+1}, \dots, x_{i+m}\}, \quad i = 0, 1, \dots, N, \tag{45}$$

where $x_{N+j} = x_{N-m+j-1}$, $j = 1, \dots, m$. By using each set of nodes X_{im} , $i = 0, 1, \dots, N$, the Abel–Goncharov interpolation operator in (8) can be denoted by P_m^i , $i = 0, 1, \dots, N$, i.e.,

$$(P_m^i[f; x_i, x_{i+1}, \dots, x_{i+m}])^{(k)} = f^{(k)}(x_k), \quad i \leq k \leq i+m, 0 \leq i \leq N, \tag{46}$$

where we have

$$(P_m^i[f; x_i, x_{i+1}, \dots, x_{i+m}])^{(k)}(x) = \sum_{j=i}^{i+m} g_{j-i}^{(i)}(x) f^{(j-i)}(x_j), \quad i = 0, 1, \dots, N, \tag{47}$$

with the Goncharov polynomials

$$g_0^{(i)}(x) = 1,$$

$$g_1^{(i)}(x) = x - x_i,$$

, ...,

$$g_k^{(i)}(x) = \frac{1}{k!} \left[x^k - \sum_{j=0}^{k-1} g_j^{(i)}(x) \binom{k}{j} \right], \quad k = 2, \dots, m. \tag{48}$$

Theorem 5. The Abel–Goncharov interpolants P_m^i , $i = 0, 1, \dots, N$, possess the degree of exactness m .

Proof. The proof is given similarly to that of Theorem 1. $\square$

Furthermore, we recall the well-known multiquadric quasi-interpolation operator $\mathcal{L}_B$, defined by Beatson and Powell in [6], as follows:

$$(\mathcal{L}_B f)(x) = f(x_0) \psi_0(x) + \sum_{i=1}^{N-1} f(x_i) \psi_i(x) + f(x_N) \psi_N(x), \quad x \in I, \tag{49}$$

where

$$\begin{aligned}
 \psi_0(x) &= \frac{1}{2} c^2 \int_{-\infty}^{x_0} \frac{1}{[(x-t)^2 + c^2]^{3/2}} dt + \frac{1}{2} c^2 \int_{x_0}^{x_1} \frac{(x_1-t)/(x_1-x_0)}{[(x-t)^2 + c^2]^{3/2}} dt \\
 &= \frac{1}{2} + \frac{\phi_1(x) - \phi_0(x)}{2(x_1 - x_0)}, \\
 \psi_N(x) &= \frac{1}{2} c^2 \int_{x_N}^{\infty} \frac{1}{[(x-t)^2 + c^2]^{3/2}} dt + \frac{1}{2} c^2 \int_{x_{N-1}}^{x_N} \frac{(t-x_{N-1})/(x_N-x_{N-1})}{[(x-t)^2 + c^2]^{3/2}} dt \\
 &= \frac{1}{2} - \frac{\phi_N(x) - \phi_{N-1}(x)}{2(x_N - x_{N-1})}, \\
 \psi_i(x) &= \frac{1}{2} c^2 \int_{x_{i-1}}^{x_{i+1}} \frac{B_i(t)}{[(x-t)^2 + c^2]^{3/2}} dt \\
 &= \frac{\phi_{i+1}(x) - \phi_i(x)}{2(x_{i+1} - x_i)} - \frac{\phi_i(x) - \phi_{i-1}(x)}{2(x_i - x_{i-1})},
 \end{aligned} \tag{50}$$

for $i = 1, 2, \dots, N-1$, where $B_i(t)$ is the piecewise linear hat function with the knots $\{x_{i-1}, x_i, x_{i+1}\}$ and satisfies $B_i(x_i) = 1$. The quasi-interpolation operator $\mathcal{L}_B$ reproduces constants, i.e.,

$$\sum_{i=0}^N \psi_i(x) = 1. \tag{51}$$

By combining the quasi-interpolation operator $\mathcal{L}_B$ with Abel–Goncharov interpolation polynomials, we construct a kind of Abel–Goncharov type multiquadric quasi-interpolation operator $\mathcal{L}_m^{AG}$ as follows:

$$(\mathcal{L}_m^{AG} f)(x) = \sum_{i=0}^N \psi_i(x) P_m^i[f; x_i, x_{i+1}, \dots, x_{i+m}](x), \quad x \in I, \tag{52}$$

where $P_m^i[f; x_i, x_{i+1}, \dots, x_{i+m}](x)$, $i = 0, 1, \dots, N$, is the Abel–Goncharov interpolation polynomials defined in (47) and $x_{N+j} = x_{N-m+j-1}$, $j = 1, 2, \dots, m$.

