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Research Article

A Simulation-Based Study for Progressive Estimation of Population Mean through Traditional and Nontraditional Measures in Stratified Random Sampling

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This study suggests a new optimal family of exponential-type estimators for estimating population mean in stratified random sampling. These estimators are based on the traditional and nontraditional measures of auxiliary information. Expressions for the bias, mean square error, and minimum mean square error of the proposed estimators are derived up to first order of approximation. It is observed that proposed estimators perform better than the traditional estimators (unbiased, combined ratio, and combined regression) and other recent estimators. A real dataset is used to highlight the applicability of proposed estimators. In addition, a simulation study is carried out to assess the performance of new family as compared to other estimators.

1. Introduction

Nowadays, it is common practice to use the auxiliary/ancillary information to boost the efficiency of estimators in survey sampling. Most of the researchers only deal with the traditional information of auxiliary variable(s) such as standard deviation, coefficient of variation, coefficient of skewness, coefficient of kurtosis, and coefficient of correlation. Having edge of this traditional information, many authors have been trying to explore new optimal estimators and families of estimators for estimating population mean under stratified random sampling. Stratified random sampling has often proved needful in improving the precision of estimators over simple random sampling, for instance, see works of Kadilar and Cingi [1, 2], Koyuncu and Kadilar [3, 4], Singh and Vishwakarma [5–7], Shabbir and Gupta [8], Haq and Shabbir [9], Singh and Solanki [10], Yadav et al. [11], Solanki and Singh [10, 12], Javed et al. [13], and Javed and Irfan [14].

The motivation behind this article is to utilize the nontraditional information as well as the traditional information of the auxiliary variable to progress the estimation of population mean in stratified random sampling. This idea is initiated first time in this article under stratified random sampling.

Nontraditional information includes quartile deviation, midrange, interquartile range, quartile average, decile mean, tri-mean, Hodges–Lehmann estimator, and L-moments of an auxiliary variable. L-moments are determined by linear combinations of the expected values of the order statistics (for detail, check the works of Hosking [15] and Shahzad et al. [16]). Furthermore, efficiency of the estimators is uncertain in the occurrence of the extreme values in the dataset. Some of the above nontraditional measures such as decile mean, Hodges–Lehmann estimator, and tri-mean are robust measures. Utilizing these measures, we can well cope with the extreme values/outliers in the dataset. In addition, L-moments also are used to reduce the negative effect of outliers on the estimators.

Rest of the article is organized in the following way. Section 2 presents the useful notations. Section 3 gives comprehensive detail of existing families of estimators. Section 4 suggests a new optimal family of estimators for estimating population mean using traditional and nontraditional measures of auxiliary variable. Expressions for the bias, mean squared error (MSE) and minimum MSE of this family are derived up to first degree of approximation in the same section. A real dataset is used in Section 5 to check the potential of new estimators as compared to existing ones. In Section 6, the performance of suggested family is evaluated by carrying out a simulation study using the same dataset used in Section 5. Section 7 contains the final discussion.

2. Useful Notations

Let us consider a finite population $U=\{U_1,U_2,U_3,\ldots,U_N\}$ of size N, and it can be stratified into L homogenous strata with h^{th} stratum containing $N_h,(h=1,2,\ldots,L)$ units subject to the restriction that $\sum_{h=1}^{L}N_h=N$. A sample of size n_h is drawn under simple random sampling without replacement (SRSWOR) from h^{th} stratum such that $\sum_{h=1}^{L}n_h=n$. Consider the N_h pairs of observations $(y_{hi},x_{hi}),i=1,2,3,\ldots,N_h$, made from h^{th} stratum for the study and auxiliary variables, respectively.

Furthermore, let $\overline{Y}=\overline{Y}_{st}=\sum_{h=1}^{L}W_h\overline{Y}_h$ and $\overline{y}_{st}=\sum_{h=1}^{L}W_h\overline{y}_h$ be the population and sample means of y, respectively, where $\overline{Y}_h=\sum_{i=1}^{N_h}(y_{hi}/N_h),\overline{y}_h=\sum_{i=1}^{n_h}(y_{hi}/n_h)$, and $W_h=(N_h/N)$ are the population mean, sample mean, and the weight of h^{th} stratum, respectively. Following the same lines, $\overline{X}_{st},\overline{x}_{st},\overline{X}_h$ and $\overline{x}_h$ can be defined for the auxiliary variable x.

To derive the expressions for the bias, mean square error (MSE), and minimum mean square error of the existing and proposed estimators, we consider the following relative error terms along with their expectations as

$$\zeta_0=\frac{\overline{y}_{st}-\overline{Y}}{\overline{Y}},\qquad \zeta_1=\frac{\overline{x}_{st}-\overline{X}}{\overline{X}}, \tag{1}$$

such that

$$E(\zeta_i)=0 \text{ for } i=0,1,$$

$$V_{mn}=\sum_{h=1}^{L}W_h^{m+n}\frac{E\left[(\overline{y}_h-\overline{Y}_h)^m(\overline{x}_h-\overline{X}_h)^n\right]}{\overline{Y}^m\overline{X}^n}. \tag{2}$$

From (2), we can write as below:

$$E(\zeta_0^2)=\frac{\sum_{h=1}^{L}W_h^2\phi_hS_{yh}^2}{\overline{Y}^2}=V_{20},$$
$$E(\zeta_1^2)=\frac{\sum_{h=1}^{L}W_h^2\phi_hS_{xh}^2}{\overline{X}^2}=V_{02}, \tag{3}$$
$$E(\zeta_0\zeta_1)=\frac{\sum_{h=1}^{L}W_h^2\phi_hS_{yxh}}{\overline{Y}\overline{X}}=V_{11},$$

where

$$\phi_h=\frac{(1-f_h)}{n_h},$$
$$f_h=\frac{n_h}{N_h},$$
$$S_{yh}^2=\frac{\sum_{i=1}^{N_h}(y_{hi}-\overline{Y}_h)^2}{N_h-1}, \tag{4}$$
$$S_{xh}^2=\sum_{i=1}^{N_h}\frac{(x_{hi}-\overline{X}_h)^2}{N_h-1},$$
$$S_{yxh}=\frac{\sum_{i=1}^{N_h}(y_{hi}-\overline{Y}_h)(x_{hi}-\overline{X}_h)}{N_h-1}.$$

Some other formulas for h^{th} stratum, under stratified random sampling are listed below:

$$\begin{aligned}
\text{Coefficient of Varaiation } C_{xh}&=\frac{S_{xh}}{\overline{X}_h},\\
\text{Coefficient of Skewness } \beta_{1xh}&=\frac{N\sum_{i=1}^{N}(x_{ih}-\overline{X}_h)^3}{(N-1)(N-2)S_{xh}^3},\\
\text{Coefficient of kurtosis } \beta_{2xh}&=\frac{N(N+1)\sum_{i=1}^{N}(x_{ih}-\overline{X}_h)^4}{(N-1)(N-2)(N-3)S_{xh}^4}-\frac{3(N-1)^2}{(N-2)(N-3)},\\
\text{Quartile deviation } QD_{xh}&=\frac{Q_{3h}-Q_{1h}}{2},\\
\text{Midrange } \mathrm{MR}_{xh}&=\frac{x_{(1)h}+x_{(N)h}}{2},\\
\text{Interquartile range } QR_{xh}&=Q_{3h}-Q_{1h},\\
\text{Quartile average } QA_{xh}&=\frac{Q_{3h}+Q_{1h}}{2},\\
\text{Tri}-\text{mean } \mathrm{TM}_{xh}&=\frac{Q_{1h}+2Q_{2h}+Q_{3h}}{4},\\
\text{Hodge}-\text{Lehmann } \mathrm{HL}_{xh}&=\text{Median}\left(\frac{x_{jh}+x_{kh}}{2}\right),\quad 1\le jh\le kh\le N,
\end{aligned} \tag{5}$$

where Q_1, Q_2, and Q_3 are the first, second, and third quartiles, respectively, $x_{(1)}$ is the minimum value, and $x_{(N)}$ is the maximum value of the data.

3. Some Existing Estimators/Classes of Estimators

This section gives a brief introduction of some well-known estimators/classes of estimators from the literature.

3.1. Usual Estimators. In stratified random sampling usual unbiased $\hat{\overline{Y}}_{st}$, combined ratio $\hat{\overline{Y}}_{CR}$ and combined regression $\hat{\overline{Y}}_{CReg}$ estimators and their MSEs are detailed below:

$$\hat{\overline{Y}}_{st} = \sum_{h=1}^{L} W_h \overline{y}_h,$$

$$\hat{\overline{Y}}_{CR} = \overline{y}_{st}\left(\frac{\overline{X}}{\overline{x}_{st}}\right),$$

$$\hat{\overline{Y}}_{CReg} = \left[\overline{y}_{st} + \hat{b}\left(\overline{X} - \overline{x}_{st}\right)\right], \text{with } \hat{b} = \frac{\sum_{h=1}^{L} W_h^2 \phi_h S_{yxh}}{\sum_{h=1}^{L} W_h^2 \phi_h S_{xh}^2},$$

$$\text{MSE}\left(\hat{\overline{Y}}_{st}\right) = \text{Var}\left(\hat{\overline{Y}}_{st}\right) = \overline{Y}^2 V_{20},$$

$$\text{MSE}\left(\hat{\overline{Y}}_{CR}\right) = \overline{Y}^2\left(V_{20} + V_{02} - 2V_{11}\right),$$

$$\text{MSE}\left(\hat{\overline{Y}}_{CReg}\right) = \overline{Y}^2 V_{20}\left(1 - \rho_{st}^2\right), \tag{6}$$

where $\rho_{st} = \sum_{h=1}^{L} W_h \rho_{yxh} = \sum_{h=1}^{L} W_h^2 \phi_h S_{yxh} / \sqrt{\sum_{h=1}^{L} W_h^2 \phi_h S_{yh}^2} \sqrt{\sum_{h=1}^{L} W_h^2 \phi_h S_{xh}^2} = V_{11}/\sqrt{V_{20}}\sqrt{V_{02}}$.

Bahl and Tuteja [17] suggested ratio and product exponential-type estimators for population mean under stratified random sampling as

$$\hat{\overline{Y}}_{BT,Re} = \overline{y}_{st} \exp\left(\frac{\overline{X} - \overline{x}_{st}}{\overline{X} + \overline{x}_{st}}\right), \tag{7}$$

$$\hat{\overline{Y}}_{BT,Pe} = \overline{y}_{st} \exp\left(\frac{\overline{x}_{st} - \overline{X}}{\overline{x}_{st} + \overline{X}}\right). \tag{8}$$

Average of (7) and (8) can be written as

$$\hat{\overline{Y}}_{BT,\text{Avg}} = \frac{\overline{y}_{st}}{2}\left\{\exp\left(\frac{\overline{X} - \overline{x}_{st}}{\overline{X} + \overline{x}_{st}}\right) + \exp\left(\frac{\overline{x}_{st} - \overline{X}}{\overline{x}_{st} + \overline{X}}\right)\right\},$$

$$\text{MSE}\left(\hat{\overline{Y}}_{BT,Re}\right) \cong \overline{Y}^2\left(V_{20} + \frac{V_{02}}{4} - V_{11}\right),$$

$$\text{MSE}\left(\hat{\overline{Y}}_{BT,Pe}\right) \cong \overline{Y}^2\left(V_{20} + \frac{V_{02}}{4} + V_{11}\right). \tag{9}$$

3.2. Koyuncu and Kadilar [3]. To estimate the population mean under stratified random sampling, a family of ratio estimators was introduced by Koyuncu and Kadilar [3] as below:

$$\hat{\overline{Y}}_K = \overline{y}_{st}\left[\frac{a_{st}\overline{X} + b_{st}}{\gamma\left(a_{st}\overline{x}_{st} + b_{st}\right) + (1 - \gamma)\left(a_{st}\overline{X} + b_{st}\right)}\right]^g, \tag{10}$$

where γ and g are suitable constants and $a_{st}\ (\neq 0)$ and b_{st} are either real numbers or functions of known parameters of the auxiliary variable such as coefficient of skewness, coefficient of kurtosis, coefficient of variation, and coefficient of correlation.

