Hindawi
Journal of Mathematics
Volume 2021, Article ID 9177818, 6 pages
https://doi.org/10.1155/2021/9177818

Research Article

Radio Labelings of Lexicographic Product of Some Graphs

Muhammad Shahbaz Aasi, Muhammad Asif, Tanveer Iqbal, and Muhammad Ibrahim

Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan, Pakistan

Correspondence should be addressed to Muhammad Shahbaz Aasi; mshahbazasi@gmail.com

Received 13 September 2021; Accepted 2 November 2021; Published 7 December 2021

Academic Editor: Antonio Di Crescenzo

Labeling of graphs has defined many variations in the literature, e.g., graceful, harmonious, and radio labeling. Secrecy of data in data sciences and in information technology is very necessary as well as the accuracy of data transmission and different channel assignments is maintained. It enhances the graph terminologies for the computer programs. In this paper, we will discuss multidistance radio labeling used for channel assignment problems over wireless communication. A radio labeling is a one-to-one mapping $\wp: V(G) \longrightarrow \mathbb{Z}^+$ satisfying the condition $|\wp(\mu) - \wp(\mu')| \geq \mathrm{diam}(G) + 1 - d(\mu, \mu')$: $\mu, \mu' \in V(G)$ for any pair of vertices μ, μ' in G. The span of labeling $\wp$ is the largest number that $\wp$ assigns to a vertex of a graph. Radio number of G, denoted by $rn(G)$, is the minimum span taken over all radio labelings of G. In this article, we will find relations for radio number and radio mean number of a lexicographic product for certain families of graphs.

1. Introduction

The notion of graph labeling was first introduced in 1966 by Rosa in [1], and since then, many different graph labelings have been defined and studied. In the 19^{th} century, for studying the channel assignment problem, the term graph labeling was used where the transmitters are used as the vertices of the graph. Two vertices (transmitters) are said to be adjacent if they are sufficiently close to each other. A model of the channel assignment problem was provided by Hale [2] in 1980. Basic notions and definitions can be found in [3].

Let $G = (V(G), E(G))$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For any $\mu, \mu' \in V(G)$, let $d(\mu, \mu')$ be the shortest length of the path between the vertices μ and μ'. A distance-two labeling is a function $\wp: V(G) \longrightarrow \{1, 2, 3, \ldots, k\}$ with span k having the maximum value k such that for any $\mu, \mu' \in V(G), \mu \neq \mu'$, the following relations are satisfied:

$$|\wp(\mu) - \wp(\mu')| \geq \begin{cases} 2, & if\ d(\mu, \mu') = 1 \\ 1, & if\ d(\mu, \mu') = 2 \end{cases}. \tag{1}$$

In 1992, Griggs and Yeh [4] extensively studied about distance-two labeling.

An assignment of positive integers to the vertices of G by $\wp$ of G is said to be a radio k-labeling if $|\wp(\mu) - \wp(\mu')| \geq k + 1 - d(\mu, \mu')$, where k is an integer, $k \geq 1$. The span of labeling $\wp$, denoted by $sp(\wp)$, is the max $\{|\wp(\mu) - \wp(\mu')|: \mu, \mu' \in V(G)\}$. Radio number of G, denoted by $rn(G)$, is the minimum span taken over all radio labelings of G. The radio k-labeling number of G is the minimum span among all radio k-labelings of G.

The study of radio k-labelings was motivated by Chartrand et al. [5] where they found the radio k-labeling number for paths. In [5], the lower and upper bounds were given for the radio k-labeling number for paths which have been improved lately by Kchikech et al. [6]. The radio k-labeling becomes a radio labeling, when $k = \mathrm{diam}(G)$. A radio labeling is a mapping from the vertices of the graph to some subsets of positive integers. The task of radio labeling is to assign to each station a positive smallest integer such that the interference in the nearest channel should be minimized. In 2001, multilevel distance labeling problem was introduced by Chartrand et al. [7].

A radio labeling is a one-to-one mapping $\wp\colon V(G) \longrightarrow \mathbb{Z}^+$ satisfying the condition

$$\left\{\left|\wp(\mu) - \wp(\mu')\right| \geq \operatorname{diam}(G) + 1 - d(\mu, \mu')\colon \mu, \mu' \in V(G)\right\}. \tag{2}$$

In [8], multilevel distance (or radio) labeling for paths and cycles are determined by Liu and Zhu. Rahim et al. in [9] discussed and determined the radio number of Helm graphs. In [8], Liu et al. calculated the radio number of path graph. The radio numbers of hypercube graphs and square cycles have been computed by Khennoufa [10] and Liu et al. [11], respectively. In [12], Naseem et al. gave a lower bound for the radio number of edge-joint graphs. Adefokun and Ajayi [13] proved that for $p \geq 4$ and q even $rn(S_p \times P_q) = pq^2/2 + q - 1$ and that for q even $rn(S_3 \times P_q) = 3q^2/2 + q$. Kim et al. [14] determined the radio numbers of P_q with $q \geq 4$ and K_p with $p \geq 3$. Lower bound has been improved by Bantva [15] for the radio number of graphs which was earlier given by Das et al. in [16]. For more results, we have [17–21].

