

## Research Article

# On Some Fixed Point Results in $E$ – Fuzzy Metric Spaces

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In the existing literature, Banach contraction theorem as well as Meir-Keeler fixed point theorem were extended to fuzzy metric spaces. However, the existing extensions require strong additional assumptions. The purpose of this paper is to determine a class of fuzzy metric spaces in which both theorems remain true without the need of any additional condition. We demonstrate the wide validity of the new class.

## 1. Introduction

The well-known Banach's contraction principle (BCP) is a classic method in nonlinear analysis and is one of the most important and heavily researched fixed point theorems. It states that if  $\mathcal{L}: \mathcal{O} \rightarrow \mathcal{O}$  is a contraction on the complete metric space  $\mathcal{O}$ , then

$\mathcal{L}$  has a unique fixed point  $u$  in  $\mathcal{O}$ , and

$$\lim_n \mathcal{L}^n \kappa = u \text{ for all } \kappa \in \mathcal{O}.$$

There exist many different concepts of a fuzzy metric space (cf. [1–5]). However, the definition of George and Veeramani [4] is reverently cited in most references in this area. The authors of [4] updated Kramosil and Michalek's definition of fuzzy metric space and obtained a Hausdorff topology for this kind of fuzzy metric space. The topology induced by a fuzzy metric space in the sense of George and Veeramani has recently been demonstrated to be metrizable in [6]. It would be very useful if the BCP remains true in fuzzy metric spaces. This has been discussed in [7, 8] and then in numerous other directions over the years (see, for

example, [9–11]). In [12], the generalized altering distance function has been defined and the Banach contraction principle in complete fuzzy metric spaces using altering distance has been extended.

In [7], in order to give the fuzzy version of the BCP, Grabiec introduced a new definition of a Cauchy sequence in fuzzy metric spaces as follows.

**Definition 1.** A sequence  $\{\kappa_n\}_n$  in a fuzzy metric space  $(\mathcal{O}, \mathcal{Q}, *)$  is Cauchy if  $\lim_n \mathcal{Q}(\kappa_{n+p}, \kappa_n, t) = 1$  for each  $t > 0$  and  $p > 0$ .

It can be seen easily that this definition of a Cauchy sequence is incorrect, for more details see [13–15], whereas, in [8], Gregori and Sapena extended the Banach fixed point theorem to fuzzy version stating the following theorem.

**Theorem 1.** Let  $(\mathcal{O}, \mathcal{Q}, *)$  be a complete fuzzy metric space in which fuzzy contractive sequences are Cauchy. Let  $\mathcal{L}: \mathcal{O} \rightarrow \mathcal{O}$  be a fuzzy contractive mapping with  $k$  serving as the contractive constant. Then,  $\mathcal{L}$  has a unique fixed point.

Here, authors added a fairly strong assumption, that is, “every contractive sequence is Cauchy.”

In this paper, we prove that the Banach contraction theorem as well as the Meir-Keeler fixed point theorem remain true in fuzzy metric spaces with only a slight modification in the definition of fuzzy spaces given by George and Veeramani. Last, in this paper, we give some results to illustrate the broad validity of our results.

Before stating the main results, we need the following definitions.

**Definition 2** (Schweizer and Sklar [16]). A binary operation  $*$ :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is called a continuous  $t$ -norm if it satisfies the following assertions:

- (T1)  $*$  is commutative and associative;
- (T2)  $*$  is continuous;
- (T3)  $a * 1 = a$  for all  $a \in [0, 1]$ ;
- (T4)  $a * b \leq c * d$  when  $a \leq c$  and  $b \leq d$ , with  $a, b, c, d \in [0, 1]$ .

Here is the definition of a fuzzy metric space given by George and Veeramani:

**Definition 3** (George and Veeramani [4]). A fuzzy metric space is an ordered triple  $(\mathcal{O}, \mathcal{Q}, *)$  in which  $\mathcal{O}$  is a non-empty set,  $*$  is a continuous  $t$ -norm, and  $\mathcal{Q}$  is a fuzzy set on  $\mathcal{O} \times \mathcal{O} \times (0, \infty) \rightarrow (0, 1]$  such that

- (T5)  $\mathcal{Q}(\kappa, \omega, t) > 0$ ;
- (T6)  $\mathcal{Q}(\kappa, \omega, t) = 1$  if and only if  $\kappa = \omega$ ;
- (T7)  $\mathcal{Q}(\kappa, \omega, t) = \mathcal{Q}(\omega, \kappa, t)$ ;
- (T8)  $\mathcal{Q}(\kappa, \omega, t) * \mathcal{Q}(\omega, z, s) \leq \mathcal{Q}(\kappa, z, t + s)$ ;
- (T9)  $\mathcal{Q}(\kappa, \omega, \cdot): (0, +\infty) \rightarrow (0, 1]$  is left continuous,

for all  $\kappa, \omega, z \in \mathcal{O}$  and  $t, s > 0$ .