Up to the first order of approximation, the bias and MSE of $\hat{\overline{Y}}_K$ are given by

$$\text{Bias}\left(\hat{\overline{Y}}_K\right) \cong \overline{Y}\left(\frac{g(g+1)}{2}\gamma^2\eta^2 V_{02} - g\gamma\eta V_{11}\right),$$

$$\text{MSE}\left(\hat{\overline{Y}}_K\right) \cong \overline{Y}^2\left(V_{20} + g^2\gamma^2\eta^2 V_{02} - 2g\gamma\eta V_{11}\right),$$

$$\text{where } \eta = \frac{a_{st}\overline{X}}{a_{st}\overline{X} + b_{st}},$$

$$\text{At } (g\gamma\eta)_{\text{opt}} = \frac{V_{11}}{V_{02}}, \text{ the minimum MSE } \hat{\overline{Y}}_K \text{ is given as,}$$

$$\text{MSE}_{\min}\left(\hat{\overline{Y}}_K\right) \cong \overline{Y}^2 V_{20}\left(1 - \rho_{st}^2\right). \tag{11}$$

3.3. Koyuncu and Kadilar [4]. Koyuncu and Kadilar [4] considered the ratio estimator of Gupta and Shabbir [18] and suggested an improved estimator defined as below:

$$\hat{\overline{Y}}_{KK} = \left[\theta_1\overline{y}_{st} + \theta_2\left(\overline{X} - \overline{x}_{st}\right)\right]\left(\frac{a_{st}\overline{X} + b_{st}}{a_{st}\overline{x}_{st} + b_{st}}\right), \tag{12}$$

where θ_1 and θ_2 are suitably chosen weights.

Given below are the expressions, up to first degree of approximation, for the bias and MSE of $\hat{\overline{Y}}_{KK}$, respectively:

$$\text{Bias}\left(\hat{\overline{Y}}_{KK}\right) \cong -\overline{Y} + \overline{X}\eta\theta_2 V_{02} + \overline{Y}\theta_1\left(1 + \eta^2 V_{02} - \eta V_{11}\right), \tag{13}$$

$$\begin{aligned}\text{MSE}\left(\hat{\overline{Y}}_{KK}\right) \cong{} & \overline{Y}^2 + \overline{X}\theta_2\left(-2\overline{Y}\eta + \overline{X}\theta_2\right)V_{02} - 2\overline{Y}\theta_1 \\ & \cdot\left\{\overline{Y} + \eta\left(\overline{Y}\eta - 2\overline{X}\theta_2\right)V_{02} + \left(-\overline{Y}\eta + \overline{X}\theta_2\right)V_{11}\right\} \\ & + \overline{Y}^2\theta_1^2\left(1 + 3\eta^2 V_{02} - 4\eta V_{11} + V_{20}\right).\end{aligned} \tag{14}$$

The suitable weights of θ_1 and θ_2 are given by

$$\theta_1 = \frac{\left(-1+\eta^2 V_{02}\right)V_{02}}{\eta^2 V_{02}^2 + V_{11}^2 - V_{02}\left(1+V_{20}\right)},$$
$$\theta_2 = -\frac{\overline{Y}\left\{V_{11} + \eta\left(\eta^2 V_{02}^2 - V_{11}^2 + V_{02}\left(-1-\eta V_{11} + V_{20}\right)\right)\right\}}{\overline{X}\left\{\eta^2 V_{02}^2 + V_{11}^2 - V_{02}\left(1+V_{20}\right)\right\}}. \quad (15)$$

Inserting the above weights of θ_1 and θ_2 in (14), we get the minimum MSE of $\widehat{\overline{Y}}_{KK}$ as

$$\mathrm{MSE}_{\min}\left(\widehat{\overline{Y}}_{KK}\right) \cong \frac{\overline{Y}^2\left(-1+\eta^2 V_{02}\right)\left(-V_{11}^2 + V_{02}V_{20}\right)}{\eta^2 V_{02}^2 + V_{11}^2 - V_{02}\left(1+V_{20}\right)}. \quad (16)$$

3.4. Shabbir and Gupta [8]. Given below is a ratio-type estimator suggested by Shabbir and Gupta [8] in stratified random sampling:

$$\widehat{\overline{Y}}_{SG} = \left[\theta_3 \overline{y}_{st} + \theta_4\left(\overline{X} - \overline{x}_{st}\right)\right]\exp\left(\frac{\overline{A} - \overline{a}_{st}}{\overline{A} + \overline{a}_{st}}\right), \quad (17)$$

where θ_3 and θ_4 are the constants to be determined. Also, we consider that

$$a_{hi} = x_{hi} + N\overline{X},$$
$$\overline{a}_{st} = \overline{x}_{st} + N\overline{X}, \quad (18)$$
$$\overline{A} = (1+N)\overline{X}.$$

Expressions for the bias and the MSE of $\widehat{\overline{Y}}_{SG}$, respectively, are given below:

$$\mathrm{Bias}\left(\widehat{\overline{Y}}_{SG}\right) \cong \frac{4(1+N)\left\{-2(1+N)\overline{Y} + \overline{X}\theta_4 V_{02}\right\} + \overline{Y}\theta_3\left\{3V_{02} + 4(1+N)\left(2+2N-V_{11}\right)\right\}}{8(1+N)^2}, \quad (19)$$

$$\mathrm{MSE}\left(\widehat{\overline{Y}}_{SG}\right) \cong \overline{Y}^2\left(\theta_3 - 1\right)^2 + \frac{\overline{Y}^2 V_{11}\theta_3}{(1+N)} - \frac{3\overline{Y}^2 V_{02}\theta_3}{4(1+N)^2} + \overline{Y}^2 V_{02}\theta_3^2 - \frac{2\overline{Y}^2 V_{11}\theta_3^2}{(1+N)} + \frac{\overline{Y}^2 V_{02}\theta_3^2}{(1+N)^2}$$
$$- \frac{\overline{X}\overline{Y}V_{02}\theta_4}{(1+N)} - 2\overline{X}\overline{Y}V_{11}\theta_3\theta_4 + \frac{2\overline{X}\overline{Y}V_{02}\theta_3\theta_4}{(1+N)} + \overline{X}^2 V_{02}\theta_4^2. \quad (20)$$

The suitable weights of θ_3 and θ_4 are given as

$$\theta_3 = \frac{V_{02}\left\{8(1+N)^2 - V_{02}\right\}}{8(1+N)^2\left\{-V_{11}^2 + V_{02}\left(1+V_{20}\right)\right\}},$$
$$\theta_4 = \frac{\overline{Y}\left[V_{02}^2 + 4(1+N)^2\left(2+2N-V_{11}\right)V_{11} + (1+N)V_{02}\left\{-V_{11}^2 + 4(1+N)\left(V_{20}-1\right)\right\}\right]}{8\overline{X}(1+N)^3\left\{-V_{11}^2 + V_{02}\left(1+V_{20}\right)\right\}}. \quad (21)$$

Putting weights of θ_3 and θ_4 in (20), we have minimum MSE of $\widehat{\overline{Y}}_{SG}$ as

$$\mathrm{MSE}_{\min}\left(\widehat{\overline{Y}}_{SG}\right) \cong \frac{\overline{Y}^2\left[V_{02}^3 + 64(1+N)^4 V_{11}^2 + 16(1+N)^2 V_{02}^2 V_{20} - 16(1+N)^2 V_{02}\left\{V_{11}^2 + 4(1+N)^2 V_{20}\right\}\right]}{64(1+N)^4\left\{V_{11}^2 - V_{02}\left(1+V_{20}\right)\right\}}. \quad (22)$$

3.5. Haq and Shabbir [9]. Haq and Shabbir [9] proposed two exponential ratio-type families of estimators detailed below:

$$\begin{aligned}\widehat{\overline{Y}}_{HS1} &= [\theta_5\overline{y}_{st} + \theta_6(\overline{X} - \overline{x}_{st})]\exp\left(\frac{a_{st}\overline{X} + b_{st}}{\gamma(a_{st}\overline{x}_{st} + b_{st}) + (1-\gamma)(a_{st}\overline{X} + b_{st})} - 1\right),\\ \widehat{\overline{Y}}_{HS2} &= [\theta_7\overline{y}_{st} + \theta_8(\overline{X} - \overline{x}_{st})]\exp\left(\frac{a_{st}\overline{X} + b_{st}}{\gamma(a_{st}\overline{x}_{st} + b_{st}) + (1-\gamma)(a_{st}\overline{X} + b_{st})} - 1\right)\\ &\times\left\{\left(\frac{1}{2}\right)\left(\frac{a_{st}\overline{X} + b_{st}}{\gamma(a_{st}\overline{x}_{st} + b_{st}) + (1-\gamma)(a_{st}\overline{X} + b_{st})} + \frac{\gamma(a_{st}\overline{x}_{st} + b_{st}) + (1-\gamma)(a_{st}\overline{X} + b_{st})}{a_{st}\overline{X} + b_{st}}\right)\right\}^2,\end{aligned} \tag{23}$$

where $\theta_5, \theta_6, \theta_7$, and θ_8 are the suitable constants.

Given below are the expressions for bias and MSE of $\widehat{\overline{Y}}_{HS1}$ and $\widehat{\overline{Y}}_{HS2}$, respectively:

$$\begin{aligned}\text{Bias}\left(\widehat{\overline{Y}}_{HS1}\right) &\cong -\overline{Y} + \overline{X}\psi\theta_6 V_{02} + \theta_5\left(\overline{Y} + \frac{3\overline{Y}\psi^2 V_{02}}{2} - \overline{Y}\psi V_{11}\right),\\ \text{Bias}\left(\widehat{\overline{Y}}_{HS2}\right) &\cong -\overline{Y} + \overline{X}\psi\theta_8 V_{02} + \theta_7\left(\overline{Y} + \frac{5\overline{Y}\psi^2 V_{02}}{2} - \overline{Y}\psi V_{11}\right),\end{aligned} \tag{24}$$

$$\begin{aligned}\text{MSE}\left(\widehat{\overline{Y}}_{HS1}\right) \cong{}& \overline{Y}^2 + \overline{X}\theta_6(-2\overline{Y}\psi + \overline{X}\theta_6)V_{02} + \overline{Y}\theta_5\{-2\overline{Y} + \psi(-3\overline{Y}\psi + 4\overline{X}\theta_6)V_{02} + 2(\overline{Y}\psi - \overline{X}\theta_6)V_{11}\}\\ &+ \overline{Y}^2\theta_5^2\left(1 + 4\psi^2 V_{02} - 4\psi V_{11} + V_{20}\right),\end{aligned} \tag{25}$$

$$\begin{aligned}\text{MSE}\left(\widehat{\overline{Y}}_{HS2}\right) \cong{}& \overline{Y}^2 + \overline{X}\theta_8(-2\overline{Y}\psi + \overline{X}\theta_8)V_{02} + \overline{Y}\theta_7\{-2\overline{Y} + \psi(-5\overline{Y}\psi + 4\overline{X}\theta_8)V_{02} + 2(\overline{Y}\psi - \overline{X}\theta_8)V_{11}\}\\ &+ \overline{Y}^2\theta_7^2\left(1 + 6\psi^2 V_{02} - 4\psi V_{11} + V_{20}\right),\end{aligned} \tag{26}$$

where $\psi = \gamma\eta$ and η is defined earlier.

