Research Article

Epi- α -Normality and Epi- β -Normality

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A topological space (Y, τ) is called epi- α -normal (epi- β -normal) if there is a coarser topology τ' on Y such that (Y, τ') is T_1 α -normal (T_1 β -normal). We investigate these properties and show some examples to explain the relationships of epi- α -normal (epi- β -normal) with other weaker versions of normality and some topological spaces.

1. Introduction

In [1], the notions of α -normality and β -normality were investigated in relative sense, and their dependency and independency on other generalizations of relative normality were proved. Some relative generalizations of β -normality are defined, their relations with different generalizations of relative normality were studied, and it is observed that some absolute generalizations of normality coincide with normality in the class of relative variant of β -normal. Also, almost β -normal spaces as a simultaneous generalization of β -normal and almost normal spaces were introduced by Das et al. [2]. Another new generalization of normality, namely, weak β -normality, in terms of θ -closed sets was introduced, which turns out to be a simultaneous generalization of β -normality and θ -normality (see [3]).

In this paper, we introduce new topological properties called epi- α -normality and epi- β -normality. We investigate these properties and show some examples to explain the relationships between epi- α -normality (epi- β -normality) with other weaker versions of normality and some topological spaces. We prove that submetrizability, T_4 imply epi- β -normality and T_1 α -normality implies epi- α -normality, and we prove the converse is not true in general. Throughout this article, a T_4 space is a T_1 -normal space, a completely regular space with T_1 is Tychonoff space, and regular space with T_1 is T_3 .

2. Epi- α -Normality and Epi- β -Normality

Recall that a topological space (Y, τ) is called epinormal [4] if there is a coarser topology τ' on Y such that (Y, τ') is T_4 . A topological space (Y, τ) is called α -normal [5] if for any two disjoint closed subsets F_1 and F_2 there exist disjoint open subsets H_1 and H_2 of Y such that $F_1 \cap H_1$ is dense in F_1 and $F_2 \cap H_2$ is dense in F_2 . A topological space (Y, τ) is called β -normal [5] if for any two disjoint closed subsets F_1 and F_2 there exist open subsets H_1 and H_2 of Y such that $F_1 \cap H_1$ is dense in F_1 , $F_2 \cap H_2$ is dense in F_2 , and $\overline{H_1} \cap \overline{H_2} = \emptyset$.

Definition 1. A topological space (Y, τ) is called epi- α -normal (epi- β -normal) if there is a coarser topology τ' on Y such that (Y, τ') is T_1 α -normal (T_1 β -normal).

Note, if we assume (Y, τ') to be just α -normal (β -normal) with out assume T_1 in the above definition, therefore any space will be epi- α -normal (epi- β -normal) as the indiscret topology.

Obviously, any T_1 α -normal space is epi- α -normal, any T_1 β -normal space is epi- β -normal, and any T_4 space is epi- β -normal just by taking $\tau = \tau'$, and any epinormal space is epi- β -normal since every normal space is β -normal.

Note that any submetrizable space [4] is epi- β -normal, but the converse is not true in general. For example, $\omega_1 + 1$ is an epi- β -normal space is Hausdorff and compact; therefore, T_4 . But it is not submetrizable (for details, see [4]).

Epi- α -normality and α -normality do not imply each other. For example, the Niemytzki plane is epi- α -normal since it is submetrizable, but it is not α -normal by the standard proof of nonnormality. On the other hand, an example of an α -normal space which is not epi- α -normal is any indiscrete space which has more than one point.

Observe that if the coarser topology (Y, τ') in Definition 1 is T_i , $i \in \{0, 1, 2\}$, then so is (Y, τ) . Since any β -normal (epi- α -normal) is Hausdorff [5], we conclude the following.

Theorem 1. *Any epi- β -normal (epi- α -normal) space is Hausdorff.*

Note that every topological space which is not Hausdorff can neither be epi- β -normal nor be epi- α -normal. For example, the Double pointed reals topological space defined in [6] is β -normal, and hence α normal but neither epi- β -normal nor epi- α -normal since it is not Hausdorff.

Theorem 2. *Every epi- β -normal space is completely Hausdorff.*

Proof. Let (Y, τ) be an epi- β -normal space. Let τ' be a coarser topology on Y ; then, (Y, τ') is T_1 β -normal. We assume that Y has more than one element and take distinct $x, y \in Y$. Since (Y, τ') is T_1 β -normal, then it is T_2 (see [5]). Choose $H_1, H_2 \in \tau$ such that $x \in H_1$, $y \in H_2$, and $H_1 \cap H_2 = \emptyset$. Now since (Y, τ') is T_1 β -normal, then by Proposition 1.1 in [5], (Y, τ') is regular, and therefore choose $U_1, U_2 \in \tau'$ such that $x \in U_1 \subseteq \overline{U_1}^{\tau'} \subseteq H_1$ and $y \in U_2 \subseteq \overline{U_2}^{\tau'} \subseteq H_2$. We have $U_1, U_2 \in \tau$ and since $\overline{A}^{\tau} \subseteq \overline{A}^{\tau'}$ for any $A \subseteq Y$, we get $\overline{U_1}^{\tau} \cap \overline{U_2}^{\tau} = \emptyset$.

