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Research Article

The Convergence of Three-Step Iterative Schemes for Generalized Φ – Hemi-Contractive Mappings and the Comparison of Their Rate of Convergence

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Charles proved the convergence of Picard-type iteration for generalized Φ−accretive nonself-mappings in a real uniformly smooth Banach space. Based on the theorems of the zeros of strongly Φ−quasi-accretive mappings and fixed points of strongly Φ−hemi-contractions, we extend the results to Noor iterative process and SP iterative process for generalized Φ−hemi-contractive mappings. Finally, we analyze the rate of convergence of four iterative schemes, namely, Noor iteration, iteration of Corollary 2, SP iteration, and iteration of Corollary 4.

1. Introduction and Preliminaries

In 2009, Charles [1] proved the convergence of Picard-type iteration for generalized Φ−accretive nonself-mappings in a real uniformly smooth Banach space. In this paper, we consider that the Noor iteration process and SP iteration process will be extended from the results of Charles [1].

In [2], Noor et al. proved the convergence of Noor iteration to fixed point of a pseudocontractive self-map defined in a real uniformly smooth Banach space. In [3], Qin and Yao showed the weak convergence of a Mann-like algorithm for nonexpansive and accretive operators. In [4], the author considers some generalized nonexpansive mappings on convex metric spaces and gives some sufficient and necessary conditions for a Noor-type iteration to approximate a common fixed point of an infinite family of uniformly quasi-sup(f_n)−Lipschitzian mappings and an infinite family of g_n−expansive mappings in convex metric spaces.

In 1953, the most general Mann iterative scheme now studied is the following: $x_0 \in K$,

$$x_{n+1} = (1 - c_n)x_n + c_n T x_n, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where $\{c_n\}_{n=1}^{\infty} \subset (0, 1)$ is a real sequence satisfying appropriate conditions. In 1974, Ishikawa [5] introduced the Ishikawa iteration process as follows: for a convex subset D of a Banach space E and a mapping T from D into itself, for any given $x_0 \in D$, the sequence $\{x_n\}$ in D is defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad n \geq 0, \end{cases} \tag{2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in [0, 1] satisfying the conditions $0 \leq \alpha_n \leq \beta_n \leq 1$, for all $n, \lim_{n \longrightarrow \infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$.

In 2000, Noor [6, 7] gave the following three-step iterative schemes for solving nonlinear operator equations in uniformly smooth Banach spaces.

Let D be a nonempty convex subset of E and let T: $D \longrightarrow D$ be a mapping. For a given $\{x_n\}$ in D, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative schemes:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T z_n, \\ z_n = (1 - \gamma_n)x_n + \gamma_n T x_n, \quad n \geq 0, \end{cases} \tag{3}$$

which is called the Noor iterative process, where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are three real sequences in [0,1] satisfying some certain conditions.

In 2011, Phuengrattana and Suantai defined the SP iteration schemes [8] as follows.

Let D be a nonempty convex subset of E and let $T: D \longrightarrow D$ be a mapping. For a given $\{x_n\}$ in D, define the sequence $\{x_n\}_{n=0}^{\infty}$ by

$$\begin{cases} x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \quad n \geq 0, \end{cases} \tag{4}$$

where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are three real sequences in [0,1] satisfying some certain conditions.

Definition 1 (see [1]). Given a gauge function ϕ, the mapping $J_\phi: E \longrightarrow 2^{E^*}$ defined by

$$J_\phi x := \left\{u^* \in E^*: \langle x, u^* \rangle = \|x\|\|u^*\|; \|u^*\| = \phi(\|x\|)\right\} \tag{5}$$

is called the duality map with gauge function ϕ, where E is any normed space. In the particular case, $\phi(t) = t$, the duality map $J = J_\phi$ is called the normalized duality map.

Proposition 1 (see [9, 10]). *If a Banach space E has a uniformly Gateaux differentiable norm, then $J: E \longrightarrow E^*$ is uniformly continuous on bounded subsets of E from the strong topology of E to the weak$*$ topology of E^*.*

Definition 2 (see [11]). Let E be an arbitrary real normed linear space. A mapping $T: D(T) \subseteq E \longrightarrow E$ is called strongly hemi-contractive if $F(T) \neq \varnothing$, and there exists $t > 1$ such that, for all $r > 0$,

$$\|x - x^*\| \leq \|(1+r)(x - x^*) - rt(Tx - x^*)\|, \tag{6}$$

holds for all $x \in D(T), x^* \in F(T)$. If $t = 1$, then T is called hemi-contractive. Finally, T is called generalized Φ−hemi-contractive with strictly increasing continuous function $\Phi: [0,\infty) \longrightarrow [0,\infty)$ such that $\Phi(0) = 0$ if, for all $x \in D(T), x^* \in F(T) \neq \varnothing$, there exists $j(x - x^*) \in J(x - x^*)$ such that

$$\langle (I-T)x - (I-T)x^*, j(x - x^*)\rangle \geq \Phi\left(\|x - x^*\|\right). \tag{7}$$

It follows from inequality (7) that T is generalized Φ−hemi-contractive if and only if

$$\langle Tx - x^*, j(x - x^*)\rangle \leq \|x - x^*\|^2 - \Phi\left(\|x - x^*\|\right), \quad \forall n \geq 0. \tag{8}$$

Definition 3 (see [1, 12]). Let $N(T) = \{x \in E: Tx = 0\} \neq \varnothing$. The mapping $T: D(T) \subseteq E \longrightarrow E$ is called generalized Φ−quasi-accretive with strictly increasing continuous function $\Phi: [0,\infty) \longrightarrow [0,\infty)$ such that $\Phi(0) = 0$ if, for all $x \in E, x^* \in N(T) \neq \varnothing$, there exists $j(x - x^*) \in J(x - x^*)$ such that

$$\langle Tx - Tx^*, j(x - x^*)\rangle \geq \Phi\left(\|x - x^*\|\right). \tag{9}$$

Definition 4 (see [13, 14]). A mapping $T: D \longrightarrow D$ is called generalized Lipschitz if there exists a constant $L > 0$ such that $\|Tx - Ty\| \leq L(1 + \|x - y\|)$, $\forall x, y \in D$.

Proposition 2 (see [1]). *If $F(T) = \{x \in E: Tx = x\} \neq \varnothing$, the mapping $T: E \longrightarrow E$ is strongly hemi-contractive if and only if $(I - T)$is strongly quasi-accretive, T is strongly ϕ−hemi-contractive if and only if $(I - T)$ is strongly ϕ−quasi-accretive, and T is generalized Φ−hemi-contractive if and only if $(I - T)$ is generalized Φ−quasi-accretive.*

Proposition 3 (see [1]). *Let E be a uniformly smooth real Banach space, and let $J: E \longrightarrow 2^{E^*}$ be a normalized duality mapping. Then,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle, \tag{10}$$

for all $x, y \in E$.

Proposition 4 (see [1]). *Let $\{\lambda_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative numbers and $\{\alpha_n\}$ be a sequence of positive numbers satisfying the conditions $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $(\gamma_n/\alpha_n) \longrightarrow 0$, as $n \longrightarrow \infty$. Let the recursive inequality*

$$\lambda_{n+1} \leq \lambda_n - \alpha_n \psi(\lambda_n) + \gamma_n, \quad n = 1, 2, \ldots, \tag{11}$$

be given, where $\psi: [0,\infty) \longrightarrow [0,\infty)$ is strictly increasing continuous function such that it is positive on $(0,\infty)$ and $\psi(0) = 0$. Then, $\lambda_n \longrightarrow 0$, as $n \longrightarrow \infty$.

2. Main Results

In this section, we will consider to extend the result of Charles [1] to the Noor iteration process and SP iteration process under the following assumptions.

First, we extend the result of Charles [1] to the Noor iteration process.

