Research Article

The Characterizing Properties of (Signless) Laplacian Permanental Polynomials of Almost Complete Graphs

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Let G be a graph with n vertices, and let $L(G)$ and $Q(G)$ denote the Laplacian matrix and signless Laplacian matrix, respectively. The Laplacian (respectively, signless Laplacian) permanental polynomial of G is defined as the permanent of the characteristic matrix of $L(G)$ (respectively, $Q(G)$). In this paper, we show that almost complete graphs are determined by their (signless) Laplacian permanental polynomials.

1. Introduction

We use G to denote a simple graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. The degree of a vertex $v \in V(G)$ is denoted by $d(v)$. The degree matrix of G , denoted by $D(G)$, is the diagonal matrix whose (i, i) th entry is $d(v_i)$. For a subgraph H of G , let $G - E(H)$ denote the subgraph obtained from G by deleting the edges of H . Let $c_i(G)$ and $p_i(G)$ denote, respectively, the numbers of i -cycles and i -vertex paths in G . Let $c_3(G_v)$ denote the number of triangles containing the vertex v of G . Let $G \cup H$ be the union of two graphs G and H which have no common vertices. For any positive integer l , let lG be the union of l disjoint copies of graph G . For convenience, the complete graph, path, cycle, and star on n vertices are denoted by K_n , P_n , C_n , and $K_{1,n-1}$, respectively.

The permanent of $n \times n$ matrix $X = (x_{ij})$ ($i, j = 1, 2, \dots, n$) is defined as

$$\text{per}(X) = \sum_{\sigma} \prod_{i=1}^n x_{i\sigma(i)}, \quad (1)$$

where the sum is taken over all permutations σ of $\{1, 2, \dots, n\}$. Valiant [1] has shown that computing the permanent is #P-complete even when restricted to $(0, 1)$ -matrices.

Let M be an $n \times n$ matrix. The permanental polynomial of M , denoted by $\pi(M, x)$, is defined to be the permanent of the characteristic matrix of M , i.e.,

$$\pi(M) = \pi(M, x) = \text{per}(xI - M), \quad (2)$$

where I is the identity matrix of size n .

Let $A(G)$ denote the adjacency matrix of G . The Laplacian matrix $L(G)$ and signless Laplacian matrix $Q(G)$ of G are defined by $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively. We call $\pi(L(G), x)$ (resp, $\pi(Q(G), x)$) the Laplacian (resp, signless Laplacian) permanental polynomial of G . The Laplacian permanental polynomial of a graph was first considered by Merris et al. [2], and the signless Laplacian permanental polynomial was first studied by Faria [3]. For more studies on (signless) Laplacian permanental polynomials, see [4–14], among others.

Two graphs G and H are called Laplacian copermanental if $\pi(L(G), x) = \pi(L(H), x)$. Analogously, signless Laplacian copermanental could be defined. A graph G is said to be determined by its Laplacian (resp, signless Laplacian) permanental polynomial if any graph Laplacian (resp, signless Laplacian) copermanental with G is isomorphic to G .

It is interesting to characterize which graph is determined by graph polynomials [15–18]. Merris et al. [2] first discussed the problem: which graph is determined by its

Laplacian permanental polynomial? Answer to the problem is very hard. Up to now, only a few results are known about the problem. Merris et al. computed the Laplacian permanental polynomials of all connected graphs on 6 vertices, and they found that there exist no nonisomorphic Laplacian copermanental graphs of such graphs. Based on the result, they stated that they do not know of a pair of nonisomorphic Laplacian copermanental graphs. Recently, Liu [19] showed that complete graph K_n and star S_n are determined by their (signless) Laplacian permanental polynomials.

Let $\mathcal{G}_n$ denote the set of graphs each of which is obtained from K_n , by removing five or fewer edges. Cámara and Haemers [20] showed that all graphs in $\mathcal{G}_n$ are determined by their characteristic polynomials of adjacency matrices of these graphs. The authors [21] proved that all graphs in $\mathcal{G}_n$ are determined by their A_α -spectra. In this paper, our interest is to discuss which graph in $\mathcal{G}_n$ is determined by its (signless) Laplacian permanental polynomial. And, we prove the following result.