Note that any space which is not completely Hausdorff cannot be epi- β -normal. For example, Prime Integer topology [6]. In this space, the set $Y = \mathbb{Z}^+$ of positive integers, where any open neighborhood $U_a(b) = \{b + na \in Y : n \in \mathbb{Z}\}$, since the closures of any two open neighborhood $U_a(b)$, $U_c(d)$ contain in common all multiples of $[a, c]$, so it is not completely Hausdorff (see [6]), and by Theorem 2, the space Y cannot be epi- β -normal.

Observe that epi- α -normality may not be necessarily completely Hausdorff. However, we have the following theorem. $\square$

Theorem 3. *If (Y, τ) is an epi- α -normal space and the coarser topology of epi- α -normality (Y, τ') is first countable, then (Y, τ) is completely Hausdorff.*

Proof. By the similar argument of Theorem 2, since every Hausdorff α -normal first countable is regular, the proof follows (see [7]).

Since every β -normal space satisfying T_1 axiom is regular (see [5]), we recall that a topological space (Y, τ) is called epi-regular [8] if there is a coarser topology τ' on Y such that (Y, τ') is T_3 , so easily we conclude the following. $\square$

Corollary 1. *Any epi- β -normal space is epi-regular.*

Any T_2 paracompact space is T_4 , so we conclude the following.

Corollary 2. *If (Y, τ) is an epi- β -normal space and the coarser topology of epi- β -normality (Y, τ') is paracompact, then (Y, τ) is T_4 .*

Also, it is well known that any T_2 compact space is T_4 , so we conclude the following.

Corollary 3. *Any epi- β -normal compact space is T_4 .*

Recall that a topological space (X, τ) is called α -regular [9, 10] if for any closed subsets F of Y and $y \in Y$ such that $y \notin F$ there exist disjoint open subsets H_1 and H_2 of Y such that $y \in H_1$ and $F \cap H_2$ is dense in F . A topological space (X, τ) is called epi- α -regular (see [9]) if there is a coarser topology τ' on Y such that (Y, τ') is T_1 α -regular. Since every α -normal space satisfying T_1 axiom is α -regular (see [9, 10]), we conclude the following.

Corollary 4. *Every epi- α -normal space is epi- α -regular.*

Theorem 4. *If Y is an epi- β -normal pseudocompact space of cardinality less than the continuum and the coarser topology of epi- β -normality is completely regular, then Y is T_4 .*

Proof. Let Y be an epi- β -normal pseudocompact space of cardinality less than the continuum and τ' be a coarser topology on Y where (Y, τ') is T_1 β -normal. Note that (Y, τ') is pseudocompact. Since any β -normal pseudocompact space is countably compact (see Theorem 2.2 in [5]), (Y, τ') is countably compact Hausdorff. Since it is of cardinality less than the continuum, then (Y, τ') is first countable.

Now, we show that $\tau' = \tau$. Let K be any closed set in (Y, τ) . Assume that the family $\mathcal{M} = \{M = \overline{U}^{\tau'} : U \in \tau \text{ and } K \subseteq U\}$. Clearly we have $K = \bigcap \{M : M \in \mathcal{M}\}$. Since the pseudocompactness is hereditary with respect to closed domain ([11], 3.10. F), M is pseudocompact in (Y, τ) for every $M \in \mathcal{M}$. Then, M is pseudocompact in (Y, τ') for every $M \in \mathcal{M}$. Since pseudocompact subsets of first countable Tychonoff space are closed [12], M is closed in (Y, τ') for every $M \in \mathcal{M}$. Hence, K is closed in (Y, τ') and then $\tau' = \tau$. Thus, (Y, τ) is T_4 . $\square$

Corollary 5. *If Y is an epi- β -normal pseudocompact first countable space and the coarser topology of epi- β -normality is completely regular, then Y is T_4 .*

Theorem 5. *If Y is an epi- β -normal space and the coarser topology of epi- β -normality is a dense subspace of a product of metrizable spaces, then Y is epi-normal.*

Proof. Assume that Y is an epi- β -normal space; then, there is a coarser topology τ' such that (Y, τ') is T_1 β -normal and it is a dense subspace of product metrizable spaces. Šćepin and Blair independently showed that every dense subspace of any product of metrizable spaces is mildly normal [5, 13, 14]. Since (Y, τ') is T_1 β -normal and mildly normal, then by

Theorem 2.4 in [5], (Y, τ') is T_1 normal and hence T_4 . Therefore, (Y, τ') is epinormal.

