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Research Article

Certain Class of Almost α-Cosymplectic Manifolds

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This paper concerned with almost α-cosymplectic manifolds satisfying conformally flat condition. Firstly, we investigate Kaehler integral submanifolds of almost α-cosymplectic manifolds. Next, we study conformally flat almost α-cosymplectic manifolds of dim $\geq$ 5 whose integral submanifolds are Kaehler. Finally, an illustrative example is constructed to verify our result.

1. Introduction

The notion of conformal flatness is one of the most primitive concepts in differential geometry. Notwithstanding this fact, most of the studies have been local character. However, Kulkarni classified conformally flat manifolds up to conformal equivalence [1].

On a Riemannian manifold M, Weyl established a tensor of type (1, 3) which vanishes whenever the metric is (locally) conformally equivalent to a flat metric. Therefore, this tensor is called the conformal curvature tensor of the metric and defined by

$$\begin{aligned} C(X,Y)Z = {} & R(X,Y)Z - \left(\frac{1}{(2n-1)}\right)[S(Y,Z)X - S(X,Z)Y \\ & + g(Y,Z)QX - g(X,Z)QY] \\ & + \left(\frac{r}{(2n(2n-1))}\right)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1}$$

for any vector fields X, Y, and Z on M. Here, we denote by R and r the Riemann curvature tensor and scalar curvature of M, respectively [2, 3]. A necessary condition being conformally flat for a Riemannian manifold is the vanishing of the Weyl curvature tensor. It is obvious that the Weyl tensor vanishes identically in 2 dimensions. In general, it is nonzero in dimensions $\geq$ 4. The metric is locally conformally flat provided that the Weyl tensor vanishes for 4 dimensions. In this case, the metric has a local coordinate system where it is proportional to a constant tensor. In dimension 3, we have

$$c(X,Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - \left(\frac{1}{2(n-2)}\right)[(\nabla_X r)Y - (\nabla_Y r)X]. \tag{2}$$

Equation (2) is a necessary and sufficient condition for three-dimensional Riemannian manifold being conformally flat. Here, c is the divergence operator of C [3].

A $(2n+1)$-dimensional Riemannian manifold M^{2n+1} is conformally flat if and only if the Weyl conformal curvature tensor C vanishes for any vector fields X, Y, and Z when $n > 2$ or the tensor L of type (1, 1) defined as

$$LX = \left(\frac{1}{2n-1}\right)\left(QX - \left(\frac{r}{4n}\right)X\right), \tag{3}$$

for any vector field X, is of Codazzi type when $n = 1$. Here, Q is the Ricci operator associated with the Ricci tensor S and r is the scalar curvature [4].

In contact metric manifolds, Okumura obtained that a conformally flat Sasakian manifold of dimension $>$ 3 is locally isometric to the unit sphere [5]. Later, this result was extended to the K-contact manifolds by Tanno for dimension $\geq$ 3. Gosh and Sharma showed that a conformally flat contact strongly pseudo-convex integrable CR manifold is locally isometric to a unit sphere if the characteristic vector field is an eigenvector of the Ricci tensor at each point [6]. Afterwards, Gosh et al. obtained that a conformally flat contact strongly pseudo-convex integrable CR manifold of dimension $>$ 3 is of constant curvature 1 [7].

Moreover, if an almost cosymplectic manifold with dimension $(2n+1)$ is conformally flat $(n \geq 2)$, then it is locally flat and cosymplectic [8]. Conversely to this, there exist conformally flat almost cosymplectic manifolds with Kaehler leaves which are not locally flat and not cosymplectic in dimension 3 [8].

Recently, Blair et al. have focused on almost contact metric manifolds with conformally flat condition [9]. The authors construct an illustrative example of 3-dimensional conformally flat almost α-Kenmotsu manifold whose sectional curvature is nonconstant. Furthermore, they consider conformally flat almost contact metric manifolds which are $*-\eta$-Einstein manifold with dim $\geq$ 5. Moreover, Wang investigated conformally flat almost Kenmotsu manifold of k-dimension 3 [4]. After these studies, we also point out that Weyl conformal curvature tensor has been studied extensively by Venkatesha et al. [10].

In this paper, we study the geometry of conformally flat almost α-cosymplectic manifolds. We aim at characterizing and classifying conformally flat almost α-cosymplectic manifolds. Then, we obtain the curvature properties of almost α-cosymplectic manifolds with Kaehler integral submanifolds and investigate almost α-cosymplectic manifolds of dim $\geq$ 5 which are Kaehler integral submanifolds. Finally, we give a concrete example of 3-dimensional almost α-Kenmotsu manifolds.

