Research Article

Enveloping Lie Superalgebras and Killing-Ricci Forms of Bol Superalgebras

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In this paper, enveloping Lie superalgebras of Bol superalgebras is introduced. The notion of Killing-Ricci forms and invariant forms of these superalgebras is investigated as generalization of the one of Bol algebras.

1. Introduction

Bol algebras were introduced [1] in the context of a study of smooth Bol loops. The algebras play the same role with respect to Bol loops as Lie algebras do with respect to Lie groups or Malcev algebras to Moufang loops [2].

A vector space V equipped with a trilinear operation $[\cdot, \cdot, \cdot]$ is called a Lie triple system if $[a, a, b] = 0$, $[a, b, c] + [b, c, a] + [c, a, b] = 0$, $[x, y, [a, b, c]] = [[x, y, a], b, c] + [a, [x, y, b], c] + [a, b, [x, y, c]]$ for all $x, y, a, b, c \in V$. A (left) Bol algebra $(V, \cdot, [\cdot, \cdot, \cdot])$ is a Lie triple system $(V, [\cdot, \cdot, \cdot])$ with an additional bilinear skew-symmetric operation $\cdot$ satisfying $[a, b, c \cdot d] = [a, b, c] \cdot d + c \cdot [a, b, d] + [c, d, a \cdot b] + (a \cdot b) \cdot (c \cdot d)$. A related notion is that of a Lie triple algebra, introduced under the name generalized Lie triple systems, by Yamaguti [3], and called later as Lie Yamaguti algebras [4].

From the standard enveloping Lie algebra of a given Bol algebra, the notion of Killing-Ricci form and invariant form for a Bol algebra are introduced and studied in [5].

A $\mathbb{Z}_2$ -graded generalization of Lie algebras, called Lie superalgebras, is considered in [6, 7], while a $\mathbb{Z}_2$ -graded generalization of Lie Yamaguti algebras called Lie Yamaguti superalgebras was first considered in [8] and generalizes Lie supertriple systems [9]. The reader may refer to [10] for applications of Lie supertriple systems in physics. As Lie Yamaguti superalgebras, Bol superalgebras first introduced in [11] may also be viewed as a generalization of Lie supertriple systems. For relations between

Malcev superalgebras and Bol superalgebras, one may refer to [12].

As a part of the general theory of superalgebras, the notion of Killing form of Lie algebras is extended to one of Lie triple systems [13], Lie superalgebras [7], Lie supertriple systems [14, 15], and next Lie Yamaguti superalgebras [16].

In this paper, we define enveloping Lie superalgebras and the Killing-Ricci form of Bol superalgebras and study this Killing-Ricci form, which could be seen as a generalization of the one of Bol algebras [5] and the Killing form of Lie supertriple systems [14, 15]. Unlike Bol algebras, in Bol superalgebras, there is an odd subspace, which is not a Bol subalgebra. This complicates the work more than Bol algebras case.

The rest of this paper is organized as follows. In section 2, we first recall some basics on Lie and Malcev superalgebras as well as Lie supertriple systems and Bol superalgebras. In Section 3, we define the notion of pseudo-superderivations (Definition 8), study their properties (Lemma 1), and introduce the notion of enveloping Lie superalgebras of Bol superalgebras (Definition 12). In Section 4, the Killing-Ricci form of Bol superalgebras is defined (Definition 13) and some of its properties are investigated (Theorem 1, Proposition 3, and Lemma 3). In the next section, the invariant form of Bol superalgebras is defined (Definition 14) and some results are obtained (Lemma 4 and Theorem 2).

Throughout this paper, all vector superspaces and superalgebras are finite dimensional over a fixed ground field $\mathbb{K}$ of characteristic 0.

2. Some Basics on Superalgebras

We recall here some useful definitions and examples of Lie supertriple systems as well as the one of Bol superalgebras. These examples are obtained from the relation between Malcev superalgebras and Bol superalgebras, which could be found in [12].

Now, let M be a linear superspace over $\mathbb{K}$, that is, a $\mathbb{Z}_2$ -graded linear space with a direct sum $M = M_0 \oplus M_1$. The elements of M_j , $j \in \mathbb{Z}_2$, are said to be homogeneous of parity j . The parity of a homogeneous element x is denoted by $\bar{x}$. For all $i, j \in \mathbb{Z}_2$, $i + j$ will always means that this sum is calculated modulo 2. If $N = N_0 \oplus N_1$ is another superspace, a linear map $f: M \rightarrow N$ is said to be of degree $r \in \mathbb{Z}_2$ if $f(M_i) \subseteq N_{i+r}$ for all $i \in \mathbb{Z}_2$. If f is of degree $r = 0$, that is,

$f(M_i) \subseteq N_i$ for all $i \in \mathbb{Z}_2$, then f is said to be an even linear map. An algebra $(A, [,])$ is called a superalgebra if the underlying vector space is $\mathbb{Z}_2$ -graded, i.e., $A = A_0 \oplus A_1$ and if furthermore $[A_i, A_j] \subseteq A_{i+j}$. For any binary operation, we will sometimes use juxtaposition in order to reduce the number of braces; i.e., for “ $\cdot$ ”, $xy \cdot z$ means $(x \cdot y) \cdot z$.

Definition 1. A Lie superalgebra is the superalgebra $(A = A_0 \oplus A_1, [,])$ satisfying the superskew-symmetry and the super-Jacobi identities that is

$$[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x], \quad (1)$$

$$[[x, y], z] + (-1)^{\bar{x}(\bar{y}+\bar{z})}[[y, z], x] + (-1)^{\bar{z}(\bar{x}+\bar{y})}[[z, x], y] = 0, \quad (2)$$

for all $x, y, z \in \mathcal{H}(A)$. In terms of the super-Jacobian,

$$SJ(x, y, z) = [[x, y], z] + (-1)^{\bar{x}(\bar{y}+\bar{z})}[[y, z], x] + (-1)^{\bar{z}(\bar{x}+\bar{y})}[[z, x], y]. \quad (3)$$

The super-Jacobi identity is written as $SJ(x, y, z) = 0$ for all $x, y, z \in \mathcal{H}(A)$.

Another class of superalgebras that is of interest in this paper is the one of Malcev superalgebras.

Definition 2 (see [17, 18]). A superalgebra $(M = M_0 \oplus M_1, [,])$ is called a Malcev superalgebra if it satisfies the following superidentities:

$$\begin{aligned} [x, y] &= -(-1)^{\bar{x}\bar{y}}[y, x] \text{ (superskew-symmetry),} \\ [[x, y], z], t &- [x, [[y, z], t]] - (-1)^{\bar{y}(\bar{z}+\bar{t})}[[x, [z, t]], y] - (-1)^{\bar{t}(\bar{y}+\bar{z})}[[x, t], y], z \\ &= (-1)^{\bar{y}\bar{z}}[[x, z], [y, t]] \text{ (super-Malcev identity),} \end{aligned} \quad (4)$$

for $x, y, z, t \in \mathcal{H}(M)$.