The following is an example of α normal Hausdorff non-regular space constructed by Murtinová in [7] where she showed that it is an example of an α normal Hausdorff non-regular space. $\square$

Example 1 (see [7]). Let $Y = \omega_1 + 1$ and define a topology τ such that ω_1 with the ordinal topology is an open subspace and a base in the point ω_1 will be the collection:

$$U_C = \{\omega_1\} \cup \{\alpha + 1 : \alpha \in C\}, \quad (1)$$

where C is a closed unbounded subset of ω_1 (Club).

The topology τ is Hausdorff since it is stronger than the order topology on $\omega_1 + 1$. This space is epi- α -normal since it is α -normal Hausdorff and it is epi- β -normal since it is stronger than the order topology on $\omega_1 + 1$, but it is not β -normal nor regular.

Murtinová in [7] proved that every first countable α -normal Hausdorff space is regular, so we have the following result.

Theorem 6. *If Y is epi- α -normal and the coarser topology of epi- α -normality (Y, τ') is first countable, then Y is epi-regular.*

Recall that a topological space (Y, τ) is called epi-completely regular [15] if there is a coarser topology τ' on Y such that (Y, τ') is Tychonoff.

Theorem 7. *If a topological space Y is epi-completely regular and the coarser topology of epi-completeness is second countable (countable), then the space is epi- β -normal.*

Theorem 8. *If a topological space Y is epi- β -normal and the coarser topology of epi- β -normal is compact second countable, then the space is submetrizable.*

The similar argument is correct for epi- α -normality.

3. Dowker's Characterisation of Epi- α -Normal and Epi- β -Normal

In light of Dowker's characterisation of α -normal and β -normal (for details, see [16]), it is natural to ask if this can be satisfied for epi- α -normal and epi- β -normal. To do this, let us start with the following lemma from [16]. Throughout this section, unless otherwise stated, a space is assumed to be an epi-regular topological space.

Lemma 1 (see [16]). *A topological space Y is α -normal if and only if for every pair F_1 and F_2 of disjoint closed subsets of Y there exists an open set U of Y such that $\overline{U \cap F_1} = F_1$ and $\overline{U \cap F_2}$ is nowhere dense in F_2 .*

Proof. Let (Y, τ) be an α -normal space and F_1, F_2 be disjoint closed subsets of Y . Then, there are two disjoint open subsets U and V such that $\overline{U \cap F_1} = F_1$ and $\overline{V \cap F_2} = F_2$. It follows that $F_2 \subseteq U^c$ which means that $F_2 \cap U = \emptyset$. Now let $\text{int}(\overline{U \cap F_2})_{F_2} \neq \emptyset$; then, there is at least an element $y \in \text{int}(\overline{U \cap F_2})_{F_2}$, so $y \in \text{int}(\overline{U \cap F_2}) \cap F_2$, which means that

$y \in \text{int}(\overline{U \cap F_2})$ and $y \in F_2$; then, there is an open set W containing y and contained in F_2 where $W \cap U \neq \emptyset$. This contradicts $F_2 \subseteq U^c$. Therefore, $\text{int}(\overline{U \cap F_2})_{F_2} = \emptyset$. Hence, $U \cap F_2$ is nowhere dense in F_2 .

Conversely, let F_1 and F_2 be disjoint closed subsets of Y ; there exists an open set U of Y such that $\overline{U \cap F_1} = F_1$ and $\overline{U \cap F_2}$ is nowhere dense in F_2 . Then, $\text{int}(\overline{U \cap F_2})_{F_2} = \emptyset$ which implies that $\text{int}(\overline{U \cap F_2}) \cap F_2 = \emptyset$. Now let $y \in F_2$; then, $y \notin \text{int}(\overline{U \cap F_2})$. It follows that for any open subset W in F_2 containing y , W is not contained in $\overline{U \cap F_2}$ and then $W \cap (\overline{U \cap F_2})^c \neq \emptyset$. Let $V = \overline{U^c}$. Then, V is an open set disjoint from U and $y \in \overline{V \cap F_2}$, and so $F_2 = \overline{V \cap F_2}$. Hence, (Y, τ) is an α -normal space.

The following theorem has the same argument of Theorem 2.3 in [16]. $\square$

Theorem 9. *Let τ_O be the order topology defined on $\omega + 1$. The product $Y \times (\omega + 1)$ is epi- α -normal if and only if*

- (1) Y is epi- α -normal.
- (2) Let (Y, τ') be the coarser topology of epi- α -normality. If $\{K_n : n \in \omega\}$ is a family of closed sets in (Y, τ') and $K = \bigcap_{n \in \omega} \text{cl}_{\tau'}(\bigcup_{i=n}^{\infty} K_i)$ and C is a closed subset of (Y, τ') disjoint from K , then there is a family $\{W_n : n \in \omega\}$ of open sets in τ' such that $W_n \cap K_n$ is τ' dense in K_n and $\bigcap_{n \in \omega} \text{cl}_{\tau'}(\bigcup_{i=n}^{\infty} W_i)$ is τ' nowhere dense in C .