2. Preliminaries

Let M^{2n+1} be a $(2n+1)$-dimensional smooth manifold endowed with a triple (ϕ, ξ, η) where ϕ is a type of (1, 1) tensor field, ξ is a vector field, and η is a 1-form on M^{2n+1} such that

$$\begin{aligned} \eta(\xi) &= 1, \\ \phi^2 &= -I + \eta \otimes \xi, \\ \phi\xi &= 0, \\ \eta \circ \phi &= 0, \\ \operatorname{rank}(\phi) &= 2n. \end{aligned} \tag{4}$$

If M^{2n+1} admits a Riemannian metric g, defined by

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \tag{5}$$

then M^{2n+1} is called almost contact structure (ϕ, ξ, η, g). Also, the fundamental 2-form Φ of M^{2n+1} is defined by $\Phi(X, Y) = g(\phi X, Y)$. If the Nijenhuis tensor vanishes, defined by

$$\begin{aligned} N_\phi(X, Y) = {} & [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] \\ & + \phi^2[X, Y] + 2d\eta(X, Y)\xi, \end{aligned} \tag{6}$$

then $(M^{2n+1}, \phi, \xi, \eta)$ is said to be normal [3]. It is obvious that a normal almost Kenmotsu manifold is said to be Kenmotsu manifold. In other words, an almost contact metric manifold is known as Kenmotsu if and only if $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$ [11]. An almost contact metric structure is cosymplectic if and only if $\nabla\eta$ and $\nabla\Phi$ are closed [12].

In the light of the above definitions, the generalization of almost Kenmotsu manifold M^{2n+1} is called almost α-Kenmotsu manifold if $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, where α is a nonzero real constant [13, 14]. If we combine almost α-Kenmotsu and almost cosymplectic manifolds, then we introduce a new notion of an almost α-cosymplectic manifold defined by $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$ for any real number α [15]. A normal almost α-cosymplectic manifold is said to be α-cosymplectic manifold, and it is either cosymplectic or α-Kenmotsu under the condition $\alpha = 0$ or $\alpha \neq 0$, respectively [16–19].

Let M^{2n+1} be an almost α-cosymplectic manifold endowed with (ϕ, ξ, η, g). D is the contact distribution of M^{2n+1} given by $D = ker\eta$. Since $d\eta = 0$, D is integrable and the $(2n)$-dimensional distribution is defined by $\phi(D) = D$. Moreover, it is clear that ξ is orthogonal to D. Assume that N is a maximal integral submanifold of D. Therefore, the ξ restricted to integral submanifold N is the normal vector of N. Thus, there exists a Hermitian structure and the tensor field ϕ induces an almost complex structure J defined by $JX = \phi X$ for any vector field X tangent to N [12, 13, 20].

Suppose that G is the Riemannian metric induced on N defined by $G(X, Y) = g(X, Y)$. Then, (J, G) has an almost Hermitian structure on N given by $G(X, Y) = G(JX, JY)$ for any vector field X and Y tangent to N. The fundamental 2-form $\Omega, \Omega(X, Y) = G(JX, Y)$ of (J, G) induced on N. This means that $\Omega(X, Y) = \Phi(X, Y)$, i.e., Ω is the pull-back of the tensor field ϕ from M^{2n+1} to N. Since Ω is closed, we obtain $d\Omega = 0$. Thus, the pair (J, G) is an almost Kaehler structure on N of D. When the structure J is complex, (J, G) becomes a Kaehler structure on N. If the structure (J, G) is Kaehler on every integral submanifolds of the distribution D, this manifold is said to be an almost α-cosymplectic manifold with Kaehler integral submanifolds.

Denote by A and h the (1, 1) tensor fields on M^{2n+1} defined by

$$\begin{aligned} A &= -\nabla\xi, \\ h &= \left(\frac{1}{2}\right) L_\xi \phi, \end{aligned} \tag{7}$$

respectively. Here, L is the Lie derivative of g. Obviously, $A(\xi) = 0$ and $h(\xi) = 0$. Thus, we the following relations for any vector fields X, Y on M^{2n+1} [16]:

$$\nabla_X \xi = -\alpha\phi^2 X - \phi h X, \tag{8}$$

$$\begin{aligned}(\phi \circ h)X + (h \circ \phi)X &= 0,\\ (\phi \circ A)X + (A \circ \phi)X &= -2\alpha\phi,\end{aligned} \tag{9}$$

$$\begin{aligned} tr(h) &= 0,\\ tr(A) &= -2\alpha n,\\ tr(\phi A) &= 0,\\ A\phi + \phi A &= -2\alpha,\end{aligned} \tag{10}$$

$$(\nabla_X A)\xi = A^2 X, tr\left(A^2\right) = \left\|A^2\right\|. \tag{11}$$

3. Curvature Properties

This section deals with the fundamental curvature equations of almost α-cosymplectic manifolds with Kaehler integral submanifolds. Let us give the basic propositions that we will use in later usage. The proof of some propositions are left to the reader for shortness.

