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Research Article

Strong Convergence of an Inertial Iterative Algorithm for Generalized Mixed Variational-like Inequality Problem and Bregman Relatively Nonexpansive Mapping in Reflexive Banach Space

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In this paper, we consider a generalized mixed variational-like inequality problem and prove a Minty-type lemma for its related auxiliary problems in a real Banach space. We prove the existence of a solution of these auxiliary problems and also prove some properties for the solution set of generalized mixed variational-like inequality problem. Furthermore, we introduce and study an inertial hybrid iterative method for solving the generalized mixed variational-like inequality problem involving Bregman relatively nonexpansive mapping in Banach space. We study the strong convergence for the proposed algorithm. Finally, we list some consequences and computational examples to emphasize the efficiency and relevancy of the main result.

1. Introduction

Throughout the paper, unless otherwise stated, let X be a reflexive Banach space with X^* as its dual and $C \neq \emptyset$ be the closed convex subset of X. In this paper, we consider the generalized mixed variational-like inequality problem (in brief, GMVLIP): find $u \in C$ such that

$$G(v, u; u) + b(u, v) - b(u, u) \geq 0, \quad \forall v \in C, \tag{1}$$

where $b\colon C \times C \longrightarrow \mathbb{R}$ and $G\colon C \times C \times C \longrightarrow \mathbb{R}$, be bifunction and trifunction, respectively, and $\mathbb{R}$ be the set of real numbers. Sol (GMVLIP equation (1)) stands for the solution of equation (1). If $b \equiv 0$, GMVLIP equation (1) is reduced to GVLIP: find $u \in C$ such that

$$G(v, u; u) \geq 0, \quad \forall v \in C, \tag{2}$$

which is introduced by Preda et al. [1] (see, for instance, [2, 3]).

If we set $G(v, u; u) = \langle Du + Au, \eta(v, u)\rangle$, where $D, A\colon C \longrightarrow X$ and $\eta\colon C \times C \longrightarrow X$, GMVLIP equation (1) is reduced to MVLIP (see for details [4]).

Further, if we set $G(v, u; u) = \langle Du\ , \eta(v, u)\rangle$ and $b \equiv 0$, GMVLIP equation (1) is reduced to VLIP: find $u \in C$ such that

$$\langle Du, \eta(v, u)\rangle \geq 0, \quad \forall v \in C, \tag{3}$$

which is presented by Parida et al. [5].

Moreover, if $\eta(v, u) = v - u$, VLIP is reduced to VIP: find $u \in C$ such that

$$\langle Du, v-u\rangle \geq 0, \quad \forall v \in C, \tag{4}$$

which is introduced by Hartmann and Stampacchia [6].

If $b \equiv 0$, $X = \mathbb{R}^n$, and $G(v,u;u) = \langle \nabla Du, \eta(v,u)\rangle$, where η is continuous and D is differentiable and η-convex, GMVLIP equation (1) is reduced to mathematical programming problem as [5]

$$\min_{u\in C} D(u). \tag{5}$$

Korpelevich [7] proposed the iterative method for VIP in 1976 on Hilbert space H as

$$\left.\begin{aligned} &u_0 \in C \subseteq H,\\ &v_n = \operatorname{proj}_C(u_n - \sigma Du_n),\\ &u_{n+1} = \operatorname{proj}_C(u_n - \sigma Dv_n), \quad n \geq 0, \end{aligned}\right\} \tag{6}$$

where $\sigma > 0$, proj_C denotes projection of H onto C, and D is monotone and Lipschitz continuous mapping. This method is called the extragradient iterative method.

Nadezkhina and Takahashi [8] proposed a hybrid extragradient algorithm involving nonexpansive mapping T on C and studied the convergence analysis in 2006 as

$$\left.\begin{aligned} &u_0 \in C \subseteq H,\\ &x_n = \operatorname{proj}_C(u_n - \sigma_n Du_n),\\ &v_n = \alpha_n u_n + (1-\alpha_n) T \operatorname{proj}_C(u_n - \sigma_n Dx_n),\\ &C_n = \left\{z \in C\colon \|v_n - z\|^2 \leq \|u_n - z\|^2\right\},\\ &D_n = \{z \in C\colon \langle u_n - z, u_0 - u_n\rangle \geq 0\},\\ &u_{n+1} = \operatorname{proj}_{C_n\cap D_n} u_0, n \geq 0. \end{aligned}\right\} \tag{7}$$

The idea considered in [8] has been generalized in [9] from Hilbert to Banach space X as

$$\left.\begin{aligned} &u_0 \in C \subseteq X,\\ &v_n = J^{-1}(\alpha_n J u_n + (1-\alpha_n) JTu_n),\\ &C_n = \{z \in K\colon \phi(z, v_n) \leq \phi(z, u_n)\},\\ &D_n = \{z \in K\colon \langle u_n - z, Ju_0 - Ju_n\rangle \geq 0\},\\ &u_{n+1} = \prod_{C_n\cap D_n} u_0, \end{aligned}\right\} \tag{8}$$

where Π_C denotes generalized projection of X onto C, ϕ is the Lyapunov function such that $\phi(u,v) = \|v\|^2 - 2\langle v, Ju\rangle + \|u\|^2$, $\forall u, v \in X$, and $J\colon X \longrightarrow 2^{X^*}$ is the normalized duality mapping with J^{-1} being its inverse. For further work, see [10–17].