The weights of $\theta_5, \theta_6, \theta_7$, and θ_8 are determined as below:

$$\begin{aligned}\theta_5 &= \frac{V_{02}\left(2 - \psi^2 V_{02}\right)}{2\left\{-V_{11}^2 + V_{02}(1 + V_{20})\right\}},\\ \theta_6 &= \frac{\overline{Y}\left\{2\psi^3 V_{02}^2 - 2V_{11}(-1 + \psi V_{11}) - \psi V_{02}(2 + \psi V_{11} - 2V_{20})\right\}}{2\overline{X}\left\{-V_{11}^2 + V_{02}(1 + V_{20})\right\}},\\ \theta_7 &= \frac{V_{02}\left(2 + \psi^2 V_{02}\right)}{2\left\{-V_{11}^2 + V_{02}\left(1 + 2\psi^2 V_{02} + V_{20}\right)\right\}},\\ \theta_8 &= \frac{\overline{Y}\left\{2\psi^3 V_{02}^2 - 2V_{11}(-1 + \psi V_{11}) + \psi V_{02}(-2 + \psi V_{11} + 2V_{20})\right\}}{2\overline{X}\left\{-V_{11}^2 + V_{02}\left(1 + 2\psi^2 V_{02} + V_{20}\right)\right\}}.\end{aligned} \tag{27}$$

By substituting values of θ_5 and θ_6 in (25) and θ_7 and θ_8 in (26), we get minimum MSE of $\widehat{\overline{Y}}_{HS1}$ and $\widehat{\overline{Y}}_{HS2}$, respectively:

$$\begin{aligned} \mathrm{MSE}_{\min}\left(\widehat{\overline{Y}}_{HS1}\right) &\cong \frac{\overline{Y}^2\left[4V_{11}^2+V_{02}\left\{\psi^4V_{02}^2-4\psi^2V_{11}^2+4\left(-1+\psi^2V_{02}\right)V_{20}\right\}\right]}{4\left\{V_{11}^2-V_{02}\left(1+V_{20}\right)\right\}}, \\ \mathrm{MSE}_{\min}\left(\widehat{\overline{Y}}_{HS2}\right) &\cong \frac{\overline{Y}^2\left[4V_{11}^2+V_{02}\left\{9\psi^4V_{02}^2-4\psi^2V_{11}^2+4\left(-1+\psi^2V_{02}\right)V_{20}\right\}\right]}{4\left\{V_{11}^2-V_{02}\left(1+2\psi^2V_{02}+V_{20}\right)\right\}}. \end{aligned} \tag{28}$$

3.6. Singh and Solanki [19]. Singh and Solanki [19] proposed a family of estimators as given below:

$$\widehat{\overline{Y}}_{SS1}=\left[\theta_9\overline{y}_{st}\left\{\frac{\gamma\left(a_{st}\overline{x}_{st}+b_{st}\right)+(1-\gamma)\left(a_{st}\overline{X}+b_{st}\right)}{\left(a_{st}\overline{X}+b_{st}\right)}\right\}^{\delta_1}+\theta_{10}\overline{y}_{st}\left\{\frac{\left(a_{st}\overline{X}+b_{st}\right)}{\gamma\left(a_{st}\overline{x}_{st}+b_{st}\right)+(1-\gamma)\left(a_{st}\overline{X}+b_{st}\right)}\right\}^{\delta_2}\right], \tag{29}$$

where δ_1 and δ_2 are suitable scalars and θ_9 and θ_{10} are the constants to be determined to make the MSE minimum. Assuming different values of δ_1, δ_2, and γ proposed family $\widehat{\overline{Y}}_{SS1}$ reduces to the ratio-type $\widehat{\overline{Y}}_{SS1R}$, product-type $\widehat{\overline{Y}}_{SS1P}$, and ratio-cum-product-type $\widehat{\overline{Y}}_{SS1RP}$ estimators.

For ratio-type, product-type, and ratio-cum-product-type estimators suitable values are $(\delta_1=0, \delta_2=1, \gamma=1)$, $(\delta_1=0, \delta_2=-1, \gamma=1)$, and $(\delta_1=1, \delta_2=1, \gamma=1)$, respectively.

For bias and MSE of $\widehat{\overline{Y}}_{SS1}$, we consider the expressions given below:

$$\mathrm{Bias}\left(\widehat{\overline{Y}}_{SS1}\right)\cong\overline{Y}\left[\theta_9\left\{1+\gamma\delta_1\eta V_{11}+\frac{\delta_1\left(\delta_1-1\right)}{2}\gamma^2\eta^2V_{02}\right\}+\theta_{10}\left\{1-\gamma\delta_2\eta V_{11}+\frac{\delta_2\left(\delta_2+1\right)}{2}\gamma^2\eta^2V_{02}\right\}-1\right], \tag{30}$$

$$\mathrm{MSE}\left(\widehat{\overline{Y}}_{SS1}\right)\cong\overline{Y}^2\left[1+\theta_9^2A_1+\theta_{10}^2A_2+2\theta_9\theta_{10}A_3-2\theta_9A_4-2\theta_{10}A_5\right], \tag{31}$$

where

$$\begin{aligned} A_1 &= \left[1+V_{20}+4\gamma\delta_1\eta V_{11}+\delta_1\left(2\delta_1-1\right)\gamma^2\eta^2V_{02}\right], \\ A_2 &= \left[1+V_{20}-4\gamma\delta_2\eta V_{11}+\delta_2\left(2\delta_2+1\right)\gamma^2\eta^2V_{02}\right], \\ A_3 &= \left[1+V_{20}+2\gamma\left(\delta_1-\delta_2\right)\eta V_{11}+\left(\frac{\gamma^2\eta^2}{2}\right)\left(\delta_1-\delta_2\right)\left(\delta_1-\delta_2-1\right)V_{02}\right], \\ A_4 &= \left[1+\gamma\delta_1\eta V_{11}+\frac{\delta_1\left(\delta_1-1\right)}{2}\gamma^2\eta^2V_{02}\right], \\ A_5 &= \left[1-\gamma\delta_2\eta V_{11}+\frac{\delta_2\left(\delta_2+1\right)}{2}\gamma^2\eta^2V_{02}\right]. \end{aligned} \tag{32}$$

The weights of θ_9 and θ_{10} are determined as below:

$$\theta_9 = \frac{A_2A_4 - A_5A_3}{A_2A_1 - A_3^2},$$
$$\theta_{10} = \frac{A_1A_5 - A_4A_3}{A_2A_1 - A_3^2}. \tag{33}$$

Substituting the above weights in (31), we get the minimum MSE as given by

$$\text{MSE}_{\min}\left(\hat{\hat{Y}}_{SS1}\right) \cong \overline{Y}^2\left[1 - \frac{A_2A_4^2 - 2A_4A_3A_5 + A_5^2A_1}{A_2A_1 - A_3^2}\right]. \tag{34}$$

3.7. Solanki and Singh [10]. Given below is the class of estimators suggested by Solanki and Singh [10]:

$$\hat{\hat{Y}}_{SS2} = \left[\theta_{11}\overline{y}_{st}\left[\frac{a_{st}\overline{X} + b_{st}}{\gamma(a_{st}\overline{x}_{st} + b_{st}) + (1-\gamma)(a_{st}\overline{X} + b_{st})}\right]^{\delta_3} + \theta_{12}\overline{y}_{st}\exp\left[\frac{\delta_4\{(a_{st}\overline{X} + b_{st}) - (a_{st}\overline{x}_{st} + b_{st})\}}{(a_{st}\overline{X} + b_{st}) + (a_{st}\overline{x}_{st} + b_{st})}\right]\right], \tag{35}$$

where θ_{11} and θ_{12} are the feasible weights to be found such that the MSE is minimal. Here, $\gamma = 1$; δ_3 and δ_4 being the constants take values (0, 1, −1) for obtaining different estimators like

(i) Ratio-type exponential estimators $\hat{\hat{Y}}_{SS2R}$ for $(\delta_3 = 0, \delta_4 = 1)$

(ii) Product-type exponential estimators $\hat{\hat{Y}}_{SS2P}$ for $(\delta_3 = 0, \delta_4 = -1)$

(iii) Ratio-ratio-type exponential estimators $\hat{\hat{Y}}_{SS2RR}$ for $(\delta_3 = 1, \delta_4 = 1)$

(iv) Product-product-type exponential estimators $\hat{\hat{Y}}_{SS2PP}$ for $(\delta_3 = -1, \delta_4 = -1)$

(v) Ratio-product-type exponential estimators $\hat{\hat{Y}}_{SS2RP}$ for $(\delta_3 = 1, \delta_4 = -1)$

(vi) Product-ratio-type exponential estimators $\hat{\hat{Y}}_{SS2PR}$ for $(\delta_3 = -1, \delta_4 = 1)$

For bias and MSE of $\hat{\hat{Y}}_{SS2}$, we consider the expressions given below:

$$\text{Bias}\left(\hat{\hat{Y}}_{SS2}\right) \cong \overline{Y}\left[\theta_{11}\left\{1 + \frac{\gamma\delta_3\eta}{2}V_{02}\left(\gamma\eta(\delta_3+1) - 2\frac{V_{11}}{V_{02}}\right)\right\} + \theta_{12}\left\{1 + \frac{\delta_4\eta}{8}V_{02}\left(\eta(\delta_4+2) - 4\frac{V_{11}}{V_{02}}\right)\right\} - 1\right], \tag{36}$$

$$\text{MSE}\left(\hat{\hat{Y}}_{SS2}\right) \cong \overline{Y}^2\left[1 + \theta_{11}^2B_1 + \theta_{12}^2B_2 + 2\theta_{11}\theta_{12}B_3 - 2\theta_{11}B_4 - 2\theta_{12}B_5\right], \tag{37}$$

where

$$B_1 = \left[1 + V_{20} + \gamma^2\eta^2\left(2\delta_3^2 + \delta_3\right)V_{02} - 4\gamma\delta_3\eta V_{11}\right],$$
$$B_2 = \left[1 + V_{20} + \frac{\eta^2\left(\delta_4^2 + \delta_4\right)}{2}V_{02} - 2\delta_4\eta V_{11}\right],$$
$$B_3 = \left[1 + V_{20} + \left(\frac{\eta^2\left[(2\gamma\delta_3 + \delta_4)^2 + 2\left(2\gamma^2\delta_3 + \delta_4\right)\right]}{8}\right)V_{02} - \eta(2\gamma\delta_3 + \delta_4)V_{11}\right],$$
$$B_4 = \left[1 + \frac{\gamma^2\eta^2\left(\delta_3^2 + \delta_3\right)}{2}V_{02} - \gamma\delta_3\eta V_{11}\right],$$
$$B_5 = \left[1 + \frac{\eta^2\left(\delta_4^2 + 2\delta_4\right)}{8}V_{02} - \frac{\delta_4\eta}{2}V_{11}\right]. \tag{38}$$