Proposition 1. *Let M^{2n+1} be an almost α-cosymplectic manifold. Then, we have*

$$\begin{aligned} R(X,Y)\xi = {} & \alpha^2[\eta(X)Y - \eta(Y)X] - \alpha[\eta(X)\phi hY\\ & - \eta(Y)\phi hX] + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y,\end{aligned} \tag{12}$$

$$R(X,Y)\xi = -(\nabla_X A)Y + (\nabla_Y A)X, \tag{13}$$

$$R(X,\xi)\xi = \alpha^2\phi^2 X + 2\alpha\phi h X - h^2 X + \phi\left(\nabla_\xi h\right)X, \tag{14}$$

$$\left(\nabla_\xi h\right)X = -\phi R(X,\xi)\xi - \alpha^2\phi X - 2\alpha h X - \phi h^2 X, \tag{15}$$

$$R(X,\xi)\xi - \phi R(\phi X,\xi)\xi = 2\left[\alpha^2\phi^2 X - h^2 X\right], \tag{16}$$

$$S(X,\xi) = -2n\alpha^2\eta(X) - \sum_{i=1}^{2n+1} g\left(\left(\nabla_{e_i}\phi h\right)e_i, X\right), \tag{17}$$

$$tr(l) = -\left[2n\alpha^2 + tr\left(h^2\right)\right], \tag{18}$$

where $l = R(.,\xi)\xi$ is the Jacobi operator with respect to ξ. By direct computations, we have the following proposition.

Proposition 2. *An almost α-cosymplectic manifold M^{2n+1} with Kaehler integral submanifolds holds the following equation:*

$$\begin{aligned} R(X,Y)\phi Z - \phi R(X,Y)Z &= g(AX,\phi Z)AY - g(AY,\phi Z)AX - g(AX,Z)\phi AY + g(AY,Z)\phi AX\\ &\quad + \eta(Z)\phi((\nabla_X A)Y - (\nabla_Y A)X) + g((\nabla_X A)Y - (\nabla_Y A)X, \phi Z)\xi\\ &= g(AX,\phi Z)AY - g(AY,\phi Z)AX - g(AX,Z)\phi AY + g(AY,Z)\phi AX - \eta(Z)\phi(R(X,Y)\xi)\\ &\quad - g(R(X,Y)\xi,\phi Z)\xi.\end{aligned} \tag{19}$$

Remark 1. The Ricci operator Q does not have to commute with the basic collineation ϕ for a contact metric manifold. Now, we give this condition for almost α-cosymplectic manifold M^{2n+1} with Kaehler integral submanifolds.

Proposition 3. *Let M^{2n+1} be an almost α-cosymplectic manifold with Kaehler integral submanifolds. The following identity is held:*

$$\begin{aligned} Q\phi - \phi Q = {} & l\phi - \phi l + 4\alpha(1-n)\phi A + 4\alpha^2(1-n)\phi X\\ & + (\eta\circ Q\phi)\xi - \eta\circ(\phi Q\xi).\end{aligned} \tag{20}$$

Proof. Using (5) and (19), one obtains

$$\begin{aligned} g(\phi R(\phi X,\phi Y)Z,\phi W) = {} & g(\phi R(Z,W)X,\phi Y) + g(AZ,\phi X)g(AW,\phi Y) - g(AW,\phi X)g(AZ,\phi Y)\\ & - g(AZ,X)g(\phi AW,\phi Y) + g(AW,X)g(\phi AZ,\phi Y) - \eta(X)g(\phi R(Z,W)\xi,\phi Y) - \eta(R(\phi X,\phi Y)Z)\eta(W).\end{aligned} \tag{21}$$

Putting $X = \phi X$ and $Y = \phi Y$ in (19), we have

$$\begin{aligned} g(R(\phi X,\phi Y)\phi Z,\phi W) - g(\phi R(\phi X,\phi Y)Z,\phi W) = {} & g(A\phi X,\phi Z)g(A\phi Y,\phi W) - g(A\phi Y,\phi Z)g(A\phi X,\phi W)\\ & - g(A\phi X,Z)g(\phi A\phi Y,\phi W) + g(A\phi Y,Z)g(\phi A\phi X,\phi W)\\ & - \eta(Z)g(\phi R(\phi X,\phi Y)\xi,\phi W).\end{aligned} \tag{22}$$