In 1967, an important technique was discovered by Bregman [18] in the light of Bregman distance function. This technique is very useful not only in design and interpretation of the iterative method but also to solve optimization and feasibility problems and to approximate equilibria, fixed point, variational inequalities, etc. (for details [19–22]).

In 2010, Reich and Sabach [23] introduced iterative algorithm on Banach space involving maximal monotone operators. In the light of Bregman projection, there were various iterative algorithms studied by researchers in this field (see, for instance, [19, 24–28]).

In 2008, Maingé [29] developed and studied an inertial Krasnosel'skiĭ–Mann algorithm as

$$\left.\begin{aligned} &t_n = u_n + \theta_n(u_n - u_{n-1}),\\ &u_{n+1} = (1-\alpha_n) t_n + \alpha_n T t_n. \end{aligned}\right\} \tag{9}$$

For further work, see [30–39].

Inspired by the work in [2, 27, 29], we establish an inertial hybrid iterative algorithm involving Bregman relatively nonexpansive mapping to find a common solution of GMVLIP equation (1) and a fixed-point problem in Banach space. Moreover, we study the convergence analysis for the main result. At last, we list some consequences and computational example to emphasize the efficiency and relevancy of the main result.

2. Preliminaries

Assume $g\colon X \longrightarrow (-\infty, +\infty]$ is a proper, convex, and lower semicontinuous mapping and $g^*\colon X^* \longrightarrow (-\infty, +\infty]$ is a Fenchel conjugate of g, defined as

$$g^*(u_0) = \sup\{\langle u_0, u\rangle - g(u)\colon u \in Y\}, \quad u_0 \in Y^*. \tag{10}$$

And, for any $w \in \operatorname{int}(\operatorname{dom} g)$, interior of the domain of g and $u \in X$, the right-hand derivative of g at w in the direction u is

$$g^0(w,u) = \lim_{\lambda\longrightarrow 0^+} \frac{g(w+\lambda u) - g(w)}{\lambda}. \tag{11}$$

A mapping g is called Gateaux differentiable at w if the above limit exists. So, $g^0(w,u)$ agrees with $\nabla g(w)$, the value of the gradient of g at w. It is called Frechet differentiable at w, if the limit is attained uniformly in $\|u\| = 1$. It is called uniformly Frechet differentiable on $C \subseteq X$, if the above limit is attained uniformly for $w \in C$ and $\|u\| = 1$.

The mapping g is called Legendre if the following holds [19]:

(i) $\operatorname{int}(\operatorname{dom} g) \neq \emptyset$, g is Gateaux differentiable on $\operatorname{int}(\operatorname{dom} g)$, and $\operatorname{dom}\nabla g = \operatorname{int}(\operatorname{dom} g)$

(ii) $\operatorname{int}(\operatorname{dom} g^*) \neq \emptyset$, g^* is Gateaux differentiable on $\operatorname{int}(\operatorname{dom} g^*)$, and $\operatorname{dom}\nabla g^* = \operatorname{int}(\operatorname{dom} g^*)$

We have the following [19]:

(i) g be Legendre iff g^* be Legendre mapping

(ii) $(\partial g)^{-1} = \partial g^*$

(iii) $\nabla g = (\nabla g^*)^{-1}$, $\operatorname{ran}\nabla g = \operatorname{dom}\nabla g^* = \operatorname{int}(\operatorname{dom} g^*)$, $\operatorname{ran}\nabla g^* = \operatorname{dom}\nabla g = \operatorname{int}(\operatorname{dom} g)$

(iv) The mappings g and g^* are strictly convex on $\operatorname{int}(\operatorname{dom} g)$ and $\operatorname{int}(\operatorname{dom} g^*)$

Definition 1 (see [18]). Let $g\colon Y \longrightarrow (-\infty, +\infty]$ be Gateaux differentiable and convex and $D_g\colon \operatorname{dom} g \times \operatorname{int}(\operatorname{dom} g) \longrightarrow [0, +\infty)$ such that

$$D_g(u,w) = g(u) - g(w) - \langle \nabla g(w), u - w\rangle, \quad w \in \text{int}(\text{dom}g), u \in \text{dom}g, \tag{12}$$

is known as Bregman distance with respect to g.

We notice that the Bregman distance is not a distance in the usual sense of term. Obviously, $D_g(w,w) = 0$, but $D_g(w,u) = 0$ may not imply $w = u$. It holds if g is the Legendre function. However, D_g is neither symmetric nor satisfy the triangle inequality. We have the following important properties of D_g [40] for $u, u_1, u_2 \in (\text{dom}g)$ and $w_1, w_2 \in \text{int}(\text{dom}g)$.