Theorem 1. *Suppose D is a nonempty closed convex subset of a real uniformly smooth Banach space E. Suppose $T: D \longrightarrow D$ is a bounded generalized Lipschitz Φ−hemi-contractive mapping and $x^* \in F(T) \neq \varnothing$. For arbitrary $x_0 \in D, \{x_n\}$ is a Noor iterative sequence defined by (3), where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0,1]$, $\lim_{n\longrightarrow\infty}\alpha_n = \lim_{n\longrightarrow\infty}\beta_n = 0, \gamma_n = o(\beta_n)$, and $\sum \alpha_n = \infty$. Then, there exists a constant $d_0 > 0$ such that if $0 < \alpha_n, \beta_n, \gamma_n \leq d_0, \{x_n\}$ converges strongly to the unique fixed point x^* of T.*

Proof. Since $T: D \longrightarrow D$ is a bounded generalized Lipschitz Φ−hemi-contractive mapping, there exists a strictly increasing continuous function $\Phi: [0,\infty) \longrightarrow [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle (I-T)x - (I-T)x^*, j(x - x^*)\rangle \geq \Phi\left(\|x - x^*\|\right), \tag{12}$$

i.e.,

$$\langle Tx - x^*, j(x - x^*)\rangle \le \|x - x^*\|^2 - \Phi\left(\|x - x^*\|\right), \quad \forall n \ge 0, \tag{13}$$

for any $x, y \in D$, $x^* \in F(T)$ and exist a constant $L > 0$ such that

$$\|Tx - Ty\| \le L(1 + \|x - y\|), \quad \forall x, y \in D. \tag{14}$$

Let r be sufficiently large such that $x_1 \in B_r(x^*)$. Define $G: = \overline{B_r(x^*)} \cap D$. Then, since T is bounded, we have that $(I - T)(G)$ is bounded.

As j is uniformly continuous on bounded subsets of E, for $\varepsilon_0: = (\Phi(r/4)/40L(1 + 2r))$, there exists a δ: $(r/2) > \delta > 0$ such that $x, y \in D(T), \|x - y\| < \delta$ implies $\|j(x) - j(y)\| < \varepsilon_0$. Set $d_0 = \min\{(\delta/2[L+ (1 + 2L)r]), \sqrt{(\Phi(r/4)/8r^2)}\}$.

Claim 1. $\{x_n\}$ is bounded.

Suffices to show that x_n is in G for all $n \ge 1$. The proof is by induction. By our assumption, $x_1 \in G$. Suppose $x_n \in G$. We prove that $x_{n+1} \in G$. Assume for contradiction that $x_{n+1} \notin G$. Then, since $x_{n+1} \in D, \forall n \ge 1$, we have that $\|x_{n+1} - x^*\| > r$. We have the following estimates:

$$\begin{aligned}
\|(x_n - x^*) - (z_n - x^*)\| &\le \gamma_n \|x_n - Tx_n\| \\
&\le \gamma_n\left(\|x_n - x^*\| + \|Tx_n - x^*\|\right) \\
&\le d_0[L + (1 + L)r] < \delta, \\
\|z_n - x^*\| &= \|(1 - \gamma_n)x_n + \gamma_n Tx_n - x^*\| \\
&\le (1 - \gamma_n)\|x_n - x^*\| + \gamma_n\|Tx_n - x^*\| \\
&\le (1 - \gamma_n)\|x_n - x^*\| + \gamma_n L\left(1 + \|x_n - x^*\|\right) \\
&\le r + d_0 L(1 + r) \\
&\le 2r.
\end{aligned} \tag{15}$$

Using (15), we obtain

$$\|Tz_n - x^*\| \le L\left(1 + \|z_n - x^*\|\right) \le L(1 + 2r). \tag{16}$$

Using (16), we obtain

$$\begin{aligned}
\|y_n - x^*\| &= \|(1 - \beta_n)x_n + \beta_n Tz_n - x^*\| \\
&\le (1 - \beta_n)\|x_n - x^*\| + \beta_n\|Tz_n - x^*\| \\
&\le r + d_0 L\left(1 + \|z_n - x^*\|\right) \\
&\le 2r
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
\|(y_n - x^*) - (x_n - x^*)\| &\le \beta_n\|Tz_n - x_n\| \\
&\le \beta_n\left(\|Tz_n - x^*\| + \|x_n - x^*\|\right) \\
&\le d_0 \cdot [L + (1 + 2L)r] < \delta.
\end{aligned} \tag{18}$$

Using (17), we obtain

$$\begin{aligned}
\|Ty_n - x^*\| &\le L\left(1 + \|y_n - x^*\|\right) \\
&\le L(1 + 2r).
\end{aligned} \tag{19}$$

Using (19), we obtain

$$\begin{aligned}
\|(x_{n+1} - x^*) - (x_n - x^*)\| &\le \alpha_n\|Ty_n - x_n\| \\
&\le \alpha_n\left(\|Ty_n - x^*\| + \|x_n - x^*\|\right) \\
&\le d_0[L + (1 + 2L)r] < \delta.
\end{aligned} \tag{20}$$

Then,

$$\begin{aligned}
\|x_n - x^*\| &\ge \|x_{n+1} - x^*\| - \alpha_n\|Ty_n - x_n\| \\
&\ge r - d_0[L + (1 + 2L)r] \\
&\ge \frac{r}{2}, \\
\|z_n - x^*\| &\ge \|x_n - x^*\| - \gamma_n\|Tx_n - x_n\| \\
&\ge \frac{r}{2} - d_0[L + (1 + L)r] \\
&\ge \frac{r}{4}. \\
\|y_n - x^*\| &\ge \|x_n - x^*\| - \beta_n\|Tz_n - x_n\| \\
&\ge \frac{r}{2} - d_0[L + (1 + 2L)r] \\
&\ge \frac{r}{4},
\end{aligned} \tag{21}$$

Therefore,

$$\begin{aligned}
\|j(x_{n+1} - x^*) - j(x_n - x^*)\| &< \varepsilon_0, \\
\|j(x_n - x^*) - j(y_n - x^*)\| &< \varepsilon_0, \\
\|j(x_n - x^*) - j(z_n - x^*)\| &< \varepsilon_0.
\end{aligned} \tag{22}$$

Using Proposition 3 and the above formulas, we obtain

$$\begin{aligned}\|z_n - x^*\|^2 &= \|(1-\gamma_n)x_n + \gamma_n T x_n - x^*\|^2 \\ &= \|(1-\gamma_n)x_n - (1-\gamma_n)x^* + \gamma_n T x_n - \gamma_n x^*\|^2 \\ &\le (1-\gamma_n)^2\|x_n - x^*\|^2 + 2\gamma_n\langle Tx_n - x^*, j(x_n - x^*)\rangle \\ &\quad + 2\gamma_n\langle Tx_n - x^*, j(z_n - x^*) - j(x_n - x^*)\rangle \\ &\le r^2 + 2\gamma_n L(1+r)\varepsilon_0,\end{aligned} \tag{23}$$

$$\begin{aligned}\|y_n - x^*\|^2 &= \|(1-\beta_n)x_n + \beta_n T z_n - x^*\|^2 \\ &\le (1-\beta_n)^2\|x_n - x^*\|^2 + 2\beta_n\langle Tz_n - x^*, j(z_n - x^*)\rangle \\ &\quad + 2\beta_n\langle Tz_n - x^*, j(y_n - x^*) - j(x_n - x^*)\rangle \\ &\quad + 2\beta_n\langle Tz_n - x^*, j(x_n - x^*) - j(z_n - x^*)\rangle \\ &\le (1-\beta_n)^2\|x_n - x^*\|^2 + 2\beta_n\left[\|z_n - x^*\|^2 - \Phi\left(\|z_n - x^*\|\right)\right] \\ &\quad + 2\beta_n\|Tz_n - x^*\|\|j(y_n - x^*) - j(x_n - x^*)\| \\ &\quad + 2\beta_n\|Tz_n - x^*\|\|j(x_n - x^*) - j(z_n - x^*)\| \\ &\le (1-\beta_n)^2 r^2 + 2\beta_n\left[\|z_n - x^*\|^2 - \Phi\left(\|z_n - x^*\|\right)\right] \\ &\quad + 4\beta_n L(1+2r)\varepsilon_0,\end{aligned} \tag{24}$$