Substituting of (21) into (22) yields immediately

$$\begin{aligned} g(R(\phi X, \phi Y)\phi Z, \phi W) = {} & g(\phi R(Z, W)X, \phi Y) + g(AZ, \phi X)g(AW, \phi Y) - g(AW, \phi X)g(AZ, \phi Y) \\ & - g(AZ, X)g(\phi AW, \phi Y) + g(AW, X)g(\phi AZ, \phi Y) - \eta(X)g(\phi R(Z, W)\xi, \phi Y) - \eta(R(\phi X, \phi Y)Z)\eta(W) \\ & + g(A\phi X, \phi Z)g(A\phi Y, \phi W) - g(A\phi Y, \phi Z)g(A\phi X, \phi W) - g(A\phi X, Z)g(\phi A\phi Y, \phi W) \\ & + g(A\phi Y, Z)g(\phi A\phi X, \phi W) - \eta(Z)g(\phi R(\phi X, \phi Y)\xi, \phi W). \end{aligned} \tag{23}$$

By the help of (5), (23) can be written as

$$\begin{aligned} g(R(\phi X, \phi Y)\phi Z, \phi W) = {} & g(R(Z, W)X, Y) - \eta(R(Z, W)X)\eta(Y) - g(AZ, X)g(AW, Y) \\ & + g(AW, X)g(AZ, Y) - \eta(X)g(R(Z, W)\xi, Y) - \eta(R(\phi X, \phi Y)Z)\eta(W) + g(A\phi X, \phi Z)g(A\phi Y, \phi W) \\ & - g(A\phi Y, \phi Z)g(A\phi X, \phi W) - \eta(Z)g(\phi R(\phi X, \phi Y)\xi, \phi W). \end{aligned} \tag{24}$$

Contracting with respect to Y and Z, we obtain

$$\begin{aligned} -\phi Q\phi X - QX = {} & -\phi l\phi X - lX + 4\alpha(1-n)A \\ & - 4\alpha^2(1-n)\phi^2 X - \eta(X)Q\xi \\ & + \sum_{i=1}^{2n+1} \eta(R(\phi X, \phi e_i)e_i)\xi. \end{aligned} \tag{25}$$

The rest of the proof follows acting ϕ on the last equation. □

Definition 1. Q and Q^* are the Ricci and $*$-Ricci operators of M^{2n+1} defined by

$$\begin{aligned} g(QX, Y) &= tr\{X \longrightarrow R(X, Y)Z\}, \\ g(Q^*X, Y) &= tr\{X \longrightarrow \phi R(X, Y)\phi Z\}, \end{aligned} \tag{26}$$

and denote by r and r^* the scalar and $*$-scalar curvatures of M^{2n+1}, where

$$\begin{aligned} r &= tr(Q), \\ r &= tr(Q^*), \end{aligned} \tag{27}$$

respectively [8].

Proposition 4. *For the Ricci and $*$-Ricci operators of almost α-cosymplectic manifold with Kaehler integral submanifolds M^{2n+1}, we have*

$$\begin{aligned} QY - Q^*Y = {} & \sum_{i=1}^{2n+1} g\left(\left(\nabla_{e_i}A\right)e_i, Y\right)\xi + \left(\nabla_\xi A\right)Y - A^2Y \\ & + 2\alpha(n-1)AY, \end{aligned} \tag{28}$$

where $\{e_1, e_2, \ldots, e_{2n}, e_{2n+1} = \xi\}$ is an orthonormal frame at any point of M^{2n+1}.

Proof. Taking trace of (19) with respect to X yields

$$\begin{aligned} g(QY, Z) - g(Q^*Y, Z) = {} & g(\phi A\phi AY, Z) + g\left(A^2Y, Z\right) + 2\alpha n g(AY, Z) \\ & - \eta(Z)\sum_{i=1}^{2n+1} g((\nabla e_i A)Y - (\nabla_Y A)e_i, e_i) + g\left(\left(\nabla_\xi A\right)Y - (\nabla_Y A)\xi, Z\right). \end{aligned} \tag{29}$$

By using the properties of A in (29), this ends the proof. □

Proposition 5. *For the scalar and $*$-scalar curvatures of almost α-cosymplectic manifold with Kaehler integral submanifolds M^{2n+1}, we obtain*

$$r^* - r = 2\|A\|^2 + 4\alpha^2 n(n-1). \tag{30}$$

Proof. (30) is a direct consequence of (29) by means of (10) and the following equation:

$$\sum_{i=1}^{2n+1} g((\nabla e_i A)\xi, e_i) = tr\left(A^2\right) = \|A\|^2. \tag{31}$$

□

Proposition 6. *For the Ricci operator of almost α-cosymplectic manifold with Kaehler integral submanifolds M^{2n+1}, we have*

$$Q\xi = -\sum_{i=1}^{2n+1} (\nabla e_i A)e_i, \qquad g(Q\xi, \xi) = -\|A\|^2. \tag{32}$$