(i) Two-point identity:

$$D_g(w_1,w_2) + D_g(w_2,w_1) = \langle \nabla g(w_1) - \nabla g(w_2), w_1 - w_2\rangle. \tag{13}$$

(ii) Three-point identity:

$$D_g(u,w_1) + D_g(w_1,w_2) - D_g(u,w_2) = \langle \nabla g(w_2) - \nabla g(w_1), u - w_1\rangle. \tag{14}$$

(iii) Four-point identity:

$$D_g(u_1,w_1) - D_g(u_1,w_2) - D_g(u_2,w_1) + D_g(u_2,w_2) = \langle \nabla g(w_2) - \nabla g(w_1), u_1 - u_2\rangle. \tag{15}$$

Definition 2 (see [23, 25]). Let $T: C \longrightarrow \text{int}(\text{dom}g)$ be a mapping and $F(T) = \{u \in C: Tu = u\}$, where $F(T)$ is the set of fixed points of T. Then, we have the following:

(i) A point $u_0 \in C$ is called an asymptotic fixed point if C contains a sequence $\{u_n\}$ with $u_n \rightharpoonup u_0$ such that $\lim_{n\longrightarrow\infty} \|Tu_n - u_n\| = 0$. We represent $\widehat{F}(T)$ as the set of asymptotic fixed points of T.

(ii) T is called Bregman quasi-nonexpansive if

$$F(T) \neq \varnothing; D_g(u_0,Tu) \leq D_g(u_0,u), \quad \forall u \in C,\ u_0 \in F(T). \tag{16}$$

(iii) T is called Bregman relatively nonexpansive if

$$\begin{aligned} &F(T) = \widehat{F}(T) \neq \varnothing; D_g(u_0,Tu) \leq D_g(u_0,u),\\ &\quad \forall u \in C,\ u_0 \in F(T). \end{aligned} \tag{17}$$

(iv) T is called Bregman firmly nonexpansive if $\forall u_1, u_2 \in C$,

$$\langle \nabla g(Tu_1) - \nabla g(Tu_2), Tu_1 - Tu_2\rangle \leq \langle \nabla g(u_1) - \nabla g(u_2), Tu_1 - Tu_2\rangle, \tag{18}$$

or, correspondingly,

$$D_g(Tu_1,Tu_2) + D_g(Tu_2,Tu_1) + D_g(Tu_1,u_1) + D_g(Tu_2,u_2) \leq D_g(Tu_1,u_2) + D_g(Tu_2,u_1). \tag{19}$$

Example 1 (see [26]). Let $A: X \longrightarrow 2^{X^*}$ be a maximal monotone mapping. If $A^{-1}(0) \neq \varnothing$ and the Legendre function $g: X \longrightarrow (-\infty,+\infty]$ is bounded on bounded subsets of X and uniformly Frechet differentiable, then the resolvent with respect to A,

$$\text{res}_A^g(u) = (\nabla g + A)^{-1} \circ \nabla \text{g}(\text{u}), \tag{20}$$

is a single-valued, closed, and Bregman relatively nonexpansive mapping from X onto $D(A)$ and $F(\mathrm{res}_A^g) = A^{-1}(0)$.

Definition 3 (see [18]). Let $g: X \longrightarrow (-\infty, +\infty]$ be a Gateaux differentiable and convex function. The Bregman projection of $w \in \mathrm{int}(\mathrm{dom}g)$ onto $C \subset \mathrm{int}(\mathrm{dom}g)$ is a unique vector $\mathrm{proj}_C^g w \in C$ with

$$D_g(\mathrm{proj}_C^g(w), w) = \inf\{D_g(u, w): u \in C\}. \tag{21}$$

Remark 1 (see [24]). (i) If X is a smooth Banach space and $g(u) = (1/2)\|u\|^2$, $\forall u \in X$, then the Bregman projection $\mathrm{proj}_C^g(u)$ reduces to $\Pi_C(u)$, generalized projection (see [41]), and it is defined as

$$\phi(\Pi_C(u), u) = \min_{v \in C} \phi(v, u), \tag{22}$$

where ϕ is a Lyapunov function. (ii) If X is a Hilbert space and $g(u) = (1/2)\|u\|^2$, $\forall u \in X$, then $\mathrm{proj}_C^g(u)$ reduces to the metric projection of u onto C.