Theorem 6. The operator $\mathcal{L}_m^{AG} f$ reproduces all univariate polynomials of degree $\leq m$.

Proof. $\mathcal{L}_m^{AG} p = p$ is proved from the following result:

$$\sum_{i=0}^N \psi_i(x) = 1, \tag{53}$$

$$P_m^i[p; x_i, x_{i+1}, \dots, x_{i+m}] = p, \quad p \in \mathbb{P}_m,$$

for $i = 0, 1, \dots, N$. $\square$

3.2. The Convergence Rate of the Operator. In order to give the convergence rate of the operator $\mathcal{L}_m^{AG}$, we apply the following notations:

$$\begin{aligned}
 I_\rho(x) &= [x - \rho, x + \rho], \quad \rho > 0, \\
 r &= \inf\{\rho > 0: \forall x \in I, I_\rho(x) \cap X \neq \emptyset\},
 \end{aligned} \tag{54}$$

$$M = \max_{x \in I} \#(I_r(x) \cap X),$$

for $X = \{x_0, x_1, \dots, x_N\}$, where $\#(\cdot)$ denotes the cardinality function. Thus, we can obtain $2r = \max\{|x_1 - x_0|, |x_2 - x_1|, \dots, |x_N - x_{N-1}|\}$, and M denotes the maximum number of points from X contained in an interval $I_r(x)$. Then, we provide the following error estimates.

Theorem 7. Suppose that c satisfies

$$c \leq Dr^l, \tag{55}$$

where D is a positive constant and l is a positive integer. If $f(x) \in C^m(I)$, then

$$\|(\mathcal{L}_m^{AG} f)(x) - f(x)\|_\infty \leq KM \|f^{(m)}\|_\infty \mathcal{E}_{l,m}(r), \tag{56}$$

where

$$\mathcal{G}_{l,m}(r) = \begin{cases} r|\ln r|, & m = 1, l = 1, \\ r, & m = 1, l > 1, \\ r^m, & m > 1, m \leq 2l - 1, \\ r^{2l-1}, & m > 1, m > 2l - 1, \end{cases} \quad (57)$$

$$\begin{aligned} d[x_i, x_{i+1}, \dots, x_{i+m}] &= \begin{cases} x_{i+m-1} - x, & x \leq x_i, \\ x_{i+m-1} - x_i, & x_i \leq x \leq x_{i+m-1}, \\ x - x_i, & x_{i+m-1} \leq x, \end{cases} \\ d^m[x_i, x_{i+1}, \dots, x_{i+m}] &= (d[x_i, x_{i+1}, \dots, x_{i+m}])^m. \end{aligned} \quad (58)$$

and K is a positive constant independent of x and X .

Based on (24), (47), and (52), we can give

Proof. Let $x_i, x_{i+1}, \dots, x_{i+m} \in I$ be fixed, $i = 0, 1, \dots, N$, $m \in \mathbb{N}$, $x_i < x_{i+1}, \dots, x_{i+m}$; for each $x \in I$, we set

$$\begin{aligned} |(L_m^{AG} f)(x) - f(x)| &= \left| \sum_{i=0}^N \psi_i(x) P_m^i[f; x_i, x_{i+1}, \dots, x_{i+m}](x) - f(x) \right| \\ &\leq \sum_{i=0}^N \psi_i(x) |P_m^i[f; x_i, x_{i+1}, \dots, x_{i+m}](x) - f(x)| \\ &\leq C(m) \|f^{(m)}\|_{\infty} \mathcal{S}_{l,m}(x), \end{aligned} \quad (59)$$

where

$$\mathcal{S}_{l,m}(x) = \sum_{i=0}^N \psi_i(x) d^m[x_i, x_{i+1}, \dots, x_{i+m}](x). \quad (60)$$

where $\cup_{j=-n}^n Q_r(x + 2rj)$ denotes the covering of I with half open intervals. Thus, for every $i \in \{0, 1, \dots, N\}$, there exists a unique $j \in \{0, 1, \dots, n\}$, such that $x_i \in T_j$. Then, we give the following inequalities as $j \geq 2$:

