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Research Article

Semi-I-Expandable Ideal Topological Spaces

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Our purpose is to introduce the notion of semi-I-expandable ideal topological spaces. Some properties of semi-I-locally finite collections are investigated. In particular, several characterizations of semi-I-expandable ideal topological spaces are established.

1. Introduction

The concept of expandable spaces was first introduced by Krajewski [1]. Moreover, Krajewski investigated the property of expanding locally finite collection to open finite collection and obtained some results relating this property to certain topological covering properties. Smith et al. [2] introduced various generalizations of the concept of expandability and investigated several characterizations of expandability properties in terms of open covers. Al-Zoubi [3] introduced the concept of s-expandable spaces as a variation of expandable spaces and showed that an extremally disconnected semiregular space is s-expandable if and only if it is expandable. Jiang and Sun [4] proved that every T_1wN-space is expandable and discussed a characterization of s-expandability for extremally disconnected spaces. Al-Zoubi [5] introduced the class of S-paracompact spaces as a generalization of paracompact spaces and investigated the relationships between S-paracompact spaces and other well-known spaces. Li and Song [6] introduced and studied S-expandable spaces which are a weaker form than S-paracompact spaces and showed that s-expandability is equivalent to S-expandability for extremally disconnected semiregular spaces. Kuratowski [7] and Vaidyanathaswamy [8] introduced and studied the concept of ideal topological spaces. Janković and Hamlett [9] developed the study in logical, systematic fashion and offered some new results, improvements of known results, and some applications. In 2002, Hatir and Noiri [10] introduced the notions of semi-I-open sets, α-I-open sets, and β-I-open sets via idealization and using these sets obtained new decomposition of continuity. In 2005, Hatir and Noiri [11] investigated some properties of semi-I-open sets and semi-I-continuous functions defined in [10] and introduced new functions via ideals, namely, semi-I-open functions and semi-I-closed functions. Açikgöz et al. [12] introduced the notion of I-submaximal ideal topological spaces and proved that every submaximal space is an I-submaximal ideal topological space. In 2009, Ekici and Noiri [13] introduced the notion of $*$-extremally disconnected ideal topological spaces and showed that $*$-extremally disconnectedness and extremally disconnectedness are equivalent to a codense ideal. In 2010, Ekici and Noiri [14] investigated several characterizations of I-submaximal ideal topological spaces and proved that semi-I-open sets and AB_I-sets are equivalent to I-submaximality and $*$-extremally disconnectedness. In 2012, Ekici and Noiri [15] introduced the notion of $*$-hyperconnected ideal topological spaces and investigated some properties of $*$-hyperconnected ideal topological spaces by utilizing semi*-I-open sets and the semi*-I-closure operator. In [16], the author investigated further characterizations of $*$-hyperconnected ideal topological spaces and studied the concept of θ-I-irreducible ideal topological spaces.

This paper is organized as follows: in Section 3, we introduce the concept of semi-I-locally finite collections. Moreover, some properties of semi-I-locally finite collections are discussed. In Section 4, we introduce the concept of semi-I-expandable ideal topological spaces. In particular, some characterizations of semi-I-expandable ideal topological spaces are investigated.

2. Preliminaries

We begin with some definitions and known results which will be used throughout this paper. In the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. In a topological space (X, τ), the closure and the interior of any subset A of X will be denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A nonempty collection I of subsets of a set X is said to be an ideal on X if I satisfies the following two properties: (i) $A \in \mathrm{I}$ and $B \subseteq A \Longrightarrow B \in \mathrm{I}$; (ii) $A \in \mathrm{I}$ and $B \in \mathrm{I} \Longrightarrow A \cup B \in \mathrm{I}$. For a topological space (X, τ) with an ideal I on X, a set operator $(.)^*\colon \mathrm{P}(X) \longrightarrow \mathrm{P}(X)$ where $\mathrm{P}(X)$ is the set of all subsets of X, called a local function [7] of A with respect to I, and τ is defined as follows: for $A \subseteq X$, $A^*(\mathrm{I}, \tau) = \{x \in X | G \cap A \notin \mathrm{I} \text{ for every } G \in \tau(x)\}$ where

$$\tau(x) = \{G \in \tau \mid x \in G\}. \tag{1}$$

A Kuratowski closure operator $\mathrm{Cl}^*(.)$ for a topology $\tau^*(\mathrm{I}, \tau)$, which is called the $*$-topology and is finer than τ, is defined by $\mathrm{Cl}^*(A) = A \cup A^*$ [9]. We shall simply write A^* for $A^*(\mathrm{I}, \tau)$ and τ^* for $\tau^*(\mathrm{I}, \tau)$. A basis $\mathrm{B}(\mathrm{I}, \tau)$ for τ^* can be described as follows: $\mathrm{B}(\mathrm{I}, \tau) = \{V - I' | V \in \tau \text{ and } I' \in \mathrm{I}\}$. However, $\mathrm{B}(\mathrm{I}, \tau)$ is not always a topology [9]. A subset A of an ideal topological space (X, τ, I) is called $*$-closed (τ^*-closed) [9] if $A^* \subseteq A$. The interior of a subset A in $(X, \tau^*(\mathrm{I}, \tau))$ is denoted by $\mathrm{Int}^*(A)$.

A subset A of an ideal topological space (X, τ, I) is called semi-I-open [10] (resp., semi*-I-open [15]) if $A \subseteq \mathrm{Cl}^*(\mathrm{Int}(A))$ (resp., $A \subseteq \mathrm{Cl}^*(\mathrm{Int}(A))$). By $s\mathrm{IO}(X, \tau)$ (resp., $s^*\mathrm{IO}(X, \tau)$), we denote the family of all semi-I-open (resp., semi*-I-open) sets of an ideal topological space (X, τ, I). The complement of a semi-I-open (resp., semi*-I-open) set is called semi-I-closed [11] (resp., semi*-I-closed [15]).