Theorem 1. *All graphs in $\mathcal{G}_n$ are determined by their (signless) Laplacian permanental polynomial.*

The rest of this paper is organized as follows. In Section 2, we present some characterizing properties of the (signless) Laplacian permanental polynomial and give some structural

properties of graphs in $\mathcal{G}_n$. In Section 3, we give the Proof of Theorem 1.

2. Preliminaries

Let $\mathcal{G}_n$ denote the set of graphs each of which is obtained from K_n by removing five or fewer edges. For $n \geq 10$, there exist exactly 45 nonisomorphic graphs each of which is obtained from K_n by removing five or fewer edges [21, 22]. These graphs are labeled by G_{ij} , $1 \leq i \leq 5, 0 \leq j \leq 25$, and illustrated in Figure 1. For some properties of graphs in $\mathcal{G}_n$, see [21, 22], among others.

Lemma 1 (see [22]). *Let $H \subseteq K_n$ be a graph with l edges and let $G = K_n - E(H)$. Then,*

$$c_3(G) = \binom{n}{3} - l(n-2) + \sum_{v \in V(H)} \binom{d(v)}{2} - c_3(H). \quad (3)$$

In [22], the first author calculated the number of triangles of some $G \in \mathcal{G}_n$, see Table 1.

Lemma 2 (see [22]). *Let $H \subseteq K_n$ be a graph with l edges and let $G = K_n - E(H)$. Then,*

$$c_4(G) = 3 \binom{n}{4} - 2l \binom{n-2}{2} + \left[2 \binom{l}{2} + (n-5) \sum_{v \in V(H)} \binom{d(v)}{2} \right] - p_4(H) + c_4(H). \quad (4)$$

In [22], the first author calculated the number of quadrangles of some $G \in \mathcal{G}_n$, see Table 2.

Lemma 3 (see [21]). *Let $c_3(G_v)$ denote the number of triangles containing the vertex v of G . Using the principle of*

inclusion-exclusion, we can obtain the following result. Let $H \subseteq K_n$ be a graph with k edges and let $G = K_n - E(H)$. Let $v \in V(G)$ and let v be an endpoint of l edges in $E(H)$. Then,

$$c_3(G_v) = \binom{n-1}{2} - (k-l) - l(n-1-l) - \binom{l}{2} + c_3(\overline{G}(v_i)) + |P_3|. \quad (5)$$

Lemma 4 (see [19]). *Let G be a graph with n vertices and m edges, and let $(d_1, d_2, \dots, d_n)$ be the degree sequence of G . Suppose that $\pi(L(G), x) = \sum_i p_i(G) x^{n-i}$. Then,*

- (i) $p_0(G) = 1$
- (ii) $p_1(G) = -2m$
- (iii) $p_2(G) = 2m^2 + m - (1/2) \sum_i d_i^2$
- (iv) $p_3(G) = -(1/2) \sum_i d_i^3 + (m+1) \sum_i d_i^2 - (4/3)m^3 - 2m^2 - 2c_3(G)$
- (v) $p_4(G) = -(1/4) \sum_i d_i^4 + ((2/3)m+1) \sum_i d_i^3 - (1/2)(2m^2 + 5m + 1) \sum_i d_i^2 + (1/8) (\sum_i d_i^2)^2 + \sum_{(v_i, v_j) \in E(G)} d_i d_j + 2 \sum_i d_i c_3(G_{v_i}) + 2c_4(G) - 4mc_3(G) + (2/3)m^4 + 2m^3 + (1/2)m^2 + (1/2)m$

Lemma 5 (see [19]). *Let G be a graph with n vertices and m edges, and let $(d_1, d_2, \dots, d_n)$ be the degree sequence of G . Suppose that $\pi(Q(G), x) = \sum_i q_i(G) x^{n-i}$. Then,*