The suitable weights of θ_{11} and θ_{12} are as below:

$$\theta_{11} = \frac{B_2B_4 - B_5B_3}{B_2B_1 - B_3^2},$$
$$\theta_{12} = \frac{B_1B_5 - B_4B_3}{B_2B_1 - B_3^2}. \qquad (39)$$

Substituting these suitable weights in (37), we have the minimum MSE as given by

$$\mathrm{MSE}_{\min}\left(\hat{\hat{Y}}_{SS2}\right) \cong \overline{Y}^2\left[1 - \frac{B_2B_4^2 - 2B_4B_3B_5 + B_5^2B_1}{B_2B_1 - B_3^2}\right]. \qquad (40)$$

3.8. Solanki and Singh [12]. Recently, Solanki and Singh [12] defined an improved estimation given as

$$\hat{\hat{Y}}_{SS3} = \left[\theta_{13}\overline{y}_{st}\left(\frac{\overline{X}_{st}^*}{\overline{x}_{st}^*}\right)^{\delta_5}\exp\left\{\frac{\delta_6\left(\overline{X}_{st}^* - \overline{x}_{st}^*\right)}{\left(\overline{X}_{st}^* + \overline{x}_{st}^*\right)}\right\} + \theta_{14}\overline{y}_{st}\left(\frac{\overline{x}_{st}^*}{\overline{X}_{st}^*}\right)^{\delta_7}\exp\left\{\frac{\delta_8\left(\overline{x}_{st}^* - \overline{X}_{st}^*\right)}{\left(\overline{X}_{st}^* + \overline{x}_{st}^*\right)}\right\}\right], \qquad (41)$$

where θ_{13} and θ_{14} are the suitably chosen weights to get minimum MSE. $\overline{X}_{st}^* = \sum_{h=1}^{L} W_h\left(a_h\overline{X}_h + b_h\right)$ and $\overline{x}_{st}^* = \sum_{h=1}^{L} W_h\left(a_h\overline{x}_h + b_h\right)$, $a_h(\neq 0), b_h$, are real number to parameters related to auxiliary variate x. Here, $\delta_5, \delta_6, \delta_7$, and δ_8 being the constants take values $(-1, 0, 1)$ for obtaining different classes of estimators $\hat{\hat{Y}}_{SS3i}$:

(i) $\hat{\hat{Y}}_{SS31}$, for $\delta_5 = -1, \delta_6 = 1, \delta_7 = 1$, and $\delta_8 = 1$

(ii) $\hat{\hat{Y}}_{SS32}$, for $\delta_5 = -1, \delta_6 = 1, \delta_7 = 1$, and $\delta_8 = 0$

(iii) $\hat{\hat{Y}}_{SS33}$, for $\delta_5 = +1, \delta_6 = 1, \delta_7 = 1$, and $\delta_8 = 1$

For bias and MSE of $\hat{\hat{Y}}_{SS3}$, we consider the expressions given below:

$$\mathrm{Bias}\left(\hat{\hat{Y}}_{SS3}\right) \cong \left[\begin{array}{c}(\theta_{13} + \theta_{14} - 1)\overline{Y} - \dfrac{t_1\theta_{13}}{2\overline{X}_{st}^*}\displaystyle\sum_{h=1}^{L} W_h^2\phi_h a_h S_{xh}^2\left\{t_3 + \frac{(t_1 - 2)}{4}a_h R^*\right\} \\ + \dfrac{t_2\theta_{14}}{2\overline{X}_{st}^*}\displaystyle\sum_{h=1}^{L} W_h^2\phi_h a_h S_{xh}^2\left\{t_3 - \frac{(t_2 - 2)}{4}a_h R^*\right\}\end{array}\right], \qquad (42)$$

where $t_1 = (2\delta_5 + \delta_6), t_2 = (2\delta_7 + \delta_8)$, and $t_3 = S_{yxh}/S_{xh}^2$ and $R^* = (\overline{Y}/\overline{X}_{st}^*)$:

$$\mathrm{MSE}\left(\hat{\hat{Y}}_{SS3}\right) \cong \overline{Y}^2\left[1 + \theta_{13}^2C_1 + \theta_{14}^2C_2 + 2\theta_{13}\theta_{14}C_3 - 2\theta_{13}C_4 - 2\theta_{14}C_5\right], \qquad (43)$$

where

$$C_1 = 1 + \frac{1}{\overline{Y}^2} \sum_{h=1}^{L} W_h^2 \phi_h \left\{ S_{yh}^2 - 2t_1 a_h R^* S_{yxh} + \frac{t_1(t_1+1)}{2} a_h^2 R^{*2} S_{xh}^2 \right\},$$
$$C_2 = 1 + \frac{1}{\overline{Y}^2} \sum_{h=1}^{L} W_h^2 \phi_h \left\{ S_{yh}^2 + 2t_2 a_h R^* S_{yxh} + \frac{t_2(t_2-1)}{2} a_h^2 R^{*2} S_{xh}^2 \right\},$$
$$C_3 = 1 + \frac{1}{\overline{Y}^2} \sum_{h=1}^{L} W_h^2 \phi_h \left\{ S_{yh}^2 + (t_2 - t_1) a_h R^* S_{yxh} + \frac{(t_2-t_1)(t_2-t_1-2)}{8} a_h^2 R^{*2} S_{xh}^2 \right\}, \quad (44)$$
$$C_4 = 1 - \frac{t_1}{2\overline{Y}} \sum_{h=1}^{L} W_h^2 \phi_h \left\{ R^* a_h S_{yxh} - \frac{(t_1+2)}{4} a_h^2 R^{*2} S_{xh}^2 \right\},$$
$$C_5 = 1 + \frac{t_2}{2\overline{Y}} \sum_{h=1}^{L} W_h^2 \phi_h \left\{ R^* a_h S_{yxh} + \frac{(t_2-2)}{4} a_h^2 R^{*2} S_{xh}^2 \right\}.$$

Given below are the weights of θ_{13} and θ_{14} for minimizing the MSE:

$$\theta_{13} = \frac{C_2 C_4 - C_5 C_3}{C_2 C_1 - C_3^2}, \qquad \theta_{14} = \frac{C_1 C_5 - C_4 C_3}{C_2 C_1 - C_3^2}. \quad (45)$$

Thus, the minimum MSE by putting the above values of θ_{13} and θ_{14} in (43) is given by

$$\mathrm{MSE}_{\min}\left(\widehat{\overline{Y}}_{SS3}\right) \cong \overline{Y}^2 \left[1 - \frac{C_1 C_5^2 - 2C_4 C_3 C_5 + C_4^2 C_2}{C_2 C_1 - C_3^2} \right]. \quad (46)$$

4. Suggested Family of Estimators

Following the lines of Shabbir et al. [20], a generalized estimator for the estimation of population mean is proposed using some traditional and nontraditional measures of an auxiliary variable. For more details of these nontraditional measures, see the works of Hettmansperger and McKean [21], Wang et al. [22], and Irfan et al. [23–26]:

$$\widehat{\overline{Y}}_P = \left[\widehat{\overline{Y}}_{BT,\mathrm{Avg}} + \theta_{15}(\overline{X} - \overline{x}_{st}) + \theta_{16} \overline{y}_{st} \right] \exp\left(\frac{a_{st}\overline{X} + b_{st}}{a_{st}\overline{x}_{st} + b_{st}} - 1 \right), \quad (47)$$

where θ_{15} and θ_{16} are the suitably chosen weights and $a_{st}(\neq 0)$ and b_{st} are either real numbers or functions of known parameters of the auxiliary variable such as standard deviation S_{xst}, coefficient of skewness β_{1xst}, coefficient of kurtosis β_{2xst}, coefficient of variation C_{xst}, coefficient of correlation ρ_{st}, quartile deviation QD_{xst}, midrange MR_{xst}, interquartile range QR_{xst}, quartile average QA_{xst}, tri-mean TM_{xst}, and Hodge–Lehmann estimator HL_{xst}.

Remark 1. Many more estimators can be generated by placing different available parameters of auxiliary variable in place of a_{st} and b_{st}. Some of them are presented in Table 1.

Using (1), the suggested class of estimators $\widehat{\overline{Y}}_P$ can be rewritten as

$$\widehat{\overline{Y}}_P = \left[\frac{\overline{Y}(1+\zeta_0)}{2} \left\{ \exp\left(-\frac{\zeta_1}{2}\left(1 + \frac{\zeta_1}{2}\right)^{-1} \right) + \exp\left(\frac{\zeta_1}{2}\left(1 + \frac{\zeta_1}{2}\right)^{-1} \right) \right\} - \theta_{15}\overline{X}\zeta_1 + \theta_{16}\overline{Y}(1+\zeta_0) \right] \cdot \exp\left(-\eta\zeta_1(1+\eta\zeta_1)^{-1}\right) \times \left\{ \frac{1}{2}\left(2 + 2\eta\zeta_1 + \eta^2\zeta_1^2\right)(1+\eta\zeta_1)^{-1} \right\}^2. \quad (48)$$

As defined earlier, $\eta = a_{st}\overline{X}/a_{st}\overline{X} + b_{st}$.

Expanding the right-hand side of (48), up to first order of approximation and subtracting $\overline{Y}$ from both sides, we obtain

Table 1: Some members of the proposed family.

Estimator	$\widehat{\overline{Y}}_P^{(i)}$	a_{st}	b_{st}
1	$\widehat{\overline{Y}}_P^{(1)}$	1	0
2	$\widehat{\overline{Y}}_P^{(2)}$	1	$\rho_{st} = \sum_{h=1}^{L} W_h\rho_h$
3	$\widehat{\overline{Y}}_P^{(3)}$	1	$C_{xst} = \sum_{h=1}^{L} W_h C_{xh}$
4	$\widehat{\overline{Y}}_P^{(4)}$	$C_{xst} = \sum_{h=1}^{L} W_h C_{xh}$	$\beta_{2xst} = \sum_{h=1}^{L} W_h\beta_{2xh}$
5	$\widehat{\overline{Y}}_P^{(5)}$	$\beta_{1xst} = \sum_{h=1}^{L} W_h\beta_{1xh}$	$C_{xst} = \sum_{h=1}^{L} W_h C_{xh}$
6	$\widehat{\overline{Y}}_P^{(6)}$	$\beta_{1xst} = \sum_{h=1}^{L} W_h\beta_{1xh}$	$\rho_{st} = \sum_{h=1}^{L} W_h\rho_h$
7	$\widehat{\overline{Y}}_P^{(7)}$	1	$\beta_{2xst} = \sum_{h=1}^{L} W_h\beta_{2xh}$
8	$\widehat{\overline{Y}}_P^{(8)}$	1	$\beta_{1xst} = \sum_{h=1}^{L} W_h\beta_{1xh}$
9	$\widehat{\overline{Y}}_P^{(9)}$	$TM_{xst} = \sum_{h=1}^{L} W_h TM_{xh}$	1
10	$\widehat{\overline{Y}}_P^{(10)}$	$TM_{xst} = \sum_{h=1}^{L} W_h TM_{xh}$	$QR_{xst} = \sum_{h=1}^{L} W_h QR_{xh}$
11	$\widehat{\overline{Y}}_P^{(11)}$	$QA_{xst} = \sum_{h=1}^{L} W_h QA_{xh}$	$S_{xst} = \sum_{h=1}^{L} W_h S_{xh}$
12	$\widehat{\overline{Y}}_P^{(12)}$	$HL_{xst} = \sum_{h=1}^{L} W_h HL_{xh}$	$\rho_{st} = \sum_{h=1}^{L} W_h\rho_h$
13	$\widehat{\overline{Y}}_P^{(13)}$	$HL_{xst} = \sum_{h=1}^{L} W_h HL_{xh}$	$QR_{xst} = \sum_{h=1}^{L} W_h QR_{xh}$
14	$\widehat{\overline{Y}}_P^{(14)}$	$MR_{xst} = \sum_{h=1}^{L} W_h MR_{xh}$	$QR_{xst} = \sum_{h=1}^{L} W_h QR_{xh}$
15	$\widehat{\overline{Y}}_P^{(15)}$	$MR_{xst} = \sum_{h=1}^{L} W_h MR_{xh}$	$QD_{xst} = \sum_{h=1}^{L} W_h QD_{xh}$

$$\begin{aligned}\widehat{\overline{Y}}_P - \overline{Y} = {} & \overline{Y}\zeta_0 + \frac{\overline{Y}\zeta_1^2}{8} - \theta_{15}\overline{X}\zeta_1 + \theta_{16}\overline{Y} + \theta_{16}\overline{Y}\zeta_0 - \eta\overline{Y}\zeta_1 - \eta\overline{Y}\zeta_0\zeta_1 + \theta_{15}\eta\overline{X}\zeta_1^2 \\ & - \theta_{16}\eta\overline{Y}\zeta_1 - \theta_{16}\eta\overline{Y}\zeta_0\zeta_1 + \frac{3}{2}\eta^2\overline{Y}\zeta_1^2 + \frac{5}{2}\theta_{16}\eta^2\overline{Y}\zeta_1^2 + \eta^2\overline{Y}\zeta_1^2.\end{aligned} \tag{49}$$