In this paper, we will consider the following class of fuzzy metric spaces.

**Definition 4** ( $E$ -fuzzy metric space). Let  $\mathcal{O}$  denote a non-empty set,  $*$  refers to a continuous  $t$ -norm, and  $\mathcal{Q}$  serves as a fuzzy set on  $\mathcal{O} \times \mathcal{O} \times (0, \infty) \rightarrow (0, 1]$  such that

- (F1)  $\mathcal{Q}(\kappa, \omega, t) > 0$ ;
- (F2)  $\mathcal{Q}(\kappa, \omega, t) = 1$  if and only if  $\kappa = \omega$ ;
- (F3)  $\mathcal{Q}(\kappa, \omega, t) = \mathcal{Q}(\omega, \kappa, t)$ ;
- (F4)  $\mathcal{Q}(\kappa, \omega, t) * \mathcal{Q}(\omega, z, s) \leq \mathcal{Q}(\kappa, z, t + s)$ ;
- (F5)  $\mathcal{Q}(\kappa, \omega, \cdot): (0, +\infty) \rightarrow (0, 1]$  is left continuous.
- (F6) For some  $r > 0$ , the family  $\{\mathcal{Q}(\kappa, \omega, \cdot): (0, r) \rightarrow (0, 1]; (\kappa, \omega) \in \mathcal{O}^2\}$  is uniformly equicontinuous,

for all  $\kappa, \omega, z \in \mathcal{O}$ , and  $t, s > 0$ . Then, the triple  $(\mathcal{O}, \mathcal{Q}, *)$  is called an  $E$ -fuzzy metric space.

**Remark 1.** Obviously, all  $E$ -fuzzy metric space is a fuzzy metric space. So, all properties in fuzzy metric spaces remain true in  $E$ -fuzzy metric spaces.

**Definition 5** (George and Veeramani [4]). Let  $(\mathcal{O}, \mathcal{Q}, *)$  be a fuzzy metric space. Then,

- (i) A sequence  $\{\kappa_n\}_n$  converges to  $\kappa \in \mathcal{O}$  if and only if  $\mathcal{Q}(\kappa_n, \kappa, t) \rightarrow 1$  as  $n \rightarrow +\infty$  for all  $t > 0$ ;
- (ii) A sequence  $\{\kappa_n\}_n$  in  $\mathcal{O}$  is a Cauchy sequence if and only if for all  $\varepsilon \in (0, 1)$  and  $t > 0$ , there exists  $n_0$  such that  $\mathcal{Q}(\kappa_n, \kappa_m, t) > 1 - \varepsilon$  for all  $m, n \geq n_0$ ;
- (iii) The fuzzy metric space is complete if every Cauchy sequence converges to some  $x \in \mathcal{O}$ .

In the sequel, we use the following essential technical lemma.

**Lemma 1.** Let  $(\mathcal{O}, \mathcal{Q}, *)$  be an  $E$ -fuzzy metric space,  $\tilde{\mathcal{Q}}$  be the continuous extension of  $\mathcal{Q}$  up to  $[0, \infty)$ , and  $\{\kappa_n\}_n$  be a sequence in  $\mathcal{O}$  such that  $\lim_{n \rightarrow \infty} \mathcal{Q}(\kappa_n, \kappa_{n+1}, t) = 1$ , for all  $t > 0$ . Then,

$$\lim_{n \rightarrow \infty} \tilde{\mathcal{Q}}(\kappa_n, \kappa_{n+1}, 0) = 1. \quad (1)$$

*Proof.* For all  $x, \omega \in \mathcal{O}$ , function  $t \mapsto \mathcal{Q}(x, \omega, t)$  is positive, continuous, and nondecreasing on  $(0, +\infty)$ , so  $\tilde{\mathcal{Q}}$  is well defined. Let  $\{t_n\}_n$  be a monotonically decreasing sequence of positive numbers, converging to 0, and  $\{\kappa_n\}_n$  be a sequence in  $\mathcal{O}$  such that  $\lim_{n \rightarrow \infty} \mathcal{Q}(\kappa_n, \kappa_{n+1}, t) = 1$ , for all  $t > 0$ , i.e., for all  $t > 0$  and for all  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$ : for all  $n \geq n_0$ ;  $1 - \mathcal{Q}(\kappa_n, \kappa_{n+1}, t) < \varepsilon$ .