Proof. By using the projection of (28) onto ξ and $g(Q^* Y, \xi) = 0$, the proof is obvious. □

Proposition 7. *An almost α-cosymplectic manifold with Kaehler integral submanifolds M^{2n+1} satisfies the following equation:*

$$\phi R(\xi, \phi Y)\xi - R(\xi, Y)\xi = -2A^2 Y - 4\alpha AY + 4\alpha^2 \phi^2 Y. \tag{33}$$

Proof. From (13), we get

$$\phi R(\xi, \phi Y)\xi - R(\xi, Y)\xi = -\phi\left(\nabla_\xi A\right)\phi Y + \phi\left(\nabla_{\phi Y} A\right)\xi + \left(\nabla_\xi A\right)Y - (\nabla_Y A)\xi. \tag{34}$$

The proof is clear from the right side of equation (34). □

Proposition 8. *Let M^{2n+1} be an almost α-Kenmotsu manifold $(n \geq 2)$. Then, it has Kaehler integral submanifolds of the distribution D if and only if it holds*

$$(\nabla_X \phi)Y = -g(\phi AX, Y)\xi + \eta(Y)\phi AX, \tag{35}$$

where $AX = \alpha\phi^2 X + \phi hX$.

Proof. By using similar technique in [8], the proof is clear. □

Proposition 9. *Let M^{2n+1} be an almost α-Kenmotsu manifold $(n \geq 2)$. If M^{2n+1} is conformally flat, then M^{2n+1} is a space of constant negative curvature $-\alpha^2$.*

Proof. By the help of [11, 15], we can complete the proof. □

4. Conformal Flatness Condition

This section is devoted to study conformally flat almost α-cosymplectic manifolds whose integral submanifolds are Kaehler.

Theorem 1. *If M^{2n+1} is a conformally flat almost α-cosymplectic manifold with Kaehler integral submanifolds $(n \geq 2)$, then ξ is the eigenvector of Ricci operator on M^{2n+1}*

Proof. Substituting ξ for X in (19), we have

$$R(\xi, Y)\phi Z - \phi R(\xi, Y)Z = \eta(Z)\phi lY - g(\phi lY, Z)\xi. \tag{36}$$

Let $\{e_i\}$ be an orthonormal basis of vector fields on M^{2n+1}. Taking $Y = Z = e_i$ in (36) for $i = 1, 2, \ldots, 2n+1$, then we get

$$\sum_{i=1}^{2n+1} R(\xi, e_i)\phi e_i = \phi Q\xi, \tag{37}$$

where $tr(\phi l) = 0$. Since M^{2n+1} is conformally flat and $n \geq 2$, C is identically zero. So, we can write

$$\begin{aligned} R(X, Y)Z = &\left(\frac{1}{(2n-1)}\right)(S(Y, Z)X - S(X, Z)Y \\ &+ g(Y, Z)QX - g(X, Z)QY) \\ &-\left(\frac{r}{2n(2n-1)}\right)(g(Y, Z)X - g(X, Z)Y). \end{aligned} \tag{38}$$

Putting $X = \xi$, $Y = e_i$, and $Z = \phi e_i$ in (38) and summing over i, we obtain

$$\sum_{i=1}^{2n+1} R(\xi, e_i)\phi e_i = \frac{1}{(2n-1)}[tr(Q\phi)\xi + (\phi Q)\xi]. \tag{39}$$

From (37) and (39), we have

$$\phi Q\xi = \frac{1}{(2n-1)}[tr(Q\phi)\xi + (\phi Q)\xi], \tag{40}$$

where $tr(Q\phi) = 0$ and $\phi Q\xi = 0$. Following from (40), we can complete the proof. □

Theorem 2. *If M^{2n+1} is a conformally flat almost α-cosymplectic manifold with Kaehler integral submanifolds $(n \geq 2)$, then we have*

$$tr(l) = \frac{r}{2} + \alpha^2 n(2n-1). \tag{41}$$

Proof. By the hypothesis, we have

$$\begin{aligned} g(Q^* Y, Z) = &\left(\frac{1}{2n-1}\right)[g(Q\phi Y, \phi Z)X + g(QY, Z) \\ &- g(Q\xi, Y)\eta(Z) - g(X, Z)QY] \\ &+\left(\frac{r}{2n(2n-1)}\right)[g(Y, Z) - \eta(Y)\eta(Z)]. \end{aligned} \tag{42}$$