Assume that

$$n = \left\lceil \frac{x_N - x_0}{2r} \right\rceil + 1,$$

$$Q_\rho(u) = (u - \rho, u + \rho], \quad u \in I, \rho > 0, \quad (61)$$

$$T_j = Q_r(x - 2rj) \cup Q_r(x + 2rj), \quad j = 0, \dots, n,$$

$$\begin{aligned} (2j-1)r &\leq |x - x_i| \leq (2j+1)r, \\ (2(j-1)-1)r &\leq |x - \tau_i| \leq (2(j+1)+1)r, \quad \text{for } \tau_i \in [x_{i-1}, x_{i+1}], \end{aligned} \quad (62)$$

$$|d[x_i, x_{i+1}, \dots, x_{i+m}](x)| \leq (2(j+m-1)+1)r.$$

We have from the definition of M

$$\begin{aligned} 1 &\leq \#(X \cap T_0) \leq M, \\ 1 &\leq \#(X \cap T_j) \leq 2M, \quad j = 1, 2, \dots, n. \end{aligned} \quad (63)$$

When $x_0 \in T_j$, $j = 2, 3, \dots, n$, we have, after some calculations,

$$\begin{aligned} \psi_0(x) &\leq \frac{1}{2} c^2 \int_{-\infty}^{x_0} \frac{1}{|x-t|^3} dt + \frac{1}{2} c^2 \frac{1}{[(x-\tau_0)^2 + c^2]^{3/2}} \int_{x_0}^{x_1} \frac{x_1-t}{x_1-x_0} dt \\ &\leq \frac{1}{4} c^2 |x-x_0|^{-2} + \frac{1}{4} c^2 (x_1-x_0) |x-\tau_0|^{-3} \\ &\leq \frac{1}{4} c^2 [(2j-1)^{-2} r^{-2} + 2(2j-3)^{-3} r^{-2}] \\ &\leq c^2 r^{-2} (2j-3)^{-2}, \end{aligned} \quad (64)$$

where $\tau_0 \in [x_0, x_1]$. When $x_N \in T_j, j = 2, 3, \dots, n$, we get in an analogous manner

$$\psi_N(x) \leq c^2 r^{-2} (2j-3)^{-2}. \quad (65)$$

When $x_i (i = 1, \dots, N-1) \in T_j, j = 2, 3, \dots, n$, we also get

$$\begin{aligned} \psi_i(x) &\leq \frac{1}{2} c^2 \frac{1}{[(x - \tau_i)^2 + c^2]^{3/2}} \int_{x_{i-1}}^{x_{i+1}} B_i(t) dt \\ &\leq \frac{1}{4} c^2 (x_{i+1} - x_{i-1}) |x - \tau_i|^{-3} \\ &\leq c^2 r^{-2} (2j-3)^{-2}, \end{aligned} \quad (66)$$

where $\tau_i \in [x_{i-1}, x_{i+1}]$. Then, we have

$$\begin{aligned} \mathcal{S}_{l,m}(x) &\leq \sum_{x_i \in T_0, T_1} \psi_i(x) d^m[x_i, x_{i+1}, \dots, x_{i+m}](x) \\ &\quad + \sum_{j=2}^n \sum_{x_i \in T_j} \psi_i(x) d^m[x_i, x_{i+1}, \dots, x_{i+m}](x) \\ &\leq \sum_{x_i \in T_0, T_1} d^m[x_i, x_{i+1}, \dots, x_{i+m}](x) \\ &\quad + \sum_{j=2}^n \sum_{x_i \in T_j} \psi_i(x) d^m[x_i, x_{i+1}, \dots, x_{i+m}](x) \\ &\leq M[(2m-1)r]^m + 2M[(2m+1)r]^m \\ &\quad + 2M \sum_{j=2}^n c^2 r^{-2} (2j-3)^{-2} [(2j+2m-1)r]^m \\ &\leq 2M[(2m+1)r]^m + ((2m+1)r)^m \\ &\quad + c^2 r^{m-2} \sum_{j=2}^n (2j-3)^{-2} (2j+2m-1)^m \Big] \\ &\leq 2M(2m+1)^m \left(2r^m + D^2 r^{m+2l-2} \sum_{j=1}^n j^{m-2} \right), \end{aligned} \quad (67)$$

where the last inequality follows the form

$$\begin{cases} 2j-3 \geq j, & j = 3, 4, \dots, n, \\ 2j+2m-1 \leq (2m+1)j, & j = 1, 2, \dots, n, \\ c \leq Dr^l. \end{cases} \quad (68)$$

□

Case 1. ($m = 1$).