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)x_n + \alpha_n T y_n - x^*\|^2 \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\langle Ty_n - x^*, j(y_n - x^*)\rangle \\ &\quad + 2\alpha_n\langle Ty_n - x^*, j(x_n - x^*) - j(y_n - x^*)\rangle \\ &\quad + 2\alpha_n\langle Ty_n - x^*, j(x_{n+1} - x^*) - j(x_n - x^*)\rangle \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\left[\|y_n - x^*\|^2 - \Phi\left(\|y_n - x^*\|\right)\right] \\ &\quad + 2\alpha_n\|Ty_n - x^*\|\|j(x_n - x^*) - j(y_n - x^*)\| \\ &\quad + 2\alpha_n\|Ty_n - x^*\|\|j(x_{n+1} - x^*) - j(x_n - x^*)\| \\ &\le (1-\alpha_n)^2 r^2 + 4\alpha_n L(1+2r)\varepsilon_0 \\ &\quad + 2\alpha_n\left[\|y_n - x^*\|^2 - \Phi\left(\|y_n - x^*\|\right)\right].\end{aligned} \tag{25}$$

Substitute (23) into (24) and then substitute (24) into (25); since $0 < \alpha_n, \beta_n, \gamma_n \le d_0$ and $d_0 = \min\{(\delta/2[L + (1+2L)r]), \sqrt{(\Phi(r/4)/8r^2)}\}$, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le (1-\alpha_n)^2 r^2 + 4\alpha_n L(1+2r)\varepsilon_0 \\ &\quad + 2\alpha_n\left[(1-\beta_n)^2 r^2 + 4\beta_n L(1+2r)\varepsilon_0\right] \\ &\quad + 2\alpha_n\left\{2\beta_n\left[r^2 + 2\gamma_n L(1+r)\varepsilon_0 - \Phi\left(\frac{r}{4}\right)\right] - \Phi\left(\frac{r}{4}\right)\right\} \\ &\le r^2 + 2\alpha_n\left[2\beta_n^2 r^2 + 10L(1+2r)\varepsilon_0 - 2\beta_n\Phi\left(\frac{r}{4}\right) - \Phi\left(\frac{r}{4}\right)\right] \\ &\le r^2 + 2\alpha_n\left[\frac{\Phi(r/4)}{2} - 2\beta_n\Phi\left(\frac{r}{4}\right) - \Phi\left(\frac{r}{4}\right)\right] \\ &\le r^2,\end{aligned} \tag{26}$$

i.e., $\|x_{n+1} - x^*\| \le r$, a contradiction. Therefore, $x_{n+1} \in G$. Thus, by induction, $\{x_n\}$ is bounded. Then, $\{y_n\}, \{z_n\}, \{Tx_n\}, \{Ty_n\}$, and $\{Tz_n\}$ are also bounded.

Claim 2. $x_n \longrightarrow x^*$.

Let $A_n = \|j(x_{n+1} - x^*) - j(x_n - x^*)\|$, $B_n = \|j(x_n - x^*) - j(y_n - x^*)\|$, $C_n = \|j(z_n - x^*) - j(x_n - x^*)\|$. Note that $x_{n+1} - x_n \longrightarrow 0$, $x_n - y_n \longrightarrow 0$, and $z_n - x_n \longrightarrow 0$ as $n \longrightarrow \infty$, and hence, by the uniform continuity of j on bounded subsets of E, we have that

$$\begin{aligned}A_n &\longrightarrow 0, \\ B_n &\longrightarrow 0, \\ C_n &\longrightarrow 0, \text{ as } n \longrightarrow \infty.\end{aligned} \tag{27}$$

Let $M_1 = \max\{\sup\|x_n - x^*\|, \sup\|Tx_n - x^*\|, \sup\|Ty_n - x^*\|, \sup\|Tz_n - x^*\|\}$; by (24)–(26), we obtain that

$$\begin{aligned}\|z_n - x^*\|^2 &\le (1-\gamma_n)^2\|x_n - x^*\|^2 + 2\gamma_n M_1 C_n \\ &\le \|x_n - x^*\|^2 + 2\gamma_n M_1 C_n,\end{aligned} \tag{28}$$

$$\begin{aligned}\|y_n - x^*\|^2 \le{}& (1-\beta_n)^2\|x_n - x^*\|^2 + 2\beta_n\left[\|z_n - x^*\|^2 - \Phi\left(\|z_n - x^*\|\right)\right] \\ &+ 2\beta_n M_1 C_n + 2\beta_n M_1 B_n,\end{aligned} \tag{29}$$

and

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \le{}& (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\left[\|y_n - x^*\|^2 - \Phi\left(\|y_n - x^*\|\right)\right] \\ &+ 2\alpha_n\|Ty_n - x^*\|\|j(x_{n+1} - x^*) - j(x_n - x^*)\| \\ &+ 2\alpha_n\|Ty_n - x^*\|\|j(x_n - x^*) - j(y_n - x^*)\| \\ \le{}& (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n M_1 A_n + 2\alpha_n M_1 B_n \\ &+ 2\alpha_n\left[\|y_n - x^*\|^2 - \Phi\left(\|y_n - x^*\|\right)\right].\end{aligned} \tag{30}$$

Taking (28) into (29) and taking (29) into (30),

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \le{}& (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n M_1 A_n + 2\alpha_n M_1 B_n \\ &+ 2\alpha_n\left[\left(1+\beta_n^2\right)\|x_n - x^*\|^2 - \Phi\left(\|y_n - x^*\|\right)\right] \\ &+ 2\alpha_n\left\{2\beta_n\left[2\gamma_n M_1 C_n - \Phi\left(\|z_n - x^*\|\right) + M_1 C_n + M_1 B_n\right]\right\} \\ \le{}& \|x_n - x^*\|^2 + 2\alpha_n\left[\frac{\alpha_n}{2}M_1^2 + \beta_n^2 M_1^2 - \Phi\left(\|y_n - x^*\|\right)\right] \\ &+ 2\alpha_n M_1 A_n + 2\alpha_n M_1 B_n + 4\alpha_n\beta_n M_1 C_n + 4\alpha_n\beta_n M_1 B_n \\ &+ 2\alpha_n\left[4\beta_n\gamma_n M_1 C_n - 2\beta_n\Phi\left(\|z_n - x^*\|\right)\right] \\ \le{}& \|x_n - x^*\|^2 + 2\alpha_n\left[Q_n - 2\beta_n\Phi\left(\|z_n - x^*\|\right) - \Phi\left(\|y_n - x^*\|\right)\right],\end{aligned} \tag{31}$$

where $Q_n = (\alpha_n/2)M_1^2 + \beta_n^2 M_1^2 + 4\beta_n\gamma_n\ M_1 C_n + 2\beta_n M_1 C_n + 2\beta_n M_1 B_n + M_1 A_n + M_1 B_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Set $\inf(\Phi(\|z_n - x^*\|)/\Phi(\|x_n - x^*\|) + 1) = W$; since Φ is a strictly increasing continuous function, then W exists. Thus,

$$\Phi\left(\|z_n - x^*\|\right) \ge W\Phi\left(\|x_n - x^*\|\right) + W \ge W\Phi\left(\|x_n - x^*\|\right). \tag{32}$$

Set $\inf(\Phi(\|y_n - x^*\|)/\Phi(\|x_n - x^*\|) + 1) = N$; since Φ is a strictly increasing continuous function, then N exists. Thus,

$$\Phi\left(\|y_n - x^*\|\right) \ge N\Phi\left(\|x_n - x^*\|\right) + N \ge N\Phi\left(\|x_n - x^*\|\right). \tag{33}$$

Then,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \|x_n - x^*\|^2 + 2\alpha_n\left[Q_n - 2\beta_n\Phi\left(\|z_n - x^*\|\right) - \Phi\left(\|y_n - x^*\|\right)\right] \\ &\le \|x_n - x^*\|^2 - 2\alpha_n(2\beta_n W - N)\Phi\left(\|x_n - x^*\|\right) + 2\alpha_n Q_n.\end{aligned} \tag{34}$$

Let $\lambda_n = \|x_n - x^*\|^2$, $S_n = 2(2\beta_n W - N)$, and $\rho_n = 2\alpha_n Q_n$; then, from inequality (34), we obtain that $\lambda_{n+1} \le \lambda_n - \alpha_n S_n \Phi(\lambda_n) + \rho_n$, where $(\rho_n/\alpha_n S_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore, the conclusion of the theorem follows from Proposition 4. Uniqueness of x^* is derived from the definition of T.