For all $r > 0$, assume $B_r: = \{z \in X: \|z\| \le r\}$. Then, a map $g: X \longrightarrow \mathbb{R}$ is said to be uniformly convex on bounded subsets of X, if $\rho_r(t) > 0$, $\forall t > 0$, where $\rho_r: [0, +\infty) \longrightarrow [0, +\infty)$ is defined as

$$\rho_r(t) = \inf_{w, v \in B_r, \|w - v\| = t, \alpha \in (0,1)} \frac{\alpha g(w) + (1 - \alpha) g(v) - g(\alpha w + (1 - \alpha) v)}{\alpha(1 - \alpha)}, \tag{23}$$

$\forall t \ge 0$. The function ρ_r is known as the gauge of uniform convexity of g. The function g is also said to be uniformly smooth on bounded subsets of X if $\lim_{t \longrightarrow 0} (\sigma_r(t)/t) = 0$, for all $r > 0$, where $\sigma_r: [0, +\infty) \longrightarrow [0, +\infty)$ is defined by

$$\sigma_r(t) = \sup_{w \in B_r, v \in S_X, \alpha \in (0,1)} \frac{\alpha g(w + (1 - \alpha) t v) + (1 - \alpha) g(w - \alpha t v) - g(w)}{\alpha(1 - \alpha)}, \tag{24}$$

$\forall t \ge 0$. The function g is said to be uniformly convex if the function $\delta_g: [0, +\infty) \longrightarrow [0, +\infty)$, defined by

$$\delta_g(t) \coloneqq \sup\left\{\frac{1}{2} g(w) + \frac{1}{2} g(v) - g\left(\frac{w + v}{2}\right): \|v - w\| = t\right\}, \tag{25}$$

satisfies that $\lim_{t \longrightarrow 0} (\sigma_r(t)/t) = 0$.

Remark 2. Let X be a Banach space, $r > 0$ be a constant, and $g: X \longrightarrow \mathbb{R}$ be a convex function which is uniformly convex on bounded subsets. Then,

$$g(\alpha w + (1 - \alpha) v) \le \alpha g(w) + (1 - \alpha) g(v) - \alpha(1 - \alpha) \rho_r(\|w - v\|), \tag{26}$$

for all $w, v \in B_r$ and $\alpha \in (0, 1)$, where ρ_r is the gauge of uniform convexity of g.

Definition 4 (see [20]). Let $g: X \longrightarrow (-\infty, +\infty]$ be a Gateaux differentiable and convex function. Then, g is called the following:

(i) Totally convex at $w \in \mathrm{int}(\mathrm{dom}g)$ if its modulus of total convexity at u, i.e., the mapping $v_g: \mathrm{int}(\mathrm{dom}g) \times [0, +\infty) \longrightarrow [0, +\infty)$ such that

$$v_g(w, s) = \inf\{D_g(v, w): v \in \mathrm{dom}g, \|v - w\| = s\}, \tag{27}$$

is positive, for $s > 0$

(ii) Totally convex if it is totally convex at each point of $w \in \mathrm{int}(\mathrm{dom}g)$

(iii) Totally convex on bounded sets if $v_g: \mathrm{int}(\mathrm{dom}g) \times [0, +\infty) \longrightarrow [0, +\infty)$ such that

$$v_g(B, s) = \inf\{v_g(w, s): w \in B \cap \mathrm{dom}g\}. \tag{28}$$

By [20] (Section 1.3, p.30), we notice that any uniformly convex function is totally convex but the converse is not true. Also, by [21] (Theorem 2.10, p.9), g is totally convex on bounded sets if and only if g is uniformly convex on bounded sets.

Definition 5 (see [20, 23]). A mapping $g: X \longrightarrow (-\infty, +\infty]$ is called the following:

(i) Coercive if $\lim_{\|u\| \longrightarrow +\infty} (g(u)/\|u\|) = +\infty$

(ii) Sequentially consistent if for any $\{u_n\}, \{v_n\} \subseteq X$ with $\{u_n\}$ bounded,

$$\lim_{n\longrightarrow\infty} D_g(v_n, u_n) = 0 \Rightarrow \lim_{n\longrightarrow\infty}\|v_n - u_n\| = 0. \tag{29}$$

Lemma 1 (see [21]). *Let g: $X \longrightarrow (-\infty, +\infty]$ be a convex function with domain at least two points. Then, g is sequentially consistent iff it is totally convex on bounded sets.*

Lemma 2 (see [42]). *Let g: $X \longrightarrow (-\infty, +\infty]$ be uniformly Frechet differentiable and bounded on $C \subseteq X$, a bounded set. Then, g is uniformly continuous on C and ∇g is uniformly continuous on C from the strong topology of X to the strong topology of X^*.*

Lemma 3 (see [23]). *Let g: $X \longrightarrow (-\infty, +\infty]$ be a Gateaux differentiable and totally convex function. If $u_0 \in X$ and $\{D_g(u_n, u_0)\}$ are bounded, then $\{u_n\}$ is also bounded.*

Lemma 4 (see [21]). *Let g: $X \longrightarrow (-\infty, +\infty]$ be a Gateaux differentiable and totally convex function on* int$(domg)$. *Let $w \in$ int$(domg)$ and $C \subseteq int(domg)$, a nonempty closed convex set. If $v \in C$, then the following statements are equivalent:*

(i) $v \in C$ is the Bregman projection of w onto C with respect to g, i.e., $v = proj_C^g(w)$

(ii) The vector v is the unique solution of the variational inequality:

$$\langle \nabla g(w) - \nabla g(v), v - u\rangle \geq 0, \quad \forall u \in C \tag{30}$$