- (i) Let $l = 1$. Then, $2r^m + D^2 r^{m+2l-2} \sum_{j=1}^n j^{m-2} = \mathcal{O}(r|\ln r|)$.
- (ii) Let $l > 1$. Then, $2r^m + D^2 r^{m+2l-2} \sum_{j=1}^n j^{m-2} = \mathcal{O}(r)$.

Case 2. ($m > 1$).

- (i) Let $m \leq 2l-1$. Then, $2r^m + D^2 r^{m+2l-2} \sum_{j=1}^n j^{m-2} = \mathcal{O}(r^m)$.

- (ii) Let $m > 2l-1$. Then, $2r^m + D^2 r^{m+2l-2} \sum_{j=1}^n j^{m-2} = \mathcal{O}(r^{2l-1})$.

In an analogous manner, we can obtain the desired error estimation as follows:

Theorem 8. Let c satisfy

$$c \leq Dr^l, \quad (69)$$

where D is a positive constant and l is a positive integer. If $f(x) \in C^{m+1}(I)$, then

$$\|(\mathcal{L}_m^{AG} f)(x) - f(x)\|_{\infty} \leq K' M \|f^{(m+1)}\|_{\infty} \mathcal{E}_{l,m}'(r), \quad (70)$$

where

$$\mathcal{E}_{l,m}'(r) = \begin{cases} r^{2l-1}, & m+1 \geq 2l-1, \\ r^{m+1}, & m+1 \leq 2l-1, \end{cases} \quad (71)$$

and K' is a positive constant independent of x and X .

4. Numerical Examples

In order to investigate the accuracy of our operators, we use the following functions on the interval $[0, 1]$ [33].

$$\text{Saddle } f_1 = \frac{1.25}{6 + 6(3x-1)^2},$$

$$\text{Sphere } f_2 = \frac{\sqrt{64 - 81(x-0.5)^2}}{9} - 0.5,$$

$$\text{Cliff } f_3 = \frac{\tanh(-9x+1)}{2} + 0.5,$$

$$\text{Gentle } f_4 = \frac{\exp(-(81/16)(x-0.5)^2)}{3},$$

$$\text{Steep } f_5 = \frac{\exp(-(81/4)(x-0.5)^2)}{3},$$

$$\begin{aligned} \text{Exponential } f_6 &= 0.75 \exp\left(-\frac{(9x-2)^2}{4}\right) + 0.75 \exp\left(-\frac{(9x+1)^2}{49}\right) \\ &\quad + 0.5 \exp\left(-\frac{(9x-7)^2}{4}\right) + 0.2 \exp(-(9x-4)^2). \end{aligned} \quad (72)$$

For each function $f_i, i = 1, 2, \dots, 6$, we will compare the numerical results of our new operator $\mathcal{L}_m^{AG}$ with the known operator $\mathcal{L}_{H_{2m-1}}$ [10] as $c = r^l$. We consider a uniform grid of 17 points for $\mathcal{L}_1^{AG}$ and $\mathcal{L}_{H_1}$, grid of 11 points for $\mathcal{L}_2^{AG}$, and grid of 8 points for $\mathcal{L}_{H_3}$ on the interval $[0, 1]$. In order to obtain the estimation as accurate as possible, we calculated the approximated functions at the points $(i/101), i = 1, 2, \dots, 100$. Tables 1–6 present the mean and maximum errors, computed for different values of parameters l and m . These results show that the approximations of Abel–Goncharov multiquadric quasi-interpolants $\mathcal{L}_m^{AG}$ are comparable with that of the multiquadric quasi-interpolants $\mathcal{L}_{H_{2m-1}}$.

TABLE 1: Saddle.

(l, m)	$\mathcal{L}_{H_{2m-1}} f_1$	$\mathcal{L}_m^{AG} f_1$		
	ϵ_{mean}	ϵ_{max}	ϵ_{mean}	ϵ_{max}
(2, 1)	0.000398	0.001875	0.000476	0.002273
(2, 2)	0.000336	0.002426	0.001960	0.009792
(3, 1)	0.000395	0.001873	0.000473	0.002270
(3, 2)	0.000330	0.002429	0.001944	0.009775
(4, 1)	0.000395	0.001873	0.000473	0.002270
(4, 2)	0.000330	0.002429	0.001944	0.009775

TABLE 2: Sphere.