Up to first order of approximation, the bias and the MSE are given by

$$\operatorname{Bias}\left(\widehat{\overline{Y}}_P\right) \cong \overline{Y}\left[V_{02}\left(\frac{1+20\eta^2}{8}\right) - \eta V_{11}\right] + \overline{X}\eta\theta_{15}V_{02} + \theta_{16}\left(\overline{Y} + \frac{5\overline{Y}\eta^2 V_{02}}{2} - \overline{Y}\eta V_{11}\right), \tag{50}$$

$$\begin{aligned}\operatorname{MSE}\left(\widehat{\overline{Y}}_P\right) \cong {} & \overline{Y}^2\eta^2V_{02} + \overline{Y}^2V_{20} - 2\overline{Y}^2\eta V_{11} + \overline{X}^2\theta_{15}^2V_{02} + 2\overline{X}\,\overline{Y}\eta\theta_{15}V_{02} - 2\overline{X}\,\overline{Y}\theta_{15}V_{11} \\ & + \overline{Y}\theta_{16}\left\{4\overline{X}\eta\theta_{15}V_{02} - 2\overline{X}\theta_{15}V_{11} + 7\eta^2\overline{Y}V_{02} - 6\eta\overline{Y}V_{11} + 2\overline{Y}V_{20} + \frac{1}{4}\overline{Y}V_{02}\right\} + \overline{Y}^2\theta_{16}^2\left\{1 + 6\eta^2V_{02} - 4\eta V_{11} + V_{20}\right\}.\end{aligned} \tag{51}$$

By minimizing (51), suitable weights of θ_{15} and θ_{16} are obtained as below:

$$\begin{aligned}\theta_{15} &= \frac{\overline{Y}\left\{2\eta V_{02}^2\left(\eta^2 + 1/4\right) + 2V_{11}\left(1 - \eta V_{11}\right) - \eta V_{02}\left(2 - \eta V_{11} - 2V_{20}\right) - 1/4V_{11}V_{02}\right\}}{2\overline{X}\left\{-V_{11}^2 + V_{02}\left(1 + 2\eta^2V_{02} + V_{20}\right)\right\}}, \\ \theta_{16} &= \frac{2V_{11}^2 - V_{02}\left(3\eta^2V_{02} + 2V_{20} + V_{02}/4\right)}{2\left\{-V_{11}^2 + V_{02}\left(1 + 2\eta^2V_{02} + V_{20}\right)\right\}}.\end{aligned} \tag{52}$$

Putting the optimal weights of θ_{15} and θ_{16} in (51), we have the minimum MSE given by

$$\mathrm{MSE}_{\min}\left(\hat{\hat{\overline{Y}}}_P\right) \cong \frac{\overline{Y}^2\left[4V_{11}^2 + V_{02}\left\{3\eta^2V_{02}^2\left(3\eta^2 + 1/2\right) - V_{11}^2\left(1+4\eta^2\right) + 1/16V_{02}^2 + 4\left(-1 + V_{02}\left(\eta^2 + 1/4\right)\right)V_{20}\right\}\right]}{4\left\{V_{11}^2 - V_{02}\left(1 + 2\eta^2V_{02} + V_{20}\right)\right\}}. \quad (53)$$

5. Application to a Dataset

To examine the performance of the proposed class of estimators, we considered a real data of Turkey (2007) used by Koyuncu and Kadilar [3] given in Table 2. In this data, let y be the number of teachers (study variable) and x be the number of students (auxiliary variable) which are recorded for primary and secondary schools at 6 regions for $N = 923$ districts. A total sample of size $n = 180$ is selected through Neyman allocation from 6 strata. (source: Ministry of Education, Republic of Turkey).

We have computed $\mathrm{MSE}_{\min}(\hat{\hat{\overline{Y}}}_K)$, $\mathrm{MSE}_{\min}(\hat{\hat{\overline{Y}}}_{KK})$, $\mathrm{MSE}_{\min}(\hat{\hat{\overline{Y}}}_{SG})$, $\mathrm{MSE}_{\min}(\hat{\hat{\overline{Y}}}_{HS1})$, $\mathrm{MSE}_{\min}(\hat{\hat{\overline{Y}}}_{HS2})$, $\mathrm{MSE}_{\min}(\hat{\hat{\overline{Y}}}_{SS1})$, $\mathrm{MSE}_{\min}(\hat{\hat{\overline{Y}}}_{SS2})$, $\mathrm{MSE}_{\min}(\hat{\hat{\overline{Y}}}_{SS3})$, and $\mathrm{MSE}_{\min}(\hat{\hat{\overline{Y}}}_P)$ for the population dataset given in Table 2 and are reported in Tables 3 and 4.

5.1. Important Findings

(i) The values of MSE of usual unbiased, combined ratio, and combined regression estimators in stratified random sampling are computed as given below:

$$\begin{aligned}\mathrm{MSE}\left(\hat{\hat{\overline{Y}}}_{st}\right) &= 2229.2662,\\ \mathrm{MSE}\left(\hat{\hat{\overline{Y}}}_{CR}\right) &= 216.4168,\\ \mathrm{MSE}\left(\hat{\hat{\overline{Y}}}_{\mathrm{CReg}}\right) &= 194.2832.\end{aligned} \quad (54)$$

(ii) It is obvious from Table 3 that MSEs of Koyuncu and Kadilar [3] class of estimators $\hat{\hat{\overline{Y}}}_K$ round about the value of $MSE(\hat{\hat{\overline{Y}}}_{CR})$.

(iii) value of $\mathrm{MSE}(\hat{\hat{\overline{Y}}}_{\mathrm{CReg}})$ is less than $\mathrm{MSE}(\hat{\hat{\overline{Y}}}_{CR})$ and $\mathrm{MSE}_{\min}(\hat{\hat{\overline{Y}}}_K)$.

(iv) It is observed from Tables 3 and 4 that the MSEs of Koyuncu and Kadilar [4] class of estimators $\hat{\hat{\overline{Y}}}_{KK}$, Shabbir and Gupta [8] class of estimators $\hat{\hat{\overline{Y}}}_{SG}$, Haq and Shabbir [9] classes of estimators $\hat{\hat{\overline{Y}}}_{HSi}$, $i = 1, 2$, and Singh and Solanki [10, 12] classes of estimators $\hat{\hat{\overline{Y}}}_{SSj}$, $j = 1, 2, 3$, are less than the MSE of combined regression estimator $\hat{\hat{\overline{Y}}}_{\mathrm{CReg}}$.

(v) Again, from Tables 3 and 4, it is valuable to mention that the proposed family of estimators $\hat{\hat{\overline{Y}}}_P$ have the least MSEs as compared to all other classes of estimators against different values of a_{st} and b_{st}.

6. Simulation Study

In this section, we carried out a simulation study using R statistical software to evaluate the behavior of proposed estimators $\hat{\hat{\overline{Y}}}_P^{(i)}$ in comparison with $\hat{\hat{\overline{Y}}}_{st}$, $\hat{\hat{\overline{Y}}}_{CR}$, $\hat{\hat{\overline{Y}}}_{\mathrm{CReg}}$, $\hat{\hat{\overline{Y}}}_K$, $\hat{\hat{\overline{Y}}}_{KK}$, $\hat{\hat{\overline{Y}}}_{SG}$, $\hat{\hat{\overline{Y}}}_{HS1}$, $\hat{\hat{\overline{Y}}}_{HS2}$, $\hat{\hat{\overline{Y}}}_{SS1}$, $\hat{\hat{\overline{Y}}}_{SS2}$, and $\hat{\hat{\overline{Y}}}_{SS3}$. Real population presented in Table 3 is used for the simulation study. Three different sample sizes $n = 180, 250$, and 350 are taken from this population on the basis of proportional allocation.

The following steps summarize the procedure of finding the average MSE of an estimator.

Step 1: select a bivariate stratified sample of size n using simple random sampling without replacement from the bivariate stratified normal population

Step 2: use sample data from Step 1 to find the MSE of all the estimators under the study

Step 3: repeat Step 1 and Step 2 10,000 times and obtain 10,000 values for MSEs

Step 4: Average of 10,000 values obtained in Step 3 are the MSE of each estimator

6.1. Findings

MSEs of $\hat{\hat{\overline{Y}}}_{st}$, $\hat{\hat{\overline{Y}}}_{CR}$, and $\hat{\hat{\overline{Y}}}_{CReg}$ for different sample sizes are under Table 5.

MSEs of the other estimators under the study are presented in Tables 6–11, and the following important considerations are made from them.

(i) It is quite obvious that the MSE values of $\hat{\hat{\overline{Y}}}_P^{(i)}$, $i = 1, 2, 3, \ldots, 15$, as compared to the MSEs of all other estimators are minimum under different sample sizes taken from the population. It proves that all the proposed estimators are more efficient.

(ii) It is also shown that, by increasing the sample size selected from the population, the MSEs decrease.

7. Discussion

In this study, we proposed a new optimal family of estimators for estimating population mean under stratified random sampling. Bias, MSE, and minimum MSE of this family of estimators are derived up to first degree of approximation. The proposed family is compared with some well-known estimators/classes of estimators under stratified random sampling such as the works of Koyuncu and Kadilar [3, 4], Shabbir and Gupta [8], Haq and Shabbir [9], Singh and Solanki [19], and Solanki and Singh [10, 12]. It is numerically inferred that the proposed family behaves optimal

Table 2: Data statistics.