Lemma 1 (see [11]). *Let (X, τ, I) be an ideal topological space and A, B subsets of X.*

(1) If $U_\alpha \in s\mathrm{IO}(X, \tau)$ for each $\alpha \in \Delta$, then $\cup\{U_\alpha\colon \alpha \in \Delta\} \in s\mathrm{IO}(X, \tau)$.

(2) If $A \in s\mathrm{IO}(X, \tau)$ and $B \in \tau$, then $A \cap B \in s\mathrm{IO}(X, \tau)$.

The semi-I-closure (resp., semi*-I-closure) of a subset A of an ideal topological space (X, τ, I), denoted by $s\mathrm{Cl}_{\mathrm{I}}(A)$ (resp., $s^*\mathrm{Cl}_{\mathrm{I}}(A)$), is defined by the intersection of all semi-I-closed (resp., semi*-I-closed) sets of X containing A [15].

Lemma 2 (see [15]). *For a subset A of an ideal topological space (X, τ, I), the following properties hold:*

(1) $sCl_{\mathrm{I}}(A) = A \cup Int^*(Cl(A))$.

(2) $s^*Cl_{\mathrm{I}}(A) = A \cup Int(Cl^*(A))$.

3. Semi-I-Locally Finite Collections

Recall that a collection $\mathrm{C} = \{C_\gamma | \gamma \in \Gamma\}$ of subsets of a topological space (X, τ) is said to be locally finite [17] if, for each $x \in X$, there exists an open set U of X containing x and U intersects C_γ at most for finitely many γ.

Definition 1. A collection $\mathrm{C} = \{C_\gamma | \gamma \in \Gamma\}$ of subsets of an ideal topological space (X, τ, I) is said to be semi-I-locally finite if, for each $x \in X$, there exists a semi-I-open set U of X containing x and U intersects C_γ at most for finitely many γ.

Definition 2. An ideal topological space (X, τ, I) is said to be semi-I-regular if, for each semi-I-closed set F and each point $x \notin F$, there exist disjoint semi-I-open sets U and V such that $x \in U$ and $F \subseteq V$.

Lemma 3. *An ideal topological space (X, τ, I) is semi-I-regular if and only if, for each semi-I-open set U and for each $x \in U$, there exists a semi-I-open set V such that $x \in V \subseteq sCl_{\mathrm{I}}(V) \subseteq U$.*

Lemma 4 (see [11]). *A subset A of an ideal topological space (X, τ, I) is semi-I-open if and only if there exists $U \in \tau$ such that $U \subseteq A \subseteq Cl^*(U)$.*

Definition 3 (see [13]). An ideal topological space (X, τ, I) is said to be $*$-extremally disconnected if the $*$-closure of every open subset U of X is open.

Lemma 5. *For an ideal topological space (X, τ, I), the following properties are equivalent:*

(1) (X, τ, I) is $*$-extremally disconnected.

(2) $s^*Cl_{\mathrm{I}}(U) = Cl^*(U)$ for each $U \in s\mathrm{IO}(X, \tau)$.

(3) $s^*Cl_{\mathrm{I}}(U)$ is $*$-closed for each $U \in s\mathrm{IO}(X, \tau)$.

Proof

(1) $\Longrightarrow$ (2): Suppose that (X, τ, I) is a $*$-extremally disconnected space. Let U be a semi-I-open set. Since U is semi-I-open, $\mathrm{Cl}^*(U) = \mathrm{Cl}^*(\mathrm{Int}(U))$ and, by Lemma 2 (2),

$$\begin{aligned} s^*\mathrm{Cl}_{\mathrm{I}}(U) &= U \cup \mathrm{Int}(\mathrm{Cl}^*(U)) \\ &= U \cup \mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(U))) \\ &= U \cup \mathrm{Cl}^*(\mathrm{Int}(U)) \\ &= \mathrm{Cl}^*(\mathrm{Int}(U)) = \mathrm{Cl}^*(U). \end{aligned} \tag{2}$$

(2) $\Longrightarrow$ (3): This is obvious.

(3) $\Longrightarrow$ (1): Let U be an open set. Then, U is semi-I-open. By (3) and Lemma 2 (2),

$$\begin{aligned} \mathrm{Int}(\mathrm{Cl}^*(U)) &= U \cup \mathrm{Int}(\mathrm{Cl}^*(U)) \\ &= s^*\mathrm{Cl}_{\mathrm{I}}(U) \\ &= \mathrm{Cl}^*(s^*\mathrm{Cl}_{\mathrm{I}}(U)) \\ &= \mathrm{Cl}^*(U \cup \mathrm{Int}(\mathrm{Cl}^*(U))) \\ &= \mathrm{Cl}^*(\mathrm{Int}(\mathrm{Cl}^*(U))) = \mathrm{Cl}^*(U), \end{aligned} \tag{3}$$

and hence $\mathrm{Cl}^*(U)$ is open. Thus, (X, τ, I) is $*$-extremally disconnected. □

Theorem 1. *Let (X, τ, I) be a $*$-extremally disconnected semi-I-regular space. Then, the collection $C = \{C_\gamma | \gamma \in \Gamma\}$ of subsets of X is semi-I-locally finite if and only if C is locally finite.*