From which it follows that for all  $t > 0$  and for all  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$ : for all  $n \geq n_0$ , and for all  $k \in \mathbb{N}$ ,

$$1 - \tilde{\mathcal{Q}}(\kappa_n, \kappa_{n+1}, 0) + \tilde{\mathcal{Q}}(\kappa_n, \kappa_{n+1}, 0) - \mathcal{Q}(\kappa_n, \kappa_{n+1}, t_k) + \mathcal{Q}(\kappa_n, \kappa_{n+1}, t_k) - \mathcal{Q}(\kappa_n, \kappa_{n+1}, t) < \frac{\varepsilon}{2}. \quad (2)$$

Therefore, for all  $t > 0$  and for all  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$ : for all  $n \geq n_0$  and for all  $k \in \mathbb{N}$ ,

$$1 - \tilde{\mathcal{Q}}(\kappa_n, \kappa_{n+1}, 0) < \frac{\varepsilon}{2} + \left| \tilde{\mathcal{Q}}(\kappa_n, \kappa_{n+1}, 0) - \mathcal{Q}(\kappa_n, \kappa_{n+1}, t_k) \right| + \left| \mathcal{Q}(\kappa_n, \kappa_{n+1}, t_k) - \mathcal{Q}(\kappa_n, \kappa_{n+1}, t) \right|. \quad (3)$$

On the other hand, by the fact that  $\lim_k \mathcal{Q}(\kappa_n, \kappa_{n+1}, t_k) = \tilde{\mathcal{Q}}(\kappa_n, \kappa_{n+1}, 0)$  and assumption (F6), we deduce that there exists  $t_0 > 0$ : for all  $\varepsilon$

$> 0$ , there exists  $n_0 \in \mathbb{N}$ , for all  $n \geq n_0$ , there exists  $k_0 \in \mathbb{N}$ , such that

$$\left| \tilde{\mathcal{Q}}(\kappa_n, \kappa_{n+1}, 0) - \mathcal{Q}(\kappa_n, \kappa_{n+1}, t_k) \right| \leq \frac{\varepsilon}{4} \quad (4)$$

$$\left| \mathcal{Q}(\kappa_n, \kappa_{n+1}, t_k) - \mathcal{Q}(\kappa_n, \kappa_{n+1}, t) \right| \leq \frac{\varepsilon}{4} \quad (5)$$

for all  $k > k_0$  and  $t < t_0$ . Hence, by relations (3)–(5), it yields for all  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$ : for all  $n \geq n_0$ ;  $1 - \tilde{\mathcal{Q}}(\kappa_n, \kappa_{n+1}, 0) < \varepsilon$ , and this means

$$\lim_n \tilde{\mathcal{Q}}(\kappa_n, \kappa_{n+1}, 0) = 1, \tag{6}$$

which achieves the proof of the lemma.  $\square$

## 2. Main Results

Now, we will present our key finding.

**Theorem 2.** *Let  $(\mathcal{O}, \mathcal{Q}, *)$  be a complete E-fuzzy metric space. Let  $\mathcal{L}: \mathcal{O} \rightarrow \mathcal{O}$  be a fuzzy contractive mapping with the contractive constant  $k$ , i.e., there exists  $k \in ]0, 1[$  such that*

$$\frac{1}{\mathcal{Q}(\mathcal{L}\kappa, \mathcal{L}\omega, t)} - 1 \leq k \left( \frac{1}{\mathcal{Q}(\kappa, \omega, t)} - 1 \right), \tag{7}$$

for all  $\kappa, \omega$  in  $\mathcal{O}$  and for all  $t > 0$ . Then,  $\mathcal{L}$  has a unique fixed point  $\kappa^*$ . Furthermore, for all  $\kappa \in \mathcal{O}$ , the sequence  $\{\mathcal{L}^n \kappa\}$  converges to  $\kappa^*$ .