By Definition 4, we know that the generalized Lipschitz mapping is the extension of nonexpansive mapping, so the following corollary follows trivially.

Corollary 1. *Suppose D is a nonempty closed convex subset of a real uniformly smooth Banach space E. Suppose $T: D \longrightarrow D$ is a bounded generalized Φ–hemi-contractive mapping and $x^* \in F(T) \neq \varnothing$. For arbitrary $x_0 \in D, \{x_n\}$ is a Noor iterative sequence defined by (3), where $\{\alpha_n\}$, $\{\beta_n\}, \{\gamma_n\} \subseteq [0,1]$, $\lim_{n\longrightarrow\infty}\alpha_n = \lim_{n\longrightarrow\infty}\beta_n = 0, \gamma_n = o(\beta_n)$, and $\sum \alpha_n = \infty$. Then, there exists a constant $d_0 > 0$ such that if $0 < \alpha_n, \beta_n, \gamma_n \leq d_0, \{x_n\}$ converges strongly to the unique fixed point x^* of T.*

The following corollary follows trivially from Definition 2 and Definition 3.

Corollary 2. *Suppose E is a real uniformly smooth Banach space. Suppose $T: E \longrightarrow E$ is a bounded generalized Φ–accretive mapping and the solution x^* of the equation $Tx = 0$ exists. For arbitrary $x_0 \in E, \{x_n\}$ in E is defined as*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n - \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n - \beta_n T z_n, \\ z_n = (1-\gamma_n)x_n - \gamma_n T x_n, \quad n \geq 0, \end{cases} \tag{35}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0,1]$, $\lim_{n\longrightarrow\infty}\alpha_n = \lim_{n\longrightarrow\infty}\beta_n = 0, \gamma_n = o(\beta_n)$, and $\sum \alpha_n = \infty$. Then, there exists a constant $d_0 > 0$ such that if $0 < \alpha_n, \beta_n, \gamma_n \leq d_0, \{x_n\}$ converges strongly to the unique solution $Tx = 0$.

Now, we extend the result of Charles [1] to the SP iteration process as follows.

Theorem 2. *Suppose D is a nonempty closed convex subset of a real uniformly smooth Banach space E. Suppose $T: D \longrightarrow D$ is a bounded generalized Lipschitz Φ–hemi-contractive mapping and $x^* \in F(T) \neq \varnothing$. For arbitrary $x_0 \in D, \{x_n\}$ is a SP iterative sequence defined by (4), where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0,1], \lim_{n\longrightarrow\infty}\alpha_n = \lim_{n\longrightarrow\infty}\beta_n = 0$, $\gamma_n = o(\beta_n)$, and $\sum \gamma_n = \infty$. Then, there exists a constant $d_0 > 0$ such that if $0 < \alpha_n, \beta_n, \gamma_n \leq d_0, \{x_n\}$ converges strongly to the unique fixed point x^* of T.*

Proof. Since $T: D \longrightarrow D$ is a bounded generalized Lipschitz Φ–hemi-contractive mapping, there exists a strictly increasing continuous function $\Phi: [0,\infty) \longrightarrow [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle (I-T)x - (I-T)x^*, j(x-x^*)\rangle \geq \Phi\left(\|x-x^*\|\right), \tag{36}$$

i.e.,

$$\langle Tx - x^*, j(x-x^*)\rangle \leq \|x-x^*\|^2 - \Phi\left(\|x-x^*\|\right), \quad \forall n \geq 0, \tag{37}$$

for any $x, y \in D$ and $x^* \in F(T)$, and there exist a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L(1 + \|x-y\|), \quad \forall x, y \in D. \tag{38}$$

Let r be sufficiently large such that $x_1 \in B_r(x^*)$. Define $G: = \overline{B_r(x^*)} \cap D$. Then, since T is bounded, we have that $(I-T)(G)$ is bounded.

As j is uniformly continuous on bounded subsets of E, for $\varepsilon_0: = (\Phi(r/4)/6[L + 3r(1+L)])$, there exists a $\delta: (r/4) > \delta > 0$ such that $x, y \in D(T), \|x-y\| < \delta$ implies $\|j(x) - j(y)\| < \varepsilon_0$. Set $d_0 = (\delta/2[L + (1+L)3r])$.

Claim 3. $\{x_n\}$ is bounded.

Suffices to show that x_n is in G for all $n \geq 1$. The proof is by induction. By our assumption, $x_1 \in G$. Suppose $x_n \in G$. We prove that $x_{n+1} \in G$. Assume for contradiction that $x_{n+1} \notin G$. Then, since $x_{n+1} \in D, \forall n \geq 1$, we have that $\|x_{n+1} - x^*\| > r$. We have the following estimates:

$$\begin{aligned} \|(x_n - x^*) - (z_n - x^*)\| &\leq \gamma_n \|x_n - Tx_n\| \\ &\leq \gamma_n\left(\|x_n - x^*\| + \|Tx_n - x^*\|\right) \\ &\leq d_0[L + (1+L)r] < \delta, \\ \|z_n - x^*\| &= \|(1-\gamma_n)x_n + \gamma_n T x_n - x^*\| \\ &\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n\|Tx_n - x^*\| \\ &\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n L\left(1 + \|x_n - x^*\|\right) \\ &\leq r + d_0 L(1+r) \\ &\leq 2r. \end{aligned} \tag{39}$$

Using (39), we obtain

$$\begin{aligned} \|Tz_n - z_n\| &\leq \|Tz_n - x^*\| + \|z_n - x^*\| \\ &\leq L + (1+L)2r. \end{aligned} \tag{40}$$

Using (40), we obtain

$$\begin{aligned} \|(z_n - x^*) - (y_n - x^*)\| &\leq \beta_n \|z_n - Tz_n\| \\ &\leq \beta_n\left(\|z_n - x^*\| + \|Tz_n - x^*\|\right) \\ &\leq \beta_n[2r + L(1+2r)] < \delta, \\ \|y_n - x^*\| &= \|(1-\beta_n)z_n + \beta_n T z_n - x^*\| \\ &\leq (1-\beta_n)\|z_n - x^*\| + \beta_n\|Tz_n - x^*\| \\ &\leq 2r + d_0 L(1+2r) \\ &\leq 3r. \end{aligned} \tag{41}$$

Using (41), we obtain

$$\begin{aligned}\|(y_n - x^*) - (x_{n+1} - x^*)\| &\le \alpha_n\left(\|y_n - x^*\| + \|Ty_n - x^*\|\right)\\ &\le \alpha_n\left[\|y_n - x^*\| + L\left(1 + \|y_n - x^*\|\right)\right]\\ &\le d_0[3r + L(1+3r)] < \delta,\\ \|Ty_n - y_n\| &= \|(Ty_n - x^*) - (y_n - x^*)\|\\ &\le L\left(1 + \|y_n - x^*\|\right) + \|y_n - x^*\|\\ &\le L + (1+L)3r.\end{aligned} \tag{42}$$

Then,

$$\begin{aligned}\|(y_n - x^*)\| &\ge \|x_{n+1} - x^*\| - \alpha_n\|Ty_n - y_n\|\\ &\ge r - d_0[L + (1+L)3r]\\ &\ge \frac{r}{2},\\ \|z_n - x^*\| &\ge \|y_n - x^*\| - \gamma_n\|Tz_n - z_n\|\\ &\ge \frac{r}{2} - d_0[L + (1+L)2r]\\ &\ge \frac{r}{4}.\end{aligned} \tag{43}$$