Proof. We need to show only necessity. Suppose that C is semi-I-locally finite. For each $x \in X$, there exists a semi-I-open set U containing x, and U intersects at most finitely many members of C. Since (X, τ, I) is semi-I-regular, there exists a semi-I-open set V such that $x \in V \subseteq sCl_I(V) \subseteq U$. Since V is semi-I-open, by Lemma 4, there exists an open set W such that $W \subseteq V \subseteq Cl^*(W)$. Since (X, τ, I) is $*$-extremally disconnected, by Lemma 5, $Cl^*(W) = Cl^*(V) = s^*Cl_I(V)$ is an open set containing x. Since $s^*Cl_I(V) \subseteq sCl_I(V)$, $Cl^*(W)$ intersects at most finitely many members of C. Thus, C is locally finite. □

Theorem 2. *Let $C = \{C_\gamma | \gamma \in \Gamma\}$ be a collection of subsets of an ideal topological space (X, τ, I). Then,*

(1) C is semi-I-locally finite if and only if $\{sCl_I(C_\gamma) | \gamma \in \Gamma\}$ is semi-I-locally finite.
(2) If C is semi-I-locally finite, then $\cup_{\gamma\in\Gamma} sCl_I(C_\gamma) = sCl_I(\cup_{\gamma\in\Gamma} C_\gamma)$.
(3) If C is semi-I-locally finite and $D_\gamma \subseteq C_\gamma$ for each $\gamma \in \Gamma$, then

$$D = \{D_\gamma | \gamma \in \Gamma\}, \tag{4}$$

is also semi-I-locally finite.

Proof. The proof is obvious. □

Definition 4 (see [18]). A subset D of an ideal topological space (X, τ, I) is called $*$-dense if $Cl^*(D) = X$.

Definition 5 (see [12]). An ideal topological space (X, τ, I) is called I-submaximal if every $*$-dense subset of X is open.

Theorem 3. *Let (X, τ, I) be an I-submaximal $*$-extremally disconnected space. Then, every semi-I-locally finite collection $C = \{C_\gamma | \gamma \in \Gamma\}$ of subsets of X is locally finite.*

Proof. It follows from Corollary 17 of [14]. □

Definition 6 (see [11]). A function $f: (X, \tau, I) \longrightarrow (Y, \sigma, J)$ is said to be I-irresolute if $f^{-1}(V)$ is semi-I-open in (X, τ, I) for every semi-J-open set V of (Y, σ, J).

Theorem 4. *Let $f: (X, \tau, I) \longrightarrow (Y, \sigma, J)$ be an I-irresolute function. If $C = \{C_\gamma | \gamma \in \Gamma\}$ is a semi-J-locally finite collection in (Y, σ, J), then*

$$\{f^{-1}(C_\gamma) | \gamma \in \Gamma\}, \tag{5}$$

is a semi-I-locally finite collection in (X, τ, I).

Proof. The proof is obvious. □

Definition 7. An ideal topological space (X, τ, I) is said to be semi-I-compact if every cover of X by semi-I-open sets has a finite subcover.

Definition 8. A function $f: (X, \tau, I) \longrightarrow (Y, \sigma, J)$ is said to be semi-(I, J)-closed if $f(F)$ is semi-J-closed in (Y, σ, J) for every semi-I-closed set F of (X, τ, I).

Lemma 6. *A function $f: (X, \tau, I) \longrightarrow (Y, \sigma, J)$ is semi-(I, J)-closed if and only if, for each $y \in Y$ and every semi-I-open set U in (X, τ, I) which contains $f^{-1}(y)$, there exists a semi-J-open subset V of (Y, σ, J) such that $y \in V$ and $f^{-1}(V) \subseteq U$.*

Proof. Suppose that f is semi-(I, J)-closed. Let $y \in Y$ and U be any semi-I-open set in (X, τ, I) such that $f^{-1}(y) \subseteq U$. Put $V = Y - f(X - U)$. Then, V is semi-J-open in (Y, σ, J) such that $y \in V$ and $f^{-1}(V) \subseteq U$.

Conversely, let F be any semi-I-closed subset of X. For each $y \in Y - f(F)$, then $f^{-1}(y) \subseteq X - F = U$. Therefore, there exists a semi-J-open subset V_y of (Y, σ, J) such that $y \in V_y$ and $f^{-1}(V_y) \subseteq U$. Put $V = \cup\{V_y | y \in Y - f(F)\}$. Then, V is semi-J-open in (Y, σ, J) such that $y \in V$ and $f^{-1}(V) \subseteq U$. Thus, $f(F)$ is semi-J-closed in (Y, σ, J). □

Theorem 5. *Let $f: (X, \tau, I) \longrightarrow (Y, \sigma, J)$ be a semi-(I, J)-closed function such that $f^{-1}(y)$ is semi-I-compact in (X, τ, I). If $C = \{C_\gamma | \gamma \in \Gamma\}$ is a semi-I-locally finite collection in (X, τ, I), then $f(C) = \{f(C_\gamma) | \gamma \in \Gamma\}$ is a semi-J-locally finite collection in (Y, σ, J).*

Proof. For each $x \in f^{-1}(y)$, there exists a semi-I-open set U_x in (X, τ, I) containing x such that U_x intersects at most finitely many members of C. Thus, $\{U_x | x \in f^{-1}(y)\}$ is a semi-I-open cover of $f^{-1}(y)$ and so there exist a finite number of points $x_1, x_2, \ldots, x_n$ of $f^{-1}(y)$ such that $f^{-1}(y) \subseteq \cup_{i=1}^{n} U_{x_i} = U$. f is semi-(I, J)-closed, so by Lemma 6 there exists a semi-J-open set V of Y containing y such that $f^{-1}(V) \subseteq U$. Thus, V intersects at most finitely many members of $f(C)$ and hence $f(C)$ is semi-J-locally finite in (Y, σ, J). □

4. Semi-I-Expandable Ideal Topological Spaces

Recall that a topological space (X, τ) is said to be *expandable* [1] if, for every locally finite collection $C = \{C_\gamma | \gamma \in \Gamma\}$ of subsets of X, there exists a locally finite collection $G = \{G_\gamma | \gamma \in \Gamma\}$ of open subsets of X such that $C_\gamma \subseteq G_\gamma$ for each $\gamma \in \Gamma$.