*Proof.* Let  $\kappa$  in  $\mathcal{O}$  and  $\kappa_n = \mathcal{L}^n \kappa$  ( $n \in \mathbb{N}$ ). Let  $t > 0$  and  $n \in \mathbb{N}$ . By inequality (7), we obtain

$$\frac{1}{\mathcal{Q}(\kappa_{n+1}, \kappa_{n+2}, t)} - 1 \leq k \left( \frac{1}{\mathcal{Q}(\kappa_n, \kappa_{n+1}, t)} - 1 \right), \tag{8}$$

for all  $t > 0$  and for all  $n$  in  $\mathbb{N}$ , which deduce that

$$\lim_{n \rightarrow \infty} \mathcal{Q}(\kappa_n, \kappa_{n+1}, t) = 1, \tag{9}$$

for all  $t > 0$ . Now, to prove that  $\{\kappa_n\}_n$  is a Cauchy sequence, we assume to the contrary. Since  $t \mapsto \mathcal{Q}(\kappa, \omega, t)$  is a non-decreasing function, there exists  $\varepsilon \in (0, 1)$  and there exists  $\xi > 0$  such that for all  $p \in \mathbb{N}$ , there exists  $n_p (\geq p) < m_p \in \mathbb{N}$  so that

$$\mathcal{Q}(\kappa_{m_p}, \kappa_{n_p}, t) \leq 1 - \varepsilon, \tag{10}$$

for all  $t < \xi$ . Let  $t_0 < \min\{\xi, r\}$ . By virtue of limit (9) and the last relation, we can write that there exists  $\varepsilon \in (0, 1)$ ; for all  $p \in \mathbb{N}$ , there exists  $n_p (\geq p) < m_p \in \mathbb{N}$ :

$$\begin{aligned} \mathcal{Q}(\kappa_{m_p}, \kappa_{n_p}, t_0) &\leq 1 - \varepsilon, \\ \mathcal{Q}(\kappa_{m_{p-1}}, \kappa_{n_p}, t_0) &> 1 - \varepsilon. \end{aligned} \tag{11}$$

Taking into account the continuity of the function  $t \mapsto \mathcal{Q}(\kappa, \omega, t)$  and the fact that  $\mathcal{Q}(\kappa_{m_{p-1}}, \kappa_{n_p}, t_0) > 1 - \varepsilon$ , we can choose  $q_0 \in \mathbb{N}$  such that

$$\mathcal{Q}(\kappa_{m_{p-1}}, \kappa_{n_p}, t_0 - \frac{1}{q_0}) > 1 - \varepsilon. \tag{12}$$

By virtue of assumptions (T4) and (F4) and relations (11) and (12), it follows that

$$\begin{aligned} 1 - \varepsilon &\geq \mathcal{Q}(\kappa_{m_p}, \kappa_{n_p}, t_0) \\ &\geq \mathcal{Q}(\kappa_{m_p}, \kappa_{m_{p-1}}, \frac{1}{q_0}) * \mathcal{Q}(\kappa_{m_{p-1}}, \kappa_{n_p}, t_0 - \frac{1}{q_0}) \\ &\geq \tilde{\mathcal{Q}}(\kappa_{m_p}, \kappa_{m_{p-1}}, 0) * (1 - \varepsilon). \end{aligned} \tag{13}$$

So, according to assumptions (T2)-(T3), limit (9), and Lemma 1, one has

$$\lim_{p \rightarrow \infty} \mathcal{Q}(\kappa_{m_p}, \kappa_{n_p}, t_0) = 1 - \varepsilon. \tag{14}$$

Suppose that for all  $p_1 \geq 0$ , there exists  $p \geq p_1$  such that  $\mathcal{Q}(\kappa_{m_{p+1}}, \kappa_{n_{p+1}}, t_0) \leq 1 - \varepsilon$  means, having in mind relations (7) and (14), that the sequence  $\{\kappa_n\}_n$  has two subsequences  $\{\kappa_{n_p}\}_p$  and  $\{\kappa_{m_p}\}_p$  verifying

$$\lim_{p \rightarrow \infty} \mathcal{Q}(\kappa_{m_p}, \kappa_{n_p}, t_0) = \lim_{p \rightarrow \infty} \mathcal{Q}(\kappa_{m_{p+1}}, \kappa_{n_{p+1}}, t_0) = 1 - \varepsilon, \tag{15}$$

(for the sake of simplicity, we have saved the same notation for the subsequence).