Definition 3

(1) A supertriple system is a couple $(S = S_0 \oplus S_1, [, ,])$ consisting of a $\mathbb{Z}_2$ -graded $\mathbb{K}$ -vector space $S = S_0 \oplus S_1$ and a $\mathbb{K}$ -trilinear map $[, ,]$, satisfying $[S_i, S_j, S_k] \subseteq S_{i+j+k}$ for all i, j, k in $\mathbb{Z}_2$ such that for all $x, y, z \in \mathcal{H}(S)$, the following equations hold:

$$\begin{aligned} [x, y, z] &= -(-1)^{\bar{x}\bar{y}}[y, x, z] \text{ (left superskew-symmetry),} \\ [x, y, z] + (-1)^{\bar{x}(\bar{y}+\bar{z})}[y, z, x] + (-1)^{\bar{z}(\bar{x}+\bar{y})}[z, x, y] &= 0, \text{ (superternary-Jacobi identity).} \end{aligned} \quad (5)$$

(2) A Lie supertriple system [9, 19] is a supertriple system $(S = S_0 \oplus S_1, [, ,])$ such that the superternary Nambu identity

$$[x, y, [u, v, w]] = [[x, y, u], v, w] + (-1)^{\bar{u}(\bar{x}+\bar{y})}[u, [x, y, v], w] + (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})}[u, v, [x, y, w]], \quad (6)$$

holds for all $x, y, u, v, w \in \mathcal{H}(S)$.

Example 1 (see [9, 12, 19]). Let $(L = L_0 \oplus L_1, [\cdot, \cdot])$ be a Lie superalgebra, then $(L, [\cdot, \cdot])$ is a Lie supertriple system where for all $x, y, z \in \mathcal{H}(L)$, $[x, y, z] := [[x, y], z]$.

Definition 4 (see [11, 12]). A Bol superalgebra is a triple $(B = B_0 \oplus B_1, \cdot, [\cdot, \cdot])$ consisting of a superspace $B = B_0 \oplus B_1$, a linear map $\cdot: B^{\otimes 2} \rightarrow B$ satisfying $B_i \cdot B_j \subseteq B_{i+j}$, and a trilinear map $[\cdot, \cdot]: B^{\otimes 3} \rightarrow B$ satisfying $[B_i, B_j, B_k] \subseteq B_{i+j+k}$, $i, j, k \in \mathbb{Z}_2$, such that the following holds:

$$(BS_1) \quad x \cdot y = -(-1)^{\overline{x}\overline{y}} y \cdot x.$$

$$(BS_2) \quad [x, y, z] = -(-1)^{\overline{x}\overline{y}} [y, x, z].$$

$$[x, y, u \cdot v] = (-1)^{\overline{u}(\overline{x}+\overline{y})} u \cdot [x, y, v] + [x, y, u] \cdot v + (-1)^{(\overline{x}+\overline{y})(\overline{u}+\overline{v})} [u, v, x \cdot y] + (x \cdot y) \cdot (u \cdot v), \quad (7)$$

holds for all $x, y, u, v \in \mathcal{H}(B)$.

$$(BS_3) [x, y, z] + (-1)^{\overline{x}(\overline{y}+\overline{z})} [y, z, x] + (-1)^{\overline{z}(\overline{x}+\overline{y})} [z, x, y] = 0.$$

$$(BS_4) [x, y, [u, v, w]] = [[x, y, u], v, w] + (-1)^{\overline{u}(\overline{x}+\overline{y})} [u, [x, y, v], w] + (-1)^{(\overline{x}+\overline{y})(\overline{u}+\overline{v})} [u, v, [x, y, w]].$$

$$(BS_5) [x, y, u \cdot v] = (-1)^{\overline{u}(\overline{x}+\overline{y})} u \cdot [x, y, v] + [x, y, u] \cdot v + (-1)^{(\overline{x}+\overline{y})(\overline{u}+\overline{v})} [u, v, x \cdot y] + (x \cdot y) \cdot (u \cdot v) \text{ holds for all } x, y, u, v, w \in \mathcal{H}(S).$$

Remark 1. A Bol superalgebra is a Lie triple supersystem $(B = B_0 \oplus B_1, [\cdot, \cdot])$ with a superskew-symmetric binary map. such that $B_i \cdot B_j \subseteq B_{i+j}$

$$[x, y, z] := \frac{1}{3} (2(x \cdot y) \cdot z - (-1)^{\overline{x}(\overline{y}+\overline{z})} (y \cdot z) \cdot x - (-1)^{\overline{z}(\overline{x}+\overline{y})} (z \cdot x) \cdot y), \quad (8)$$

for all $x, y, z \in \mathcal{H}(M)$ becomes a Bol superalgebra $(M = M_0 \oplus M_1, \cdot, [\cdot, \cdot])$. This allows us to get the following example of Bol superalgebra from an example of a Malcev superalgebra [20].

Example 2. Let $(L^2(2, 2), \cdot)$ be a non-Lie Malcev superalgebra [20] defined with respect to a basis (e_1, e_2, e_3, e_4) , where $L^2(2, 2)_0 = \text{span}(e_1, e_2)$ and $L^2(2, 2)_1 = \text{span}(e_3, e_4)$ with the nonzero products given by $e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_3, e_1 \cdot e_4 = -e_4$, and $e_2 \cdot e_3 = -e_4$. Then, $(L^2(3, 1), \cdot, [\cdot, \cdot])$ is a four-dimensional Bol superalgebra with $L^2(2, 2)_0 = \text{span}(e_1, e_2)$ and $L^2(2, 2)_1 = \text{span}(e_3, e_4)$, where the nonzero products are given by $e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_3, e_1 \cdot e_4 = -e_4, e_2 \cdot e_3 = -e_4$ and $[e_1, e_2, e_1] = -e_2, [e_1, e_3, e_1] = -e_3, [e_1, e_4, e_1] = -e_4$.

From [12], we also get the following example.

Example 3 (see [12]). There is a four-dimensional Bol superalgebra $(L^2(3, 1), \cdot, [\cdot, \cdot])$ with $L^2(3, 1)_0 = \text{span}(e_1, e_2, e_3)$ and $L^2(3, 1)_1 = \text{span}(e_4)$, where the nonzero products are given by $e_1 \cdot e_3 = e_1, e_2 \cdot e_3 = e_1 + e_2, e_3 \cdot e_4 = e_4, e_4 \cdot e_4 = e_1$ and $[e_1, e_3, e_3] = e_1, [e_2, e_3, e_3] = 2e_1 + e_2, [e_3, e_4, e_3] = -e_4$.

Definition 5. Let $B = B_0 \oplus B_1$ be a Bol superalgebra. A graded subspace $H = H_0 \oplus H_1$ of B is a subsuperalgebra of B if $H_i \cdot H_j \subseteq H_{i+j}$ and $[H_i, H_j, H_k] \subseteq H_{i+j+k}$, for all $i, j, k \in \mathbb{Z}_2$.