Definition 9. An ideal topological space (X, τ, I) is said to be semi-I-expandable (resp., ω_0-semi-I-expandable) if, for every semi-I-locally finite collection $C = \{C_\gamma | \gamma \in \Gamma\}$ (resp., $|\Gamma| \leq \omega_0$) of subsets of X, there exists a locally finite collection $G = \{G_\gamma | \gamma \in \Gamma\}$ of open subsets of X such that $C_\gamma \subseteq G_\gamma$ for each $\gamma \in \Gamma$.

Now, we have the following characterizations of semi-I-expandable ideal topological spaces in terms of coverings.

Theorem 6. *An ideal topological space* (X, τ, I) *is semi-*I*-expandable if every semi-*I*-open cover of* X *has a locally finite open refinement.*

Proof. Let $\mathrm{C} = \{C_\gamma | \gamma \in \Gamma\}$ be a semi-I-locally finite collection of semi-I-closed subsets of X. Let Γ_0 be any finite subset of Γ and let $V_\lambda = X - \cup\{C_\gamma | \gamma \neq \lambda\}$, $\lambda \in \Gamma_0$. Then, V_λ is semi-I-open for each $\lambda \in \Gamma_0$ and V_λ intersects at most finitely many members of C for each $\lambda \in \Gamma_0$. The family $\mathrm{V} = \{V_\lambda | \lambda \in \Gamma_0\}$ is a cover of X. For each $x \in X$, there exists a semi-I-open subset V_x of X such that V_x intersects at most many members of C, say $C_{\gamma(1)}, C_{\gamma(2)}, \ldots, C_{\gamma(n)}$. By assumption V has a locally finite open refinement $\mathrm{W} = \{W_\beta | \beta \in \Omega\}$. Put $U_\gamma = \cup\{W' \in \mathrm{W} | W' \cap C_\gamma \neq \varnothing\}$ for each $\gamma \in \Gamma$. Then, $C_\gamma \subseteq U_\gamma$ and U_γ is open for each $\gamma \in \Gamma$. Now, it suffices to show that $\mathrm{U} = \{U_\gamma | \gamma \in \Gamma\}$ is locally finite. Let $x \in X$. Then, there exists an open set G_x which contains x and intersects only finite many members of W. Thus, $G_x \cap U_\gamma \neq \varnothing$ if and only if $G_x \cap W_\beta \neq \varnothing$ and $W_\beta \cap C_\gamma \neq \varnothing$ for some $\beta \in \Omega$. Since W is a refinement of V, W_β is contained in some V_λ which intersects only finite many C_γ. Thus, U is locally finite. This shows that (X, τ, I) is semi-I-expandable. □

Theorem 7. *Let* (X, τ, I) *be a* $*$*-extremally disconnected space. Then,* (X, τ, I) *is semi-*I*-expandable if every semi-*I*-open cover* U *of* X *has a locally finite semi-*I*-open refinement* V.

Proof. Let $\mathrm{C} = \{C_\gamma | \gamma \in \Gamma\}$ be a semi-I-locally finite collection of semi-I-closed subsets of X. As in the proof of Theorem 6, we construct a locally finite semi-I-open collection $\mathrm{U} = \{U_\gamma | \gamma \in \Gamma\}$ of subsets of X such that $C_\gamma \subseteq U_\gamma$ for each $\gamma \in \Gamma$. Since U_γ is semi-I-open in (X, τ, I) for each $\gamma \in \Gamma$, we choose $V_\gamma \in \tau$ such that $V_\gamma \subseteq U_\gamma \subseteq \mathrm{Cl}^*(V_\gamma)$. Since (X, τ, I) is $*$-extremally disconnected, the collection $\mathrm{V} = \{\mathrm{Cl}^*(V_\gamma) | \gamma \in \Gamma\}$ is open locally finite in (X, τ, I) and $C_\gamma \subseteq U_\gamma \subseteq \mathrm{Cl}^*(V_\gamma)$ for each $\gamma \in \Gamma$. Thus, (X, τ, I) is semi-I-expandable. □

Theorem 8. *Let* (X, τ, I) *be a* $*$*-extremally disconnected space. Then,* (X, τ, I) *is* ω_0*-semi-*I*-expandable if and only if every countable semi-*I*-open cover* U *of* X *has a locally finite semi-*I*-open refinement* V.

Proof. It follows from Theorem 7.