In [22], Ali et al. proposed a formula for finding a lower bound for $rn(G)$, for graphs with small diameter. It is sometimes very useful to determine how many pairs $(\mu_s, \mu_{(s+1)})$ with $\wp(\mu_{(s+1)}) - \wp(\mu_s) = 1$ we can have. If there can be atmost 'y' such pairs in a graph G, then

$$rn(G) \geq y + 2(q - 1 - y) + 1. \tag{3}$$

In this paper, firstly, we determine the radio number and then radio mean number for the lexicographic product of path with path, path with cycle, and cycle with cycle. Finally, we present computer programs for finding such radio labelings of these families of graphs.

2. Applications

Labeling of graphs is one of the most popular parameters due to its diverse applications in real life. Radio labeling process proved as an efficient way of determining the time of communication for sensor networks. For giving valuable mathematical models, it has a wide scope of applications such as coding theory, electrical switchboards, circuit design, communication network addressing, channel assignment process, social networks, astronomy, demand and supply scenario, radar, database management, X-ray crystallography, and data security.

3. Lexicographic Product of Graphs

The lexicographic product was first studied by Hausdorff in 1914 [23]. The lexicographic product of two graphs G_1 and G_2 is denoted by $G_1[G_2]$ which is a graph with (Figure 1)

(1) The vertex set of the Cartesian product $V(G_1) \times V(G_2)$, and

(2) Distinct vertices (μ, μ') and (μ_0, μ_0') are adjacent in $G_1[G_2]$ iff

  (a) $\mu\mu_0 \in E(G_1)$, or

  (b) $\mu = \mu_0$ and $\mu'\mu_0' \in E(G_2)$.

4. Main Results

In this section, we discuss the radio labelings and compute the radio number for the lexicographic product of path with path $P_p[P_q]$ and path with cycle $P_p[C_q]$ for $p = 2, 3$. Moreover, we also presented a computer program for computing the radio number of these families of graphs.

4.1. Results of Radio Labeling. Let P_q be the path with q vertices. The lexicographic product of P_1 with P_q is isomorphic to graph P_q. The radio number of paths is investigated by Liu et al. in [8] as stated in the following result.

Theorem 4.1 (see [8]). *For any $q \geq 3$,*

$$rn\left(P_1\left[P_q\right]\right) = \operatorname{rn}\left(P_q\right) = \begin{cases} 2k(k-1) + 1, & if\ q = 2k, \\ 2k^2 + 2, & if\ q = 2k + 1, \end{cases} \tag{4}$$

$$\begin{aligned} rn\left(P_2\left[P_q\right]\right) &= \begin{cases} 2q, & \text{if q} = 1, 2, \\ 2q + 2, & \text{if q} = 3, \\ 2q + 1, & \text{if q} \geq 4, \end{cases} \\ rn\left(P_2\left[C_q\right]\right) &= \begin{cases} 2q, & \text{if q} = 3, \\ 2q + 3, & \text{if q} = 4, \\ 2q + 1, & \text{if q} \geq 5. \end{cases} \end{aligned} \tag{5}$$

We have a result for lower bound of $rn(P_p[P_q])$ for $p = 2, 3$ and $q \geq 4$.

Theorem 4.2. *For all $q \geq 4$, $rn(P_p[P_q]) \geq pq + 1$.*

Proof. In order to prove that the value stated above is a lower bound for the radio number, we will use the idea of distance-two labeling, i.e., expression 1.