In [12], we proved that any Malcev superalgebra $(M = M_0 \oplus M_1)$ equipped with a trilinear operation $[\cdot, \cdot]$ where

Definition 6. A subsuperalgebra $H = H_0 \oplus H_1$ of a Bol superalgebra B is an invariant subsuperalgebra (resp., an ideal) of B if $[B, B, H] \subseteq H$ (resp., $BH \subseteq H$ and $[B, B, H] \subseteq H$).

If H is an ideal of B , it is an invariant subsuperalgebra of B . Obviously, the center of a Bol superalgebra B defined by $Z(B) = \{x \in B, xy = 0 \text{ and } [x, y, z] = 0, \forall y, z \in B\}$ is an ideal of B .

Definition 7. Let $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$ be two Bol superalgebras. An even linear map $f: A \rightarrow B$ is called a morphism of Bol superalgebras if $f(xy) = f(x)f(y)$ and $f([x, y, z]) = [f(x), f(y), f(z)]$ for all $x, y, z \in B_0 \cup B_1$.

Recall [6] that if $V = V_0 \oplus V_1$ is a vector superspace, then the set of the linear mappings of V into itself which are homogeneous of degree r is denoted by $\text{End}_r(V) = \{f \in \text{End}(V), f(V_i) \subseteq V_{r+i}\}$, and we obtain an associative superalgebra $\text{End}(V) = \text{End}_0(V) \oplus \text{End}_1(V)$. The bracket $[f, g] = fg - (-1)^{\overline{f}\overline{g}} gf$ makes $\text{End}(V)$ into a Lie superalgebra which we denote by $l(V)$ or $l(m, n)$ where $m = \dim V_0$ and $n = \dim V_1$. Let $e_1, \dots, e_m, e_{m+1}, \dots, e_{m+n}$ be a basis of V . In this basis, the matrix of $f \in l(m, n)$ is expressed as $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, α being an $(m \times m)$ -, δ an $(n \times n)$ -, β an $(m \times n)$ -, and γ an $(n \times m)$ - matrix. The matrices of even elements have the form $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ and those of odd ones $\begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$. For $f = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, the supertrace of M is defined by

$\text{str}(M) = \text{tr}\alpha - \text{tr}\delta$ and does not depend on the choice of a homogeneous basis. We have $\text{str}([f, g]) = 0$; that is, $\text{str}(fg) = (-1)^{\overline{f}\overline{g}}\text{str}(gf)$ and $\text{str}(hfh^{-1}) = \text{str}(f)$.

3. Enveloping Lie Superalgebras of a Bol Superalgebra

As derivations for algebras, superderivations of different superalgebras are an important subject of study in superalgebras and diverse area. They appear in many fields of mathematics and physics. In particular, they allow the

construction of new superalgebras structures. In the case of Bol superalgebras, instead of superderivations, we have the notion of pseudo-superderivations. They generalize pseudoderivations for Bol algebras [2] and superderivations for Lie supertriple systems [19] and allow the construction of enveloping Lie superalgebras of Bol superalgebras.

Definition 8. Let $B = B_0 \oplus B_1$ be a Bol superalgebra. A linear map $P \in \text{End}_r(B)$ is called a pseudo-superderivation of companion $a \in B_r$ ($r \in \mathbb{Z}_2$) if, for any $x, y, z \in B_0 \cup B_1$:

$$P([x, y, z]) = [P(x), y, z] + (-1)^{r\overline{x}}[x, P(y), z] + (-1)^{r(\overline{x}+\overline{y})}[x, y, P(z)], \quad (9)$$

$$P(xy) = (-1)^{r\overline{x}}xP(y) + P(x)y + (-1)^{r(\overline{x}+\overline{y})}[x, y, a] + a \cdot xy. \quad (10)$$

Remark 2. Note that a pseudo-superderivation P can have more than one companion. Let denote the set of all companions of a pseudo-superderivation P by $\text{Com}(P)$.

Let $pS_r(B)$ be the set of all pseudo-superderivations of degree r and $pS(B) = pS_0(B) \oplus pS_1(B)$. Furthermore, let $PS_r(B) = \{(P, a), P \in pS_r(B), a \in \text{Com}(P)\}$ and $PS(B) = P_{S_0(B)} \oplus PS_1(B)$.

For any pseudo-superderivations $P, Q \in pS_0(B) \cup pS_1(B)$ of a Bol superalgebra B , consider the supercommutator given by $[P, Q] := PQ - (-1)^{\overline{P}\overline{Q}}QP$. Then, we have the following.

Lemma 1. Let $(P, a) \in PS_r(B)$, $(Q, b) \in PS_r(B)$, $(R, c) \in PS_s(B)$, and $\lambda \in \mathbb{K}$. Then, the following holds:

- (1) $(P, a) + \lambda(Q, b) := (P + \lambda Q, a + \lambda b) \in PS_r(B)$.
- (2) $[(P, a), (R, c)] := ([P, R], P(c) - (-1)^{rs}R(a) - ac) \in PS_{r+s}(B)$.

Proof. Let $(P, a) \in PS_r(B)$, $(Q, b) \in PS_r(B)$, $(R, c) \in PS_s(B)$, and $\lambda \in \mathbb{K}$. The first statement is a straightforward computation. For the second statement, pick $x, y, z, u, v \in B_0 \cup B_1$. Then, using repeatedly 9 for P and R , we prove 9 for $[P, R]$ as follows:

$$\begin{aligned} [P, R]([x, y, w]) &= PR([x, y, w]) - (-1)^{rs}RP([x, y, w]) \\ &= P([R(x), y, z] + (-1)^{s\overline{x}}[x, R(y), z] + (-1)^{s(\overline{x}+\overline{y})}[x, y, R(z)]) \\ &\quad - (-1)^{rs}R([P(x), y, z] + (-1)^{r\overline{x}}[x, P(y), z] + (-1)^{r(\overline{x}+\overline{y})}[x, y, P(z)]) \\ &= [PR(x), y, z] + (-1)^{r(\overline{x}+s)}[R(x), P(y), z] + (-1)^{r(s+\overline{x}+\overline{y})}[R(x), y, P(z)] \\ &\quad + (-1)^{s\overline{x}}[P(x), R(y), z] + (-1)^{(r+s)\overline{x}}[x, PR(y), z] \\ &\quad + (-1)^{s\overline{x}+r(\overline{x}+\overline{y}+s)}[x, R(y), P(z)] + (-1)^{s(\overline{x}+\overline{y})}[P(x), y, R(z)] \\ &\quad + (-1)^{s(\overline{x}+\overline{y})+r\overline{x}}[x, P(y), R(z)] + (-1)^{(r+s)(\overline{x}+\overline{y})}[x, y, PR(z)] \\ &\quad - (-1)^{rs}\{[RP(x), y, z] + (-1)^{s(\overline{x}+r)}[P(x), R(y), z] \\ &\quad + (-1)^{s(r+\overline{x}+\overline{y})}[P(x), y, R(z)] + (-1)^{r\overline{x}}[R(x), P(y), z] + (-1)^{(r+s)\overline{x}}[x, RP(y), z] \\ &\quad + (-1)^{r\overline{x}+s(\overline{x}+\overline{y}+r)}[x, P(y), R(z)] + (-1)^{r(\overline{x}+\overline{y})}[R(x), y, P(z)] \\ &\quad + (-1)^{r(\overline{x}+\overline{y})+s\overline{x}}[x, R(y), P(z)] + (-1)^{(r+s)(\overline{x}+\overline{y})}[x, y, RP(z)]\} \\ &= [[P, R](x), y, z] + (-1)^{(r+s)\overline{x}}[x, [P, R](y), z] + (-1)^{(r+s)(\overline{x}+\overline{y})}[x, y, [P, R](z)]. \end{aligned} \quad (11)$$