Conversely, let (X, τ, I) be ω_0-semi-I-expandable and let $\mathrm{U} = \{U_n | n \in \mathbb{N}\}$ be a countable semi-I-open cover of X. For each n, put $W_n = \cup_{n_0 \leq n} U_{n_0}$. Then,

$$\mathrm{W} = \{W_n | n \in \mathbb{N}\} \tag{6}$$

is an increasing semi-I-open cover of X, and hence the collection

$$\mathrm{C} = \{C_n = X - W_n | n \in \mathbb{N}\} \tag{7}$$

is semi-I-locally finite in (X, τ, I). Therefore, there exists a locally finite collection $\mathrm{G} = \{G_n | n \in \mathbb{N}\}$ of open subsets of (X, τ, I) such that $C_n \subseteq G_n$ for each n. Now, for each $n \in \mathbb{N}$, put $V_n = U_n - \cup_{n_0 < n}(X - G_{n_0})$. Then, by Lemma 1, V_n is semi-I-open in (X, τ, I) and $V_n \subseteq U_n$ for each n. Finally, since G is locally finite, it is easy to show that the collection $\mathrm{V} = \{V_n | n \in \mathbb{N}\}$ is a locally finite refinement of U. □

Theorem 9. *Let* $f\colon (X, \tau, \mathrm{I}) \longrightarrow (Y, \sigma, \mathrm{J})$ *be an* I*-irresolute* (I, J)*-closed surjection and* $f^{-1}(y)$ *be compact for each* $y \in Y$. *If* (X, τ, I) *is semi-*I*-expandable, then* (Y, σ, J) *is semi-*J*-expandable.*

Proof. Let $\mathrm{C} = \{C_\gamma | \gamma \in \Gamma\}$ be a semi-J-locally finite collection in (Y, σ, J). By Theorem 4, the collection $f^{-1}(\mathrm{C}) = \{f^{-1}(C_\gamma) | \gamma \in \Gamma\}$ is semi-I-locally finite in (X, τ, I), and so there exists a locally finite collection $\mathrm{G} = \{G_\gamma | \gamma \in \Gamma\}$ of open subsets of X such that $f^{-1}(C_\gamma) \subseteq G_\gamma$ for each $\gamma \in \Gamma$. Put $V_\gamma = Y - f(X - G_\gamma)$ for each $\gamma \in \Gamma$. It is easy to see that V_γ is open and $F_\gamma \subseteq V_\gamma$ for each $\gamma \in \Gamma$. Finally, we show that the collection $\mathrm{V} = \{V_\gamma | \gamma \in \Gamma\}$ of subsets of Y is locally finite. Let $y \in Y$. For each $x \in f^{-1}(y)$, there exists an open set U_x containing x such that U_x intersects at most finitely many members of G. Therefore, the collection $\{U_x | x \in f^{-1}(y)\}$ is an open cover of $f^{-1}(y)$, and so there exist a finite number of points $x_1, x_2, \ldots, x_n$ of $f^{-1}(y)$ such that $f^{-1}(y) \subseteq \cup_{i=1}^n U_{x_i} = U$. Since f is (I, J)-closed, there exists an open set V_y containing y and $f^{-1}(V) \subseteq U$. Since $f^{-1}(V_\gamma) \subseteq G_\gamma$ for each $\gamma \in \Gamma$ and G is locally finite in (X, τ, I), V_y intersects at most finitely many members of V which means that V is locally finite in (Y, σ, J). Thus, (Y, σ, J) is semi-J-expandable. □

Theorem 10. *Let* $f\colon (X, \tau, \mathrm{I}) \longrightarrow (Y, \sigma, \mathrm{J})$ *be a continuous semi-*(I, J)*-closed surjection and let* $f^{-1}(y)$ *be semi-*I*-compact in* (X, τ, I) *for each* $y \in Y$. *If* (Y, σ, J) *is semi-*J*-expandable, then* (X, τ, I) *is semi-*I*-expandable.*

Proof. Let $\mathrm{C} = \{C_\gamma \in \gamma\}$ be a semi-I-locally finite collection of subsets of (X, τ, I). By Theorem 5, $\{f(C_\gamma) | \gamma \in \Gamma\}$ is a semi-J-locally finite collection in (Y, σ, J). Then, there exists a locally finite collection $\mathrm{G} = \{G_\gamma | \gamma \in \Gamma\}$ of open subsets of (Y, σ, J) such that $f(C_\gamma) \subseteq G_\gamma$ for each $\gamma \in \Gamma$. Then, $C_\gamma \subseteq f^{-1}(f(C_\gamma)) \subseteq f^{-1}(G_\gamma)$ and $\{f^{-1}(G_\gamma) | \gamma \in \Gamma\}$ is an open locally finite collection in (X, τ, I). Thus, (X, τ, I) is semi-I-expandable. □

Theorem 11. *For an ideal topological space* (X, τ, I), *the following properties are equivalent:*

(1) (X, τ, I) *is semi-*I*-expandable.*

(2) (X, τ, I) *is expandable and every semi-*I*-locally finite collection of* X *is locally finite.*

Proof

(1) $\Longrightarrow$ (2): Let (X, τ, I) be semi-I-expandable and $C = \{C_\gamma | \gamma \in \Gamma\}$ a semi-I-locally finite collection of subsets of X. Then, there exists a locally finite collection $G = \{G_\gamma | \gamma \in \Gamma\}$ of open subsets of X such that $C_\gamma \subseteq G_\gamma$ for each $\gamma \in \Gamma$. Observe that local finiteness of G implies that the family C must have been locally finite.