Now, we suppose that there exists  $p_1 \geq 0$  such that  $\mathcal{Q}(\kappa_{m_{p+1}}, \kappa_{n_{p+1}}, t_0) > 1 - \varepsilon$  for all  $p \geq p_1$ . We claim that  $\lim_p \mathcal{Q}(\kappa_{m_{p+1}}, \kappa_{n_{p+1}}, t_0) = 1 - \varepsilon$ . Suppose not, i.e., there exists  $\alpha > 0$  and two subsequences  $\{\kappa_{n_p}\}_p$  and  $\{\kappa_{m_p}\}_p$  verifying

$$\mathcal{Q}(\kappa_{m_{p+1}}, \kappa_{n_{p+1}}, t_0) > \alpha + (1 - \varepsilon), \tag{16}$$

for all  $p \in \mathbb{N}$ .

Having  $q \in \mathbb{N}$  satisfying  $\mathcal{Q}(\kappa_{m_{p+1}}, \kappa_{n_{p+1}}, t_0 - (1/q)) > \alpha + (1 - \varepsilon)$ , we obtain

$$\begin{aligned} 1 - \varepsilon &\geq \mathcal{Q}(\kappa_{m_p}, \kappa_{n_p}, t_0) \\ &\geq \mathcal{Q}(\kappa_{m_p}, \kappa_{m_{p+1}}, \frac{1}{2q}) * \mathcal{Q}(\kappa_{m_{p+1}}, \kappa_{n_{p+1}}, t_0 - \frac{1}{q}) \\ &\quad * \mathcal{Q}(\kappa_{n_{p+1}}, \kappa_{n_p}, \frac{1}{2q}) \\ &\geq \tilde{\mathcal{Q}}(\kappa_{m_p}, \kappa_{m_{p+1}}, 0) * [\alpha + (1 - \varepsilon)] * \tilde{\mathcal{Q}}(\kappa_{n_{p+1}}, \kappa_{n_p}, 0) \\ &\rightarrow \alpha + (1 - \varepsilon), \end{aligned} \tag{17}$$

as  $p \rightarrow \infty$ .

This is a contradiction. Then,

$$\lim_p \mathcal{Q}(\kappa_{m_{p+1}}, \kappa_{n_{p+1}}, t_0) = 1 - \varepsilon. \tag{18}$$

Relations (14), (15), and (18) drive to a clear contradiction with condition (7). So,  $\{\kappa_n\}_n$  is a Cauchy sequence in the complete fuzzy metric space  $\mathcal{O}$  and we deduce that there exists  $\kappa^* \in \mathcal{O}$  such that

$$\lim_n \mathcal{Q}(\kappa_n, \kappa^*, t) = 1, \quad (19)$$

for all  $t > 0$ , and by relation (7), we obtain

$$\frac{1}{\mathcal{Q}(\mathcal{L}\kappa_n, \mathcal{L}\kappa^*, t)} - 1 \leq k \left( \frac{1}{\mathcal{Q}(\kappa_n, \kappa^*, t)} - 1 \right), \quad (20)$$

for all  $n \in \mathbb{N}$  and for all  $t > 0$ . Passing to the limit, having in mind the limit in (19), it follows that  $\mathcal{Q}(\kappa^*, \mathcal{L}\kappa^*, t) = 1$ , which, with assumption (F2) and relation (7), means that  $\kappa^*$  is the unique fixed point of mapping  $\mathcal{L}$ . This achieves the proof.  $\square$

**Theorem 3.** (fuzzy Meir-Keeler fixed point theorem). Let  $(\mathcal{O}, \mathcal{Q}, *)$  be a complete E-fuzzy metric space. Let  $\mathcal{L}: \mathcal{O} \rightarrow \mathcal{O}$  be a fuzzy Meir-Keeler type mapping, i.e., for all  $\varepsilon \in (0, 1)$ , there exists  $\delta > 0$  such that

$$\varepsilon - \delta < \mathcal{Q}(\kappa, \omega, t) \leq \varepsilon \implies \mathcal{Q}(\mathcal{L}\kappa, \mathcal{L}\omega, t) > \varepsilon, \quad (21)$$

for all  $\kappa, \omega$  in  $\mathcal{O}$  and for all  $t > 0$ . Then,  $\mathcal{L}$  has a unique fixed point  $\kappa^*$ . Furthermore, for all  $\kappa \in \mathcal{O}$ , the sequence  $\{\mathcal{L}^n \kappa\}$  converges to  $\kappa^*$ .