The case of epi- β -normal spaces is similar to the case of epi- α -normal spaces, but it is more complicated. First, we introduce the following lemma induced from [16].

Lemma 2. *Let Y be a T_1 space. If $Y \times (\omega + 1)$ is epi- β -normal, then*

- (1) Y is β -normal.
- (2) Let (Y, τ') be the coarser topology of epi- β -normality. If $\{K_n : n \in \omega\}$ is a family of closed sets in (Y, τ') and $K = \bigcap_{n \in \omega} \text{cl}_{\tau'}(\bigcup_{i=n}^{\infty} K_i)$ and C is a closed subset of (Y, τ') disjoint from K , then there is a family $\{W_n : n \in \omega\}$ of open sets in τ' such that $W_n \cap K_n$ is τ' dense in K_n and $\bigcap_{n \in \omega} \text{cl}_{\tau'}(\bigcup_{i=n}^{\infty} W_i)$ is disjoint from C .

Recall that a topological space is said to be α -countably paracompact [16] if for every decreasing sequence $\{F_n : n \in \omega\}$ of closed subsets of Y satisfying $\bigcap_{n \in \omega} F_n = \emptyset$ there exists a sequence $\{W_n : n \in \omega\}$ of open subsets of Y such that $W_n \cap F_n$ is dense in F_n for $n \in \omega$ and $\bigcap_{n \in \omega} \overline{W_n} = \emptyset$.

With the definition of α -countably paracompact from [16] and Lemma 2, it is not hard to prove the following corollary.

Corollary 6. *If $Y \times (\omega + 1)$ is epi- β -normal, then Y is epi- β -normal and α -countably paracompact.*

With Lemma 2, we are ready to introduce an important result of epi- β -normal spaces which has the same argument of [16].

Theorem 10. Let τ_ω be the order topology defined on $\omega + 1$. The product $Y \times (\omega + 1)$ is epi- β -normal if and only if

- (1) Y is epi- β -normal.
- (2) Condition of Lemma 2 is satisfied.
- (3) Let (Y, τ') be the coarser topology of epi- β -normality. For each decreasing sequence $\{F_n: n \in \omega\}$ of closed sets in (Y, τ') satisfying $\bigcap_{n \in \omega} F_n = \emptyset$, there is a sequence $\{V_n: n \in \omega\}$ of open subset of (Y, τ') where $F_n \subset V_n$ for $n \in \omega$ and $\bigcap_{n \in \omega} \text{cl}_{\tau'}(V_n)$ is nowhere dense in the relative topology of F_0 with respect to the coarser topology (Y, τ') .

Corollary 7. Let Y be epi- α -normal and countably paracompact. Then, $Y \times (\omega + 1)$ is epi- α -normal.

Corollary 8. Let Y be epi- β -normal and countably paracompact. Then, $Y \times (\omega + 1)$ is epi- β -normal.

4. Properties of Epi- α -Normality and Epi- β -Normality

Let Y be any topological space. Let $Y' = Y \times \{1\}$. Note that $Y \cap Y' = \emptyset$. Let $A(Y) = Y \cup Y'$. For an element $y \in Y$, $(y, 1) \in A(Y)$ denoted by y' , and for a subset $A \subseteq Y$, we have $A' = \{y': y \in A\} = A \times \{1\} \subseteq Y'$. For each $y' \in Y'$, let $\mathcal{B}(y') = \{\{y'\}\}$. For each $y \in Y$, let $\mathcal{B}(y) = \{U \cup (U' \setminus \{y'\}): U \text{ is open in } Y \text{ and } y \in U\}$. We define a basic neighborhood $\mathcal{B} = \{\mathcal{B}(y): y \in Y\} \cup \{\mathcal{B}(y'): y' \in Y'\}$ for a unique topology defined on $A(Y)$. This generated topological space $A(Y)$ is called Alexandroff duplicate of Y (for revision, see [17]).

Theorem 11. Let Y be an epi- α -normal space, and so is its Alexandroff duplicate $A(Y)$.

Proof. Let Y be any epi- α -normal space and let τ be a topology on Y ; since Y is epi- α -normal, then there is a coarser topology τ' on Y such that (Y, τ') is T_1 α -normal. Since by Theorem 0.1 in [18] α -normality is preserved by the Alexandroff duplicate space and also T_1 , then $A(Y, \mathcal{S}')$ is also T_1 α -normal, and it is coarser than $A(Y, \mathcal{S})$ by the topology of the Alexandroff duplicate. Hence, $(A(Y), \mathcal{S})$ is epi- α -normal.

We do not know if the above theorem is valid for epi- β -normal space. $\square$

Theorem 12. β -normality and α -normality are additive properties.

Proof. We prove the case for β -normality, and that for α -normality will be similar.