(2) $\Longrightarrow$ (1): It follows directly from the definition of semi-I-expandability. □

Theorem 12. *Let (X, τ, I) be an ideal topological space. If, for every semi-I-locally finite collection C, there exists a locally finite open cover U of X such that each element of U meets only finitely many elements of C, then (X, τ, I) is semi-I-expandable.*

Proof. Let $C = \{C_\gamma | \gamma \in \Gamma\}$ be a semi-I-locally finite collection of subsets of X and $U = \{U_\delta | \delta \in \nabla\}$ the locally finite open cover of X such that each element of U intersects only finitely many elements of C. Put $W_\gamma = \cup \{U_\delta \in U | U_\delta \cap C_\gamma \neq \emptyset\}$, $\gamma \in \Gamma$. Clearly, $C_\gamma \subseteq W_\gamma$ and W_γ is open for each $\gamma \in \Gamma$. We claim that

$$W = \{W_\gamma | \gamma \in \Gamma\} \tag{8}$$

is locally finite. For each $x \in X$, since U is locally finite, there exists an open neighborhood N_x of x which meets only finitely many members of U. Then,

$$N_x \cap W_\gamma \neq \emptyset \tag{9}$$

if and only if $N_x \cap U_\delta \neq \emptyset$ and $U_\delta \cap C_\gamma \neq \emptyset$ for some $\delta \in \nabla$. Since U_δ meets only finitely many members of C, N_x intersects only finite many members of W. Thus, W is locally finite and hence (X, τ, I) is semi-I-expandable. □

Corollary 1. *Let (X, τ, I) be a $*$-extremally disconnected space. If, for every semi-I-locally finite collection C, there exists a locally finite semi-I-open cover U of X such that each element of U meets only finitely many elements of C, then (X, τ, I) is semi-I-expandable.*

Definition 10. A collection C of subsets of an ideal topological space (X, τ, I) is said to be σ-semi-I-locally finite if $C = \cup_{n=1}^{\infty} C_n$, where each C_n is semi-I-locally finite.

Theorem 13. *Let (X, τ, I) be a $*$-extremally disconnected space. Then, (X, τ, I) is semi-I-expandable if and only if every semi-I-open cover of X with a σ-semi-I-locally finite refinement has a locally finite semi-I-open refinement.*

Proof. Suppose that every semi-I-open cover of X with a σ-semi-I-locally finite refinement has a locally finite semi-I-open refinement. Let $C = \{C_\gamma | \gamma \in \Gamma\}$ be a semi-I-locally finite collection of semi-I-closed subsets of X. Define

$$B_n = \{\Omega \subseteq \Gamma | |\Omega| = n\}, \tag{10}$$

for each $n \in \mathbb{N}$, $U_n = \{X - \cup_{\gamma \notin \Omega} C_\gamma | \Omega \in B_n\}$, $U_0 = \{X - \cup_{\gamma \in \Gamma} C_\gamma\}$. Then, $U = \cup_{n=1}^{\infty} U_n$ is a semi-I-open cover of X. For each $U' \in U$, U' intersects only finitely many members of C. Now define $G_n = \{x \in X | \text{ord}(x, C) = n\}$, for each $n \in \mathbb{N}$ and $V_0 = U_0$. For each n, let $V_n = \{(\cap_{\gamma \in \Omega} C_\gamma) \cap G_n | \Omega \in B_n\}$. Since C is semi-I-locally finite, $V = \cup_{i=0}^{\infty} V_i$ is a σ-semi-I-locally refinement of U. Hence, U has a locally finite semi-I-open refinement W. Thus, each element of W meets only finitely many members of C; by Corollary 1, (X, τ, I) is semi-I-expandable.

Conversely, let (X, τ, I) be a semi-I-expandable and $*$-extremally disconnected space. Let $U = \{U_\lambda | \lambda \in \Omega\}$ be a semi-I-open cover of X with a σ-semi-I-locally finite refinement $L = \cup_{n=1}^{\infty} L_n$, where $L_n = \{L_{(\gamma,n)} | \gamma \in \Gamma_n\}$ is semi-I-locally finite for each n. We will prove that U has a locally finite semi-I-open refinement. Since (X, τ, I) is semi-I-expandable, for each n, there exists a locally finite open collection $\{G_{(\gamma,n)} | \gamma \in \Gamma_n\}$ such that $L_{(\gamma,n)} \subseteq G_{(\gamma,n)}$ for each $\gamma \in \Gamma_n$. Since L is a refinement of U, for any $L_{(\gamma,n)} \in L$, there exists $\lambda(\gamma, n) \in \Omega$ such that $L_{(\gamma,n)} \subseteq U_{\lambda(\gamma,n)}$. For each $n \in \mathbb{N}$ and $\gamma \in \Gamma_n$, let $W_{(\gamma,n)} = G_{(\gamma,n)} \cap U_{\lambda(\gamma,n)}$, $W_n = \{W_{(\gamma,n)} | \gamma \in \Gamma_n\}$. It follows from Lemma 1 that $W = \cup_{n=1}^{\infty} W_n$ is a σ-locally finite semi-I-open refinement of U. Put $S_n = \cup \{W_{(\gamma,n)} | \gamma \in \Gamma_n\}$ for each $n \in \mathbb{N}$. By the definition of semi-I-open sets, S_n is semi-I-open, and hence $\{S_n | n \in \mathbb{N}\}$ is a countable semi-I-open cover of X. Since (X, τ, I) is semi-I-expandable, (X, τ, I) is ω_0-semi-I-expandable. By Theorem 8, there exists a locally finite semi-I-open refinement $\{V_n | n \in \mathbb{N}\}$, and we may assume that $V_n \subseteq S_n$. Since V_n is semi-I-open for each n, there exists an open set G_n such that $G_n \subseteq V_n \subseteq Cl^*(G_n)$. Since (X, τ, I) is $*$-extremally disconnected, $\{Cl^*(G_n) | n \in \mathbb{N}\}$ is a locally finite collection of open subsets of X. Let $G = \{Cl^*(G_n) \cap W_{(\gamma,n)} | \gamma \in \Gamma_n, n \in \mathbb{N}\}$. It is easy to check that G is a locally finite semi-I-open refinement of U. □