The order of the graph $P_p[P_q]$ is pq for $p = 2, 3$ and there exists $pq - 2$, such pairs with labeling difference equals to 1. So, 3 implies that

$$\begin{aligned} rn\left(P_p\left[P_q\right]\right) &\geq (pq - 2) + 2[(pq) - 1 - (pq - 2)] \\ &= pq - 2 + 2[pq - 1 - pq + 2] + 1 \\ &= pq - 2 + 2pq - 2 - 2pq + 4 + 1 \\ &= pq + 1, \\ rn\left(P_p\left[P_q\right]\right) &\geq pq + 1. \end{aligned} \tag{6}$$

□

Theorem 4.3. *For all $q \geq 4$, $rn(P_2[P_q]) \leq 2q + 1$.*

Proof. The vertex set is partitioned in two disjoint sets V_l and V_r. Each partition is given as$V_l = V_l^1 \cup V_l^2$ and $V_r = V_r^1 \cup V_r^2$. For $t = l, r$, $V_t^1 = \left\{v_t^1, v_t^2, v_t^3, \ldots, v_t^{\lceil q/2 \rceil}\right\}$ and $V_t^2 = \left\{v_t^{\lceil q/2 \rceil + 1}, v_t^{\lceil q/2 \rceil + 2}, v_t^{\lceil q/2 \rceil + 3}, \ldots, v_t^q\right\}$. Define a mapping $\wp\colon V(P_2[P_q]) \longrightarrow \mathbb{N}$ as follows:

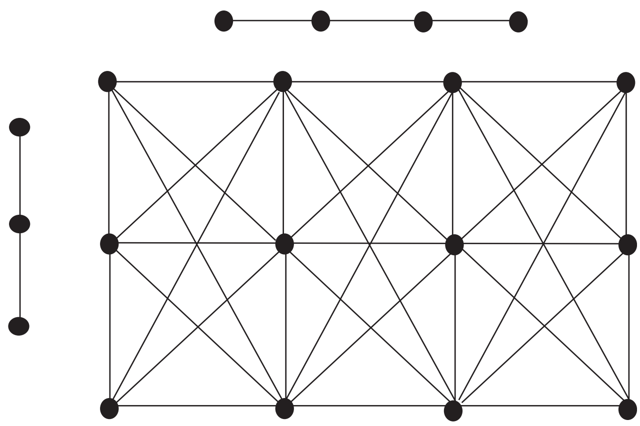

Figure 1: $P_4[P_3]$.

$$\begin{aligned}
&\wp\left(v_l^{\left\lceil \frac{q}{2}\right\rceil - s+1}\right) = q + 2s \text{ for } s = 1,2,3,\ldots,\left\lceil \frac{q}{2}\right\rceil\\
&\wp\left(v_l^{q-s+1}\right) = q + 1 + 2s \text{ for } s = 1,2,3,\ldots,q-\left\lceil \frac{q}{2}\right\rceil.\\
&\wp\left(v_c^{\left\lceil \frac{q}{2}\right\rceil - s+1}\right) = 2s - 1 \text{ for } s = 1,2,3,\ldots,\left\lceil \frac{q}{2}\right\rceil\\
&\wp\left(v_c^{q-s+1}\right) = 2s \text{ for } s = 1,2,3,\ldots,q-\left\lceil \frac{q}{2}\right\rceil.\\
&\wp\left(v_r^{\left\lceil \frac{q}{2}\right\rceil - s+1}\right) = 2(q+s) \text{ for } s = 1,2,3,\ldots,\left\lceil \frac{q}{2}\right\rceil\\
&\wp\left(v_r^{q-s+1}\right) = 2(q+s)+1 \text{ for } s = 1,2,3,\ldots,q-\left\lceil \frac{q}{2}\right\rceil.
\end{aligned} \tag{7}$$

Claim: the mapping $\wp$ is a valid radio labeling. We must show that condition 2 for radio labeling holds for all pair of vertices $\mu, \eta \in V(P_2[P_q])$.

Case 1: suppose μ and η are any two vertices in V_l, then two subcases can be obtained.

Case 1.1: let μ and η be any two distinct vertices in V_l^1, then $\mu = v_k$ and $\eta = v_l$, $1 \le k \ne l \le \lceil q/2 \rceil$; therefore, $\wp(\mu) = q + 2k$ and. $\wp(\eta) = q + 2l$. Also, we note that $d(\mu,\eta) \ge 1$; hence, $d(\mu,\eta) + |\wp(\mu) - \wp(\eta)| \ge 1 + 2 = 3$ $d(\mu,\eta) + |2(k-l)| \ge 3$.

Case 1.2: let μ and η be any two distinct vertices in V_l^2, then $\mu = v_k$ and $\eta = v_l$, $1 \le k \ne l \le q - \lceil q/2 \rceil$; therefore, $\wp(\mu) = q + 1 + 2k$ and $\wp(\eta) = q + 1 + 2l$. Also, we note that $d(\mu,\eta) \ge 1$; hence, $d(\mu,\eta) + |\wp(\mu) - \wp(\eta)| \ge 1 + 2 = 3$ $d(\mu,\eta) + |2(k-l)| \ge 3$.