- (i) $q_0(G) = 1$
- (ii) $q_1(G) = -2m$
- (iii) $q_2(G) = 2m^2 + m - (1/2) \sum_i d_i^2$
- (iv) $q_3(G) = -(1/3) \sum_i d_i^3 + (m+1) \sum_i d_i^2 - (4/3)m^3 - 2m^2 - 2c_3(G)$
- (v) $q_4(G) = -(1/4) \sum_i d_i^4 + ((2/3)m+1) \sum_i d_i^3 - (1/2)(2m^2 + 5m + 1) \sum_i d_i^2 + (1/8) (\sum_i d_i^2)^2 + \sum_{(v_i, v_j) \in E(G)} d_i d_j - 2 \sum_i d_i c_3(G_{v_i}) + 2c_4(G) + 4mc_3(G) + (2/3)m^4 + 2m^3 + (1/2)m^2 + (1/2)m$

FIGURE 1: The graphs obtained from K_n by deleting five or fewer edges drawn as lines in a disk.TABLE 1: The numbers of triangles of some graphs in $\mathcal{G}_n$.

Graph	$c_3(G)$
G_{30}	$\binom{n}{3} - 3n + 9$
G_{32}, G_{33}	$\binom{n}{3} - 3n + 8$
G_{41}, G_{47}	$\binom{n}{3} - 4n + 11$
G_{44}, G_{46}, G_{410}	$\binom{n}{3} - 4n + 12$
G_{50}, G_{51}, G_{514}	$\binom{n}{3} - 5n + 12$
G_{52}	$\binom{n}{3} - 5n + 11$
G_{54}	$\binom{n}{3} - 5n + 20$
$G_{53}, G_{57}, G_{518}, G_{519}, G_{522}$	$\binom{n}{3} - 5n + 14$
G_{31}	$\binom{n}{3} - 3n + 7$
G_{40}	$\binom{n}{3} - 4n + 14$

TABLE 1: Continued.

Graph	$c_3(G)$
G_{43}	$\binom{n}{3} - 4n + 9$
G_{42}, G_{45}, G_{48}	$\binom{n}{3} - 4n + 10$
G_{59}, G_{524}	$\binom{n}{3} - 5n + 17$
$G_{55}, G_{512}, G_{520}, G_{523}$	$\binom{n}{3} - 5n + 15$
$G_{58}, G_{511}, G_{513}, G_{517}$	$\binom{n}{3} - 5n + 13$
$G_{56}, G_{510}, G_{515}, G_{516}, G_{525}$	$\binom{n}{3} - 5n + 16$

By Lemmas 4 and 5, we have the following.

Corollary 1. Let G be a graph with n vertices and m edges, and let $(d_1, d_2, \dots, d_n)$ be the degree sequence of G . Suppose that $\pi(L(G), x) = \sum_i p_i(G) x^{n-i}$ and $\pi(Q(G), x) = \sum_i q_i(G) x^{n-i}$. Then,

TABLE 2: The numbers of quadrangles of some graphs in $\mathcal{G}_n$.

Graph	$c_4(G)$
G_{32}	$3\binom{n}{4} - 3n^2 + 18n - 27$
G_{42}	$3\binom{n}{4} - 4n^2 + 22n - 22$
G_{50}	$3\binom{n}{4} - 5n^2 + 27n - 21$
G_{53}	$3\binom{n}{4} - 5n^2 + 29n - 32$
G_{55}	$3\binom{n}{4} - 5n^2 + 30n - 38$
G_{511}	$3\binom{n}{4} - 5n^2 + 28n - 26$
G_{58}, G_{514}	$3\binom{n}{4} - 5n^2 + 28n - 25$
G_{512}, G_{525}	$3\binom{n}{4} - 5n^2 + 31n - 44$
G_{520}	$3\binom{n}{4} - 5n^2 + 30n - 40$
G_{41}, G_{45}	$3\binom{n}{4} - 4n^2 + 23n - 27$
G_{48}	$3\binom{n}{4} - 4n^2 + 22n - 23$
G_{51}	$3\binom{n}{4} - 5n^2 + 27n - 20$
G_{57}	$3\binom{n}{4} - 5n^2 + 29n - 30$
G_{523}	$3\binom{n}{4} - 5n^2 + 30n - 39$
G_{517}	$3\binom{n}{4} - 5n^2 + 28n - 27$
G_{519}, G_{522}	$3\binom{n}{4} - 5n^2 + 29n - 33$
G_{518}	$3\binom{n}{4} - 5n^2 + 30n - 37$

(i) $p_3(G) - q_3(G) = 4c_3(G)$

(ii) $p_4(G) - q_4(G) = 4\sum_i d_i c_3(G_{v_i}) - 8mc_3(G)$

For convenience, we calculate the value $\sum_i d_i c_3(G_{v_i})$ of some graphs in $\mathcal{G}_n$ see Table 3.