Thus, we get (9) for $[P, R]$. To prove, (10) for $[P, R]$, note that if we use repeatedly (10) for P , we get

$$\begin{aligned}
P(c \cdot xy) &= (-1)^{rs} c \cdot P(xy) + P(c) \cdot xy + (-1)^{r(\bar{x}+\bar{y}+s)} [c, xy, a] + a \cdot (c \cdot xy) \\
&= +(-1)^{r(\bar{x}+s)} c \cdot xP(y) + (-1)^{rs} c \cdot P(x)y + (-1)^{r(\bar{x}+\bar{y}+s)} c \cdot [x, y, a] \\
&\quad + (-1)^{rs} c \cdot (a \cdot xy) + P(c) \cdot xy + (-1)^{r(\bar{x}+\bar{y}+s)} [c, xy, a] + a \cdot (c \cdot xy),
\end{aligned} \tag{12}$$

and therefore, by (9) and (12) for P and R , we get

$$\begin{aligned}
[P, R](xy) &= PR(xy) - (-1)^{rs} RP(xy) \\
&= P\left((-1)^{s\bar{x}} xR(y) + R(x)y + (-1)^{s(\bar{x}+\bar{y})} [x, y, c] + c \cdot xy\right) \\
&\quad - (-1)^{rs} \left\{ R\left((-1)^{r\bar{x}} xP(y) + P(x)y + (-1)^{r(\bar{x}+\bar{y})} [x, y, a] + a \cdot xy\right) \right\} \\
&= (-1)^{(r+s)\bar{x}} xPR(y) + (-1)^{s\bar{x}} P(x)R(y) + (-1)^{r(\bar{x}+\bar{y}+s)+s\bar{x}} [x, R(y), a] \\
&\quad + (-1)^{s\bar{x}} a \cdot xR(y) + (-1)^{r(\bar{x}+s)} R(x)Py + PR(x)P(y) + (-1)^{r(\bar{x}+\bar{y}+s)} [R(x), y, a] \\
&\quad + a \cdot R(x)y + (-1)^{s(\bar{x}+\bar{y})} [P(x), y, c] + (-1)^{s(\bar{x}+\bar{y})+r\bar{x}} [x, P(y), c] \\
&\quad + (-1)^{(r+s)(\bar{x}+\bar{y})} [x, y, P(c)] + (-1)^{r(\bar{x}+s)} c \cdot xP(y) + (-1)^{rs} c \cdot P(x)y \\
&\quad + (-1)^{r(\bar{x}+\bar{y}+s)} c \cdot [x, y, a] + (-1)^{rs} c \cdot (a \cdot xy) + P(c) \cdot xy + (-1)^{r(\bar{x}+\bar{y}+s)} [c, xy, a] \\
&\quad + a \cdot (c \cdot xy) - (-1)^{rs} \left\{ (-1)^{(r+s)\bar{x}} xRP(y) + (-1)^{r\bar{x}} R(x)P(y) \right. \\
&\quad + (-1)^{s(\bar{x}+\bar{y}+r)+r\bar{x}} [x, P(y), c] + (-1)^{r\bar{x}} c \cdot xP(y) + (-1)^{s(\bar{x}+r)} P(x)Ry + RP(x)(y) \\
&\quad + (-1)^{s(\bar{x}+\bar{y}+r)} [P(x), y, c] + c \cdot P(x)y + (-1)^{r(\bar{x}+\bar{y})} [R(x), y, a] \\
&\quad + (-1)^{r(\bar{x}+\bar{y})+s\bar{x}} [x, R(y), a] + (-1)^{(r+s)(\bar{x}+\bar{y})} [x, y, R(a)] + (-1)^{s(\bar{x}+r)} a \cdot xR(y) \\
&\quad + (-1)^{rs} a \cdot R(x)y + (-1)^{s(\bar{x}+\bar{y}+r)} a \cdot [x, y, c] + (-1)^{rs} a \cdot (c \cdot xy) + R(a) \cdot xy \\
&\quad \left. + (-1)^{s(\bar{x}+\bar{y}+r)} [a, xy, c] + c \cdot (a \cdot xy) \right\} \\
&= (-1)^{(r+s)\bar{x}} x \cdot [P, R](y) + [P, R](x) \cdot y + (-1)^{(r+s)(\bar{x}+\bar{y})} [x, y, P(c) - (-1)^{rs} R(a)] \\
&\quad + (P(c) - (-1)^{rs} R(a)) \cdot xy + (-1)^{r(\bar{x}+\bar{y}+s)} [c, xy, a] - (-1)^{s(\bar{x}+\bar{y})} [a, xy, c] \\
&\quad + (-1)^{r(\bar{x}+\bar{y}+s)} c[x, y, a] - (-1)^{s(\bar{x}+\bar{y})} a[x, y, c] \\
&= (-1)^{(r+s)\bar{x}} x \cdot [P, R](y) + [P, R](x) \cdot y + (-1)^{(r+s)(\bar{x}+\bar{y})} [x, y, P(c) - (-1)^{rs} R(a)] \\
&\quad + (P(c) - (-1)^{rs} R(a)) \cdot xy - (-1)^{(r+s)(\bar{x}+\bar{y})+rs} [xy, c, a] - (-1)^{s(\bar{x}+\bar{y})} [a, xy, c] \\
&\quad - (-1)^{(r+s)(\bar{x}+\bar{y})} [x, y, a]c - (-1)^{s(\bar{x}+\bar{y})} a[x, y, c] \text{ (by } (BS_1) \text{ and } (BS_2)) \\
&= (-1)^{(r+s)\bar{x}} x \cdot [P, R](y) + [P, R](x) \cdot y + (-1)^{(r+s)(\bar{x}+\bar{y})} [x, y, P(c) - (-1)^{rs} R(a) - ac] \\
&\quad + (P(c) - (-1)^{rs} R(a) - ac) \cdot xy \text{ (by } (BS_1) - (BS_3) \text{ and } (BS_5)),
\end{aligned} \tag{13}$$

and then, we get (10). This ends the proof.