Let $\{Y_\alpha\}_{\alpha \in \Lambda}$ be a family of β -normal spaces and A_1, A_2 be two disjoint closed subsets of the sum $\bigoplus_{\alpha \in \Lambda} Y_\alpha$. By Proposition 2.2.1 in [11], the intersections $A_1 \cap Y_\alpha$ and $A_2 \cap Y_\alpha$ are closed in Y_α for each $\alpha \in \Lambda$. From β -normality of Y_α , it follows that there are two open sets U_α and V_α in Y_α where

$$\begin{aligned} \overline{A_1 \cap Y_\alpha \cap U_\alpha} &= A_1 \cap Y_\alpha, \\ \overline{A_2 \cap Y_\alpha \cap V_\alpha} &= A_2 \cap Y_\alpha, \\ \overline{U \cap A_1} &= \emptyset. \end{aligned} \quad (2)$$

Let $U = \bigcup_{\alpha \in \Lambda} U_\alpha$ and $V = \bigcup_{\alpha \in \Lambda} V_\alpha$; then, clearly

$$\begin{aligned} \overline{A_1 \cap U} &= \overline{\bigcup_{\alpha \in \Lambda} (A_1 \cap U_\alpha)} = \bigcup_{\alpha \in \Lambda} \overline{A_1 \cap U_\alpha} = \overline{A_1} = A_1, \\ \overline{A_2 \cap V} &= \overline{\bigcup_{\alpha \in \Lambda} (A_2 \cap V_\alpha)} = \bigcup_{\alpha \in \Lambda} \overline{A_2 \cap V_\alpha} = \overline{A_2} = A_2, \\ \overline{U \cap V} &= \overline{\bigcup_{\alpha \in \Lambda} U_\alpha \cap \bigcup_{\alpha \in \Lambda} V_\alpha} = \bigcup_{\alpha \in \Lambda} (\overline{U_\alpha \cap V_\alpha}) = \emptyset. \end{aligned} \quad (3)$$

Since U and V are open in $\bigoplus_{\alpha \in \Lambda} Y_\alpha$, the sum $\bigoplus_{\alpha \in \Lambda} Y_\alpha$ is β -normal. $\square$

Theorem 13. Epi- β -normality and epi- α -normal space are additive properties.

Proof. Let (Y_α, τ_α) be epi- β -normal spaces for each $\alpha \in \Lambda$. Let τ'_α be a topology on Y_α coarser than τ_α such that (Y_α, τ'_α) is T_1 β -normal. Since T_1 and β -normality is also additive by Theorem 12. Then, and the sum $\bigoplus_{\alpha \in \Lambda} (Y_\alpha, \tau'_\alpha)$ is T_1 and β -normal, and its topology is coarser than the topology on $\bigoplus_{\alpha \in \Lambda} (Y_\alpha, \tau_\alpha)$. The case for epi- α -normality is similar. $\square$

Theorem 14. Let Y be a β -normal (α -normal) space, and $h: Y \longrightarrow Z$ is an onto, continuous, open, and closed function. Then, Z is β -normal (α -normal).

Proof. We prove the case for β -normality, and that for α -normality will be similar. Let A_1 and A_2 be disjoint closed subsets of Z . Then, by continuity of h , $h^{-1}(A_1)$ and $h^{-1}(A_2)$ are disjoint closed subsets of Y . Since Y is a β -normal space [5], there exists an open subset G of Y such that $\overline{h^{-1}(A_1) \cap G} = h^{-1}(A_1)$ and $h^{-1}(A_2) \cap \overline{G} = \emptyset$. Since $h^{-1}(A_2) \cap \overline{G} = \emptyset$, then $A_2 \cap h(\overline{G}) = \emptyset$. Observe that $h(\overline{G})$ is a closed set containing the open set $h(G)$ and $\overline{h(G)} \subseteq h(\overline{G})$. Then, $\overline{h(G)} \cap A_2 = \emptyset$. Now it is sufficient to show that $\overline{A_1 \cap h(G)} = A_1$. Let $z \in A_1$ and W be an open set containing z ; then, $h^{-1}(z) \subseteq h^{-1}(A_1) \cap h^{-1}(W)$. Since $\overline{h^{-1}(A_1) \cap G} = h^{-1}(A_1)$, $h^{-1}(A_1) \cap G \cap h^{-1}(W) \neq \emptyset$. Hence, by surjectivity of h , $A_1 \cap h(G) \cap W = h(h^{-1}(A_1) \cap h(G) \cap h^{-1}(W)) \supseteq h(h^{-1}(A_1) \cap G \cap h^{-1}(W)) \neq \emptyset$ as required. $\square$

Corollary 9. Let (Y, τ) be an epi- β -normal (epi- α -normal) space, and $g: (Y, \tau) \longrightarrow (Z, \mathcal{S})$ is an onto, continuous, open, and closed function. Then, Z is epi- β -normal (epi- α -normal).