Case 2: suppose μ and η are any two vertices in V_c, then two subcases can be obtained.

Case 2.1: let μ and η be any two distinct vertices in V_c^1, then $\mu = v_k$ and $\eta = v_l$, $1 \le k \ne l \le \lceil q/2 \rceil$; therefore, $\wp(\mu) = 2k - 1$ and $\wp(\eta) = 2l - 1$. Also, we note that $d(\mu,\eta) \ge 1$; hence, $d(\mu,\eta) + |\wp(\mu) - \wp(\eta)| \ge 1 + 2 = 3$ $d(\mu,\eta) + |2(k-l)| \ge 3$.

Case 2.2: let μ and η be any two distinct vertices in V_c^2, then $\mu = v_k$ and $\eta = v_l$, $1 \le k \ne l \le q - \lceil q/2 \rceil$; therefore, $\wp(\mu) = 2k$ and $\wp(\eta) = 2l$. Also, we note that $d(\mu,\eta) \ge 1$; hence, $d(\mu,\eta) + |\wp(\mu) - \wp(\eta)| \ge 1 + 2 = 3$ $d(\mu,\eta) + |2(k-l)| \ge 3$.

Case 3: suppose μ and η are any two vertices in V_r, then two subcases can be obtained.

Case 3.1: let μ and η be any two distinct vertices in V_r^1, then $\mu = v_k$ and $\eta = v_l$, $1 \le k \ne l \le \lceil q/2 \rceil$. therefore, $\wp(\mu) = 2(q+k)$ and $\wp(\eta) = 2(q+l)$. Also, we note that $d(\mu,\eta) \ge 1$; hence, $d(\mu,\eta) + |\wp(\mu) - \wp(\eta)| \ge 1 + 2 = 3$ $d(\mu,\eta) + |2(k-l)| \ge 3$.

Case 3.2: let μ and η be any two distinct vertices in V_r^2, then $\mu = v_k$ and $\eta = v_l$, $1 \le k \ne l \le q - \lceil q/2 \rceil$; therefore, $\wp(\mu) = 2(q+k) + 1$ and $\wp(\eta) = 2(q+l) + 1$. Also, we note that $d(\mu,\eta) \ge 1$; hence, $d(\mu,\eta) + |\wp(\mu) - \wp(\eta)| \ge 1 + 2 = 3$ $d(\mu,\eta) + |2(k-l)| \ge 3$. □

Theorem 4.4

$$rn\left(P_3\left[P_q\right]\right) = \begin{cases} 4, & \text{if } q = 1,\\ 3q + 2, & \text{if } q = 2,3,\\ 3q + 1, & \text{if } q \ge 4, \end{cases} \tag{8}$$

$$rn\left(P_3\left[C_q\right]\right) = \begin{cases} 3(q+1), & \text{if } q = 3,\\ 3q + 2, & \text{if } q = 4,\\ 3q + 1, & \text{if } q \ge 5, \end{cases} \tag{9}$$

Theorem 4.5. *For all $q \ge 4$, $rn(P_3[P_q]) \le 3q + 1$.*

Proof. The vertex set is partitioned in three disjoint sets V_l, V_c, and V_r. Each partition is further partitioned in two disjoint sets, i.e., $V_l = V_l^1 \cup V_l^2$, $V_c = V_c^1 \cup V_c^2$ and $V_r = V_r^1 \cup V_r^2$. For $t = l, c, r$, $V_t^1 = \left\{v_t^1, v_t^2, v_t^3, \ldots, v_t^{\lceil q/2 \rceil}\right\}$ and $V_t^2 = \left\{v_t^{\lceil q/2 \rceil + 1}, v_t^{\lceil q/2 \rceil + 2}, v_t^{\lceil q/2 \rceil + 3}, \ldots, v_t^q\right\}$. Define a mapping $\wp\colon V(P_3[P_q]) \longrightarrow \mathbb{N}$ as follows:

$$\begin{aligned}
&\wp\left(v_l^{\lceil q/2 \rceil - s+1}\right) = q + 2s \text{ for } s = 1,2,3,\ldots,\lceil q/2 \rceil\\
&\wp\left(v_l^{q-s+1}\right) = q + 1 + 2s \text{ for } s = 1,2,3,\ldots,q-\left\lceil \frac{q}{2}\right\rceil.\\
&\wp\left(v_c^{\left\lceil \frac{q}{2}\right\rceil - s+1}\right) = 2s - 1 \text{ for } s = 1,2,3,\ldots,\left\lceil \frac{q}{2}\right\rceil\\
&\wp\left(v_c^{q-s+1}\right) = 2s \text{ for } s = 1,2,3,\ldots,q-\left\lceil \frac{q}{2}\right\rceil.\\
&\wp\left(v_r^{\lceil q/2 \rceil - s+1}\right) = 2(q+s) \text{ for } s = 1,2,3,\ldots,\left\lceil \frac{q}{2}\right\rceil\\
&\wp\left(v_r^{q-s+1}\right) = 2(q+s)+1 \text{ for } s = 1,2,3,\ldots,q-\left\lceil \frac{q}{2}\right\rceil.
\end{aligned} \tag{10}$$