*Proof.* Let  $\kappa \in \mathcal{O}$  and  $\kappa_n = \mathcal{L}^n \kappa$  ( $n \in \mathbb{N}$ ) and  $t > 0$ . Obviously, we have

$$\mathcal{Q}(\kappa, \mathcal{L}\kappa, t) - \delta < \mathcal{Q}(\kappa, \mathcal{L}\kappa, t) \leq \mathcal{Q}(\kappa, \mathcal{L}\kappa, t), \quad (22)$$

for all  $\delta > 0$ , and due to relation (21), we obtain  $\mathcal{Q}(\mathcal{L}^2 \kappa, \mathcal{L}\kappa, t) > \mathcal{Q}(\kappa, \mathcal{L}\kappa, t)$ . Recursively, we obtain a sequence  $\{\mathcal{Q}(\kappa_n, \kappa_{n+1}, t)\}_n$  in  $[0, 1]$  verifying

$$\mathcal{Q}(\kappa_n, \kappa_{n+1}, t) < \mathcal{Q}(\kappa_{n+1}, \kappa_{n+2}, t), \quad (23)$$

for all  $n$  in  $\mathbb{N}$ . It is a bounded increasing sequence. Then, there exists a function  $u: (0, \infty) \rightarrow [0, 1]$  such that

$$\lim_{n \rightarrow +\infty} \mathcal{Q}(\kappa_n, \kappa_{n+1}, t) = \sup_{n \in \mathbb{N}} \mathcal{Q}(\kappa_n, \kappa_{n+1}, t) = u(t), \quad (24)$$

for all  $t > 0$ . We claim that  $u(t) = 1$ , for all  $t > 0$ . Suppose not, i.e., there exists  $t_0 > 0$  such that  $u(t_0) \in (0, 1)$ . By the limit in (24), for all  $\delta \in (0, u(t_0))$ , there exists  $n_0 \in \mathbb{N}$  such that

$$u(t_0) - \delta < \mathcal{Q}(\kappa_n, \kappa_{n+1}, t_0) \leq u(t_0), \quad (25)$$

for all  $n \geq n_0$ , which, with condition (21), implies that  $\mathcal{Q}(\kappa_{n+1}, \kappa_{n+2}, t_0) > u(t_0)$ . This is a clear contradiction with (24). Therefore,

$$\lim_n \mathcal{Q}(\kappa_n, \kappa_{n+1}, t) = 1, \quad (26)$$

for all  $t > 0$ . Now, we follow, exactly, the same lines as in the proof of Theorem 2 to deduce that  $\{\kappa_n\}_n$  is a Cauchy sequence in the complete fuzzy metric space  $\mathcal{O}$ , which deduce that there exists  $\kappa^* \in \mathcal{O}$  such that

$$\lim_n \mathcal{Q}(\kappa^*, \kappa_n, t) = 1. \quad (27)$$

On the other hand, for all  $n \in \mathbb{N}$  and all  $\delta \in (0, \mathcal{Q}(\kappa^*, \kappa_n, t))$ , we have

$$\mathcal{Q}(\kappa^*, \kappa_n, t) - \delta < \mathcal{Q}(\kappa^*, \kappa_n, t) \leq \mathcal{Q}(\kappa^*, \kappa_n, t). \quad (28)$$

Condition (21) assures that

$$1 \geq \mathcal{Q}(\mathcal{L}\kappa^*, \mathcal{L}\kappa_n, t) > \mathcal{Q}(\kappa^*, \kappa_n, t), \quad (29)$$

which, with the limit in (27), gives  $\lim_n \mathcal{Q}(\mathcal{L}\kappa^*, \kappa_n, t) = 1$ , and finally

$$\kappa^* = \mathcal{L}\kappa^*. \quad (30)$$

For the uniqueness, we assume that there exists  $\omega^* (\neq \kappa^*) \in \mathcal{O}$  such that  $\omega^* = \mathcal{L}\omega^*$ . It is clear that for all  $\delta \in (0, \mathcal{Q}(\kappa^*, \omega^*, t))$ ,  $\mathcal{Q}(\kappa^*, \omega^*, t) - \delta < \mathcal{Q}(\kappa^*, \omega^*, t) \leq \mathcal{Q}(\kappa^*, \omega^*, t)$ .