Proof. We only prove the case for epi- β -normality. Let (Y, τ) be any epi- β -normal space, and let τ' be a coarser topology on Y such that (Y, τ') is β -normal, T_1 . Since $g: Y \longrightarrow Z$ is an onto, continuous, open, and closed function, then by Theorem 14 $(Z, \mathcal{S}')$, where $\mathcal{S}' = \{g(U): U \in \tau'\}$, is β -normal, and it is obviously T_1 and $\mathcal{S}'$ is coarser than $\mathcal{S}$. Hence, $(Z, \mathcal{S})$ is epi- β -normal. $\square$

Corollary 10. *Epi- β -normality and epi- α -normality are topological properties.*

Theorem 15. *Let (Y, τ) be an epi- β -normal (epi- α -normal) space, and (A, τ'_A) is a closed subspace of the coarser topology of epi- β -normality (epi- α -normal) (Y, τ') ; then, the subspace (A, τ_A) is epi- β -normal (epi- α -normal).*

Proof. This is straightforward since any closed subspace of an β -normal (α -normal) space is β -normal (α -normal) (see Proposition 1.3 in [5]). $\square$

Definition 2. A space (Y, τ) is hereditarily epi- α -normal if there is a coarser topology τ' such that (Y, τ') is T_1 hereditarily α -normal. A space (Y, τ) is hereditarily epi- β -normal if there is a coarser topology τ' such that (Y, τ') is T_1 hereditarily β -normal.

Note that any Hausdorff hereditarily α -normal space is hereditarily epi- α -normal and any Hausdorff hereditarily β -normal space is hereditarily epi- β -normal. The following is clear.

Corollary 11. *A space (Y, τ) is hereditarily epi- α -normal if and only if every subspace of (Y, τ) is epi- α -normal.*

Theorem 16. *For each topological space (Y, τ) , the following cases are equivalent:*

- (1) *The space (Y, τ) is hereditarily epi- α -normal.*
- (2) *Any subspace (Z, τ_Z) in which (Z, τ'_Z) is an open subspace of a coarser topological space (Y, τ') of (Y, τ) is epi- α -normal.*
- (3) *For every pair of sets, A_1, A_2 of (Y, τ) , which are separated in (Y, τ') , there exist disjoint open sets U_1, U_2 in (Y, τ) such that $A_1 \cap U_1$ is dense in A_1 and $A_2 \cap U_2$ is dense in A_2 with respect to (Y, τ') (we may call this epi- α -separated).*

Proof

(1 $\Rightarrow$ 2) It is clear.

(2 $\Rightarrow$ 3) Let A_1, A_2 be a pair of sets of (Y, τ) which are separated in (Y, τ') . Here is an error in set $Z = Y, (cl_{\tau'}(A_1) \cap cl_{\tau'}(A_2))$ with $A_1, A_2 \subseteq Z$. Since $cl_{\tau'_Z}(A_1) \cap cl_{\tau'_Z}(A_2) = \emptyset$ and by hypothesis that (Z, τ'_Z) is T_1 α -normal, there exist two disjoint open sets $U_1, U_2 \subseteq (Z, \tau'_Z)$ such that

$$\begin{aligned} cl_{\tau'_Z}(U_1 \cap cl_{\tau'_Z}(A_1)) &= cl_{\tau'_Z}(A_1), \\ cl_{\tau'_Z}(U_2 \cap cl_{\tau'_Z}(A_2)) &= cl_{\tau'_Z}(A_2). \end{aligned} \quad (4)$$

But (Z, τ'_Z) is an open subspace of (Y, τ') , and thus U_1 and U_2 are open in (Y, τ') , and since (Y, τ') is coarser than (Y, τ) , then U_1 and U_2 are open in (Y, τ) as needed.

(3 $\Rightarrow$ 1) Let (Z, τ_Z) be a subspace of an epi- α -normal space (Y, τ) ; then, there is a coarser topology τ' such

that (Y, τ') is T_1 α -normal. It is enough to show that (Z, τ'_Z) is T_1 α -normal as a subspace of (Y, τ') . Since (Y, τ') is T_2 , then so is (Z, τ'_Z) .

Let A_1 and A_2 be a pair of disjoint closed subsets of (Z, τ'_Z) . Note that A_1 and A_2 are separated in (Y, τ') . By hypothesis, there exist open disjoint subsets U_1, U_2 of (Y, τ) such that $A_1 \cap U_1$ is dense in A_1 and $A_2 \cap U_2$ is dense in A_2 with respect to (Y, τ') . Now $U_1 \cap Z$ and $U_2 \cap Z$ are open in (Z, τ'_Z) , and clearly $A_1 \cap (U_1 \cap Z)$ is dense in A_1 and $A_2 \cap (U_2 \cap Z)$ is dense in A_2 with respect to the space (Z, τ'_Z) . That is, (Z, τ'_Z) is α -normal. But (Z, τ'_Z) is coarser than (Z, τ_Z) ; then, (Z, τ_Z) is epi- α -normal as a subspace of (Y, τ) . Hence, (Y, τ) is hereditarily epi- α -normal.