Therefore,

$$\begin{aligned}\|j(x_{n+1} - x^*) - j(y_n - x^*)\| &< \varepsilon_0,\\ \|j(y_n - x^*) - j(z_n - x^*)\| &< \varepsilon_0,\\ \|j(z_n - x^*) - j(x_n - x^*)\| &< \varepsilon_0.\end{aligned} \tag{44}$$

Using Proposition 3 and the above formulas, we obtain

$$\begin{aligned}\|z_n - x^*\|^2 &= \|(1-\gamma_n)x_n + \gamma_n Tx_n - x^*\|^2\\ &= \|(1-\gamma_n)x_n - (1-\gamma_n)x^* + \gamma_n Tx_n - \gamma_n x^*\|^2\\ &\le \|x_n - x^*\|^2 - 2\gamma_n\langle (I-T)x_n, j(x_n - x^*)\rangle\\ &\quad + 2\gamma_n\langle (I-T)x_n, j(z_n - x^*) - j(x_n - x^*)\rangle\\ &\le \|x_n - x^*\|^2 - 2\gamma_n\Phi\left(\|x_n - x^*\|\right)\\ &\quad + 2\gamma_n[L + (1+L)r]\varepsilon_0\\ &\le r^2 + 2\gamma_n[L + (1+L)r]\varepsilon_0,\end{aligned} \tag{45}$$

$$\begin{aligned}\|y_n - x^*\|^2 &= \|(1-\beta_n)z_n + \beta_n Tz_n - x^*\|^2\\ &= \|(z_n - x^*) - \beta_n(I-T)z_n\|^2\\ &\le \|z_n - x^*\|^2 - 2\beta_n\langle (I-T)z_n, j(z_n - x^*)\rangle\\ &\quad - 2\beta_n\langle (I-T)z_n, j(y_n - x^*) - j(z_n - x^*)\rangle\\ &\le \|z_n - x^*\|^2 - 2\beta_n\Phi\left(\|z_n - x^*\|\right)\\ &\quad + 2\beta_n\|z_n - Tz_n\|\|j(y_n - x^*) - j(z_n - x^*)\|\\ &\le \|z_n - x^*\|^2 - 2\beta_n\Phi\left(\|z_n - x^*\|\right)\\ &\quad + 2\beta_n[L + (1+L)2r]\varepsilon_0,\end{aligned} \tag{46}$$

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)y_n + \alpha_n Ty_n - x^*\|^2\\ &\le \|y_n - x^* - \alpha_n(I-T)y_n\|^2\\ &\le \|y_n - x^*\|^2 - 2\alpha_n\langle (I-T)y_n, j(y_n - x^*)\rangle\\ &\quad - 2\alpha_n\langle (I-T)y_n, j(x_{n+1} - x^*) - j(y_n - x^*)\rangle\\ &\le \|y_n - x^*\|^2 - 2\alpha_n\Phi\left(\|y_n - x^*\|\right)\\ &\quad + 2\alpha_n\|y_n - Ty_n\|\|j(x_{n+1} - x^*) - j(y_n - x^*)\|\\ &\le \|y_n - x^*\|^2 - 2\alpha_n\Phi\left(\|y_n - x^*\|\right)\\ &\quad + 2\alpha_n[L + (1+L)3r]\varepsilon_0.\end{aligned} \tag{47}$$

Substitute (45) into (46) and then substitute (46) into (47); since $0 < \alpha_n, \beta_n, \gamma_n \le d_0$ and $d_0 = (\delta/2[L + (1+L)3r])$, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le r^2 + 2\gamma_n[L + (1+L)r]\varepsilon_0 - 2\beta_n\Phi\left(\frac{r}{4}\right)\\ &\quad + 2\beta_n[L + (1+L)2r]\varepsilon_0 - 2\alpha_n\Phi\left(\frac{r}{2}\right)\\ &\quad + 2\alpha_n[L + (1+L)3r]\varepsilon_0\\ &\le r^2 + 2\alpha_n\left\{[L + (1+L)3r]\varepsilon_0 + \frac{\beta_n}{\alpha_n}[L + (1+L)2r]\varepsilon_0\right\}\\ &\quad + 2\alpha_n\left\{\frac{\gamma_n}{\alpha_n}[L + (1+L)r]\varepsilon_0 - \Phi\left(\frac{r}{2}\right) - \frac{\beta_n}{\alpha_n}\Phi\left(\frac{r}{4}\right)\right\}\\ &\le r^2 + 2\alpha_n\left[\frac{\Phi(r/4)}{2} - \Phi\left(\frac{r}{4}\right) - \frac{\beta_n}{\alpha_n}\Phi\left(\frac{r}{4}\right)\right]\\ &\le r^2,\end{aligned} \tag{48}$$

i.e., $\|x_{n+1} - x^*\| \le r$, a contradiction. Therefore, $x_{n+1} \in G$. Thus, by induction, $\{x_n\}$ is bounded. Then, $\{y_n\}, \{z_n\}, \{Tx_n\}, \{Ty_n\}$, and $\{Tz_n\}$ are also bounded.

Claim 4. $x_n \longrightarrow x^*$.

Let $A_n = \|j(x_{n+1} - x^*) - j(y_n - x^*)\|, B_n = \|j(y_n - x^*) - j(z_n - x^*)\|, C_n = \|j(z_n - x^*) - j(x_n - x^*)\|$; note that $x_{n+1} - y_n \longrightarrow 0, y_n - z_n \longrightarrow 0$, and $z_n - x_n \longrightarrow 0$, as $n \longrightarrow \infty$, and hence, by the uniform continuity of j on bounded subsets of E, we have that

$$\begin{aligned} A_n &\longrightarrow 0, \\ B_n &\longrightarrow 0, \\ C_n &\longrightarrow 0, \text{ as } n \longrightarrow \infty. \end{aligned} \tag{49}$$

Let $M_1 = \max\{\sup\|y_n - Ty_n\|, \sup\|z_n - Tz_n\|, \sup\|x_n - Tx_n\|, \sup\|Tz_n - x^*\|\}$; by (46)–(48), we obtain that

$$\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 - 2\gamma_n\Phi(\|x_n - x^*\|) + 2\gamma_n M_1 C_n, \tag{50}$$

$$\|y_n - x^*\|^2 \le \|z_n - x^*\|^2 - 2\beta_n\Phi(\|z_n - x^*\|) + 2\beta_n M_1 B_n, \tag{51}$$

$$\|x_{n+1} - x^*\|^2 \le \|y_n - x^*\|^2 - 2\alpha_n\Phi(\|y_n - x^*\|) + 2\alpha_n M_1 A_n. \tag{52}$$

Taking (50) into (51) and taking (51) into (52),

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \|x_n - x^*\|^2 - 2\gamma_n\Phi(\|x_n - x^*\|) + 2\gamma_n M_1 C_n \\ &\quad - 2\beta_n\Phi(\|z_n - x^*\|) - 2\alpha_n\Phi(\|y_n - x^*\|) \\ &\quad + 2\beta_n M_1 B_n + 2\alpha_n M_1 A_n \\ &\le \|x_n - x^*\|^2 - 2\gamma_n\Phi(\|x_n - x^*\|) - 2\alpha_n\Phi(\|y_n - x^*\|) \\ &\quad + 2\gamma_n\left[\frac{\alpha_n}{\gamma_n}M_1 A_n + \frac{\beta_n}{\gamma_n}M_1 B_n + M_1 C_n\right] \\ &\quad - 2\beta_n\Phi(\|z_n - x^*\|), \end{aligned} \tag{53}$$

where $Q_n = (\alpha_n/\gamma_n)M_1 A_n + (\beta_n/\gamma_n)M_1 B_n + M_1 C_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Set $\inf(\Phi(\|z_n - x^*\|)/\Phi(\|x_n - x^*\|) + 1) = W$; since Φ is a strictly increasing continuous function, then W exists. Thus,

$$\Phi(\|z_n - x^*\|) \ge W\Phi(\|x_n - x^*\|) + W \ge W\Phi(\|x_n - x^*\|). \tag{54}$$