Lemma 6 (see [19]). *The following can be deduced from the (signless) Laplacian permanental polynomial of a graph G :*

TABLE 3: The value of $\sum_i d_i c_3(G_{v_i})$ of some graphs in $\mathcal{G}_n$.

Graph	$\sum_i d_i c_3(G_{v_i})$
G_{30}	$(1/2)n^4 - 2n^3 - (19/2)n^2 + 56n - 66$
G_{41}	$(1/2)n^4 - 2n^3 - (27/2)n^2 + 70n - 66$
G_{42}	$(1/2)n^4 - 2n^3 - (27/2)n^2 + 65n - 50$
G_{50}	$(1/2)n^4 - 2n^3 - (35/2)n^2 + 79n - 48$
G_{53}	$(1/2)n^4 - 2n^3 - (35/2)n^2 + 89n - 79$
G_{58}	$(1/2)n^4 - 2n^3 - (35/2)n^2 + 84n - 62$
G_{519}	$(1/2)n^4 - 2n^3 - (35/2)n^2 + 89n - 80$
G_{511}	$(1/2)n^4 - 2n^3 - (35/2)n^2 + 84n - 61$
G_{55}	$(1/2)n^4 - 2n^3 - (35/2)n^2 + 94n - 94$
G_{512}	$(1/2)n^4 - 2n^3 - (35/2)n^2 + 96n - 105$
G_{518}	$(1/2)n^4 - 2n^3 - (35/2)n^2 + 91n - 88$
G_{513}	$(1/2)n^4 - 2n^3 - (35/2)n^2 + 86n - 69$
G_{32}	$(1/2)n^4 - 2n^3 - (19/2)n^2 + 53n - 60$
G_{45}	$(1/2)n^4 - 2n^3 - (27/2)n^2 + 67n - 60$
G_{48}	$(1/2)n^4 - 2n^3 - (27/2)n^2 + 65n - 52$
G_{51}	$(1/2)n^4 - 2n^3 - (35/2)n^2 + 79n - 46$
G_{57}	$(1/2)n^4 - 2n^3 - (35/2)n^2 + 89n - 75$
G_{514}	$(1/2)n^4 - 2n^3 - (35/2)n^2 + 81n - 56$
G_{522}	$(1/2)n^4 - 2n^3 - (35/2)n^2 + 89n - 78$
G_{517}	$(1/2)n^4 - 2n^3 - (35/2)n^2 + 84n - 63$
G_{523}	$(1/2)n^4 - 2n^3 - (35/2)n^2 + 94n - 96$
G_{525}	$(1/2)n^4 - 2n^3 - (35/2)n^2 + 99n - 112$
G_{520}	$(1/2)n^4 - 2n^3 - (35/2)n^2 + 94n - 95$

(i) *The number of vertices*

(ii) *The number of edges*

(iii) *The sum of the squares of degree of vertices*

We recorded the results of the sum of squares of degrees of some graphs in advance, see Table 4.

3. The Proof of Theorem 1

We give some lemmas to prove Theorem 1 before. First, by Lemma 6 and Table 4, we obtain a result as follows.

Lemma 8. *Graphs $K_n, G_{10}, G_{20}, G_{21}, G_{31}, G_{33}, G_{34}, G_{40}, G_{43}, G_{46}, G_{49}, G_{52}, G_{54}$, and G_{521} are determined by the (signless) Laplacian permanental polynomial, respectively.*

Lemma 9. *Graphs G_{30} and G_{32} are determined by the (signless) Laplacian permanental polynomial.*

Proof. By Table 4, we know that $\sum_{i=1}^n d_i^2(G_{30}) = \sum_{i=1}^n d_i^2(G_{32}) = n^3 - 2n^2 - 11n + 24$.