Now, we can prove the following result. $\square$

Proposition 1. Let $B = B_0 \oplus B_1$ be a Bol superalgebra. Then, $(PS(B), [,])$ is a Lie superalgebra.

Proof. The proof follows by Lemma 1 and the fact that $pS(B)$ is a Lie superalgebra since it is straightforward to

check that $pS(B)$ is a subsuperalgebra of the Lie superalgebra $\text{End}(B)$. $\square$

Definition 9. $PS(B)$ is called the enlarged Lie superalgebra of pseudo-superderivations of a Bol superalgebra B .

Let $B = B_0 \oplus B_1$ be a Bol superalgebra. For any $x, y \in B_0 \cup B_1$, denote by $D_{x,y}$, the endomorphism of B defined by $D_{x,y}(z) := [x, y, z]$ for all $z \in B$. We have, for any

$x, y \in B_0 \cup B_1$, $r \in \mathbb{Z}_2$, $D_{x,y}(B_r) \in B_{r+\bar{x}+\bar{y}}$; that is, $D_{x,y}$ is a linear map of degree $\bar{x} + \bar{y}$. Moreover, it comes from (BS_4) and (BS_5) that

$$\begin{aligned} D_{x,y}([u, v, w]) &= [D_{x,y}(u), v, w] + (-1)^{\bar{u}(\bar{x}+\bar{y})} [u, D_{x,y}(v), w] + (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})} [u, v, D_{x,y}(w)] \\ D_{x,y}(u \cdot v) &= (-1)^{\bar{u}(\bar{x}+\bar{y})} u \cdot D_{x,y}(v) + D_{x,y}(u) \cdot v + (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})} [u, v, x \cdot y] + (x \cdot y) \cdot (u \cdot v), \end{aligned} \quad (14)$$

for any x, y, u, v in $B_0 \cup B_1$. It follows that $D_{x,y}$ is pseudo-superderivation of degree $\bar{x} + \bar{y}$ and companion xy , called inner pseudo-superderivation of B .

Now, one can reformulate the definition of a Bol superalgebra in the following manner.

Definition 10. A vector superspace $B = B_0 \oplus B_1$ equipped with a bilinear operation $(x, y) \mapsto x \cdot y$ satisfying $B_i \cdot B_j \subseteq B_{i+j}$ and a trilinear operation $(x, y, z) \mapsto [x, y, z]$ satisfying $[B_i, B_j, B_k] \subseteq B_{i+j+k}$; $i, j, k \in \mathbb{Z}_2$, is called a Bol superalgebra if the following holds:

$$\begin{aligned} (BS_1) \quad x \cdot y &= -(-1)^{\bar{x}\bar{y}} y \cdot x. \\ (BS_2) \quad [x, y, z] &= -(-1)^{\bar{x}\bar{y}} [y, x, z]. \\ (BS_3) \quad [x, y, z] + (-1)^{\bar{x}(\bar{y}+\bar{z})} [y, z, x] + (-1)^{\bar{z}(\bar{x}+\bar{y})} [z, x, y] &= 0, \text{ and the endomorphism } D_{x,y}: z \mapsto [x, y, z] \text{ is its} \\ &\text{pseudo-superderivation with a companion } xy \text{ for all} \\ &x, y \in B_0 \cup B_1 \text{ and } z \in B_0 \cup B_1. \end{aligned}$$

Let $ipS_r(B, B)$ be the vector space spanned by all inner pseudo-superderivations

$$D_{x,y} (x, y \in B_0 \cup B_1 \text{ and } \bar{x} + \bar{y} = r \in \mathbb{Z}_2).$$

We can define naturally a $\mathbb{Z}_2$ -gradation by setting $ipS(B, B) = ipS_0(B, B) \oplus ipS_1(B, B)$. Evidently, $ipS(B, B)$ is a subsuperalgebra of the Lie superalgebra $pS(B)$. Accordingly, $IPS(B, B)$ can be introduced as the set of all pairs (P, c) ,

where $P \in ipS(B, B)$ and $c \in \text{Com}(P)$. Evidently, $IPS(B, B)$ is a subsuperalgebra of the Lie superalgebra $PS(B)$.

Definition 11. $IPS(B, B)$ is called the enlarged Lie superalgebra of inner pseudo-superderivations.

Let $B = B_0 \oplus B_1$ be a Bol superalgebra and $H = H_0 \oplus H_1$ be a subsuperalgebra of $PS(B)$ such that $IPS(B, B) \subseteq H$. For $i \in \mathbb{Z}_2$, let $\mathcal{L}_i(B) = B_i \oplus H_i$ and define a new superbracket operation in $\mathcal{L}^H(B) = \mathcal{L}_0(B) \oplus \mathcal{L}_1(B) = B \oplus H$ as follows: for any $x, y \in B_0 \cup B_1$, $(P, a), (Q, b) \in H_0 \cup H_1$:

$$[x, y] := (D_{x,y}, xy), \quad (15)$$

$$[(P, a), x] := -(-1)^{\bar{a}\bar{x}} [x, (P, a)] := P(x), \quad (16)$$

$$[(P, a), (Q, b)] := ([P, Q], P(b) - (-1)^{\bar{a}\bar{b}} Q(a) - ab). \quad (17)$$

Then, we have the following.

Proposition 2. $(\mathcal{L}^H(B), [,])$ is a Lie superalgebra.

Proof. It is clear that the operation $[,]$ is supersymmetric. For the Jacobi superidentity, there are many cases to distinguish. First, for all x, y, u in $B_0 \cup B_1$, $(P, a), (Q, b)$ in $H_0 \cup H_1$, we get

$$\begin{aligned} &([[x, y], (P, a)] + (-1)^{\bar{x}(\bar{y}+\bar{a})} [[y, (P, a)], x] + (-1)^{\bar{a}(\bar{x}+\bar{y})} [[(P, a), x], y])(u) \\ &= [x, y, P(u)] - (-1)^{\bar{a}(\bar{x}+\bar{y})} P([x, y, u]) - (-1)^{\bar{x}(\bar{a}+\bar{y})+\bar{a}\bar{y}} [P(y), x, u] + (-1)^{\bar{a}(\bar{x}+\bar{y})} [P(x), y, u] \\ &= -(-1)^{\bar{a}(\bar{x}+\bar{y})} (-(-1)^{\bar{a}(\bar{x}+\bar{y})} [x, y, P(u)] + P([x, y, u]) - (-1)^{\bar{a}\bar{x}} [x, P(y), u] - [P(x), y, u]) \\ &= 0 \text{ (by (9))}, \end{aligned} \quad (18)$$

$$\text{i.e., } [[x, y], (P, a)] + (-1)^{\bar{x}(\bar{y}+\bar{a})} [[y, (P, a)], x] + (-1)^{\bar{a}(\bar{x}+\bar{y})} [[(P, a), x], y] = 0.$$

$$\begin{aligned} &([(P, a), x], (Q, b)] + (-1)^{\bar{a}(\bar{x}+\bar{b})} [[x, (Q, b)], (P, a)] + (-1)^{\bar{b}(\bar{a}+\bar{x})} [[(Q, b), (P, a)], x]) \\ &= PQ(x) - (-1)^{\bar{a}\bar{b}} QP(x) + (-1)^{\bar{a}(\bar{x}+\bar{b})} [Q(x), (P, a)] - (-1)^{\bar{x}\bar{b}} [P(x), (Q, b)] \\ &= PQ(x) - (-1)^{\bar{a}\bar{b}} QP(x) - PQ(x) + (-1)^{\bar{a}\bar{b}} QP(x) = 0. \end{aligned} \quad (19)$$