(iii) The vector v is the unique solution of the inequality:

$$D_g(u, v) + D_g(v, w) \leq D_g(u, w), \quad \forall u \in C \tag{31}$$

Lemma 5 (see [25]). *Let g: $X \longrightarrow (-\infty, +\infty]$ be Legendre and T: $C \longrightarrow C$ be Bregman quasi nonexpansive mapping with respect to g. Then, $F(T)$ is closed and convex.*

Lemma 6 (see [23]). *Let g: $X \longrightarrow (-\infty, +\infty]$ be Gateaux differentiable and totally convex function, $u_0 \in X$, and $C \subseteq X$, a nonempty closed convex set. Suppose that $\{u_n\}$ is bounded and any weak subsequential limit of $\{u_n\}$ belongs to C. If $D_g(u_n, u_0) \leq D_g(proj_C^g u_0, u_0)$, then $\{u_n\}$ strongly converges to $proj_C^g u_0$.*

Lemma 7 (see [43]). *Let C be a nonempty subset of a Hausdroff topological vector space X^* and let f: $C \longrightarrow 2^X$ be a KKM mapping. If $f(v)$ is closed in X^* for all $v \in C$ and compact for some $v \in C$, then $\bigcap_{v\in C} f(v) \neq \emptyset$.*

Definition 6 (see [1]). A function G: $C \times C \times C \longrightarrow \mathbb{R}$ is said to be generalized relaxed α-monotone if for any $u, v \in C$, we have

$$G(v, u; v) - G(v, u; u) \geq \alpha(u, v), \tag{32}$$

where

$$\lim_{t\longrightarrow 0} \frac{\alpha(u, tv + (1-t)u)}{t} = 0. \tag{33}$$

Remark 3

(i) If $G(v, u; w) = \langle Aw, \eta(v, u)\rangle$, where η: $C \times C \longrightarrow X$, we say that the mapping A is a generalized η-α monotone

(ii) In Definition 6, let $G(v, u; w) = \langle Aw, \eta(v, u)\rangle$ and $\alpha(u, v) = \beta(v - u)$, where β: $C \longrightarrow R$ with $\beta(tw) = t^p\beta(w)$, for $t > 0$ and $p > 1$, then we say that A is called a relaxed η-α monotone mapping

(iii) In case (ii), if $\eta(v, u) = v - u$ for all $u, v \in C$, then Definition 6 reduces to $\langle Av - Au, v - u\rangle \geq \beta(v - u)$ for all $u, v \in C$ and A is called a relaxed α-monotone mapping

(iv) In case (iii), if $\beta(w) = k\|w\|^p$, where $k > 0$ is a constant, then Definition 6 reduces to $\langle Av - Au, v - u\rangle \geq k\|v - u\|^p$ for all $u, v \in C$ and A is called a p-monotone mapping

(v) If $\alpha \equiv 0$, then (iii) reduces to $\langle Av - Au, v - u\rangle \geq 0$ for all $u, v \in C$ and A is called a monotone mapping

We construct an example for generalized relaxed α-monotone mapping as follows.

Example 2. Consider $X = X^*$, $C = (-\infty, \infty)$, and

$$G(v, u; w) = \begin{cases} -cw((v - u), & v < u, \\ cw(v - u), & v \geq u, \end{cases} \tag{34}$$

where $c > 0$ is a constant. Thus, G is generalized relaxed α-monotone with

$$\alpha(u, v) = \begin{cases} -c(v - u)^2, & v < u, \\ c(v - u)^2, & v \geq u. \end{cases} \tag{35}$$

Assumption 1. Let b: $C \times C \longrightarrow \mathbb{R}$ satisfy the following:

(i) b is skew-symmetric, i.e., $b(u, u) - b(u, v) - b(v, u) + b(v, v) \geq 0$, $\forall u, v \in C$

(ii) b is convex in the second argument

(iii) b is continuous

3. Existence of Solutions and Resolvent Operator

For $w \in C$, assume the auxiliary problems (in short, AP) related to GMVLIP equation (1): find $u \in C$ such that

$$G(v, u; u) + \langle \nabla g(u) - \nabla g(w), v - u\rangle + b(u, v) - b(u, u) \geq 0, \quad \forall v \in C, \tag{36}$$

and find $u \in C$ such that

$$G(v,u;v) + \langle \nabla g(u) - \nabla g(w), v-u\rangle + b(u,v) - b(u,u) \geq \alpha(u,v), \quad \forall v \in C. \tag{37}$$

We have the Minty-type lemma as follows.

Lemma 8. *Let g: $X \longrightarrow (-\infty,+\infty]$ be Gateaux differentiable and coercive function, and let b: $C\times C \longrightarrow \mathbb{R}$ satisfy Assumption 1 (ii). Assume G: $C\times C\times C \longrightarrow \mathbb{R}$ with the following cases:*

(i) $G(v,u;.)$ is hemicontinuous

(ii) $G(.,u;w)$ is convex

(iii) $G(u,u;w) = 0$

(iv) G is a generalized relaxed α-monotone

Then, AP equation (36) and AP equation (37) are equivalent.