By Lemma 4 (iv) and Table 1, we have

$$p_3(G_{30}) - p_3(G_{32}) = -\frac{1}{3} \left(\sum_i d_i^3(G_{30}) - \sum_i d_i^3(G_{32}) \right) + 2(c_3(G_{30}) - c_3(G_{32})) = 4. \quad (6)$$

TABLE 4: The sum of squares of degrees of some graphs in $\mathcal{G}_n$.

Graph	$\sum_{1 \leq i \leq n} d_i^2$
G_{20}	$n^3 - 2n^2 - 7n + 14$
G_{30}, G_{32}	$n^3 - 2n^2 - 11n + 24$
G_{33}	$n^3 - 2n^2 - 11n + 22$
G_{40}	$n^3 - 2n^2 - 15n + 36$
G_{42}, G_{48}	$n^3 - 2n^2 - 15n + 28$
G_{44}, G_{410}	$n^3 - 2n^2 - 15n + 32$
G_{49}	$n^3 - 2n^2 - 15n + 24$
G_{52}	$n^3 - 2n^2 - 19n + 32$
G_{54}	$n^3 - 2n^2 - 19n + 50$
$G_{56}, G_{512}, G_{515}, G_{525}$	$n^3 - 2n^2 - 19n + 42$
$G_{58}, G_{511}, G_{514}, G_{517}$	$n^3 - 2n^2 - 19n + 36$
G_{510}, G_{524}	$n^3 - 2n^2 - 19n + 44$
G_{21}	$n^3 - 2n^2 - 7n + 12$
G_{31}	$n^3 - 2n^2 - 11n + 20$
G_{34}	$n^3 - 2n^2 - 11n + 18$
G_{41}, G_{45}, G_{47}	$n^3 - 2n^2 - 15n + 30$
G_{43}	$n^3 - 2n^2 - 15n + 26$
G_{46}	$n^3 - 2n^2 - 15n + 34$
G_{50}, G_{51}	$n^3 - 2n^2 - 19n + 34$
$G_{53}, G_{57}, G_{513}, G_{519}, G_{522}$	$n^3 - 2n^2 - 19n + 38$
$G_{55}, G_{518}, G_{520}, G_{523}$	$n^3 - 2n^2 - 19n + 40$
G_{521}	$n^3 - 2n^2 - 19n + 30$
G_{59}, G_{516}	$n^3 - 2n^2 - 19n + 46$

Furthermore, by Lemma 5 (v) and Tables 1–3, we have

$$\begin{aligned}
& q_4(G_{32}) - q_4(G_{30}) \\
&= -\frac{1}{4} \left(\sum_i d_i^4(G_{32}) - \sum_i d_i^4(G_{30}) \right) + \left(\frac{2}{3}m + 1 \right) \\
&\quad \cdot \left(\sum_i d_i^3(G_{32}) - \sum_i d_i^3(G_{30}) \right) \\
&\quad + \left(\sum_{(v_i, v_j) \in E(G_{32})} d_i d_j - \sum_{(v_i, v_j) \in E(G_{30})} d_i d_j \right) \\
&\quad - 2 \left(\sum_i d_i c_3(G_{v_{32}}) + \sum_i d_i c_3(G_{v_{30}}) \right) \\
&\quad + 2(c_4(G_{32}) - c_4(G_{30})) + 4m(c_3(G_{32}) - c_3(G_{30})) = -6.
\end{aligned} \tag{7}$$

These imply that G_{30} and G_{32} are not (signless) Laplacian copermanental. $\square$

Lemma 10. *The following statements hold:*

- (i) *Graphs G_{41} , G_{45} , and G_{47} are not pairwise (signless) Laplacian copermanental*

- (ii) *Graphs G_{42} and G_{48} are not (signless) Laplacian copermanental*
 (iii) *Graphs G_{44} and G_{410} are not (signless) Laplacian copermanental*