Next, the other cases when the three elements are in $B_0 \cup B_1$ or $H_0 \cup H_1$ follow from (HBS_3) and the fact that H is a Lie superalgebra. $\square$

Definition 12. An enveloping superalgebra of a Bol superalgebra B is a Lie superalgebra $\mathcal{L}^H(B)$ defined above. Taking $H = PS(B)$, we obtain the maximal enveloping superalgebra, and taking $H = IPS(B, B)$, we obtain the minimal (standard) enveloping superalgebra.

The following result will be used in the last section.

Lemma 2. Let K be an ideal of a Bol superalgebra B . Then, $\mathcal{K} = K \oplus IPS(B, K)$ is an ideal of the standard enveloping superalgebra $L = B \oplus IPS(B, B)$.

Proof. It suffices to prove that $[\mathcal{K}, L] \subseteq \mathcal{K}$, which is a straightforward computation. $\square$

4. Killing-Ricci Forms of Bol Superalgebras

The definition of the Killing-Ricci form given here for Bol superalgebras stems from [5], where the Killing-Ricci form for Bol algebras is defined following [21] as the restriction of the Killing form of the standard enveloping Lie algebra of the given Bol algebra to this later. Let $B = B_0 \oplus B_1$ be an n -dimensional Bol superalgebra and $L(B) = (B_0 \oplus IPS_0(B, B)) \oplus (B_1 \oplus IPS_1(B, B)) := B \oplus IPS(B, B)$ its standard enveloping Lie superalgebra. Let α be the Killing form of $L(B)$ and β , the restriction of α to $B \times B$. From [5], we introduce the following definition.

Definition 13. The form β is called the Killing-Ricci form of the Bol superalgebra B .

For any $x, y \in B_0 \cup B_1$, define the endomorphism $R_{x,y}$ of the vector superspace B by $R_{x,y}(z) = (-1)^{\bar{z}(\bar{x}+\bar{y})}[z, x, y] = (-1)^{\bar{z}(\bar{x}+\bar{y})}D_{z,x}(y)$ for all $z \in B_0 \cup B_1$. It is clear that $R_{x,y}$ is of degree $\bar{x} + \bar{y}$.

The next result gives an explicit expression of β .

Theorem 1. For every $x, y \in B_0 \cup B_1$,

$$\beta(x, y) = \text{str}(R_{x,y} + (-1)^{\bar{x}\bar{y}}R_{y,x}). \quad (20)$$

Proof. Let $\{e_i\}$, $\{f_j\}$, $\{u_i\}$, and $\{v_i\}$ be bases for $B_0, B_1, IPS_0(B, B)$, and $IPS_1(B, B)$, respectively. It suffices to prove (20) for all elements x, y in the basis, i.e., $\beta(e_i, e_j) = \alpha(e_i, e_j)$, $\beta(e_i, f_j) = \alpha(e_i, f_j)$, and $\beta(f_i, f_j) = \alpha(f_i, f_j)$. For these bases, we express the operations of B and $ISP(B, B)$, using the tensor notation (i.e., repeated indices imply summation), as follows:

$$\begin{aligned} D_{e_i, e_j} &= R_{ij}^m u_m, \\ D_{e_i, f_j} &= S_{ij}^m v_m, \\ D_{f_i, f_j} &= T_{ij}^m u_m, \\ [u_m, e_i] &= u_m(e_i) = C_{mi}^j e_j, \\ [v_m, e_i] &= v_m(e_i) = H_{mi}^j f_j, \\ [u_m, f_i] &= u_m(f_i) = K_{mi}^j f_j, \\ [v_m, f_i] &= v_m(f_i) = L_{mi}^j e_j. \end{aligned} \quad (21)$$

Since $R_{e_i, f_j}(B_0) \subseteq B_1$ and $R_{e_i, f_j}(B_1) \subseteq B_0$, we have $\text{str}(R_{e_i, f_j} + R_{f_j, e_i}) = 0$ and then $\beta(e_i, f_j) = 0 = \alpha(e_i, f_j)$ (consistency property of α). Hence, it remains to show that $\beta(e_i, e_j) = \alpha(e_i, e_j)$ and $\beta(f_i, f_j) = \alpha(f_i, f_j)$. The identities (15) and (16) imply the following:

$$\begin{aligned} L_{e_i} L_{e_j}(e_k) &= [e_i, [e_j, e_k]] = [e_i, D_{e_j, e_k}] = D_{e_k, e_j}(e_i) = R_{kj}^m C_{mi}^t e_t, \\ L_{e_i} L_{e_j}(f_k) &= [e_i, [e_j, f_k]] = [e_i, D_{e_j, f_k}] = -D_{e_j, f_k}(e_i) = -S_{jk}^m H_{mi}^t f_t, \\ L_{e_i} L_{e_j}(u_m) &= [e_i, [e_j, u_m]] = -[e_i, u_m(e_j)] = -C_{mj}^k [e_i, e_k] = -C_{mj}^k R_{ik}^t u_t, \\ L_{e_i} L_{e_j}(v_m) &= [e_i, [e_j, v_m]] = -[e_i, v_m(e_j)] = -H_{mj}^k [e_i, e_k] = -H_{mj}^k S_{ik}^t v_t. \end{aligned} \quad (22)$$

Thus, from the relation $\beta(e_i, e_j) = \alpha(e_i, e_j) = \text{str}(L_{e_i} L_{e_j})$, we get

$$\beta(e_i, e_j) = R_{kj}^m C_{mi}^k + S_{jk}^m H_{mi}^k - C_{mj}^k R_{ik}^m + H_{mj}^k S_{ik}^m. \quad (23)$$

From the other hand, we have

$$\begin{aligned} R_{e_i, e_j}(e_k) &= D_{e_k, e_i}(e_j) = -D_{e_i, e_k}(e_j) = -R_{ik}^m u_m(e_j) = -R_{ik}^m C_{mj}^t e_t, \\ R_{e_i, e_j}(f_k) &= D_{f_k, e_i}(e_j) = -D_{e_i, f_k}(e_j) = -S_{ik}^m v_m(e_j) = -S_{ik}^m H_{mj}^t f_t. \end{aligned} \quad (24)$$

By interchanging i and j , we have

$$R_{e_j, e_i}(e_k) = -R_{jk}^m C_{mi}^t e_t = R_{kj}^m C_{mi}^t e_t, \quad \text{and } R_{e_j, e_i}(f_k) = -R_{jk}^m H_{mi}^t f_t. \quad (25)$$