Claim: the mapping $\wp$ is a valid radio labeling. We must show that condition 2 for radio labeling holds for all pair of vertices $\mu, \eta \in V(P_3[P_q])$.

Case 1: suppose μ and η are any two vertices in V_l, then two subcases can be obtained.

Case 1.1: let μ and η be any two distinct vertices in V_l^1, then $\mu = v_k$ and $\eta = v_l$, $1 \leq k \neq l \leq \lceil q/2 \rceil$; therefore, $\wp(\mu) = q + 2k$ and $\wp(\eta) = q + 2l$. Also, we note that $d(\mu, \eta) \geq 1$; hence, $d(\mu, \eta) + |\wp(\mu) - \wp(\eta)| \geq 1 + 2 = 3$ $d(\mu, \eta) + |2(k - l)| \geq 3$.

Case 1.2: let μ and η be any two distinct vertices in V_l^2, then $\mu = v_k$ and $\eta = v_l$, $1 \leq k \neq l \leq q - \lceil q/2 \rceil$; therefore, $\wp(\mu) = q + 1 + 2k$ and $\wp(\eta) = q + 1 + 2l$. Also, we note that $d(\mu, \eta) \geq 1$; hence, $d(\mu, \eta) + |\wp(\mu) - \wp(\eta)| \geq 1 + 2 = 3$ $d(\mu, \eta) + |2(k - l)| \geq 3$.

Case 2: suppose μ and η are any two vertices in V_c, then two subcases can be obtained.

Case 2.1: let μ and η be any two distinct vertices in V_c^1, then $\mu = v_k$ and $\eta = v_l$, $1 \leq k \neq l \leq \lceil q/2 \rceil$; therefore, $\wp(\mu) = 2k - 1$ and $\wp(\eta) = 2l - 1$. Also, we note that $d(\mu, \eta) \geq 1$; hence, $d(\mu, \eta) + |\wp(\mu) - \wp(\eta)| \geq 1 + 2 = 3$ $d(\mu, \eta) + |2(k - l)| \geq 3$.

Case 2.2: let μ and η be any two distinct vertices in V_c^2, then $\mu = v_k$ and $\eta = v_l$, $1 \leq k \neq l \leq q - \lceil q/2 \rceil$; therefore, $\wp(\mu) = 2k$ and $\wp(\eta) = 2l$. Also, we note that $d(\mu, \eta) \geq 1$; hence, $d(\mu, \eta)$ $+|\wp(\mu) - \wp(\eta)| \geq 1 + 2 = 3d(\mu, \eta) + |2(k - l)| \geq 3$.

Case 3: suppose μ and η are any two vertices in V_r, then two subcases can be obtained.

Case 3.1: let μ and η be any two distinct vertices in V_r^1, then $\mu = v_k$ and $\eta == v_l$, $1 \leq k \neq l \leq \lceil q/2 \rceil$; therefore, $\wp(\mu) = 2(q + k)$ and $\wp(\eta) = 2(q + l)$. Also, we note that $d(\mu, \eta) \geq 1$; hence, $d(\mu, \eta) + |\wp(\mu) - \wp(\eta)| \geq 1 + 2 = 3$ $d(\mu, \eta) + |2(k - l)| \geq 3$.

Case 3.2: let μ and η be any two distinct vertices in V_r^2, then $\mu = v_k$ and $\eta = v_l$, $1 \leq k \neq l \leq q - \lceil q/2 \rceil$; therefore, $\wp(\mu) = 2(q + k) + 1$ and $\wp(\eta) = 2(q + l) + 1$. Also, we note that $d(\mu, \eta) \geq 1$; hence, $d(\mu, \eta) + |\wp(\mu) - \wp(\eta)| \geq 1 + 2 = 3$ $d(\mu, \eta) + |2(k - l)| \geq 3$. □