Lemma 7 (see [19]). *Let (X, τ, I) be an ideal topological space, $A \subseteq U \subseteq X$ and $U \in \tau$. If A is semi-$I_{|U}$-open in $(U, \tau_{|U}, I_{|U})$, then A is semi-I-open in (X, τ, I).*

Theorem 14. *Let (X, τ, I) be a semi-I-expandable ideal topological space. If U is clopen subset of (X, τ, I), then $(U, \tau_{|U}, I_{|U})$ is semi-$I_{|U}$-expandable.*

Proof. Let U be a clopen subset of a semi-I-expandable space (X, τ, I). Let $C = \{C_\gamma | \gamma \in \Gamma\}$ be a semi-I-locally finite collection of subsets of $(U, \tau_{|U}, I_{|U})$. Since U is clopen in (X, τ, I), C is semi-I-locally finite in (X, τ, I) for each $x \in X$. Then, either $x \in U$ or $x \notin U$. If $x \in U$, then there exists $V \in sI_{|U}O(U, \tau_{|U})$ containing x such that V intersects at most finitely many members of C. Since U is open, $V \in sIO(X, \tau)$, by Lemma 7, and hence C is a semi-I-locally finite collection in (X, τ, I). If $x \notin U$, then $X - U$ is semi-I-open in (X, τ, I) containing x which intersects no member of C. Thus, C is a semi-I-locally finite collection of the semi-

I-expandable ideal topological space (X, τ, I), so there exists a locally finite collection of open subsets of (X, τ, I), say $\mathrm{G} = \{G_\gamma | \gamma \in \Gamma\}$ such that $C_\gamma \subseteq G_\gamma$ for each $\gamma \in \Gamma$. Now, consider $\mathrm{G}_0 = \{G_\gamma \cap U | \gamma \in \Gamma\}$. Then, G_0 is a locally finite collection of open subsets of U such that $C_\gamma \subseteq U \cap G_\gamma$ for each $\gamma \in \Gamma$. Thus, $(U, \tau_{|U}, \mathrm{I}_{|U})$ is semi-$\mathrm{I}_{|U}$-expandable. □

Recall that a topological space (X, τ) is said to be *paracompact* [20] if every open cover of X has a locally finite open refinement.

Definition 11. An ideal topological space (X, τ, I) is called semi-I-paracompact if every open cover of X has a locally finite semi-I-open refinement.

Theorem 15. *Let (X, τ, I) be a ω_0-semi-*I*-expandable $*$-extremally disconnected space. Then, (X, τ, I) is semi-*I*-paracompact if every open cover of X has a σ-locally finite semi-*I*-open refinement.*

Proof. Let $\mathrm{U} = \{U_\gamma | \gamma \in \Gamma\}$ be an open cover and $\mathrm{V} = \{V_{(\gamma,n)} | \gamma \in \Gamma_n\}$ be a locally finite collection of semi-I-open subsets of X. Let $W_n = \cup_{\gamma\in\Gamma_n} V_{(\gamma,n)}$ for each $n \in \mathbb{N}$. Then, W_n is semi-I-open by the definition of semi-I-open sets. $\{W_n | n \in \mathbb{N}\}$ is a countable semi-I-open cover of X and, hence, by Theorem 8 $\{W_n | n \in \mathbb{N}\}$ has a locally finite semi-I-open refinement $\mathrm{H} = \{H_n | n \in \mathbb{N}\}$, and we may assume that $H_n \subseteq W_n$ for each $n \in \mathbb{N}$. Since H_n is semi-I-open for each n, there exists an open set G_n such that $G_n \subseteq H_n \subseteq \mathrm{Cl}^*(G_n)$. Thus, $\{\mathrm{Cl}^*(G_n) | n \in \mathbb{N}\}$ is a locally finite collection of open subsets since (X, τ, I) is a $*$-extremally disconnected space. Therefore, $\mathrm{G} = \{\mathrm{Cl}^*(G_n) \cap V_{(\gamma,n)} | \gamma \in \Gamma_n, n \in \mathbb{N}\}$ is a locally finite semi-I-open refinement of U by Lemma 1 (2). Thus, (X, τ, I) is semi-I-paracompact. □

Definition 12. A function $f\colon (X, \tau, \mathrm{I}) \longrightarrow (Y, \sigma, \mathrm{J})$ is said to be semi-(I, J)-open if $f(G)$ is semi-J-open in (Y, σ, J) for every semi-I-open set G of (X, τ, I).

Theorem 16. *Let $f\colon (X, \tau, \mathrm{I}) \longrightarrow (Y, \sigma, \mathrm{J})$ be a continuous, semi-(I, J)-open, and closed surjection such that $f^{-1}(y)$ is compact for each $y \in Y$. If (X, τ, I) is semi-*I*-paracompact, then (Y, σ, J) is semi-*J*-paracompact.*

Proof. Let $\mathrm{U} = \{U_\gamma | \gamma \in \Gamma\}$ be an open cover of (Y, σ, J). Then,

$$f^{-1}(\mathrm{U}) = \{f^{-1}(U_\gamma) | \gamma \in \Gamma\} \tag{11}$$

is an open cover of the semi-I-paracompact space (X, τ, I) and so $f^{-1}(\mathrm{U})$ has a locally finite semi-I-open refinement, say $\mathrm{V} = \{V_\alpha | \alpha \in \nabla\}$. Since f is semi-(I, J)-open, the collection $f(\mathrm{V}) = \{f(V_\alpha) | \alpha \in \nabla\}$ is a semi-J-open refinement of U. Finally, we shall show that the collection $f(\mathrm{V})$ is locally finite in (Y, σ, J). Let $y \in Y$. For each $x \in f^{-1}(y)$, there exists an open set G_x containing x such that G_x intersects at most finitely many members of V. The collection

$$\{G_x | x \in f^{-1}(y)\} \tag{12}$$

is an open cover of $f^{-1}(y)$, and therefore there exists a finite subset K_0 of $f^{-1}(y)$ such that $f^{-1}(y) \subseteq \cup_{x\in K_0} G_x$. Since f is closed, there exists an open set W_y containing y such that $f^{-1}(W_y) \subseteq \cup_{x\in K_0} G_x$. Then, $f^{-1}(W_y)$ intersects at most finitely many members of V. Therefore, W_y intersects at most finitely many members of $f(\mathrm{V})$. Thus, $f(\mathrm{V})$ is locally finite in (Y, σ, J). □