It is officious to observe that an analogous of Theorem 17 is not satisfied for epi- β -normality. Only parts (1) and (2) hold for hereditarily epi- β -normal spaces. For example $X = [0, 1]$ with the usual topology is Hausdorff hereditarily β -normal space hence is hereditarily epi- β -normal but does not satisfies condition (3) of Theorem 17 (for details, see [19]). $\square$

Theorem 17. *For each topological space (Y, τ) , the following cases are equivalent:*

- (1) *The space (Y, τ) is hereditarily epi- β -normal.*
- (2) *Any subspace (A, τ_A) of (Y, τ) in which (A, τ'_A) is an open subspace of a coarser topological space (Y, τ') of (Y, τ) is epi- β -normal.*

Proof. The implication (1 $\Rightarrow$ 2) is clear. For (2 $\Rightarrow$ 1), let (A, τ_A) be any subspace of an epi- β -normal space (Y, τ) . Then, there is a coarser topology τ' such that (Y, τ') is T_1 β -normal. It is enough to show that the coarser space (A, τ'_A) is T_1 β -normal as a subspace of (Y, τ') . Since (Y, τ') is T_1 , then so is (A, τ'_A) .

Now let F_1 and F_2 be any disjoint closed subsets of (A, τ'_A) . Clearly, $F_1 = cl_{\tau'_A}(F_1) \cap A$ and $F_2 = cl_{\tau'_A}(F_2) \cap A$. Consider $Z = Y, (cl_{\tau'_A}(F_1) \cap cl_{\tau'_A}(F_2))$; then, (Z, τ'_Z) is an open subspace of (Y, τ') with F_1 and F_2 being disjoint closed subsets of (Z, τ'_Z) , and it follows that (Z, τ_Z) as a subspace of (Y, τ') is epi- β -normal. So, there exist open sets U_1, U_2 in (Z, τ_Z) such that $U_1 \cap F_1$ is dense in F_1 and $U_2 \cap F_2$ is dense in F_2 with respect to the topology τ'_Z , and $cl_{\tau'_Z}(U_1) \cap cl_{\tau'_Z}(U_2) = \emptyset$. Since $A \subseteq Z$, we have that $U_1 \cap A$ and $U_2 \cap A$ are open disjoint subsets of A with $F_1 \cap U_1 \cap A$ being dense in F_1 and $F_2 \cap U_2 \cap A$ being dense in F_2 with respect to τ'_A and $cl_{\tau'_A}(U_1 \cap A) \cap cl_{\tau'_A}(U_2 \cap A) = \emptyset$. That is, (A, τ'_A) is β -normal, and hence Y is epi-hereditarily β -normal. $\square$

5. Relations with Other Topological Properties

Theorem 18

- (1) *A space is epinormal if and only if it is epi-mildly normal and the coarser topology of epi-mildly normal is β -normal.*

- (2) A space is epinormal if and only if it is epi- β -normal and the coarser topology of epi- β -normal is mildly normal.

Proof. We prove (1), and the proof of (2) is similar.

(1) ($\longrightarrow$) (2) The first direction is clear. Now, we prove the second direction.

($\longrightarrow$) Let Y be epi-mildly normal such that there is a coarser topology τ' in which (Y, τ') is T_2 mildly normal and β -normal; therefore, by Theorem 2.4 in [5], (Y, τ') is T_2 normal. Hence, (Y, τ) is epinormal.

However, epi-mild normality does not imply epi- β -normality nor epi- α -normality, and here is an example. $\square$

Example 2. Let $(\mathbb{N}, \mathcal{RPS})$ be the relatively prime integer topology $\mathcal{RPS}$ [6]; then, $(\mathbb{N}, \mathcal{RPS})$ is Hausdorff not completely Hausdorff. Hence, it is not epi- β -normal space. Also, the space $(\mathbb{N}, \mathcal{RPS})$ is first countable, so it is not epi- α -normal. However, $(\mathbb{N}, \mathcal{RPS})$ is epi-mildly normal space (see [20]).

Recall that a topological space Y is called partially normal [21] if for any two disjoint subsets A_1 and A_2 of Y , where A_1 is closed domain and A_2 is π -closed, there exist two disjoint open subsets H_1 and H_2 of Y such that $A_1 \subseteq H_1$ and $A_2 \subseteq H_2$. The topology on Y generated by the family of all open domain is denoted by τ_s . The space (Y, τ_s) is called the semi-regularization of Y [8].

Theorem 19. If (Y, τ) is a seminormal and partial normal space and τ_s is T_1 , then (Y, τ) is epi- β -normal.