Hence, by (21),  $\mathcal{Q}(\mathcal{L}\kappa^*, \mathcal{L}\omega^*, t) > \mathcal{Q}(\kappa^*, \omega^*, t)$  or  $\mathcal{Q}(\kappa^*, \omega^*, t) > \mathcal{Q}(\kappa^*, \omega^*, t)$ , a contradiction, and this achieves the proof.  $\square$

Now, we give the following corollary.  $\square$

**Corollary 1.** Let  $(\mathcal{O}, d)$  be a complete metric space, and  $\mathcal{L}$  a Meir-Keeler mapping on  $\mathcal{O}$ , i.e., for each  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $\kappa, \omega \in \mathcal{O}$ ,

$$\varepsilon \leq d(\kappa, \omega) < \varepsilon + \delta \implies d(\mathcal{L}(\kappa), \mathcal{L}(\omega)) < \varepsilon. \quad (31)$$

Let  $\mathcal{Q}$  be a function on  $\mathcal{O} \times \mathcal{O} \times (0, +\infty)$  defined by

$$\mathcal{Q}(\kappa, \omega, t) = \frac{t + 1}{t + 1 + d(\kappa, \omega)}. \quad (32)$$

Then,

- (1)  $(\mathcal{O}, \mathcal{Q}, \cdot)$  is an E-fuzzy metric space, where  $\cdot$  is the product t-norm.
- (2) For all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$\varepsilon - \delta < \mathcal{Q}(\kappa, \omega, t) \leq \varepsilon \implies \mathcal{Q}(\mathcal{L}\kappa, \mathcal{L}\omega, t) > \varepsilon, \quad (33)$$

for all  $\kappa, \omega$  in  $\mathcal{O}$  and for all  $t > 0$ .

*Proof.*  $(\mathcal{O}, \mathcal{Q}, \cdot)$  is a fuzzy metric space (see [4]) and  $\{t \mapsto \mathcal{Q}(\kappa, \omega, t); \kappa, \omega \in \mathcal{O}\}$  is a set of functions with common Lipschitz constant "1." So, it is uniformly equicontinuous. This means that  $(\mathcal{O}, \mathcal{Q}, \cdot)$  is an E-fuzzy metric space. For the second assumption, it suffices to see that

For all  $\varepsilon > 0$ ,  $\delta \in (0, \varepsilon)$ ,  $t > 0$  and all  $\kappa, \omega \in \mathcal{O}$ , we have

$$\begin{aligned} \varepsilon - \delta < \mathcal{Q}(\kappa, \omega, t) \leq \varepsilon &\iff \varepsilon - \delta < \frac{t + 1}{t + 1 + d(\kappa, \omega)} \leq \varepsilon \\ &\iff \varepsilon - \delta < \frac{1}{1 + (1/(t + 1))d(\kappa, \omega)} \leq \varepsilon \\ &\iff (t + 1) \left( \frac{1}{\varepsilon} - 1 \right) \leq d(\kappa, \omega) < (t + 1) \left( \frac{1}{\varepsilon - \delta} - 1 \right). \end{aligned} \quad (34)$$

Let  $\varepsilon_0 = (t + 1)((1/\varepsilon) - 1)$  and  $\delta_{\varepsilon_0} > 0$  such that  $\varepsilon_0 \leq d(\kappa, \omega) < \varepsilon_0 + \delta_{\varepsilon_0} \implies d(\mathcal{L}\kappa, \mathcal{L}\omega) < \varepsilon_0$ . (35)