Set $\inf(\Phi(\|y_n - x^*\|)/\Phi(\|x_n - x^*\|) + 1) = N$; since Φ is a strictly increasing continuous function, then N exists. Thus,

$$\Phi(\|y_n - x^*\|) \ge N\Phi(\|x_n - x^*\|) + N \ge N\Phi(\|x_n - x^*\|). \tag{55}$$

Then,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \|x_n - x^*\|^2 + 2\gamma_n Q_n - 2\gamma_n\Phi(\|x_n - x^*\|) \\ &\quad - 2\beta_n W\Phi(\|x_n - x^*\|) - 2\alpha_n N\Phi(\|x_n - x^*\|) \\ &\le \|x_n - x^*\|^2 + 2\gamma_n Q_n \\ &\quad - 2\gamma_n\Phi(\|x_n - x^*\|)\left[1 + \frac{\beta_n}{\gamma_n}W + \frac{\alpha_n}{\gamma_n}N\right]. \end{aligned} \tag{56}$$

Let $\lambda_n = \|x_n - x^*\|^2, S_n = 2(1 + (\beta_n/\gamma_n)W + (\alpha_n/\gamma_n)N)$ and $\rho_n = 2\gamma_n Q_n$; then, from inequality (56), we obtain that $\lambda_{n+1} \le \lambda_n - \gamma_n S_n \Phi(\lambda_n) + \rho_n$, where $(\rho_n/\gamma_n S_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore, the conclusion of the theorem follows from Proposition 4. Uniqueness of x^* is derived from the definition of T.

By Definition 4, we know that the generalized Lipschitz mapping is the extension of nonexpansive mapping, so the following corollary follows trivially.

Corollary 3. *Suppose D is a nonempty closed convex subset of a real uniformly smooth Banach space E. Suppose $T: D \longrightarrow D$ is a bounded generalized Φ–hemi-contractive mapping and $x^* \in F(T) \ne \varnothing$. For arbitrary $x_0 \in D, \{x_n\}$ be a SP iterative sequence defined by (4), where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1], \lim_{n\longrightarrow\infty}\alpha_n = \lim_{n\longrightarrow\infty}\beta_n = 0$, $\gamma_n = o(\beta_n)$, and $\sum\gamma_n = \infty$. Then, there exists a constant $d_0 > 0$ such that if $0 < \alpha_n, \beta_n, \gamma_n \le d_0, \{x_n\}$ converges strongly to the unique fixed point x^* of T.*

The following corollary follows trivially from Definition 2 and Definition 3.

Corollary 4. *Suppose E is a real uniformly smooth Banach space. Suppose $T: E \longrightarrow E$ is a bounded generalized Φ–accretive mapping and the solution x^* of the equation $Tx = 0$ exists. For arbitrary $x_0 \in E, \{x_n\}$ in E is defined as*

$$\begin{cases} x_{n+1} = (1 - \alpha_n)y_n - \alpha_n Ty_n, \\ y_n = (1 - \beta_n)z_n - \beta_n Tz_n, \\ z_n = (1 - \gamma_n)x_n - \gamma_n Tx_n, \quad n \ge 0, \end{cases} \tag{57}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1], \lim_{n\longrightarrow\infty}\alpha_n = \lim_{n\longrightarrow\infty}\beta_n = 0$, $\gamma_n = o(\beta_n)$, and $\sum\gamma_n = \infty$. Then, there exists a constant $d_0 > 0$ such that if $0 < \alpha_n, \beta_n, \gamma_n \le d_0, \{x_n\}$ converges strongly to the unique solution $Tx = 0$.

3. Experiments

In this section, we analyze the rate of convergence of four iterative schemes, namely, Noor iteration, iteration of

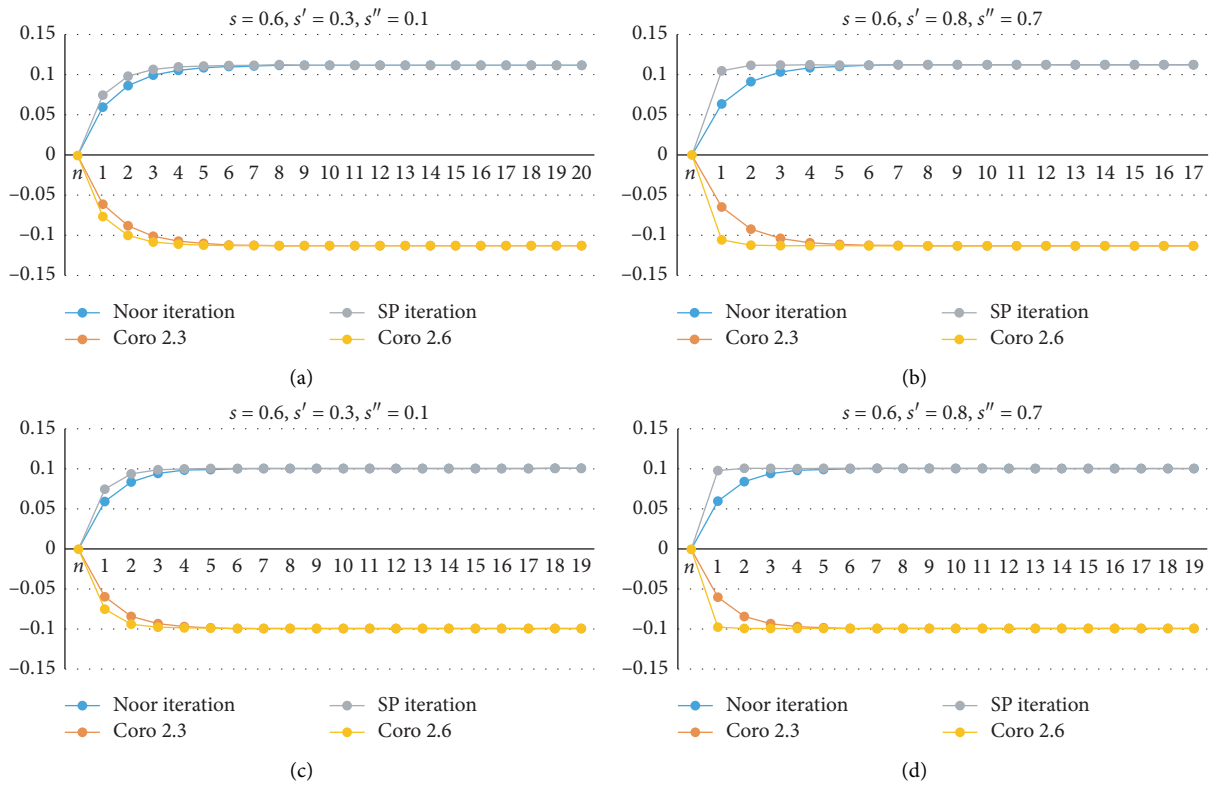

FIGURE 1: (a, b) Graphs of quadratic functions in Tables 1 and 2, respectively. (c, d) Graphs of cubic functions in Tables 3 and 4, respectively.

TABLE 1: Iterations of quadratic function $s = 0.6, s' = 0.3, s'' = 0.1$.

	$s = 0.6, s' = 0.3, s'' = 0.1$			
	Noor	Co2.3	SP	Co2.6
n	$F(x)$	$F(x)$	$F(x)$	$F(x)$
1	0.0605411	−0.0605411	0.0756347	−0.0756347
2	0.0874702	−0.0874702	0.0992939	−0.0992939
3	0.100248	−0.100248	0.107705	−0.107705
4	0.106495	−0.106495	0.11082	−0.11082
5	0.109594	−0.109594	0.11199	−0.11199
6	0.111142	−0.111142	0.112432	−0.112432
7	0.111918	−0.111918	0.1126	−0.1126
8	0.112308	−0.112308	0.112663	−0.112663
9	0.112503	−0.112503	0.112687	−0.112687
10	0.112602	−0.112602	0.112696	−0.112696
11	0.112652	−0.112652	0.1127	−0.1127
12	0.112676	−0.112676	0.112701	−0.112701
13	0.112689	−0.112689	0.112701	−0.112701
14	0.112695	−0.112695	**0.112702**	**−0.112702**
15	0.112698	−0.112698	0.112702	−0.112702
16	0.1127	−0.1127	0.112702	−0.112702
17	0.112701	−0.112701	0.112702	−0.112702
18	0.112701	−0.112701	0.112702	−0.112702
19	0.112701	−0.112701	0.112702	−0.112702
20	**0.112702**	**−0.112702**	0.112702	−0.112702
...	...	...	...	...
25	**0.112702**	**−0.112702**	**0.112702**	**−0.112702**

Table 2: Iterations of quadratic function $s = 0.6, s' = 0.8, s'' = 0.7$.