4.2. Computing Radio Number of Lexicographic Product of Graphs by Using Computer Language. This computer code has been composed by using Python language.

```
import numpy as np
import math as mt
def main():
m = int(input('m = Enter the number of vertices (either 2 or 3) = '))
n = int(input('n = Enter the number of vertices (n> = 5) = '))
name3 = input('Type rnPP for lexico of two path graphs, Type rnPC for radio number of path and cycles, Type exist to quit the program: ')
while name3 ! = 'exit':
if name3 = = 'rnPP':
print('Executing rnPP')
rnpp(n, m)
elif name3 = = 'rnPC':
print('Executing rnPC')
rnpc(n, m)
else:
print('Input error: Enter the correct input value.')
name3 = input('Enter rnPP for lexico of two path graphs, rnPC for radio number of path and cycles, or exist to quit the program: ')
def rnpc(n, m):
if m = = 2:
q1 = mt.ceil(n/2)
l = np.zeros(n, dtype = int)
r = np.zeros(n, dtype = int)
for i in range(0, q1, 1):
l[q1-1-i] = 2*i
r[q1-1-i] = (n+1) + 2*i
for j in range(1, n-q1+1, 1):
l[n-j] = 2*j-1
r[n-j] = n+2*j
for lc, rc in zip(l, r):
print(lc, rc)
elif m = = 3:
q2 = mt.ceil(n/2)
l = np.zeros(n, dtype = int)
r = np.zeros(n, dtype = int)
c = np.zeros(n, dtype = int)
for i in range(0, q2, 1):
l[q2-1-i] = (n+1) + 2*i
r[q2-1-i] = 2*(n + i) + 1
c[q2-1-i] = 2*i
for j in range(1, n-q2+1, 1):
l[n-j] = n + 2*j
r[n-j] = 2 * (n + j)
c[n-j] = 2 * j-1
for lc, cc, rc in zip(l, c, r):
print(lc, cc, rc)
else:
print('Try again! Enter either 2 or 3 for the value of m.')
exit()
def rnpp(n, m):
if m = = 2:
q1 = mt.ceil(n/2)
l = np.zeros(n, dtype = int)
```

```
r = np.zeros(n, dtype = int)
for i in range(0, q1, 1):
l[q1-1-i] = 2*i + 1
r[q1-1-i] = (n+2) + 2*i
for j in range(1, n-q1+1, 1):
l[n-j] = 2*j
r[n-j] = (n+1) + 2*j
for lc, rc in zip(l, r):
print(lc, rc)
elif m = = 3:
q2 = mt.ceil(n/2)
l = np.zeros(n, dtype = int)
r = np.zeros(n, dtype = int)
c = np.zeros(n, dtype = int)
for i in range(0, q2, 1):
l[q2-1-i] = (n+2) + 2*i
r[q2-1-i] = 2*(n + i+1)
c[q2-1-i] = 2*i+1
for j in range(1, n-q2+1, 1):
l[n-j] = n + 2*j+1
r[n-j] = 2 * (n + j) + 1
c[n-j] = 2 * j
for lc, cc, rc in zip(l, c, r):
print(lc, cc, rc)
else:
print('Try again! Enter either 2 or 3 for the value of m.')
exit()
main()
```

5. Results of Radio Mean Labeling

Ponraj et al. [24] discussed the radio mean labeling. In this section, we discuss the radio mean labeling and compute the radio mean number for the lexicographic product of path with path $P_p[P_q]$ and path with cycle $P_p[C_q]$ for $p = 2, 3$. Moreover, we also presented a computer program for computing the radio number of these families of graphs.

Definition 5.1. Radio mean labeling of a connected graph G is a one-to-one map $\wp$ from the vertex set $V(G)$ to the set of natural numbers $\mathbb{N}$ such that for two distinct vertices μ and μ' of G,

$$d(\mu, \mu') + \left\lceil \frac{\wp(\mu) + \wp(\mu')}{2} \right\rceil \geq 1 + \operatorname{diam}(G). \tag{11}$$

The radio mean number of $\wp$, denoted by $\operatorname{rmn}(\wp)$, is the maximum number assigned to any vertex of G. The radio mean number of G, $\operatorname{rmn}(G)$ is the minimum value of $\operatorname{rmn}(\wp)$ taken over all radio mean labeling $\wp$ of G.