Values	h^{th} stratum					
	1	2	3	4	5	6
N_h	127	117	103	170	205	201
n_h	31	21	29	38	22	39
$\bar{Y}_h$	703.74	413.00	573.17	424.66	267.03	393.84
$\bar{X}_h$	20804.59	9211.79	14309.3	9478.85	5569.95	12997.59
S_{yh}	883.835	644.922	1033.467	810.585	403.654	711.723
S_{xh}	30486.751	15180.769	27549.697	18218.931	8497.776	23094.141
S_{yxh}	25237153.52	9747942.85	28294397.04	14523885.53	3393591.75	15864573.97
C_{xh}	1.465	1.648	1.925	1.922	1.526	1.777
ρ_h	0.936	0.996	0.994	0.983	0.989	0.965
$\beta_{2h(x)}$	4.593	18.543	15.446	10.162	21.947	23.114
$\beta_{1h(x)}$	2.164	3.867	3.748	3.121	4.084	4.411
QD_{xh}	12736.8	3023.5	5698.75	2742.25	2025.5	5545.5
MR_{xh}	77193.5	54644	85780.5	56386.5	34057	89530.5
QR_{xh}	25473.5	6047	11397.5	5484.5	4051	11091
QA_{xh}	15489.3	5104.5	7704.25	3899.25	3468.5	8105.5
TM_{xh}	10853.1	4705.75	5994.13	3321.38	3137.75	7116.25
HL_{xh}	7913.75	4883.25	7124.5	3899.5	3173.25	8074

Table 3: Minimum MSEs of different estimators based on real data.

a_{st}	b_{st}	$\hat{\hat{\bar{Y}}}_{K}$	$\hat{\hat{\bar{Y}}}_{KK}$	$\hat{\hat{\bar{Y}}}_{SG}$	$\hat{\hat{\bar{Y}}}_{HS1}$	$\hat{\hat{\bar{Y}}}_{HS2}$	$\hat{\hat{\bar{Y}}}_{SS1R}$	$\hat{\hat{\bar{Y}}}_{SS1P}$	$\hat{\hat{\bar{Y}}}_{SS1RP}$	$\hat{\hat{\bar{Y}}}_{P}$
1	0	216.4168	194.0826	194.0852	183.4821	115.8697	193.4183	175.9254	192.8467	**102.9573**
1	$\sum_{h=1}^{L} W_h\rho_h$	216.3768	194.0826	194.0852	183.4853	115.8949	193.4192	175.9254	192.8484	**102.9843**
1	$\sum_{h=1}^{L} W_hC_{xh}$	216.3470	194.0826	194.0852	183.4877	115.9136	193.4198	175.9254	192.8496	**103.0045**
$\sum_{h=1}^{L} W_hC_{xh}$	$\sum_{h=1}^{L} W_h\beta_{2xh}$	216.0228	194.0826	194.0852	183.5136	116.1178	193.4264	175.9254	192.8633	**103.2247**
$\sum_{h=1}^{L} W_h\beta_{1xh}$	$\sum_{h=1}^{L} W_hC_{xh}$	216.3976	194.0826	194.0852	183.4836	115.8818	193.4187	175.9254	192.8475	**102.9702**
$\sum_{h=1}^{L} W_h\beta_{1xh}$	$\sum_{h=1}^{L} W_h\rho_h$	216.4058	194.0826	194.0852	183.4830	115.8766	193.4186	175.9254	192.8472	**102.9647**
1	$\sum_{h=1}^{L} W_h\beta_{2xh}$	215.7475	194.0826	194.0852	183.5357	116.2921	193.4320	175.9254	192.8749	**103.4126**
1	$\sum_{h=1}^{L} W_h\beta_{1xh}$	216.2676	194.0826	194.0852	183.4940	115.9635	193.4214	175.9254	192.8530	**103.0583**
$\sum_{h=1}^{L} W_hTM_{xh}$	1	216.4168	194.0826	194.0852	183.4821	115.8697	193.4183	175.9254	192.8467	**102.9573**
$\sum_{h=1}^{L} W_hTM_{xh}$	$\sum_{h=1}^{L} W_hQR_{xh}$	216.3449	194.0826	194.0852	183.4878	115.9149	193.4198	175.9254	192.8497	**103.0059**
$\sum_{h=1}^{L} W_hQA_{xh}$	$\sum_{h=1}^{L} W_hS_{xh}$	216.3012	194.0826	194.0852	183.4913	115.9423	193.4207	175.9254	192.8516	**103.0355**
$\sum_{h=1}^{L} W_hHL_{xh}$	$\sum_{h=1}^{L} W_h\rho_h$	216.4168	194.0826	194.0852	183.4821	115.8697	193.4183	175.9254	192.8467	**102.9573**
$\sum_{h=1}^{L} W_hHL_{xh}$	$\sum_{h=1}^{L} W_hQR_{xh}$	216.3457	194.0826	194.0852	183.4878	115.9144	193.4198	175.9254	192.8497	**103.0054**
$\sum_{h=1}^{L} W_hMR_{xh}$	$\sum_{h=1}^{L} W_hQR_{xh}$	216.4105	194.0826	194.0852	183.4826	115.8737	193.4185	175.9254	192.8470	**102.9615**
$\sum_{h=1}^{L} W_hMR_{xh}$	$\sum_{h=1}^{L} W_hQD_{xh}$	216.4136	194.0826	194.0852	183.4823	115.8717	193.4184	175.9254	192.8468	**102.9594**

Bold values indicate minimum MSE.

Table 4: Minimum MSEs of different estimators based on real data.

a_{st}	b_{st}	$\hat{\hat{\bar{Y}}}_{SS2R}$	$\hat{\hat{\bar{Y}}}_{SS2P}$	$\hat{\hat{\bar{Y}}}_{SS2RR}$	$\hat{\hat{\bar{Y}}}_{SS2PP}$	$\hat{\hat{\bar{Y}}}_{SS2RP}$	$\hat{\hat{\bar{Y}}}_{SS2PR}$	$\hat{\hat{\bar{Y}}}_{SS31}$	$\hat{\hat{\bar{Y}}}_{SS32}$	$\hat{\hat{\bar{Y}}}_{SS33}$	$\hat{\hat{\bar{Y}}}_{P}$
1	0	194.1562	185.5207	193.6484	143.7752	193.6484	191.9754	108.8591	143.7752	172.0985	**102.9573**
1	$\sum_{h=1}^{L} W_h\rho_h$	194.1560	185.5200	193.6489	143.7805	193.6489	191.9743	108.8704	143.7804	172.1103	**102.9843**
1	$\sum_{h=1}^{L} W_hC_{xh}$	194.1558	185.5195	193.6492	143.7844	193.6492	191.9735	108.8793	143.7844	172.1196	**103.0045**
$\sum_{h=1}^{L} W_hC_{xh}$	$\sum_{h=1}^{L} W_h\beta_{2xh}$	194.1539	185.5141	193.6532	143.8273	193.6532	191.9647	107.0285	132.7917	161.5885	**103.2247**
$\sum_{h=1}^{L} W_h\beta_{1xh}$	$\sum_{h=1}^{L} W_hC_{xh}$	194.1561	185.5203	193.6486	143.7777	193.1521	191.9749	178.5658	214.5583	231.1787	**102.9702**
$\sum_{h=1}^{L} W_h\beta_{1xh}$	$\sum_{h=1}^{L} W_h\rho_h$	194.1562	185.5205	193.6485	143.7767	193.1515	191.9751	178.5631	214.5571	231.1754	**102.9647**
1	$\sum_{h=1}^{L} W_h\beta_{2xh}$	194.1522	185.5095	193.6566	143.8638	193.6566	191.9571	109.0538	143.8638	172.3019	**103.4126**
1	$\sum_{h=1}^{L} W_h\beta_{1xh}$	194.1554	185.5182	193.6502	143.7949	193.1561	191.9714	108.9023	143.7949	172.1437	**103.0583**

Table 4: Continued.

a_{st}	b_{st}	$\hat{\hat{Y}}_{SS2R}$	$\hat{\hat{Y}}_{SS2P}$	$\hat{\hat{Y}}_{SS2RR}$	$\hat{\hat{Y}}_{SS2PP}$	$\hat{\hat{Y}}_{SS2RP}$	$\hat{\hat{Y}}_{SS2PR}$	$\hat{\hat{Y}}_{SS31}$	$\hat{\hat{Y}}_{SS32}$	$\hat{\hat{Y}}_{SS33}$	$\hat{\hat{Y}}_{P}$
$\sum_{h=1}^{L} W_h TM_{xh}$	1	194.1562	185.5207	193.6484	143.7752	193.1515	191.9754	676.9615	703.3570	656.1403	**102.9573**
$\sum_{h=1}^{L} W_h TM_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	194.1558	185.5195	193.6493	143.7847	193.1537	191.9735	676.9765	703.3632	656.1786	**103.0059**
$\sum_{h=1}^{L} W_h QA_{xh}$	$\sum_{h=1}^{L} W_h S_{xh}$	194.1556	185.5187	193.6498	143.7904	193.155	191.9723	820.1003	844.5858	772.9658	**103.0355**
$\sum_{h=1}^{L} W_h HL_{xh}$	$\sum_{h=1}^{L} W_h \rho_h$	194.1562	185.5207	193.6484	143.7752	193.1515	191.9754	476.1121	506.1138	486.5211	**102.9573**
$\sum_{h=1}^{L} W_h HL_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	194.1558	185.5195	193.6493	143.7846	193.1537	191.9735	476.1289	506.1211	486.5537	**103.0054**
$\sum_{h=1}^{L} W_h MR_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	194.1562	185.5206	193.6485	143.7761	193.1517	191.9753	383.5274	415.2221	407.5903	**102.9615**
$\sum_{h=1}^{L} W_h MR_{xh}$	$\sum_{h=1}^{L} W_h QD_{xh}$	194.1562	185.5206	193.6484	143.7756	193.1516	191.9753	383.5266	415.2217	407.5889	**102.9594**

Bold values indicate minimum MSE.

Table 5: MSEs of $\hat{\hat{Y}}_{st}$, $\hat{\hat{Y}}_{CR}$, and $\hat{\hat{Y}}_{CReg}$ for different sample sizes.