Proof

- (i) By Lemma 4 (iv) and Table 1, we get that $p_3(G_{41}) - p_3(G_{45}) = 4$, $p_3(G_{41}) - p_3(G_{47}) = 2$ and $p_3(G_{45}) - p_3(G_{47}) = 2$. Furthermore, by Corollary 1 (i), Table 1, and the equations above, we obtain that $q_3(G_{41}) - q_3(G_{45}) = 4 - 4(c_3(G_{41}) - c_3(G_{45})) = 0$, $q_3(G_{41}) - q_3(G_{47}) = 2 - 4(c_3(G_{41}) - c_3(G_{47})) = 2$, and $q_3(G_{45}) - q_3(G_{47}) = 2 - 4(c_3(G_{45}) - c_3(G_{47})) = 6$. Furthermore, by Lemma 5 (v) and Tables 1–3, we have $q_4(G_{41}) - q_4(G_{45}) = -2$. These mean that graphs G_{41} , G_{45} , and G_{47} are not pairwise (signless) Laplacian copermanental.
- (ii) From Table 4, we obtain that $\sum_{i=1}^n d_i^2(G_{42}) = \sum_{i=1}^n d_i^2(G_{48}) = n^3 - 2n^2 - 15n + 28$. By Lemma 4 (v) and Tables 1–3, we have $p_4(G_{48}) - p_4(G_{42}) = -7$. By Corollary 1 (ii), $p_4(G_{48}) - p_4(G_{42}) = -7$, Tables 1 and 3, and the equation above, we obtain that $q_4(G_{48}) - q_4(G_{42}) = -7 - 4\sum_i d_i c_3(G_{v_{48}}) - \sum_i d_i c_3(G_{v_{42}}) + 8m(c_3(G_{48}) - c_3(G_{42})) = 1$. This implies that G_{42} and G_{48} are not (signless) Laplacian copermanental.
- (iii) Similarly, by Lemma 4 (iv) and Table 1, we have $p_3(G_{44}) - p_3(G_{410}) = -2$. By Corollary 1 (i), Table 1 and the equation above, we obtain $q_3(G_{44}) - q_3(G_{410}) = -2 - 4(c_3(G_{44}) - c_3(G_{410})) = -2$. These indicate that G_{44} and G_{410} are not (signless) Laplacian copermanental. $\square$

Lemma 11. *The following statements hold:*

- (i) *Graphs G_{50} and G_{51} are not (signless) Laplacian copermanental*
 (ii) *Graphs G_{53} , G_{57} , G_{513} , G_{519} , and G_{522} are not pairwise (signless) Laplacian copermanental*
 (iii) *Graphs G_{58} , G_{511} , G_{514} , and G_{517} are not pairwise (signless) Laplacian copermanental*
 (iv) *Graphs G_{55} , G_{518} , G_{520} , and G_{523} are not pairwise (signless) Laplacian copermanental*
 (v) *Graphs G_{59} and G_{516} are not (signless) Laplacian copermanental*
 (vi) *Graphs G_{510} and G_{524} are not (signless) Laplacian copermanental*
 (vii) *Graphs G_{56} , G_{512} , G_{515} , and G_{525} are not pairwise (signless) Laplacian copermanental*