Then, we get

$$\text{str}\left(R_{e_i, e_j} + (-1)^{\bar{e}_i \bar{e}_j} R_{e_j, e_i}\right) = -R_{ik}^m C_{mj}^k + S_{ik}^m H_{mj}^k + R_{kj}^m C_{mi}^k + R_{jk}^m H_{mi}^k. \quad (26)$$

From (23) and (26), we obtain

$$\beta(e_i, e_j) = \text{str}\left(R_{e_i, e_j} + (-1)^{\bar{e}_i \bar{e}_j} R_{e_j, e_i}\right). \quad (27)$$

Again, the identities (15) and (16) imply the following:

$$\begin{aligned} L_{f_i} L_{f_j}(e_k) &= [f_i, [f_j, e_k]] = -[f_i, D_{e_k, f_j}] = -D_{e_k, f_j}(f_i) = -S_{kj}^m v_m(f_i) \\ &= -S_{kj}^m T_{mi}^t e_t, \\ L_{f_i} L_{f_j}(f_k) &= [f_i, [f_j, f_k]] = [f_i, D_{f_j, f_k}] = -D_{f_j, f_k}(f_i) = -T_{jk}^m u_m(f_i) \\ &= -T_{jk}^m K_{mi}^t f_t, \\ L_{f_i} L_{f_j}(u_k) &= [f_i, [f_j, u_k]] = -[f_i, u_k(f_j)] = -K_{kj}^m [f_i, f_m] = -K_{kj}^m D_{f_i, f_m} \\ &= -K_{kj}^m T_{im}^s u_s, \\ L_{f_i} L_{f_j}(v_k) &= [f_i, [f_j, v_k]] = [f_i, v_k(f_j)] = L_{kj}^m [f_i, e_m] = -L_{kj}^m D_{e_m, f_i} \\ &= -L_{kj}^m S_{mi}^s v_s. \end{aligned} \quad (28)$$

Again, from the relation $\beta(f_i, f_j) = \alpha(f_i, f_j) = \text{str}(L_{f_i} L_{f_j})$, we get

$$\beta(f_i, f_j) = -S_{kj}^m L_{mi}^k + T_{jk}^m K_{mi}^k - K_{kj}^m T_{im}^k + L_{kj}^m S_{mi}^k. \quad (29)$$

From the other hand, we get

$$\begin{aligned} R_{f_i, f_j}(e_k) &= D_{e_k, f_i}(f_j) = S_{ki}^m v_m(f_j) = S_{ki}^m L_{mj}^s e_s, \quad \text{and,} \\ R_{f_i, f_j}(f_k) &= D_{f_k, f_i}(f_j) = T_{ki}^m u_m(f_j) = T_{ki}^m K_{mj}^s f_s. \end{aligned} \quad (30)$$

By interchanging i and j , we get

$$R_{f_j, f_i}(e_k) = S_{kj}^m L_{mi}^s e_s, \quad \text{and } R_{f_j, f_i}(f_k) = T_{kj}^m K_{mi}^s f_s = T_{jk}^m K_{mi}^s f_s. \quad (31)$$

Next, we get

$$\begin{aligned} \text{str}\left(R_{f_i, f_j} + (-1)^{\bar{f}_i \bar{f}_j} R_{f_j, f_i}\right) &= \text{str}\left(R_{f_i, f_j} - R_{f_j, f_i}\right), \\ &= S_{ki}^m L_{mj}^k - T_{ki}^m K_{mj}^k - S_{kj}^m L_{mi}^k + T_{jk}^m K_{mi}^k \\ &= S_{mi}^k L_{kj}^m - T_{mi}^k K_{kj}^m - S_{kj}^m L_{mi}^k + T_{jk}^m K_{mi}^k, \end{aligned} \quad (32)$$

and therefore from (29) and (32), we obtain

$$\beta(f_i, f_j) = \text{str}\left(R_{f_i, f_j} + (-1)^{\bar{f}_i \bar{f}_j} R_{f_j, f_i}\right), \quad (33)$$

where by relation (20) is proved. $\square$

Remark 3. Recall that if $(B, \cdot, [\cdot, \cdot])$ is a (left) Bol algebra, then the Killing-Ricci form on B is defined as $\beta(x, y) = \text{tr}(R_{x, y} + R_{y, x})$ which we deduce from the one for (right) Bol algebra [5] where $R_{x, y}(z) = [z, x, y]$. So, if a Bol superalgebra B is reduced to a Bol algebra, β as in Theorem 1 is the Killing-Ricci form of the Bol algebra B .

Proposition 3. Let $B = B_0 \oplus B_1$ be a Bol superalgebra with a Killing-Ricci form β . Then, the following holds:

- (1) $\beta(x, y) = (-1)^{\bar{x}\bar{y}} \beta(y, x)$ for all $x, y \in B_0 \cup B_1$ (supersymmetry).
- (2) $\beta(B_0, B_1) = 0$ (consistence).
- (3) $\beta(A(x), A(y)) = \beta(x, y)$, $A \in \text{Aut}(B)$.
- (4) $\beta([x, y, z], u) = -(-1)^{\bar{z}(\bar{x}+\bar{y})} \beta(z, [x, y, u])$, for all $x, y, z, u \in B_0 \cup B_1$.

Proof. We know [6] that if α is a Killing form of any Lie superalgebra, then α satisfies the relations (1) – (3) of the proposition above. Then, the relations (1) – (3) follow from the fact that the Killing-Ricci form β is the restriction to B of

the Killing form of the standard enveloping superalgebra $L(B)$ of B . For the relation (4), pick $x, y, z, u \in B_0 \cup B_1$ and denote by α the Killing form of $L(B)$. Since $L(B)$ is a Lie superalgebra, then α satisfies

$$\alpha([x, y], z) = -(-1)^{\overline{x}\overline{y}}\alpha(y, [x, z]). \quad (34)$$

Then,

$$\begin{aligned} \beta([x, y, z], u) &= \alpha([x, y], [z, u]) = -(-1)^{\overline{x}(\overline{y}+\overline{z})} \\ \alpha(z, [[x, y], u]) &= -(-1)^{\overline{z}(\overline{x}+\overline{y})}\beta(z, [x, y, u]). \end{aligned} \quad (35)$$

Hence, the proposition is proved. $\square$

Remark 4. If we consider the Bol superalgebra B as a Lie supertriple system with the ternary operation $[^v]$, that we denote by B_S , then the relation (4) in Proposition 3 says that the Killing-Ricci form of B is an invariance form of B_S .

The following result will be used below.

Lemma 3. Let K be a Killing-Ricci form of a Bol superalgebra B . Then, the following conditions are equivalent:

$$\begin{aligned} K([x, y, z], u) &= -(-1)^{\overline{z}(\overline{x}+\overline{y})}K(z, [x, y, u]), \\ K([x, y, z], u) &= -(-1)^{\overline{y}(\overline{z}+\overline{u})}K(x, [z, u, y]), \\ K(x, [y, z, u]) &= (-1)^{\overline{x}\overline{y}+\overline{z}\overline{u}}K(y, [x, u, z]), \end{aligned} \quad (36)$$

for all $x, y, z, u \in B_0 \cup B_1$.