It follows that

$$r^* = \left(\frac{1}{2n-1}\right)[r - 2g(Q\xi, \xi)]. \tag{43}$$

Then, making use of (32) and (43), we get

$$r^* = \left(\frac{1}{2n-1}\right)\left(r + 2\|A\|^2\right). \tag{44}$$

Since $n \geq 2$, from (30) and (44), we have

$$\begin{aligned} r^* &= -2\alpha^2 n, \\ \|A\|^2 &= -\frac{r}{2} - \alpha^2 n(2n-1). \end{aligned} \tag{45}$$

Taking into account (32) and (42), we deduce

$$g(Q^*Y,Z) - g(Q^*Z,Y) = \frac{1}{(2n-1)}\left[\sum_{i=1}^{2n+1} g((\nabla e_i A)e_i, Y)\eta(Z) - g((\nabla e_i A)e_i, Z)\eta(Y)\right]. \tag{46}$$

On the other hand, using (28) and (30) in (46), we have

$$g(Q^*Y,Z) - g(Q^*Z,Y) = \sum_{i=1}^{2n+1} g((\nabla e_i A)e_i, Y)\eta(Z) - g((\nabla e_i A)e_i, Z)\eta(Y). \tag{47}$$

From (46) and (47), it follows that

$$\sum_{i=1}^{2n+1} (\nabla e_i A)e_i = \sum_{i=1}^{2n+1} g((\nabla e_i A)e_i, \xi)\xi. \tag{48}$$

Taking into account (45) and (48), one obtains

$$\sum_{i=1}^{2n+1} (\nabla e_i A)e_i = \|A\|^2\xi = \left(\frac{-r}{2}\right)\xi - \alpha^2 n(2n-1)\xi. \tag{49}$$

Finally, making use of (32) and (49) and Theorem 1, the proof ends. □

Theorem 3. *A conformally flat almost α-cosymplectic manifold M^{2n+1} $(n \geq 2)$ with Kaehler integral submanifolds satisfies the following equation:*

$$\begin{aligned} Q &= \left(\frac{r}{4n}\right) + \left(\frac{2n-1}{4n}\right) r\eta\otimes\xi + \left(\frac{\alpha^2}{2}\right)(2n-1)^2\eta\otimes\xi \\ &\quad + \alpha(2n-1)A + \left(\frac{\alpha^2}{2}\right)(2n-1). \end{aligned} \tag{50}$$

Proof. *From (42), we have*

$$\begin{aligned} Q^*Y &= \left(\frac{1}{2n-1}\right)(-\phi Q\phi Y + QY) - \left(\frac{1}{2n(2n-1)}\right) rY \\ &\quad - \left(n - \frac{1}{2n(2n-1)}\right) r\eta(Y)\xi - \alpha^2 n\eta(Y)\xi. \end{aligned} \tag{51}$$

Putting $X = \xi$ in (13), we get

$$R(\xi,Y)\xi = -\left(\nabla_\xi A\right)Y + A^2 Y. \tag{52}$$

On the other hand, using (38) and (41), it follows that

$$\begin{aligned} R(\xi,Y)\xi &= -\left(\frac{1}{2n-1}\right)QY - \left(n - \frac{1}{2n(2n-1)}\right) rY \\ &\quad + \left(\frac{1}{2n}\right) r\eta(Y)\xi + 2\alpha^2 n\eta(Y)\xi - \alpha^2 nY. \end{aligned} \tag{53}$$

The last two equalities lead to

$$\begin{aligned} \left(\nabla_\xi A\right)Y &= A^2 Y + \left(\frac{1}{2n-1}\right)QY + \left(n - \frac{1}{2n(2n-1)}\right) rY \\ &\quad - \left(\frac{1}{2n}\right) r\eta(Y)\xi - 2\alpha^2 n\eta(Y)\xi + \alpha^2 nY. \end{aligned} \tag{54}$$

In view of (28), (49), (51), and (54), one finds

$$\begin{aligned} (2n-3)QY + \phi Q\phi Y &= \left(\frac{n-2}{2n}\right) rY + \left(\frac{(n-1)^2}{n}\right) r\eta(Y)\xi \\ &\quad + 2\alpha^2 n(n-2)(2n-1)\eta(Y)\xi \\ &\quad + \alpha^2 n(2n-1)Y + 2\alpha(n-1)(2n-1)AY. \end{aligned} \tag{55}$$

Substituting ϕY for Y in (55) and using (41), we have

$$\begin{aligned} (2n-3)\phi Q\phi Y + QY &= -\left(\frac{n-2}{2n}\right) rY + \left(\frac{n-1}{n}\right) r\eta(Y)\xi \\ &\quad + 2\alpha^2(2-n)(2n-1)\eta(Y)\xi \\ &\quad + \alpha^2(3n-4)(2n-1)Y \\ &\quad + 2\alpha(n-1)(2n-1)AY. \end{aligned} \tag{56}$$

Finally, this proof ends using (55) and (56).