	$s = 0.6, s' = 0.8, s'' = 0.7$			
	Noor	Co2.3	SP	Co2.6
n	$F(x)$	$F(x)$	$F(x)$	$F(x)$
1	0.0642255	−0.0642255	0.104921	−0.104921
2	0.091647	−0.091647	0.111992	−0.111992
3	0.103513	−0.103513	0.112636	−0.112636
4	0.108682	−0.108682	0.112696	−0.112696
5	0.110942	−0.110942	0.112701	−0.112701
6	0.111931	−0.111931	**0.112702**	**−0.112702**
7	0.112364	−0.112364	0.112702	−0.112702
8	0.112554	−0.112554	0.112702	−0.112702
9	0.112637	−0.112637	0.112702	−0.112702
10	0.112673	−0.112673	0.112702	−0.112702
11	0.112689	−0.112689	0.112702	−0.112702
12	0.112696	−0.112696	0.112702	−0.112702
13	0.112699	−0.112699	0.112702	−0.112702
14	0.112701	−0.112701	0.112702	−0.112702
15	0.112701	−0.112701	0.112702	−0.112702
16	0.112701	−0.112701	0.112702	−0.112702
17	**0.112702**	**−0.112702**	0.112702	−0.112702
...	...	...	...	...
25	**0.112702**	**−0.112702**	**0.112702**	**−0.112702**

Table 3: Iterations of cubic function $s = 0.6, s' = 0.3, s'' = 0.1$.

	$s = 0.6, s' = 0.3, s'' = 0.1$			
	Noor	Co2.3	SP	Co2.6
n	$F(x)$	$F(x)$	$F(x)$	$F(x)$
1	0.0600162	−0.0599838	0.0748305	−0.0747695
2	0.0842313	−0.0837704	0.0940842	−0.0932201
3	0.0941177	−0.0930929	0.0991753	−0.0976483
4	0.0981802	−0.0967241	0.100534	−0.0987014
5	0.0998545	−0.0981347	0.100898	−0.0989512
6	0.100545	−0.098682	0.100996	−0.0990104
7	0.100831	−0.0988944	0.101022	−0.0990245
8	0.100948	−0.0989767	0.101029	−0.0990278
9	0.100997	−0.0990086	**0.101031**	−0.0990286
10	0.101017	−0.099021	0.101031	−0.0990288
11	0.101025	−0.0990258	0.101031	−0.0990288
12	0.101029	−0.0990277	0.101031	−0.0990288
13	0.10103	−0.0990284	0.101031	**−0.0990289**
14	**0.101031**	−0.0990287	0.101031	−0.0990289
15	0.101031	−0.0990288	0.101031	−0.0990289
16	0.101031	−0.0990288	0.101031	−0.0990289
17	0.101031	−0.0990288	0.101031	−0.0990289
18	0.101031	−0.0990288	0.101031	−0.0990289
19	0.101031	**−0.0990289**	0.101031	−0.0990289
...	...	...	...	...
25	**0.101031**	**−0.0990289**	**0.101031**	**−0.0990289**

Corollary 2, SP iteration, and iteration of Corollary 4, iterative schemes for complex space by using Visual Studio. The results obtained are extensions of some recent results of Rana et al. [15] and Chugh et al. [16].

We take $\alpha_n = s, \beta_n = s'$, and $\gamma_n = s''$ and derive the fixed points of the following polynomial functions:

Quadratic functions $= z^2 + c$

Cubic functions $= z^3 + c$

Biquadratic functions $= z^4 + c$

Recently, Rana et al. [15] drew a comparative analysis of Picard, Mann, and Ishikawa iterative schemes by starting with $z = (0, 0)$ and $c = 0.1$ in complex space. In this paper, we will continue the comparative study in complex space by taking the same z and c, for Noor iteration, iteration of

Table 4: Iterations of cubic function $s = 0.6, s' = 0.8, s'' = 0.7$.

	$s = 0.6, s' = 0.8, s'' = 0.7$			
	Noor	Co2.3	SP	Co2.6
n	$F(x)$	$F(x)$	$F(x)$	$F(x)$
1	0.0603104	−0.059696	0.0982125	−0.0969962
2	0.0846007	−0.0834203	0.100946	−0.0989904
3	0.0943985	−0.0928372	0.101029	−0.0990281
4	0.0983532	−0.0965731	**0.101031**	−0.0990288
5	0.0999499	−0.0980549	0.101031	**−0.0990289**
6	0.100595	−0.0986426	0.101031	−0.0990289
7	0.100855	−0.0988757	0.101031	−0.0990289
8	0.10096	−0.0989681	0.101031	−0.0990289
9	0.101003	−0.0990048	0.101031	−0.0990289
10	0.10102	−0.0990193	0.101031	−0.0990289
11	0.101027	−0.0990251	0.101031	−0.0990289
12	0.101029	−0.0990273	0.101031	−0.0990289
13	0.10103	−0.0990283	0.101031	−0.0990289
14	**0.101031**	−0.0990286	0.101031	−0.0990289
15	0.101031	−0.0990288	0.101031	−0.0990289
16	0.101031	−0.0990288	0.101031	−0.0990289
17	0.101031	−0.0990288	0.101031	−0.0990289
18	0.101031	−0.0990288	0.101031	−0.0990289
19	0.101031	**−0.0990289**	0.101031	−0.0990289
...	...	...	...	...
25	**0.101031**	**−0.0990289**	**0.101031**	**−0.0990289**

Table 5: Iterations of biquadratic function $s = 0.6, s' = 0.1, s'' = 0.1$.

	$s = 0.6, s' = 0.1, s'' = 0.1$			
	Noor	Co2.3	SP	Co2.6
n	$F(x)$	$F(x)$	$F(x)$	$F(x)$
1	0.06	−0.06	0.0676001	−0.0676001
2	0.0840101	−0.0840101	0.0895219	−0.0895219
3	0.0936363	−0.0936363	0.0966522	−0.0966522
4	0.0975019	−0.0975019	0.0989758	−0.0989758
5	0.0990556	−0.0990556	0.0997335	−0.0997335
6	0.0996802	−0.0996802	0.0999807	−0.0999807
7	0.0999314	−0.0999314	0.100061	−0.100061
8	0.100032	−0.100032	0.100088	−0.100088
9	0.100073	−0.100073	0.100096	−0.100096
10	0.100089	−0.100089	0.100099	−0.100099
11	0.100096	−0.100096	**0.1001**	**−0.1001**
12	0.100099	−0.100099	0.1001	−0.1001
13	**0.1001**	**−0.1001**	0.1001	−0.1001
...	...	...	...	...
25	**0.1001**	**−0.1001**	**0.1001**	**−0.1001**

Corollary 2, SP iteration, and iteration of Corollary 4 and, hence, extend the results of Rana et al. [15] and Chugh et al. [16].

Quadratic functions are provided in Tables 1 and 2, and their corresponding graphs are shown in Figures 1(a) and 1(b), respectively. Cubic functions are provided in Tables 3 and 4, and their corresponding graphs are shown in Figures 1(c) and 1(d). Biquadratic functions are provided in Tables 5–8, and their corresponding graphs are shown in Figures 1(a)–1(d).

Table 6: Iterations of biquadratic function $s = 0.6, s' = 0.3, s'' = 0.1$.