Proof. It is enough to show that (Y, τ_s) is β -normal. Let A_1 and A_2 be any disjoint closed sets in (Y, τ_s) , then $Y - A_2$ is an open set containing the closed set A_1 . By seminormality of (Y, τ) , there is an open set U_1 such that $A_1 \subseteq U_1 \subseteq \text{int}(\overline{U_1}) \subseteq Y - A_2$. Since $\text{int}(\overline{U_1})$ is regularly open and by using Theorem 2.11 in [20], there exists an open set U_2 in (Y, τ_s) such that $A_1 \subseteq U_2 \subseteq \overline{U_2} \subseteq \text{int}(\overline{U_1}) \subseteq Y - A_2$. Therefore, $A_1 \cap \overline{U_2} \subseteq \overline{U_2} \subseteq \text{int}(\overline{U_1}) \subseteq Y - A_2$; then, (Y, τ_s) is β -normal. Hence, (Y, τ) is epi- β -normal.

Note that epi- β -normality does not imply normality. For example, Either-or Topology [6]. It is normal because the only disjoint closed sets are the ground set and the empty set. But it is not epi- β -normal because it is not completely Hausdorff. Also, epi- β -normality does not imply almost normality; a good example for this is right order topology [6], where the topology is generated by basis of the form $P_a = \{y: y > a, a \in \mathbb{R}\}$. This topology is almost normal because there are no nonempty disjoint closed sets. But it is not epi- β -normal because it is not completely Hausdorff space.

Moreover, epi- β -normality does not imply partial normality. a basic open neighborhood for the irrational point is isolated, and a basic open neighborhood for a rational point is the same as in the usual topology and consider $\mathcal{M} \times \mathbb{P}$ denote the Michael product, where $\mathbb{P}$ with the usual topology, see [11]. Since all minimal completely regular spaces

are compact, we have the following theorem. Observe that $\mathcal{M} \times \mathbb{P}$ is epi- β -normal, by taking the coarser topology $\mathcal{U} \times \mathbb{P}$ of $\mathcal{M} \times \mathbb{P}$. Then, $\mathcal{M} \times \mathbb{P}$ is Hausdorff β -normal, since it is metrizable. Since all minimal completely regular spaces are compact [[23], 3.3]. $\square$

Theorem 20. In the class of minimal Tychonoff spaces, any epi-mildly normal space is epi- β -normal.

Since any nontrivial Mrówka space $\Psi(A)$ is epinormal (see [24]) and it is not α -normal (see [5]), we conclude the following.

Theorem 21. Any nontrivial Mrówka space $\Psi(A)$ is epi- α -normal non- α -normal.

Since any almost normal T_1 is almost regular and every almost regular semiregular space is regular, then we have the following theorem.

Theorem 22. In semiregular space, every epi-almost normal space is epi- β -normal.

Recall that a Hausdorff space Y is called H -closed if Y is a closed subspace of every Hausdorff containing it (see [[11], 3.12.5]). Since a regular space is H -closed if and only if is compact and since every epi- β -normal is epi-regular (see Corollary 1), we will have the following.

Corollary 12. Let (Y, τ) be epi- β -normal compact space; then, the coarser topology of epi- β -normality (Y, τ') is H -closed.

Since any regular Lindelöf space is normal, then the following is true.

Corollary 13. Let (Y, τ) be epi- β -normal, and the coarser topology of epi- β -normality (Y, τ') is Lindelöf; then, (Y, τ) is epinormal.

Recall that a topological space Y is called C -paracompact [25] if there exist a paracompact space Z and a bijective function $h: Y \longrightarrow Z$ such that the restriction $h|_K: K \longrightarrow h(K)$ is a homeomorphism for each compact subspace $K \subseteq Y$.

Theorem 23. Let (Y, τ) be a C_2 -paracompact and Fréchet; then, (Y, τ) is epi- β -normal.

Proof. Let (Y, τ) be C_2 -paracompact and Fréchet; then, (Y, τ) is epinormal by Theorem 2.16 in [25], and hence it is epi- β -normal. $\square$

Theorem 24. If (Y, τ) is a Hausdorff nearly compact space, then (Y, τ) is epi- β -normal.

Proof. Let (Y, τ) be a Hausdorff nearly compact space, and let τ_s be the semi-regularization of τ , then τ_s is a Hausdorff nearly compact space. Therefore, τ_s is T_4 and hence T_2 and β -normal. Thus, (Y, τ) is epi- β -normal.

Recall that a topological space Y is called extremally disconnected if the closure of any open subset of Y is open [22].

Since every α -normal, extremely disconnected space is normal [5], then we have the following. $\square$

Corollary 14. *If Y is an $\text{epi-}\alpha$ -normal space and the coarser topology of $\text{epi-}\alpha$ -normality is extremely disconnected, then Y is epinormal .*

Epi- β -normality does not imply extreme disconnectedness. For example, weak parallel topology [6] is $\text{epi-}\beta$ -normal but not extremely disconnected.

Corollary 15. *Any $\text{epi-}\beta$ -normal zero-dimensional space is semiregular.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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