Theorem 4. *Let M^{2n+1} be a conformally flat almost α-cosymplectic manifold with Kaehler integral submanifolds $(n \geq 2)$. Then, the following relation is held:*

$$Q\phi - \phi Q = l\phi - \phi l - 4\alpha(n-1)\phi A + 4\alpha^2(n-1)\phi X. \quad (57)$$

Proof. The proof follows from (20) and Theorem 1 in [8]. □

Theorem 5. *Let M^{2n+1} be an almost α-cosymplectic manifold with Kaehler integral submanifolds $(n \geq 2)$. If M^{2n+1} is conformally flat, then M^{2n+1} is either locally flat and cosymplectic or M^{2n+1} is an α-Kenmotsu manifold with constant negative curvature $-\alpha^2$.*

Proof. Let M^{2n+1} be an almost α-cosymplectic manifold with Kaehler integral submanifolds. Assume additionally that M^{2n+1} is conformally flat and $n \geq 2$. At certain places of the main idea of the proof, we are inspired by the paper of Dacko and Olszak [8]. Also, the almost cosymplectic case $(\alpha = 0)$ is clear by means of Theorem 1 in [8].

Now, we shall prove the assertion of our theorem in case of $\alpha \neq 0$. Since $n \geq 3$, (50) holds. Thus, the auxiliary tensor L has the following shape:

$$LX = QX - \left(\frac{1}{4n}\right)rX. \quad (58)$$

From (50) and (58), L can be written as

$$\begin{aligned} LX = {} & \left(2n - \frac{1}{4n}\right)r\eta(X)\xi + \left(\frac{\alpha^2}{2}\right)(2n-1)^2\eta(X)\xi \\ & + \alpha(2n-1)AX + \left(\frac{\alpha^2}{2}\right)(2n-1)X. \end{aligned} \quad (59)$$

In view of (7), (10), and (59) with $X = \xi$, we obtain

$$\begin{aligned} & \left(\frac{2n-1}{4n}\right)(\xi(r)\eta(Y)\xi - Y(r)\xi) + \alpha(2n-1) \\ & \cdot \left(\left(\nabla_\xi A\right)Y\right) + \left(\frac{2n-1}{4n}\right)r + \left(\frac{\alpha^2}{2}\right) \\ & \cdot (2n-1)^2 - (\alpha(2n-1)A)AY = 0, \end{aligned} \quad (60)$$

where $\xi(r)\eta(Y)\xi = Y(r)\xi$. So, the last equation reduces to

$$\begin{aligned} 0 = {} & \alpha(2n-1)\left(\left(\nabla_\xi A\right)Y\right) + \left(\frac{2n-1}{4n}\right)r \\ & + \left(\frac{\alpha^2}{2}\right)(2n-1)^2 - \alpha(2n-1)A)AY. \end{aligned} \quad (61)$$

Replacing X and Y by ξ in (38) and using (50), we find that

$$lX = -\left(\frac{tr(l)}{2n}\right)\phi^2 X + \alpha\phi hX. \quad (62)$$

Then, using (52) and (62), we have

$$\left(\nabla_\xi A\right)Y = -\left(\frac{tr(l)}{2n}\right)\phi^2 X + \alpha\phi hX + A^2 Y. \quad (63)$$

Following from (61) and (63), we get

$$\left(\left(\frac{tr(l)}{2n}\right) + \alpha^2\right)\phi hX = 0. \quad (64)$$

Also, using (18) and (64), one obtains

$$-\left(\frac{tr\left(h^2\right)}{2n}\right)\phi hX = 0, \quad (65)$$

which reduces to $h = 0$. Finally, we can use Proposition 9 to complete the proof. □

Remark 2. It is noted that Theorem 5 generalizes the result of Dacko and Olszak [8].

Example. Considering $M^3 = \{(x, y, z) \in R^3\}$ such that (x, y, z) are the standard coordinates in R, the vector fields are

$$\begin{aligned} E_1 &= \lambda_2 e^{-\alpha z}\left(\frac{\partial}{\partial x}\right) + \lambda 1 e^{-\alpha z}\left(\frac{\partial}{\partial y}\right), \\ E_2 &= -\lambda_1 e^{-\alpha z}\left(\frac{\partial}{\partial x}\right) + \lambda_2 e^{-\alpha z}\left(\frac{\partial}{\partial y}\right), \\ E_3 &= \left(\frac{\partial}{\partial z}\right), \end{aligned} \quad (66)$$

where g_1 and g_2 are given by $g_1(z) = \lambda_2 e^{-\alpha z}$ and $g_2(z) = \lambda_1 e^{-\alpha z}$ with $\lambda_1^2 + \lambda_2^2 \neq 0, \alpha \neq 0$ for constants λ_1, λ_2, and α. Also, the set of $\{E_1, E_2, E_3\}$ is linearly independent at each point of M^3. Let g be the Riemannian tensor product given by

$$g = \left(g_1^2 + g_2^2\right)^{-1}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y) + \mathrm{d}z \otimes \mathrm{d}z. \quad (67)$$