5. Semi-I($*$)-Paracompact Subsets

We begin this section by introducing the concept of semi-I($*$)-paracompact subsets. In particular, some properties of semi-I($*$)-paracompact subsets are discussed.

Definition 13. A subset A of an ideal topological space (X, τ, I) is said to be semi-I($*$)-paracompact set if every cover A by open subsets of (X, τ, I) has a locally finite semi-I-open refinement in (X, τ, I).

Recall that a subset A of an ideal topological space (X, τ, I) is called I_g-*closed* [21] if $A^* \subseteq U$, whenever U is open and $A \subseteq U$.

Lemma 8 (see [22]). *For a subset A of an ideal topological space (X, τ, I), the following properties are equivalent:*

(a) A is I_g-closed.

(b) $Cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

(c) For all $x \in Cl^*(A)$, $Cl(\{x\}) \cap A \neq \varnothing$.

(d) $Cl^*(A) - A$ contains no nonempty closed set.

(e) $A^* - A$ contains no nonempty closed set.

Theorem 17. *Every I_g-closed subset of a semi-*I*-paracompact space is semi-*I($*$)*-paracompact.*

Proof. Let (X, τ, I) be a semi-I-paracompact space and let A be a I_g-closed subset of X. Let $\mathrm{U} = \{U_\gamma | \gamma \in \Gamma\}$ be any cover of A by open subsets of X. Since $A \subseteq \cup_{\gamma\in\Gamma} U_\gamma$ and A is I_g-closed, by Lemma 8, $\mathrm{Cl}^*(A) \subseteq \cup_{\gamma\in\Gamma} U_\gamma$. For each $x \notin \mathrm{Cl}^*(A)$, there exists an open set G_x of X containing x such that $G_x \cap A = \varnothing$. Now, put $\mathrm{U}_0 = \{U_\gamma | \gamma \in \Gamma\} \cup \{G_x | x \notin \mathrm{Cl}^*(A)\}$. Then, U_0 is an open cover of the semi-I-paracompact space (X, τ, I). Let $\mathrm{V} = \{V_\lambda | \lambda \in \nabla\}$ be a locally finite semi-I-open refinement of U_0. Then, for each $\lambda \in \nabla$, either $V_\lambda \subseteq U_{\gamma(\lambda)}$ for some $\gamma(\lambda)$ or $V_\lambda \subseteq G_{x(\lambda)}$ for some $x(\lambda)$. Put $\nabla_0 = \{\lambda \in \nabla | V_\lambda \subseteq U_{\gamma(\lambda)}\}$. Then, $\mathrm{V}_0 = \{V_\lambda | \lambda \in \nabla_0\}$ is a locally finite semi-I-open refinement of U and $A \subseteq \cup_{\lambda\in\nabla_0} V_\lambda$. Thus, A is semi-I($*$)-paracompact. □

Theorem 18. *Every open semi-*I($*$)*-paracompact subset of an ideal topological space (X, τ, I) is semi-*I*-paracompact.*

Proof. Let U be an open semi-I($*$)-paracompact subset of an ideal topological space (X, τ, I). Let $\mathrm{V} = \{V_\gamma | \gamma \in \Gamma\}$ be any open cover of U by open subsets of the subspace $(U, \tau_{|U}, I_{|U})$. Since U is open, V is a cover of U by open subsets of X and so V has a locally finite semi-I-open refinement of W in (X, τ, I). Then, $W'_0 = \{W' \cap U | W' \in W\}$ is a locally finite semi-I-open refinement of V in $(U, \tau_{|U}, I_{|U})$ and the result follows. □

Theorem 19. *Let U be a clopen subspace of an ideal topological space* (X, τ, I). *Then, U is semi-*I$(*)$*-paracompact if and only if it is semi-*I*-paracompact.*

Proof. It follows from Theorem 18.

Conversely, let $V = \{V_\gamma | \gamma \in \Gamma\}$ be any open cover of U by open subsets of (X, τ, I). Then, $V_0 = \{U \cap V_\gamma | \gamma \in \Gamma\}$ is an open cover of the semi-$I_{|U}$-paracompact subspace $(U, \tau_{|U}, I_{|U})$ and so V_0 has a locally finite semi-I-open refinement W in $(U, \tau_{|U}, I_{|U})$. By Lemma 7, $W' \in sIO(X, \tau)$ for every $W' \in W$. To show that W is locally finite in (X, τ, I), let $x \in X$. If $x \in U$, then there exists $G_x \in \tau_{|U} \subseteq \tau$ containing x such that G_x intersects at most finitely many members of W. Otherwise, $X - U$ is an open set containing x which intersects no member of W. Therefore, W is locally finite in (X, τ, I) such that $U \subseteq \cup\{W' | W' \in W\}$. Thus, U is semi-I$(*)$-paracompact. □

Corollary 2. *Every clopen subspace of a semi-*I*-paracompact space is semi-*I*-paracompact.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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