Proof. By Table 4, we have that

$$\begin{aligned}
 \sum_{i=1}^n d_i^2(G_{50}) &= \sum_{i=1}^n d_i^2(G_{51}) = n^3 - 2n^2 - 19n + 34, \\
 \sum_{i=1}^n d_i^2(G_{53}) &= \sum_{i=1}^n d_i^2(G_{57}) = \sum_{i=1}^n d_i^2(G_{513}) = \sum_{i=1}^n d_i^2(G_{519}) = \sum_{i=1}^n d_i^2(G_{522}) = n^3 - 2n^2 - 19n + 38, \\
 \sum_{i=1}^n d_i^2(G_{55}) &= \sum_{i=1}^n d_i^2(G_{518}) = \sum_{i=1}^n d_i^2(G_{520}) = \sum_{i=1}^n d_i^2(G_{523}) = n^3 - 2n^2 - 19n + 40, \\
 \sum_{i=1}^n d_i^2(G_{56}) &= \sum_{i=1}^n d_i^2(G_{512}) = \sum_{i=1}^n d_i^2(G_{515}) = \sum_{i=1}^n d_i^2(G_{525}) = n^3 - 2n^2 - 19n + 42, \\
 \sum_{i=1}^n d_i^2(G_{58}) &= \sum_{i=1}^n d_i^2(G_{511}) = \sum_{i=1}^n d_i^2(G_{514}) = \sum_{i=1}^n d_i^2(G_{517}) = n^3 - 2n^2 - 19n + 36, \\
 \sum_{i=1}^n d_i^2(G_{59}) &= \sum_{i=1}^n d_i^2(G_{516}) = n^3 - 2n^2 - 19n + 46, \\
 \sum_{i=1}^n d_i^2(G_{56}) &= \sum_{i=1}^n d_i^2(G_{512}) = \sum_{i=1}^n d_i^2(G_{515}) = \sum_{i=1}^n d_i^2(G_{525}) = n^3 - 2n^2 - 19n + 42, \\
 \sum_{i=1}^n d_i^2(G_{58}) &= \sum_{i=1}^n d_i^2(G_{511}) = \sum_{i=1}^n d_i^2(G_{514}) = \sum_{i=1}^n d_i^2(G_{517}) = n^3 - 2n^2 - 19n + 36, \\
 \sum_{i=1}^n d_i^2(G_{59}) &= \sum_{i=1}^n d_i^2(G_{516}) = n^3 - 2n^2 - 19n + 46, \\
 \sum_{i=1}^n d_i^2(G_{510}) &= \sum_{i=1}^n d_i^2(G_{524}) = n^3 - 2n^2 - 19n + 44.
 \end{aligned} \tag{8}$$

(i) By Lemma 4 (v) and Tables 1–3, we have $p_4(G_{50}) - p_4(G_{51}) = -7$. By Corollary 1 (ii), Tables 1 and 3, and the equation above, we get that $q_4(G_{50}) - q_4(G_{51}) = 1$. These mean that G_{50} and G_{51} are not (signless) Laplacian copermanental.

(ii) By Lemma 4 (iv) and (v) and Tables 1–3, we get that $p_4(G_{53}) - p_4(G_{57}) = -14$, $p_3(G_{53}) - p_3(G_{513}) = 4$, $p_3(G_{53}) - p_3(G_{519}) = 2$, $p_3(G_{53}) - p_3(G_{522}) = 2$, $p_3(G_{57}) - p_3(G_{513}) = 4$, $p_3(G_{57}) - p_3(G_{519}) = 2$, $p_3(G_{57}) - p_3(G_{522}) = 2$, $p_3(G_{513}) - p_3(G_{519}) = -2$, $p_3(G_{513}) - p_3(G_{522}) = -2$, and $p_4(G_{519}) - p_4(G_{522}) = -5$.

By Corollary 1, Tables 1 and 3, and the equations above, we obtain that $p_4(G_{519}) - p_4(G_{522}) = -5$, $q_4(G_{53}) - q_4(G_{513}) = -4$, $q_3(G_{53}) - q_3(G_{519}) = 2$, $q_3(G_{53}) - q_3(G_{522}) = 2$, $q_4(G_{57}) - q_4(G_{513}) = -6$, $q_3(G_{57}) - q_3(G_{519}) = 2$, $q_3(G_{57}) - q_3(G_{522}) = 2$, $q_3(G_{513}) - q_3(G_{519}) = 2$, $q_3(G_{513}) - q_3(G_{522}) = 2$, and $q_4(G_{519}) - q_4(G_{522}) = 3$. These imply that G_{53} , G_{57} , G_{513} , G_{519} , and G_{522} are not pairwise (signless) Laplacian copermanental.