	MSE$(\hat{\hat{Y}}_{st})$	MSE$(\hat{\hat{Y}}_{CR})$	MSE$(\hat{\hat{Y}}_{CReg})$
$n = 180$	2234.0170	208.6489	191.4028
$n = 250$	1600.3790	155.6986	136.6418
$n = 350$	971.9409	95.6147	84.1167

Table 6: Minimum MSEs of different estimators based on the simulation study ($n = 180$).

a_{st}	b_{st}	$\hat{\hat{Y}}_{K}$	$\hat{\hat{Y}}_{KK}$	$\hat{\hat{Y}}_{SG}$	$\hat{\hat{Y}}_{HS1}$	$\hat{\hat{Y}}_{HS2}$	$\hat{\hat{Y}}_{SS1R}$	$\hat{\hat{Y}}_{SS1P}$	$\hat{\hat{Y}}_{SS1RP}$	$\hat{\hat{Y}}_{P}$
1	0	208.7024	182.4218	182.4243	171.9060	104.0812	181.7569	163.3593	181.1334	**91.1038**
1	$\sum_{h=1}^{L} W_h \rho_h$	208.9233	182.8256	182.8281	172.3738	105.0166	182.1669	163.8692	181.5503	**92.1283**
1	$\sum_{h=1}^{L} W_h C_{xh}$	208.1726	182.0508	182.0533	171.5916	104.1605	181.3923	163.0362	180.7751	**91.2563**
$\sum_{h=1}^{L} W_h C_{xh}$	$\sum_{h=1}^{L} W_h \beta_{2xh}$	206.1250	180.3394	180.3419	169.9015	102.4944	179.6982	161.1364	179.0981	**89.5657**
$\sum_{h=1}^{L} W_h \beta_{1xh}$	$\sum_{h=1}^{L} W_h C_{xh}$	207.2181	181.6087	181.6112	171.2167	104.2205	180.9596	162.6734	180.3525	**91.4022**
$\sum_{h=1}^{L} W_h \beta_{1xh}$	$\sum_{h=1}^{L} W_h \rho_h$	208.0968	181.9132	181.9157	171.4806	104.2386	181.2527	163.0222	180.6335	**91.3737**
1	$\sum_{h=1}^{L} W_h \beta_{2xh}$	207.6863	181.5076	181.5100	171.1143	104.1185	180.8584	162.5738	180.2520	**91.2628**
1	$\sum_{h=1}^{L} W_h \beta_{1xh}$	208.9447	182.9924	182.9949	172.5532	105.3055	182.3371	164.0466	181.7253	**92.4328**
$\sum_{h=1}^{L} W_h TM_{xh}$	1	208.6489	182.8849	182.8874	172.3985	104.8140	182.2277	163.8243	181.6142	**91.8843**
$\sum_{h=1}^{L} W_h TM_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	208.9039	182.8772	182.8797	172.4226	105.0771	182.2188	163.9184	181.6032	**92.1900**
$\sum_{h=1}^{L} W_h QA_{xh}$	$\sum_{h=1}^{L} W_h S_{xh}$	209.1184	182.5700	182.5725	172.0577	104.2553	181.9039	163.516	181.2788	**91.2757**
$\sum_{h=1}^{L} W_h HL_{xh}$	$\sum_{h=1}^{L} W_h \rho_h$	208.7805	182.6141	182.6166	172.1151	104.4292	181.9516	163.5377	181.3314	**91.4787**
$\sum_{h=1}^{L} W_h HL_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	207.4471	181.8892	181.8914	173.1457	117.5424	181.6167	162.7754	181.3831	**105.7468**
$\sum_{h=1}^{L} W_h MR_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	208.2513	181.6575	181.66	171.1876	103.6541	180.9891	162.7212	180.3609	**90.7320**
$\sum_{h=1}^{L} W_h MR_{xh}$	$\sum_{h=1}^{L} W_h QD_{xh}$	207.2467	181.1462	181.1487	170.6938	103.2264	180.4891	162.1121	179.8724	**90.3161**

Bold values indicate minimum MSE.

Table 7: Minimum MSEs of different estimators based on the simulation study ($n = 180$).

a_{st}	b_{st}	$\hat{\hat{Y}}_{SS2R}$	$\hat{\hat{Y}}_{SS2P}$	$\hat{\hat{Y}}_{SS2RR}$	$\hat{\hat{Y}}_{SS2PP}$	$\hat{\hat{Y}}_{SS2RP}$	$\hat{\hat{Y}}_{SS2PR}$	$\hat{\hat{Y}}_{SS31}$	$\hat{\hat{Y}}_{SS32}$	$\hat{\hat{Y}}_{SS33}$	$\hat{\hat{Y}}_{P}$
1	0	182.3466	173.2652	182.0005	130.4476	181.4683	179.9313	94.73656	130.4476	160.545	**91.1038**
1	$\sum_{h=1}^{L} W_h \rho_h$	182.7497	173.7173	182.4081	131.1713	181.8815	180.3422	95.68471	131.166	161.1143	**92.1283**
1	$\sum_{h=1}^{L} W_h C_{xh}$	181.9698	172.9086	181.6336	130.2683	181.1066	179.5505	94.70196	130.261	160.3244	**91.2563**
$\sum_{h=1}^{L} W_h C_{xh}$	$\sum_{h=1}^{L} W_h \beta_{2xh}$	180.2450	171.0819	179.9328	128.2003	179.4204	177.7797	92.33457	128.1317	158.6125	**89.5657**
$\sum_{h=1}^{L} W_h \beta_{1xh}$	$\sum_{h=1}^{L} W_h C_{xh}$	181.5270	172.4999	181.1971	130.0811	180.6785	179.112	94.72428	130.0791	160.0748	**91.4022**
$\sum_{h=1}^{L} W_h \beta_{1xh}$	$\sum_{h=1}^{L} W_h \rho_h$	181.8389	172.8398	181.4947	130.4089	180.9661	179.4442	95.01959	130.4074	160.221	**91.3737**
1	$\sum_{h=1}^{L} W_h \beta_{2xh}$	181.4267	172.4007	181.0958	129.978	180.5776	179.0127	94.42132	129.8881	159.7761	**91.2628**
1	$\sum_{h=1}^{L} W_h \beta_{1xh}$	182.9171	173.8882	182.5766	131.3816	182.0538	180.5082	95.89918	131.3617	161.3055	**92.4328**

Table 7: Continued.

a_{st}	b_{st}	$\hat{\hat{Y}}_{SS2R}$	$\hat{\hat{Y}}_{SS2P}$	$\hat{\hat{Y}}_{SS2RR}$	$\hat{\hat{Y}}_{SS2PP}$	$\hat{\hat{Y}}_{SS2RP}$	$\hat{\hat{Y}}_{SS2PR}$	$\hat{\hat{Y}}_{SS31}$	$\hat{\hat{Y}}_{SS32}$	$\hat{\hat{Y}}_{SS33}$	$\hat{\hat{Y}}_{P}$
$\sum_{h=1}^{L} W_h TM_{xh}$	1	182.8072	173.7227	182.4678	130.9729	181.9436	180.3827	95.33219	130.9729	161.1424	**91.8843**
$\sum_{h=1}^{L} W_h TM_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	182.8017	173.7682	182.4597	131.2176	181.9338	180.3930	95.71564	131.208	161.1659	**92.1900**
$\sum_{h=1}^{L} W_h QA_{xh}$	$\sum_{h=1}^{L} W_h S_{xh}$	182.4934	173.4168	182.1482	130.6215	181.6146	180.0789	94.8953	130.6059	160.6594	**91.2757**
$\sum_{h=1}^{L} W_h HL_{xh}$	$\sum_{h=1}^{L} W_h \rho_h$	182.5333	173.444	182.1942	130.6498	181.6645	180.1084	94.9620	130.6483	160.8088	**91.4787**
$\sum_{h=1}^{L} W_h HL_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	181.6013	172.2849	181.7109	132.9205	181.5076	178.6688	94.22969	129.8961	160.2185	**105.7468**
$\sum_{h=1}^{L} W_h MR_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	181.5821	172.5648	181.2344	129.9974	180.6984	179.1876	94.48528	129.9965	159.8306	**90.7320**
$\sum_{h=1}^{L} W_h MR_{xh}$	$\sum_{h=1}^{L} W_h QD_{xh}$	181.0637	171.9921	180.7301	129.3164	180.2037	178.6427	93.73929	129.316	159.4238	**90.3161**

Bold values indicate minimum MSE.

Table 8: Minimum MSEs of different estimators based on the simulation study (n = 250).

a_{st}	b_{st}	$\hat{\hat{Y}}_{K}$	$\hat{\hat{Y}}_{KK}$	$\hat{\hat{Y}}_{SG}$	$\hat{\hat{Y}}_{HS1}$	$\hat{\hat{Y}}_{HS2}$	$\hat{\hat{Y}}_{SS1R}$	$\hat{\hat{Y}}_{SS1P}$	$\hat{\hat{Y}}_{SS1RP}$	$\hat{\hat{Y}}_{P}$
1	0	155.6986	130.9425	130.9435	125.1831	87.54561	130.5094	121.2922	130.0932	**80.3467**
1	$\sum_{h=1}^{L} W_h \rho_h$	155.6335	130.9868	130.9878	125.2404	87.70134	130.5545	121.3769	130.14	**80.5204**
1	$\sum_{h=1}^{L} W_h C_{xh}$	155.188	130.6177	130.6186	124.8729	87.3303	130.1873	120.9742	129.7744	**80.1478**
$\sum_{h=1}^{L} W_h C_{xh}$	$\sum_{h=1}^{L} W_h \beta_{2xh}$	155.2285	130.7593	130.7602	125.0268	87.56625	130.3357	121.0766	129.9308	**80.3851**
$\sum_{h=1}^{L} W_h \beta_{1xh}$	$\sum_{h=1}^{L} W_h C_{xh}$	155.3389	130.8714	130.8724	125.1137	87.48645	130.4407	121.2061	130.0281	**80.2890**
$\sum_{h=1}^{L} W_h \beta_{1xh}$	$\sum_{h=1}^{L} W_h \rho_h$	155.6699	131.0058	131.0068	125.244	87.60117	130.572	121.3724	130.1561	**80.4011**
1	$\sum_{h=1}^{L} W_h \beta_{2xh}$	155.0061	130.2404	130.2413	124.5235	87.1653	129.8145	120.6219	129.4065	**79.9985**
1	$\sum_{h=1}^{L} W_h \beta_{1xh}$	155.2016	130.5104	130.5113	124.7728	87.28445	130.0795	120.8933	129.6659	**80.1095**
$\sum_{h=1}^{L} W_h TM_{xh}$	1	155.5360	131.0069	131.0078	125.2573	87.69095	130.5764	121.3634	130.1638	**80.5060**
$\sum_{h=1}^{L} W_h TM_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	155.8067	131.277	131.278	125.5701	88.33712	130.8469	121.7497	130.4349	**81.2152**
$\sum_{h=1}^{L} W_h QA_{xh}$	$\sum_{h=1}^{L} W_h S_{xh}$	154.6152	130.1677	130.1686	124.4597	87.16355	129.7406	120.5834	129.3311	**80.0265**
$\sum_{h=1}^{L} W_h HL_{xh}$	$\sum_{h=1}^{L} W_h \rho_h$	155.7181	130.6542	130.6551	124.9063	87.34791	130.2172	121.0742	129.7966	**80.1639**
$\sum_{h=1}^{L} W_h HL_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	155.0446	130.6161	130.6169	125.8356	95.03376	130.4127	120.9906	130.2424	**88.5004**
$\sum_{h=1}^{L} W_h MR_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	155.8021	131.0947	131.0957	125.365	87.96272	130.6616	121.5326	130.2458	**80.8098**
$\sum_{h=1}^{L} W_h MR_{xh}$	$\sum_{h=1}^{L} W_h QD_{xh}$	154.8427	130.0278	130.0287	124.2956	86.82202	129.5947	120.4377	129.178	**79.6540**

Bold values indicate minimum MSE.