Proof. Let $u \in C$ be a solution of AP equation (36) and by the concept of G, we obtain

$$\begin{aligned}&G(v,u;v) + \langle \nabla g(u) - \nabla g(w), v-u\rangle + b(u,v) - b(u,u)\\ &\geq G(v,u;u) + \langle \nabla g(u) - \nabla g(w), v-u\rangle + b(u,v) - b(u,u) + \alpha(u,v)\\ &\geq \alpha(u,v),\end{aligned} \tag{38}$$

which shows that $u \in C$ is a solution of AP equation (37).

Conversely, let $u \in C$ be a solution of AP equation (37). Then,

$$G(v,u;v) + \langle \nabla g(u) - \nabla g(w), v-u\rangle + b(u,v) - b(u,u) \geq \alpha(u,v), \quad \forall v \in C. \tag{39}$$

For any $v \in C$, let $v_t = tv + (1-t)u$, $t \in (0,1]$, and we get $v_t \in C$. By equation (39), we have

$$G(v_t,u;v_t) + \langle \nabla g(u) - \nabla g(w), v_t-u\rangle + b(u,v_t) - b(u,u) \geq \alpha(u,v_t). \tag{40}$$

Using conditions (ii) and (iii), we obtain

$$G(v_t,u;v_t) \leq t\psi(v,u;v_t) + (1-t)\psi(u,u;v_t) = t\psi(v,u;v_t). \tag{41}$$

By Assumption 1 (ii), we have

$$\langle \nabla g(u) - \nabla g(w), v_t-u\rangle = t\langle \nabla g(u) - \nabla g(w), v-u\rangle \tag{42}$$

and

$$b(u,v_t) \leq tb(u,v) + (1-t)b(u,u). \tag{43}$$

Using equations (40)–(43), we have

$$\begin{aligned}&tG(v,u;v_t) + t\langle \nabla g(u) - \nabla g(w), v-u\rangle + tb(u,v) + (1-t)b(u,u) - b(u,u)\\ &\geq G(v_t,u;v_t) + \langle \nabla g(u) - \nabla g(w), v_t-u\rangle + b(u,v_t) - b(u,u)\\ &tG(v,u;v_t) + t\langle \nabla g(u) - \nabla g(w), v-u\rangle + tb(u,v) - tb(u,u) \geq \alpha(u,v_t).\end{aligned} \tag{44}$$

Hence,

$$G(v,u;v_t) + \langle \nabla g(u) - \nabla g(w), v-u\rangle + b(u,v) - b(u,u) \geq \frac{\alpha(u,v_t)}{t}. \tag{45}$$

Let $t \longrightarrow 0$, and by condition (i), we obtain

$$G(v,u;u) + \langle \nabla g(u) - \nabla g(w), v-u\rangle + b(u,v) - b(u,u) \geq 0. \tag{46}$$

Thus, $u \in C$ be a solution of AP equation (95). □

Theorem 1. *Let $g: X \longrightarrow (-\infty,+\infty]$ be a Gateaux differentiable and coercive function, $b: C\times C \longrightarrow \mathbb{R}$ satisfy Assumption 1 (ii)-(iii), and $\alpha: C\times C \longrightarrow \mathbb{R}$ be a bifunction. Consider $G: C\times C\times C \longrightarrow \mathbb{R}$ and for any $u,v,w \in C$, assume the following:*

(i) $G(v,u;.)$ is hemicontinuous
(ii) $G(.,u;w)$ is convex and lower semicontinuous
(iii) $G(u,v;w) + G(v,u;w) = 0$
(iv) G is a generalized relaxed α-monotone
(v) $\alpha(.,v)$ is lower semicontinuous

Then, AP equation (36) has solution.

Proof. Let $F_w, G_w: C \longrightarrow 2^C$, for any $w \in C$, be two set-valued mappings with

$$F_w(v) = \{u \in C: G(v,u;u) + \langle \nabla g(u) - \nabla g(w), v-u\rangle + b(u,v) - b(u,u) \geq 0\}, \quad \forall v \in C, \tag{47}$$

and

$$G_w(v) = \{u \in C: G(v,u;v) + \langle \nabla g(u) - \nabla g(w), v-u\rangle + b(u,v) - b(u,u) \geq \alpha(u,v)\}, \quad \forall v \in C. \tag{48}$$

Obviously, $\overline{u} \in C$ solves AP equation (36) if and only if $\overline{u} \in \cap_{v\in C} F_w(v)$. Hence, $\cap_{v\in C} F_w(v) \neq \emptyset$. Next, we prove that F_w is a KKM mapping. On the contrary, let F_w be not a KKM mapping; then, $\exists \{v_1, v_2, \ldots, v_m\} \subset C$ such that $co\{v_1, v_2, \ldots, v_m\} \nsubseteq \cup_{i=1}^m F_w(v_i)$; this means there exists a $u_0 \in co\{v_1, v_2, \ldots, v_m\}$, $u_0 = \sum_{i=1}^m t_i v_i$ where $t_i \geq 0$, $i = 1, 2, \ldots m$, $\sum_{i=1}^m t_i = 1$, but $u_0 \notin \cup_{i=1}^m F_w(v_i)$. Then,

$$G(v_i,u_0;u_0) + \langle \nabla g(u_0) - \nabla g(w), v_i-u_0\rangle + b(u_0,v_i) - b(u_0,u_0) < 0. \tag{49}$$