	$s = 0.6, s' = 0.3, s'' = 0.1$			
	Noor	Co2.3	SP	Co2.6
n	$F(x)$	$F(x)$	$F(x)$	$F(x)$
1	0.0600005	−0.0600005	0.0748011	−0.0748011
2	0.0840163	−0.0840163	0.0936851	−0.0936851
3	0.0936439	−0.0936439	0.0984714	−0.0984714
4	0.0975076	−0.0975076	0.0996866	−0.0996866
5	0.099059	−0.099059	0.0999953	−0.0999953
6	0.0996821	−0.0996821	0.100074	−0.100074
7	0.0999324	−0.0999324	0.100094	−0.100094
8	0.100033	−0.100033	0.100099	−0.100099
9	0.100073	−0.100073	**0.1001**	**−0.1001**
10	0.10009	−0.10009	0.1001	−0.1001
11	0.100096	−0.100096	0.1001	−0.1001
12	0.100099	−0.100099	0.1001	−0.1001
13	**0.1001**	**−0.1001**	0.1001	−0.1001
...	...	...	...	...
25	**0.1001**	**−0.1001**	**0.1001**	**−0.1001**

Table 7: Iterations of biquadratic function $s = 0.6, s' = 0.5, s'' = 0.4$.

	$s = 0.6, s' = 0.5, s'' = 0.4$			
	Noor	Co2.3	SP	Co2.6
n	$F(x)$	$F(x)$	$F(x)$	$F(x)$
1	0.0600038	−0.0600038	0.0880149	−0.0880149
2	0.0840261	−0.0840261	0.0986334	−0.0986334
3	0.0936535	−0.0936535	0.0999221	−0.0999221
4	0.0975142	−0.0975142	0.100079	−0.100079
5	0.0990629	−0.0990629	0.100098	−0.100098
6	0.0996841	−0.0996841	**0.1001**	**−0.1001**
7	0.0999334	−0.0999334	0.1001	−0.1001
8	0.100033	−0.100033	0.1001	−0.1001
9	0.100074	−0.100074	0.1001	−0.1001
10	0.10009	−0.10009	0.1001	−0.1001
11	0.100096	−0.100096	0.1001	−0.1001
12	0.100099	−0.100099	0.1001	−0.1001
13	**0.1001**	**−0.1001**	0.1001	−0.1001
...	...	...	...	...
25	**0.1001**	**−0.1001**	**0.1001**	**−0.1001**

Table 8: Iterations of biquadratic function $s = 0.6, s' = 0.8, s'' = 0.7$.

	$s = 0.6, s' = 0.8, s'' = 0.7$			
	Noor	Co2.3	SP	Co2.6
n	$F(x)$	$F(x)$	$F(x)$	$F(x)$
1	0.0600246	−0.0600246	0.0976546	−0.0976546
2	0.0840529	−0.0840529	0.10004	−0.10004
3	0.093674	−0.093674	0.100099	−0.100099
4	0.0975268	−0.0975268	**0.1001**	**−0.1001**
5	0.0990697	−0.0990697	0.1001	−0.1001
6	0.0996876	−0.0996876	0.1001	−0.1001
7	0.0999351	−0.0999351	0.1001	−0.1001
8	0.100034	−0.100034	0.1001	−0.1001
9	0.100074	−0.100074	0.1001	−0.1001
10	0.10009	−0.10009	0.1001	−0.1001
11	0.100096	−0.100096	0.1001	−0.1001
12	0.100099	−0.100099	0.1001	−0.1001
13	**0.1001**	**−0.1001**	0.1001	−0.1001
...	...	...	...	...
25	**0.1001**	**−0.1001**	**0.1001**	**−0.1001**

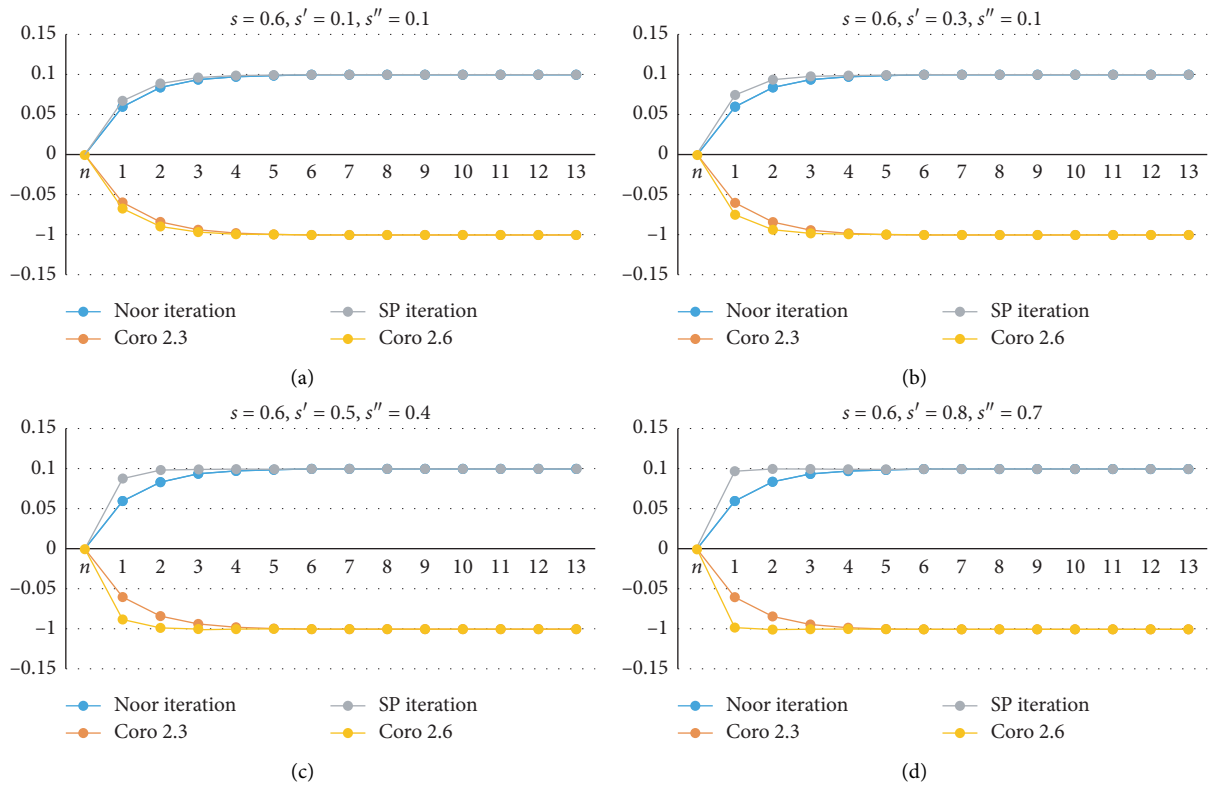

Figure 2: (a–d) Graphs of biquadratic functions given in Tables 5–8, respectively.

4. Conclusion

Keeping in mind comparative analysis drawn by [15, 16], Tables 1–8, we conclude that

(i) In case of quadratic polynomial, $F(x)$ of Noor and Co2.3 iterative schemes and SP and Co2.6 iterative schemes are opposite to each other; they show equivalence,and the speed of convergence of iterative schemes is compared as follows: SP > Noor; Co2.6 > Co2.3.

(ii) In case of cubic polynomial, the speed of convergence of iterative schemes is compared as follows: SP > Noor; Co2.6 > Co2.3; Noor > Co2.3; SP > Co2.6.

(iii) In case of biquadratic polynomial, $F(x)$ of Noor and Co2.3 iterative schemes and SP and Co2.6 iterative schemes are opposite to each other; they show equivalence, and the speed of convergence of iterative schemes is compared as follows: SP > Noor; Co2.6 > Co2.3 (Figure 2).

(iv) In the case of biquadratic polynomial, the most important discovery is that, as long as the value of s is set, the convergence rate of Noor and the convergence rate of Co2.3 iterative schemes will not change.

Data Availability

The data used to support the findings of this study are included within the article and are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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