Let η be the 1-form defined by $\eta(X) = g(X, E_3)$ and ϕ be the (1, 1) tensor field defined by

$$\begin{aligned} \phi(e_1) &= e_2, \\ \phi(e_2) &= -e_1, \\ \phi(e_3) &= 0. \end{aligned} \quad (68)$$

Furthermore, we can calculate

$$\begin{aligned} & [E_1, E_3] = \alpha e_1, \\ & [E_2, E_3] = \alpha e_2, \\ & [E_1, E_2] = 0, \\ & \nabla_{E_1}E_1 = -\alpha E_3, \quad \nabla_{E_2}E_1 = -E_3, \quad \nabla_{E_3}E_1 = 0, \\ & \nabla_{E_1}E_2 = -E_3, \quad \nabla_{E_2}E_2 = -\alpha E_3, \quad \nabla_{E_3}E_2 = 0, \\ & \nabla_{E_1}E_3 = \alpha E_1, \quad \nabla_{E_2}E_3 = \alpha E_2, \quad \nabla_{E_3}E_3 = 0. \end{aligned} \quad (69)$$

Thus, we can check the only nonzero components of Φ. Namely, we get

$$\Phi\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) = -\left(g_1^2 + g_2^2\right)^{-1} = -\left(\lambda_1^2 + \lambda_2^2\right)^{-1} e^{2\alpha z}. \tag{70}$$

Since $\eta = dz$, it implies that $d\Phi = 2\alpha(\eta\wedge\Phi)$ on M^3. Moreover, Nijenhuis torsion tensor of ϕ vanishes.

For the curvature operator R, the nonzero components are as follows:

$$\begin{aligned} R(E_1,E_2)E_1 &= \alpha(\alpha E_2 - E_1), R(E_1,E_3)E_2 \\ &= \alpha E_3, R(E_1,E_2)E_2 = \alpha(E_2 - \alpha E_1), \\ R(E_1,E_3)E_1 &= \alpha^2 E_3, R(E_1,E_3)E_3 = -\alpha^2 E_1, \\ R(E_2,E_3)E_1 &= \alpha E_3, \\ R(E_2,E_3)E_3 &= -\alpha^2 E_2, R(E_2,E_3)E_2 = \alpha^2 E_3, \\ R(E_3,E_2)E_1 &= -\alpha E_3. \end{aligned} \tag{71}$$

Clearly, g is not locally flat.

For the Ricci tensor S, assume $S_{ij} = S(E_i, E_j)$; then, we obtain $S_{ii} = -2\alpha^2$ and $r = -6\alpha^2$. Consequently, the Ricci operator Q satisfies the equations

$$\begin{aligned} QE_1 &= bE_1 + \alpha E_2, \\ QE_2 &= \alpha E_1 + bE_2, \\ QE_3 &= bE_3, \end{aligned} \tag{72}$$

with $b = -2\alpha^2$. For the auxiliary operator L of M^3, we have

$$\begin{aligned} LE_1 &= aE_1 + \alpha E_2, \\ LE_2 &= \alpha E_1 + aE_2, \\ LE_3 &= aE_3, \end{aligned} \tag{73}$$

with $a = -(2\alpha^2 + (r/4)), r = 4(b - a)$ where L is defined by $L = (Q - (r/4)I)$.

To obtain the conformal flatness of g, it remains to verify the Codazzi condition for L. Namely,

$$0 = -L\left[E_i, E_j\right] - \nabla_{E_j} LE_i + \nabla_{E_i} LE_j, \tag{74}$$

for $1 \le i < j \le 3$. It is seen that the Codazzi condition does not hold. Thus, the manifold M^3 is not conformally flat and has constant sectional curvature $K = -\alpha^2$.

5. Conclusion and Discussion

A Riemannian manifold is conformally flat if each point has a neighborhood that can be mapped to flat space by a conformal transformation [2, 3]. There exist conformally flat contact metric manifolds which are not constant curvature [9]. However, this is an open problem in dimensions ≥ 5. In recent years, some authors have studied this area for almost contact metric manifolds [4, 8, 9].

This paper deals with the conformally flat almost α-cosymplectic manifolds given by Kaehler integral submanifolds. Our main target is to make some generalizations and classifications on such manifolds, and certain results are proved in the last two sections. [20].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this study.

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