By Theorem 1 (ii)-(iii), we get

$$\begin{aligned} 0 &= G(u_0,u_0;u_0) + \langle \nabla g(u_0) - \nabla g(w), u_0-u_0\rangle + b(u_0,u_0) - b(u_0,u_0) \\ &\leq \sum_{i=1}^m t_i G(v_i,u_0;u_0) + \sum_{i=1}^m t_i \langle \nabla g(u_0) - \nabla g(w), v_i-u_0\rangle + \sum_{i=1}^m t_i b(u_0,v_i) - \sum_{i=1}^m t_i b(u_0,u_0) \\ &= \sum_{i=1}^m t_i \left[G(v_i,u_0,u_0) + \langle \nabla g(u_0) - \nabla g(w), v_i-u_0\rangle + b(u_0,v_i) - b(u_0,u_0)\right] \\ &< 0, \end{aligned} \tag{50}$$

which is a contradiction. Thus, F_w is a KKM mapping.

Next, we prove that $F_w(v) \subset G_w(v)$, $\forall v \in C$. Let $u \in F_w(v)$, for any $v \in C$; then,

$$G(v,u;u) + \langle \nabla g(u) - \nabla g(w), v-u\rangle + b(u,v) - b(u,u) \geq 0. \tag{51}$$

Using the concept of G, we obtain

$$\begin{aligned} &G(v,u;v) + \langle \nabla g(u) - \nabla g(w), v-u\rangle + b(u,v) - b(u,u) \\ &\geq G(v,u;u) + \langle \nabla g(u) - \nabla g(w), v-u\rangle + b(u,v) - b(u,u) + \alpha(u,v) \\ &\geq \alpha(u,v). \end{aligned} \tag{52}$$

Thus, $F_w(v) \subset G_w(v), \forall v \in C$, which yields that $G_w(v)$ is a KKM mapping.

Let $\{u_n\}$ be any sequence in $G_w(v)$ with $u_n \longrightarrow u$ as $n \longrightarrow \infty$. Then,

$$G(v, u_n; v) + \langle \nabla g(u_n) - \nabla g(w), v - u_n \rangle + b(u_n, v) - b(u_n, u_n) \geq \alpha(u_n, v). \tag{53}$$

Since g is Gateaux differentiable function, ∇g is norm-to-weak $*$ continuous. By (ii) and (iii) and lower semi-continuity of α, we have

$$\begin{aligned}
\alpha(u, v) + G(u, v; v) &\leq \lim_{n \longrightarrow \infty} \inf \alpha(u_n, v) + \lim_{n \longrightarrow \infty} \inf G(u_n, v; v) \\
&\leq \lim_{n \longrightarrow \infty} \inf\{\alpha(u_n, v) + G(u_n, v; v)\} \\
&\leq \lim_{n \longrightarrow \infty} \sup\{\alpha(u_n, v) + G(u_n, v; v)\} \\
&= \lim_{n \longrightarrow \infty} \sup\{\alpha(u_n, v) - G(v, u_n, v)\} \\
&\leq \langle \nabla g(u) - \nabla g(w), v - u \rangle + b(u, v) - b(u, u),
\end{aligned} \tag{54}$$

which yields that $G(v, u; v) + \langle \nabla g(u) - \nabla g(w), v - u \rangle + b(u, v) - b(u, u) \geq \alpha(u, v)$. Thus, $u \in G_w(v)$ and $G_w(v)$ are the closed subset of C, $\forall v \in C$. As C is closed convex and bounded subset in X, it is weakly compact. Thus, $G_w(v)$ is also compact. By Lemmas 7 and 10, we have $\cap_{v \in C} F_w(v) = \cap_{v \in C} G_w(v) \neq \varnothing$. Therefore, AP equation (36) has a solution. □

The resolvent of G: $C \times C \times C \longrightarrow \mathbb{R}$ with respect to b is the operator $\mathrm{res}^f_{G,b}$: $X \longrightarrow 2^C$, defined as follows:

$$\mathrm{res}^g_{G,b}(u) = \{w \in C: G(v, w; w) + \langle \nabla g(w) - \nabla g(u), v - w \rangle + b(w, v) - b(w, w) \geq 0, \forall v \in C\}, \quad \forall u \in X. \tag{55}$$

We obtain some properties of the resolvent operator $\mathrm{res}^g_{G,b}$. First, we show that $\mathrm{res}^g_{G,b}(u) \neq \varnothing$ for $u \in X$ and dom $(\mathrm{res}^g_{G,b}) = X$ under some suitable conditions. □