(iii) Similarly, by Lemma 4 (iv) and (v) and Tables 1–3, we have $p_3(G_{58}) - p_3(G_{511}) = 2$, $p_3(G_{58}) - p_3(G_{514}) = 4$, $p_3(G_{58}) - p_3(G_{517}) = 2$, $p_3(G_{511}) - p_3$

$(G_{514}) = 2$, $p_3(G_{511}) - p_3(G_{517}) = 0$, $p_4(G_{517}) - p_4(G_{511}) = -7$, and $p_3(G_{514}) - p_3(G_{517}) = -2$. Furthermore, by Corollary 1, Tables 1 and 3, and the equations above, we get that $q_3(G_{58}) - q_3(G_{511}) = 2$, $q_4(G_{58}) - q_4(G_{514}) = -6$, $q_3(G_{58}) - q_3(G_{517}) = 2$, $q_3(G_{511}) - q_3(G_{514}) = -2$, $q_3(G_{514}) - q_3(G_{517}) = 2$, and $q_4(G_{517}) - q_4(G_{511}) = 1$. These imply that graphs G_{58} , G_{511} , G_{514} , and G_{517} are not pairwise (signless) Laplacian copermanental.

(iv) Similarly, by Lemma 4 (iv) and (v) and Tables 1–3, we have $p_3(G_{55}) - p_3(G_{518}) = 2$, $p_3(G_{55}) - p_3(G_{520}) = 2$, $p_3(G_{55}) - p_3(G_{523}) = 0$, $p_3(G_{518}) - p_3(G_{520}) = 0$, $p_3(G_{518}) - p_3(G_{523}) = -2$, $p_3(G_{520}) - p_3(G_{523}) = -2$, $p_4(G_{55}) - p_4(G_{523}) = 7$, and $p_4(G_{518}) - p_4(G_{520}) = -1$. By Corollary 1, Tables 1 and 3, and the equations above, we know that $q_3(G_{55}) - q_3(G_{518}) = -2$, $q_3(G_{55}) - q_3(G_{520}) = 2$, $q_4(G_{55}) - q_4(G_{523}) = -1$, $q_3(G_{518}) - q_3(G_{520}) = 4$, $q_3(G_{518}) - q_3(G_{523}) = 2$, and $q_3(G_{520}) - q_3(G_{523}) = -2$. These mean that graphs G_{55} , G_{518} , G_{520} , and G_{523} are not pairwise (signless) Laplacian copermanental.

(v) By Lemma 4 (iv) and Table 1, we get that $p_3(G_{59}) - p_3(G_{516}) = 6$. By Corollary 1 (i), Table 1,

and the equation above, we have $q_3(G_{59}) - q_3(G_{516}) = 2$. These mean that graphs G_{59} and G_{56} are not (signless) Laplacian copermanental.

(vi) By Lemma 4 (iv) and Table 1, we have that $p_3(G_{510}) - p_3(G_{524}) = -6$. By Corollary 1 (i), Table 1, and the equation above, we obtain that $q_3(G_{510}) - q_3(G_{524}) = -2$. Obviously, graphs G_{510} and G_{524} are not (signless) Laplacian copermanental.

(vii) By Lemma 4 (iv) and Table 1, we have $p_3(G_{56}) - p_3(G_{512}) = 8$, $p_3(G_{56}) - p_3(G_{515}) = 6$, $p_3(G_{56}) - p_3(G_{525}) = 4$, $p_3(G_{512}) - p_3(G_{515}) = -2$, $p_3(G_{512}) - p_3(G_{525}) = -4$, and $p_3(G_{515}) - p_3(G_{525}) = -2$. Furthermore, by Corollary 1, Tables 1–3, and the equations above, we have $q_3(G_{56}) - q_3(G_{512}) = 4$, $q_3(G_{56}) - q_3(G_{515}) = 6$, $q_3(G_{56}) - q_3(G_{525}) = 4$, $q_3(G_{512}) - q_3(G_{515}) = 2$, $q_3(G_{512}) - q_3(G_{525}) = 6$, $q_3(G_{515}) - q_3(G_{525}) = -2$, and $q_4(G_{512}) - q_4(G_{525}) = 24$. Obviously, graphs G_{56} , G_{512} , G_{515} , and G_{525} are not pairwise (signless) Laplacian copermanental. $\square$

Proof of Theorem 1. From Lemmas 6–11, we directly obtain Theorem 1. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest or personal relationships that could have appeared to influence the work reported in this paper.

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