Now, we choose  $\delta$  in (34) such that  $(t + 1)((1/(\varepsilon - \delta)) - 1) < (t + 1)((1/\varepsilon) - 1) + \delta_{\varepsilon_0}$ . Therefore, using relations (34) and (35), it follows that

$$\begin{aligned} \varepsilon - \delta < \mathcal{Q}(\kappa, \omega, t) \leq \varepsilon &\implies (t + 1)\left(\frac{1}{\varepsilon} - 1\right) < d(\kappa, \omega) \leq (t + 1)\left(\frac{1}{\varepsilon} - 1\right) + \delta_{\varepsilon_0} \\ &\implies d(\mathcal{L}\kappa, \mathcal{L}\omega) < (t + 1)\left(\frac{1}{\varepsilon} - 1\right) \\ &\implies \frac{t + 1}{t + 1 + d(\mathcal{L}\kappa, \mathcal{L}\omega)} > \varepsilon \\ &\implies \mathcal{Q}(\mathcal{L}\kappa, \mathcal{L}\omega, t) > \varepsilon, \end{aligned} \tag{36}$$

and this achieves the proof.  $\square$

**Theorem 4.** Consider the integral operator  $\mathcal{L}$  on  $C([0, I], \mathbb{R})$  as

$$\mathcal{L}\kappa(r) = g(r) + \int_0^r F(r, s, \kappa(s))ds. \tag{41}$$

### 3. Application

The purpose of this section is to give an example of the existence of a solution for an integral equation, where we can apply Theorem 2 to get its solution. For such integral equations, we refer the reader to [17] where the authors provide a common solution for a system of two integral equations.

Consider the integral equation,

$$\kappa(r) = g(r) + \int_0^r F(r, s, \kappa(s))ds, \quad \text{for all } r \in [0, I], I > 0, \tag{37}$$

and Banach space  $C([0, I], \mathbb{R})$  of all continuous functions defined on  $[0, I]$  equipped with supremum norm

$$\|\kappa\| = \sup_{r \in [0, I]} |\kappa(r)|, \quad \kappa \in C([0, I], \mathbb{R}), \tag{38}$$

with induced metric

$$d(\kappa, \omega) = \sup_{r \in [0, I]} |\kappa(r) - \omega(r)|. \tag{39}$$

Now, consider the fuzzy metric space with product  $t$ -norm as

$$\mathcal{Q}(\kappa, \omega, t) = \frac{t}{t + d(\kappa, \omega)}, \quad \text{for all } \kappa, \omega \in C([0, I], \mathbb{R}), t > 0. \tag{40}$$

According to George and Veeramani, standard fuzzy metric space and the corresponding metric space have same topologies. So, fuzzy metric space defined in (40) is complete.

Suppose that there exists  $f: [0, I] \times [0, I] \rightarrow [0, \infty)$  such that  $f \in L^1([0, I], \mathbb{R})$  and suppose that  $F$  satisfies the following condition:

$$|F(s, r, \kappa(r)) - F(s, r, \omega(r))| \leq f(r, s)|\kappa(s) - \omega(s)|, \tag{42}$$

for all  $\kappa, \omega \in C([0, I], \mathbb{R})$  and for all  $r, s \in [0, I]$  where

$$\sup_{r \in [0, I]} \int_0^r f(r, s)ds \leq k < 1. \tag{43}$$

Then, the integral equation (37) has a unique solution.

*Proof.* Let  $\kappa, \omega \in C([0, I], \mathbb{R})$  and consider

$$\begin{aligned} &|\mathcal{L}\kappa(r) - \mathcal{L}\omega(r)| \\ &\leq \int_0^r |F(r, s, \kappa(s)) - F(r, s, \omega(s))|ds \\ &\leq \int_0^r f(r, s)|\kappa(s) - \omega(s)|ds \\ &\leq d(\kappa, \omega) \int_0^r f(r, s)ds \\ &\leq kd(\kappa, \omega). \end{aligned} \tag{44}$$

So,

$$d(\mathcal{L}\kappa, \mathcal{L}\omega) \leq kd(\kappa, \omega). \tag{45}$$

Using (40), we can write

$$\begin{aligned}
\frac{1}{\mathcal{Q}(\kappa, \omega, t)} - 1 &= \frac{d(\kappa, \omega)}{t} \\
\frac{1}{\mathcal{Q}(\mathcal{L}\kappa, \mathcal{L}\omega, t)} - 1 &= \frac{t + d(\mathcal{L}\kappa, \mathcal{L}\omega) - t}{t} \\
&= \frac{d(\mathcal{L}\kappa, \mathcal{L}\omega)}{t} \\
&\leq k \frac{d(\kappa, \omega)}{t} \\
&\leq k \left( \frac{1}{\mathcal{Q}(\kappa, \omega, t)} - 1 \right).
\end{aligned} \tag{46}$$

Since all the conditions of Theorem 2 hold, (37) has a unique solution.  $\square$

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

### Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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