Theorem 5.2. *For $p = 2, 3$ and $q \geq 1$, $rmn(P_p[P_q]) = pq$.*

Proof. Let $V(P_p[P_q]) = \cup_{t=1}^{p} V_t^s$ for $p = 2, 3$ and $1 \leq s \leq q$ and $E(P_p[P_q]) = \{v_t^s v_t^{s+1}\colon 1 \leq t \leq p; 1 \leq s \leq q\} \cup \{v_t^s v_{t+1}^{s'}\colon 1 \leq s, s' \leq q\}$. It is clear that $\operatorname{diam}(P_p[P_q]) = 2$. We define a vertex labeling $\wp\colon V(P_p[P_q]) \longrightarrow \mathbb{N}$ as follows: $\wp(v_t^s) = ps - p + t$ for $1 \leq t \leq p$ and $1 \leq s \leq q$. Now, we check the radio mean condition.

$$d(\mu, \mu') + \left\lceil \frac{\wp(\mu) + \wp(\mu')}{2} \right\rceil \geq 1 + diam\left(P_p\left[P_q\right]\right), \tag{12}$$

for all $\mu, \mu' \in V(P_p[P_q])$.

Case 1: the vertex labeling for the pair (v_t^s, v_t^{s+1}) for a fixed t, $1 \leq t \leq p$ and $1 \leq s \leq q-1$, is given as $\wp(v_t^s) = ps - p + t$ and $\wp(v_t^{s+1}) = p(s+1) - p + t = ps + t$. Here, $d(v_t^s, v_t^{s+1}) = 1$. So, $d(v_t^s, v_t^{s+1}) + \lceil ps - p + t + ps + t/2 \rceil = 1 + \lceil 2ps - p + 2j/2 \rceil \geq 1 + 2 \geq 3$.

Case 2: check the pair $(v_t^s, v_{t+1}^{s'})$ for a fixed t, $1 \leq t \leq p-1$ and $1 \leq s, s' \leq q$. $\wp(v_t^s) = ps - p + t$, $\wp(v_{t+1}^{s'}) = ps' - p + t + 1$, and $d(v_t^s, v_{t+1}^{s'}) = 1$. So, $d(v_t^s, v_{t+1}^{s'}) + \lceil (ps - p + t) + (ps' - p + t + 1)/2 \rceil = 1 + \lceil p(s + s' - 2) + 2t + 1/2 \rceil \geq 1 + 2 \geq 3$.

Case 3: check the pair $(v_t^s, v_t^{s'})$ for a fixed t, $1 \leq t \leq p$ and $s' = s + 2$, for $1 \leq s \leq q-2$. $\wp(v_t^s) = ps - p + t$, $\wp(v_t^{s'}) = ps' - p + t$, and $d(v_t^s, v_t^{s'}) = 2$. So, $d(v_t^s, v_t^{s'}) + \lceil (ps - p + t) + (ps' - p + t)/2 \rceil = 1 + \lceil p(s + s') - 2(p - t)/2 \rceil \geq 1 + 2 \geq 3$. □

5.1. Computing Radio Mean Number of Lexicographic Product of Graphs by Using Computer Language. This computer code has been composed by using Python language.

```
import numpy as np
print('Program to calculate the Radio Mean Labelling')
m = int(input('m = Enter the number of vertices (either 2 or 3) = '))
n = int(input('n = Enter the number of vertices (n> = 1)='))
if m = = 2:
lt = np.zeros(n, dtype = int)
rt = np.zeros(n, dtype = int)
for j in range(1, m+1, 1):
if j = = 1:
for i in range(1, n+1, 1):
lt[i-1] = m*i - m + j.
else:
for i in range(1, n+1, 1):
rt[i-1] = m*i - m + j.
for lc, rc in zip(lt, rt):
print(lc, rc)
```

```
elif m = = 3:
lt = np.zeros(n, dtype = int)
rt = np.zeros(n, dtype = int)
ct = np.zeros(n, dtype = int)
for j in range(1, m+1, 1):
if j = = 1:
for i in range(1, n+1, 1):
lt[i-1] = m∗i - m + j
elif j = = 2:
for i in range(1, n+1, 1):
ct[i-1] = m∗i - m + j
else:
for i in range(1, n+1, 1):
rt[i-1] = m∗i - m + j
for lc, cc, rc in zip(lt, ct, rt):
print(lc, cc, rc)
else:
print('Error! The input value of m is either 2 or 3. Try again.')
```

6. Conclusion

In this paper, we have discussed the radio number and radio mean number of lexicographic product of graphs, namely, $P_2[P_q]$, $P_3[P_q]$, $P_2[C_q]$, and $P_3[C_q]$ for $q \geq 5$. We also computed the exact value of radio number and radio mean number of these families. Moreover, in this paper, we have presented their computer codes and also two open problems for future work have been given.

7. Open Problems

(1) Determining the radio number of $P_p[P_q]$ for $p \geq 4$.

(2) Determining the radio mean number of $P_p[P_q]$ for $p \geq 4$.