(l, m)	$\mathcal{L}_{H_{2m-1}} f_2$	$\mathcal{L}_m^{AG} f_2$		
	ϵ_{mean}	ϵ_{max}	ϵ_{mean}	ϵ_{max}
(2, 1)	0.000449	0.000895	0.000526	0.002686
(2, 2)	0.000035	0.000134	0.000394	0.002127
(3, 1)	0.000445	0.000896	0.000522	0.002687
(3, 2)	0.000032	0.000132	0.000392	0.002128
(4, 1)	0.000445	0.000896	0.000522	0.002687
(4, 2)	0.000032	0.000132	0.000392	0.002128

TABLE 3: Cliff.

(l, m)	$\mathcal{L}_{H_{2m-1}} f_3$	$\mathcal{L}_m^{AG} f_3$		
	ϵ_{mean}	ϵ_{max}	ϵ_{mean}	ϵ_{max}
(2, 1)	0.001952	0.014873	0.002110	0.029450
(2, 2)	0.004444	0.050405	0.010699	0.110462
(3, 1)	0.001943	0.014863	0.002101	0.029426
(3, 2)	0.004342	0.050377	0.010604	0.110363
(4, 1)	0.001943	0.014863	0.002101	0.029426
(4, 2)	0.004342	0.050377	0.010604	0.110363

TABLE 4: Gentle.

(l, m)	$\mathcal{L}_{H_{2m-1}} f_4$	$\mathcal{L}_m^{AG} f_4$		
	ϵ_{mean}	ϵ_{max}	ϵ_{mean}	ϵ_{max}
(2, 1)	0.000545	0.001645	0.000654	0.001975
(2, 2)	0.000185	0.000741	0.002037	0.005950
(3, 1)	0.000542	0.001642	0.000650	0.001975
(3, 2)	0.000181	0.000739	0.002021	0.005949
(4, 1)	0.000542	0.001642	0.000650	0.001975
(4, 2)	0.000181	0.000739	0.002021	0.005949

TABLE 5: Steep.

(l, m)	$\mathcal{L}_{H_{2m-1}} f_5$	$\mathcal{L}_m^{AG} f_5$		
	ϵ_{mean}	ϵ_{max}	ϵ_{mean}	ϵ_{max}
(2, 1)	0.001700	0.006483	0.002150	0.008659
(2, 2)	0.001917	0.009080	0.012294	0.037863
(3, 1)	0.001692	0.006477	0.002138	0.008651
(3, 2)	0.001893	0.009100	0.012200	0.037811
(4, 1)	0.001692	0.006477	0.002138	0.008651
(4, 2)	0.001893	0.009100	0.012200	0.037811

TABLE 6: Exponential.

(l, m)	$\mathcal{L}_{H_{2m-1}} f_6$		$\mathcal{L}_m^{AG} f_6$	
	$\varepsilon_{\text{mean}}$	ε_{max}	$\varepsilon_{\text{mean}}$	ε_{max}
(2, 1)	0.004663	0.015048	0.006739	0.026089
(2, 2)	0.010839	0.035808	0.041908	0.152439
(3, 1)	0.004645	0.015044	0.006714	0.026077
(3, 2)	0.010731	0.035689	0.041734	0.152347
(4, 1)	0.004645	0.015044	0.006714	0.026077
(4, 2)	0.010731	0.035689	0.041733	0.152347

5. Conclusions

In this study, by combining multiquadric quasi-interpolant $\mathcal{L}_B$ with the Abel–Goncharov univariate operator, we construct a kind of Abel–Goncharov multiquadric quasi-interpolants $\mathcal{L}_m^{AG} f$. Meanwhile, we have also proven that the operators $\mathcal{L}_m^{AG} f$ possess the m^{th} degree polynomial reproduction property and good convergence capacity, so that it is convenient for people in various applications. Moreover, the associated algorithm is easily implemented.

In our future work, the univariate Abel–Goncharov type multiquadric quasi-interpolants can be extended to the multivariate case. Moreover, we can also apply the operators to fit scattered data.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The work was supported by the Open Project of Key Laboratory of Symbolic Computation and Knowledge Engineering of Ministry of Education, Jilin University (Grant no. 93K172019K13).

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