Table 9: Minimum MSEs of different estimators based on the simulation study (n = 250).

a_{st}	b_{st}	$\hat{\hat{Y}}_{SS2R}$	$\hat{\hat{Y}}_{SS2P}$	$\hat{\hat{Y}}_{SS2RR}$	$\hat{\hat{Y}}_{SS2PP}$	$\hat{\hat{Y}}_{SS2RP}$	$\hat{\hat{Y}}_{SS2PR}$	$\hat{\hat{Y}}_{SS31}$	$\hat{\hat{Y}}_{SS32}$	$\hat{\hat{Y}}_{SS33}$	$\hat{\hat{Y}}_{P}$
1	0	130.9376	126.3729	130.671	104.1672	130.317	129.8108	85.72231	104.1672	118.2598	**80.3467**
1	$\sum_{h=1}^{L} W_h \rho_h$	130.9843	126.4391	130.7156	104.3089	130.3628	129.8644	85.91894	104.3061	118.3264	**80.5204**
1	$\sum_{h=1}^{L} W_h C_{xh}$	130.612	126.0493	130.3477	103.8761	129.9964	129.4838	85.45359	103.8723	117.9719	**80.1478**
$\sum_{h=1}^{L} W_h C_{xh}$	$\sum_{h=1}^{L} W_h \beta_{2xh}$	130.751	126.1653	130.4933	103.9616	130.1484	129.61	85.45669	103.9258	118.1125	**80.3851**
$\sum_{h=1}^{L} W_h \beta_{1xh}$	$\sum_{h=1}^{L} W_h C_{xh}$	130.8666	126.2933	130.6011	104.0666	130.2499	129.7363	85.6053	104.0656	118.207	**80.2890**
$\sum_{h=1}^{L} W_h \beta_{1xh}$	$\sum_{h=1}^{L} W_h \rho_h$	131.0036	126.4477	130.7337	104.2572	130.3797	129.8818	85.81913	104.2564	118.3116	**80.4011**
1	$\sum_{h=1}^{L} W_h \beta_{2xh}$	130.2341	125.6814	129.9731	103.5868	129.6259	129.1058	85.13934	103.5396	117.5702	**79.9985**
1	$\sum_{h=1}^{L} W_h \beta_{1xh}$	130.5056	125.9563	130.2402	103.8306	129.8883	129.382	85.4301	103.8202	117.8577	**80.1095**
$\sum_{h=1}^{L} W_h TM_{xh}$	1	131.0025	126.4398	130.7368	104.2592	130.3856	129.8752	85.83846	104.2592	118.3623	**80.5060**
$\sum_{h=1}^{L} W_h TM_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	131.2748	126.7694	131.0072	104.829	130.6564	130.1634	86.59144	104.824	118.7145	**81.2152**
$\sum_{h=1}^{L} W_h QA_{xh}$	$\sum_{h=1}^{L} W_h S_{xh}$	130.162	125.6269	129.8997	103.5951	129.5513	129.0399	85.28246	103.587	117.5998	**80.0265**
$\sum_{h=1}^{L} W_h HL_{xh}$	$\sum_{h=1}^{L} W_h \rho_h$	130.6518	126.1239	130.3805	104.0301	130.0228	129.54	85.66646	104.0293	117.9576	**80.1639**
$\sum_{h=1}^{L} W_h HL_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	130.5198	125.8475	130.4821	105.5124	130.3329	129.135	85.57302	103.9372	118.0266	**88.5004**

Table 9: Continued.

a_{st}	b_{st}	$\widehat{\overline{Y}}_{SS2R}$	$\widehat{\overline{Y}}_{SS2P}$	$\widehat{\overline{Y}}_{SS2RR}$	$\widehat{\overline{Y}}_{SS2PP}$	$\widehat{\overline{Y}}_{SS2RP}$	$\widehat{\overline{Y}}_{SS2PR}$	$\widehat{\overline{Y}}_{SS31}$	$\widehat{\overline{Y}}_{SS32}$	$\widehat{\overline{Y}}_{SS33}$	$\widehat{\overline{Y}}_{P}$
$\sum_{h=1}^{L} W_h MR_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	131.0921	126.571	130.8231	104.5418	130.4693	129.9783	86.23798	104.5414	118.4748	**80.8098**
$\sum_{h=1}^{L} W_h MR_{xh}$	$\sum_{h=1}^{L} W_h QD_{xh}$	130.0233	125.4884	129.7565	103.4035	129.4021	128.9063	85.05503	103.4033	117.3831	**79.6540**

Bold values indicate minimum MSE.

Table 10: Minimum MSEs of different estimators based on the simulation study (n = 350).

a_{st}	b_{st}	$\widehat{\overline{Y}}_{K}$	$\widehat{\overline{Y}}_{KK}$	$\widehat{\overline{Y}}_{SG}$	$\widehat{\overline{Y}}_{HS1}$	$\widehat{\overline{Y}}_{HS2}$	$\widehat{\overline{Y}}_{SS1R}$	$\widehat{\overline{Y}}_{SS1P}$	$\widehat{\overline{Y}}_{SS1RP}$	$\widehat{\overline{Y}}_{P}$
1	0	95.6147	80.9165	80.9167	78.7783	64.7223	80.7549	77.4339	80.6023	**62.0382**
1	$\sum_{h=1}^{L} W_h \rho_h$	95.7970	81.1781	81.1783	79.0419	65.0101	81.0168	77.7030	80.8649	**62.3304**
1	$\sum_{h=1}^{L} W_h C_{xh}$	95.8381	81.2087	81.2089	79.0722	65.0369	81.0475	77.7319	80.8957	**62.3564**
$\sum_{h=1}^{L} W_h C_{xh}$	$\sum_{h=1}^{L} W_h \beta_{2xh}$	95.9324	81.3256	81.3258	79.1991	65.2374	81.1666	77.8540	81.0173	**62.5658**
$\sum_{h=1}^{L} W_h \beta_{1xh}$	$\sum_{h=1}^{L} W_h C_{xh}$	95.4390	80.8880	80.8883	78.7582	64.7632	80.7277	77.4170	80.5766	**62.0907**
$\sum_{h=1}^{L} W_h \beta_{1xh}$	$\sum_{h=1}^{L} W_h \rho_h$	95.5746	80.9982	80.9984	78.8672	64.8656	80.8375	77.5279	80.6861	**62.1920**
1	$\sum_{h=1}^{L} W_h \beta_{2xh}$	95.4697	80.9304	80.9306	78.8167	64.9358	80.7731	77.4707	80.6256	**62.2776**
1	$\sum_{h=1}^{L} W_h \beta_{1xh}$	95.2259	80.7899	80.7901	78.6602	64.6624	80.6310	77.3026	80.4815	**61.9877**
$\sum_{h=1}^{L} W_h TM_{xh}$	1	95.4491	80.9784	80.9786	78.8435	64.8153	80.8183	77.4931	80.6676	**62.1367**
$\sum_{h=1}^{L} W_h TM_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	95.4708	80.9427	80.9429	78.8079	64.7784	80.7825	77.4580	80.6317	**62.0986**
$\sum_{h=1}^{L} W_h QA_{xh}$	$\sum_{h=1}^{L} W_h S_{xh}$	95.8493	81.2139	81.2141	79.0756	65.0261	81.0530	77.7292	80.9015	**62.3421**
$\sum_{h=1}^{L} W_h HL_{xh}$	$\sum_{h=1}^{L} W_h \rho_h$	95.2618	80.6549	80.6551	78.5217	64.4964	80.4942	77.1748	80.3426	**61.8179**
$\sum_{h=1}^{L} W_h HL_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	95.3351	80.8718	80.8720	79.0984	67.6261	80.7971	77.4132	80.7369	**65.1968**
$\sum_{h=1}^{L} W_h MR_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	95.3490	80.7720	80.7722	78.6452	64.6708	80.6114	77.3103	80.4601	**62.0023**
$\sum_{h=1}^{L} W_h MR_{xh}$	$\sum_{h=1}^{L} W_h QD_{xh}$	95.4295	80.9228	80.9230	78.7999	64.8580	80.7628	77.4671	80.6122	**62.1960**

Bold values indicate minimum MSE.

Table 11: Minimum MSEs of different estimators based on the simulation study (n = 350).

a_{st}	b_{st}	$\widehat{\overline{Y}}_{SS2R}$	$\widehat{\overline{Y}}_{SS2P}$	$\widehat{\overline{Y}}_{SS2RR}$	$\widehat{\overline{Y}}_{SS2PP}$	$\widehat{\overline{Y}}_{SS2RP}$	$\widehat{\overline{Y}}_{SS2PR}$	$\widehat{\overline{Y}}_{SS31}$	$\widehat{\overline{Y}}_{SS32}$	$\widehat{\overline{Y}}_{SS33}$	$\widehat{\overline{Y}}_{P}$
1	0	80.9234	79.2725	80.8145	71.2534	80.6842	80.5234	64.6608	71.2534	76.1766	**62.0382**
1	$\sum_{h=1}^{L} W_h \rho_h$	81.1856	79.5382	81.0763	71.5339	80.9464	80.7866	64.9508	71.5329	76.4450	**62.3304**
1	$\sum_{h=1}^{L} W_h C_{xh}$	81.2162	79.5679	81.1069	71.5607	80.9771	80.8169	64.9747	71.5593	76.4743	**62.3564**
$\sum_{h=1}^{L} W_h C_{xh}$	$\sum_{h=1}^{L} W_h \beta_{2xh}$	81.3327	79.6859	81.2251	71.7037	81.0974	80.9320	65.1163	71.6908	76.5992	**62.5658**
$\sum_{h=1}^{L} W_h \beta_{1xh}$	$\sum_{h=1}^{L} W_h C_{xh}$	80.8951	79.2493	80.7868	71.2598	80.6576	80.4959	64.6915	71.2595	76.1724	**62.0907**
$\sum_{h=1}^{L} W_h \beta_{1xh}$	$\sum_{h=1}^{L} W_h \rho_h$	81.0054	79.3602	80.8967	71.3700	80.7673	80.6066	64.8009	71.3698	76.2789	**62.1920**
1	$\sum_{h=1}^{L} W_h \beta_{2xh}$	80.9370	79.2952	80.8309	71.3489	80.7047	80.5366	64.7841	71.3320	76.2272	**62.2776**
1	$\sum_{h=1}^{L} W_h \beta_{1xh}$	80.7961	79.1415	80.6895	71.1302	80.5617	80.3930	64.5396	71.1264	76.0767	**61.9877**
$\sum_{h=1}^{L} W_h TM_{xh}$	1	80.9853	79.3323	80.8772	71.3154	80.7484	80.5837	64.7264	71.3154	76.2575	**62.1367**
$\sum_{h=1}^{L} W_h TM_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	80.9496	79.2969	80.8415	71.2807	80.7126	80.5483	64.6881	71.2788	76.2159	**62.0986**
$\sum_{h=1}^{L} W_h QA_{xh}$	$\sum_{h=1}^{L} W_h S_{xh}$	81.2212	79.5689	81.1123	71.5478	80.9828	80.8204	64.9479	71.5449	76.4741	**62.3421**
$\sum_{h=1}^{L} W_h HL_{xh}$	$\sum_{h=1}^{L} W_h \rho_h$	80.6616	79.0115	80.5535	71.0028	80.4239	80.2613	64.4191	71.0025	75.9286	**61.8179**
$\sum_{h=1}^{L} W_h HL_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	80.8480	79.1645	80.8219	71.8426	80.7687	80.3565	64.7315	71.2769	76.1725	**65.1968**
$\sum_{h=1}^{L} W_h MR_{xh}$	$\sum_{h=1}^{L} W_h QR_{xh}$	80.7793	79.1383	80.6707	71.1661	80.5413	80.3818	64.6116	71.1659	76.0604	**62.0023**
$\sum_{h=1}^{L} W_h MR_{xh}$	$\sum_{h=1}^{L} W_h QD_{xh}$	80.9301	79.2917	80.8218	71.3356	80.6930	80.5328	64.7949	71.3355	76.2251	**62.1960**

Bold values indicate minimum MSE.

as compared to other estimators. A simulation study is also carried out in support of efficient proposed estimators. So, to get more enhanced results in practice under stratified random sampling, our suggested family of estimators is recommended.

The possible extensions of this work are to estimate the (1) finite population mean using robust quantile regression and L-moments characteristics of an auxiliary information under stratified ranked set sampling, (2) finite population parameters including median, variance, and proportions using L-moments under different sampling designs, and (3) population mean in the presence of nonsampling errors using L-moments and calibration approach.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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