Data Availability

To support this study, no data were used.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] A. Rosa, *On Certain Valuations of the Vertices of a Graph. 1967 Theory Of Graphs*, pp. 349–355, International Symposium, Rome, Italy, 1966.

[2] W. K. Hale, "Frequency assignment: theory and applications," *Proceedings of the IEEE*, vol. 68, pp. 1497–1514, 1980.

[3] B. Andrsfai, *Introductory Graph Theory*, The Institute of Physics, London, UK, 1978.

[4] J. R. Griggs and R. K. Yeh, "Labeling graphs with a condition at distance 2," *SIAM Journal on Discrete Mathematics*, vol. 5, pp. 586–595, 1992.

[5] G. Chartrand, L. Nebeský, and P. Zhang, "Radio k-colorings of paths," *Discussiones Mathematicae Graph Theory*, vol. 24, pp. 5–21, 2004.

[6] M. Kchikech, R. Khenoufa, and O. Togni, "Radio k-labelings for cartesian products of graphs," *Discussiones Mathematicae Graph Theory*, vol. 28, pp. 165–178, 2008.

[7] G. Chartrand, D. Erwin, P. Zhang, and F. Harary, "Radio labelings of graphs," *Bulletin of the Institute of Combinatorics and its Applications*, vol. 33, pp. 77–85, 2001.

[8] D. D. F. Liu and X. Zhu, "Multilevel distance labelings for paths and cycles," *SIAM Journal on Discrete Mathematics*, vol. 19, pp. 610–621, 2005.

[9] M. T. Rahim and I. Tomescu, "Multi-level distance labelings for Helm graphs," *Ars Combinatoria*, vol. 104, pp. 513–523, 2012.

[10] R. Khennoufa and O. Togni, "The radio antipodal and radio numbers of the hypercube," *Ars Combinatoria*, vol. 102, pp. 447–461, 2011.

[11] D. D. F. Liu and M. Xie, "Radio number for square cycles," *Congressus Numerantium*, vol. 169, pp. 105–125, 2004.

[12] A. Naseem, K. Shabbir, and H. Shaker, "The radio number of edge-joint graphs," *Ars Combinatoria*, vol. 139, pp. 337–315, 2018.

[13] T. C. Adefokun and D. O. Ajayi, "On radio number of stacked-book graphs," pp. 1–9, 2019, https://arxiv.org/abs/1901.00355.

[14] B. M. Kim, W. Hwang, and B. C. Song, "Radio number for the product of a path and a complete graph," *Journal of Combinatorial Optimization*, vol. 30, no. 1, pp. 139–149, 2015.

[15] D. Bantva, "A lower bound for the radio number of graphs, Algorithms and Discrete Applied Mathematics," vol. 11394, Springer, Lecture Notes in Computer Science, , pp. 161–173, 2019.

[16] S. Das, S. Ghosh, S. Nandi, and S. Sen, "A lower bound technique for radio k-coloring," *Discrete Mathematics*, vol. 340, pp. 855–861, 2017.

[17] C. Y. Jung, W. Nazeer, S. Nazeer, A. Rafiq, and S. M. Kang, "Radio number for cross product $P_n(P_2)$," *International Journal of Pure and Applied Mathematics*, vol. 97, pp. 515–525, 2014.

[18] I. Kousar, S. Nazeer, and W. Nazeer, "Multilevel distance labelings for generalized petersen graph $P(n, 3)$," *Science International*, vol. 27, pp. 1767–1777, 2015.

[19] S. Nazeer, I. Kousar, and W. Nazeer, "Multilevel distance labeling for the graphs having diameter equal to diameter of cycle," *Science International*, vol. 26, pp. 519–525, 2014.

[20] S. Nazeer, I. Kousar, and W. Nazeer, "Radio number for prism related graphs D_n^*," *Science International*, vol. 26, pp. 551–555, 2014.

[21] S. Nazeer, I. Kousar, and W. Nazeer, "Radio and radio antipodal labelings for circulant graphs $G(4k+2; 1; 2)$," *Journal of Applied Mathematics and Informatics*, vol. 33, pp. 173–183, 2015.

[22] A. Ahmad and R. M. Ghemici, "Radio labeling of some ladder related graphs," *Mathematical Reports*, vol. 19, no. 9, pp. 107–119, 2017.

[23] F. Hausdorff, *Grundzge der Mengenlehre*, Leipzig, Germany, 1914.

[24] R. Ponraj, S. S. Narayanan, and R. Kala, "Radio mean labeling of graphs," *AKCE International Journal of Graphs and Combinatorics*, vol. 12, pp. 224–228, 2015.