Proof. We know that, in any Lie supertriple system with an invariant form, the relations (39)–(41) are inequivalent [14]. The proof then follows from the fact that the Killing-Ricci form of B is an invariant form of B_S . $\square$

5. Invariant Forms of Bol Superalgebras

In this section, we introduce the concept of invariant forms of Bol superalgebras as generalization of those of Bol algebras and Lie supertriple systems.

Definition 14. An invariant form b of a Bol superalgebra $B = B_0 \oplus B_1$ is a supersymmetric bilinear form on B satisfying the identities

$$b(xy, z) = -(-1)^{\overline{x}\overline{y}}b(y, xz), \quad (37)$$

$$b([x, y, z], u) = -(-1)^{\overline{y}(\overline{z}+\overline{u})}b(x, [z, u, y]), \quad (38)$$

for all $x, y, z, u \in B_0 \cup B_1$.

Remark 5. If B is reduced to a Lie supertriple system (resp., a (left) Bol algebra), then b is reduced to an invariant form to a Lie supertriple system [14, 15], (resp., a (left) Bol algebra) which can be deduced from one of the (right) Bol algebras [5].

Let b be an invariant form of a Bol superalgebra $B = B_0 \oplus B_1$. Then, b is an invariant form of B_S , and by Lemma 3, b satisfies the following equivalent conditions:

$$b([x, y, z], u) = -(-1)^{\overline{z}(\overline{x}+\overline{y})}b(z, [x, y, u]), \quad (39)$$

$$b([x, y, z], u) = -(-1)^{\overline{y}(\overline{z}+\overline{u})}b(x, [z, u, y]), \quad (40)$$

$$b(x, [y, z, u]) = (-1)^{\overline{x}\overline{y}+\overline{z}\overline{u}}b(y, [x, u, z]), \quad (41)$$

for all $x, y, z, u \in B_0 \cup B_1$.

Definition 15. Let b be an invariant form of a Bol superalgebra B and V be a subset of B . The orthogonal $V^\perp$ of V with respect to b is defined by $V^\perp = \{x \in B, b(x, y) = 0, \forall y \in V\}$. The invariant form b is nondegenerate if $B^\perp = \{0\}$.

Lemma 4. Let b be an invariant form of a Bol superalgebra B . Then, the following holds:

- (1) $(B + [B, B, B])^\perp = Z(B)$ if b is nondegenerate.
- (2) If I is an ideal of B then $I^\perp$ is an ideal of B . In particular, $B^\perp$ is an ideal of B .

Proof. Pick a homogeneous element u in $(B + [B, B, B])^\perp$. Then, for any x, y, z in $B_0 \cup B_1$, we get $b(u, xy) = 0$ and $b(u, [x, y, z]) = 0$. This implies by (37) and (38) that $(-1)^{\overline{u}\overline{x}}b(xu, y) = 0$ and $(-1)^{\overline{z}(\overline{x}+\overline{y})}b([u, z, x], y) = 0$. As b is nondegenerate, we obtain $xu = 0$ and $[u, z, x] = 0$ for any x, z in $B_0 \cup B_1$, and this gives $u \in Z(B)$.

Conversely, if $u \in Z(B)$, we have for all x, y, z, v, w in B , $b(u, xy + [z, v, w]) = b(u, xy) + b(u, [z, v, w]) = 0$ and $u \in (B + [B, B, B])^\perp$ whence $(B + [B, B, B])^\perp = Z(B)$.

Now, suppose that I is an ideal of B that is $BI \subseteq I$, $[B, I, B] \subseteq I$; then for any homogeneous elements $x, y \in B$, $u \in I^\perp$, and $v \in I$, we get $b(xu, v) = -(-1)^{\overline{u}\overline{x}}b(u, xv) = 0$ and $b([x, u, y], v) = -(-1)^{\overline{x}\overline{u}}b([u, x, y], v) = (-1)^{\overline{x}(\overline{u}+\overline{y}+\overline{v})}b(u, [y, v, x]) = 0$ by (21). Then, $BI^\perp \subseteq I^\perp$ and $[B, I^\perp, B] \subseteq I^\perp$, i.e., $I^\perp$ is an ideal of B .

We can now prove the following result. $\square$

Theorem 2. Let $B = B_0 \oplus B_1$ be a Bol superalgebra such that $\text{str}(D_{x,y}L_z) = 0$ for all $x, y, z \in B$. Then, the Killing-Ricci form β of B is nondegenerate if and only if the standard enveloping superalgebra $L(B) = B \oplus \text{IPS}(B, B)$ is a semisimple Lie superalgebra.

Proof. Let α be the Killing form of the Lie superalgebra $L(B)$. Then, for all $x, y, z \in B$, $\alpha(D_{x,y}, z) = \text{str}(L_{D_{x,y}}L_z) = \text{str}(D_{x,y}L_z)$, that is, $\text{str}(D_{x,y}L_z) = 0$ if and only if

$$\alpha(D_{x,y}, z) = 0. \quad (42)$$

Then, by (42) and the invariance of α , we have for $x, y, u, v \in B_0 \cup B_1$,

$$\begin{aligned}
\alpha(D_{x,y}, D_{u,v}) &= \alpha([x, y], [u, v]), \\
&= -(-1)^{\bar{x}\bar{y}} \alpha(y, [x, [u, v]]) \\
&= (-1)^{\bar{x}(\bar{u}+\bar{v}+\bar{y})} \alpha(y, [[u, v], x]) \\
&= (-1)^{\bar{x}(\bar{u}+\bar{v}+\bar{y})} \beta(y, [u, v, x]).
\end{aligned} \tag{43}$$

Thus, if β is nondegenerate, the restriction of α to $IPS(B, B) \times IPS(B, B)$ is nondegenerate, and therefore, α is nondegenerate.

Now by contradiction, suppose that β is degenerate. Then, by Lemma 4, $B^\perp$ is a nonzero ideal of B , and therefore, $B^\perp \oplus IPS(B, B^\perp)$ is a nonzero ideal of $L(B)$ by Lemma 2.

By the identities (42) and (43), we obtain

$$\begin{aligned}
\alpha(B^\perp \oplus IPS(B, B^\perp), B \oplus IPS(B, B)) &= \alpha(B^\perp, B) + \alpha(B^\perp, IPS(B, B)) \\
&\quad + \alpha(IPS(B, B^\perp), B) + \alpha(IPS(B, B^\perp), IPS(B, B)) \\
&= \alpha(B^\perp, B) + \alpha(B, [B^\perp, B, B]) = 0.
\end{aligned} \tag{44}$$

It follows that α is nondegenerate, and therefore, $L(B)$ is nondegenerate, which ends the theorem. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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