Lemma 9. *Let g: $X \longrightarrow (-\infty, +\infty]$ be a coercive and Gateaux differentiable function. If G: $C \times C \times C \longrightarrow \mathbb{R}$ satisfies all conditions of Theorem 1 and b: $C \times C \longrightarrow \mathbb{R}$ satisfies Assumption 1, then dom $(\mathrm{res}^g_{G,b}) = X$.*

Proof. First, we prove that for any $\xi \in X^* \exists u \in C$ such that

$$G(v, u; u) + b(u, v) - b(u, u) + g(v) - g(u) - \langle \xi, v - u \rangle \geq 0, \tag{56}$$

for any $v \in C$. As g is coercive, the function h: $X \times X \longrightarrow (-\infty, +\infty]$ defined by

$$h(u, v) = g(v) - g(u) - \langle \xi, v - u \rangle \tag{57}$$

satisfies

$$\lim_{\|u - v\| \longrightarrow +\infty} \frac{h(u, v)}{\|u - v\|} = -\infty, \tag{58}$$

for each fixed $v \in C$. By Theorem 1 in [44], equation (56) holds. Now, we show that equation (56) yields

$$G(v, u; u) + b(u, v) - b(u, u) + \langle \nabla g(u), v - u \rangle - \langle \xi, v - u \rangle \geq 0, \tag{59}$$

for any $v \in C$. Assume $v_t = tv + (1 - t)u$ and $t \in (0, 1]$; we get $v_t \in C$. By equation (59) and the concept of G, we get

$$G(v_t, u; v_t) + b(u, v_t) - b(u, u) + \langle \nabla g(u), v_t - u \rangle - \langle \xi, v_t - u \rangle \geq \alpha(u, v_t), \tag{60}$$

$$\begin{aligned}
&G(tv + (1 - t)u, u; v_t) + b(u, tv + (1 - t)u) - b(u, u) \\
&\quad + g(tv + (1 - t)u) - g(v) - \langle \xi, tv + (1 - t)u - u \rangle \geq \alpha(u, v_t), \forall v \in C.
\end{aligned} \tag{61}$$

Since

$$g(tv+(1-t)u)-g(v)\leq \langle \nabla g(tv+(1-t)u), tv+(1-t)u-u\rangle, \tag{62}$$

we get from equation (61), Theorem 1 (ii), and Assumption 1 (ii) that

$$\begin{aligned} &tG(v,u;v_t)+(1-t)G(u,u;v_t)+tb(u,v)+(1-t)b(u,u)-b(u,u)\\ &\quad+\langle \nabla g(tv+(1-t)u), tv+(1-t)u-u\rangle-\langle \xi, tv+(1-t)u-u\rangle \geq \alpha(u,v_t), \quad \forall \overline{v}\in C. \end{aligned} \tag{63}$$

From Lemma 10 (iii), we have

$$tG(v,u;v_t)+tb(u,v)-tb(u,u)+\langle \nabla g(tv+(1-t)u), t(v-u)\rangle-\langle \xi, t(v-u)\rangle \geq \alpha(u,v_t) \tag{64}$$

and

$$t\left[G(v,u;v_t)+b(u,v)-b(u,u)+\langle \nabla g(tv+(1-t)u), (v-u)\rangle-\langle \xi, (v-u)\rangle\right]\geq \alpha(u,v_t). \tag{65}$$

Therefore,

$$G(v,u;v_t)+b(u,v)-b(u,u)+\langle \nabla g(tv+(1-t)u), (v-u)\rangle-\langle \xi, (v-u)\rangle \geq \frac{\alpha(u,v_t)}{t}, \quad \forall \overline{v}\in C. \tag{66}$$

As g is a Gateaux differentiable function, ∇g is norm-to-weak $*$ continuous. Taking $t\longrightarrow 0$, we have

$$G(v,u;u)+b(u,v)-b(u,u)+\langle \nabla g(u), (v-u)\rangle-\langle \xi, (v-u)\rangle \geq 0, \quad \forall \overline{v}\in C. \tag{67}$$

Thus, for any $u\in X$, let $\xi=\nabla g(\overline{u})$; we have $\overline{u}\in C$ such that

$$G(v,u;u)+b(u,v)-b(u,u)+\langle \nabla g(u), (v-u)\rangle-\langle \nabla g(\overline{u}), (v-u)\rangle \geq 0, \quad \forall \overline{v}\in C, \tag{68}$$

i.e.,

$$G(v,u;u)+b(u,v)-b(u,u)+\langle \nabla g(u)-\nabla g(\overline{u}), (v-u)\rangle \geq 0, \quad \forall \overline{v}\in C, \tag{69}$$

that is, $u\in \mathrm{res}^g_{G,b}(u)$. Hence, dom $(\mathrm{res}^g_{G,b})=X$. □ □

Lemma 10. *Let G: $C\times C\times C\longrightarrow \mathbb{R}$ satisfy all conditions of Theorem 1, and let b: $C\times C\longrightarrow \mathbb{R}$ satisfy Assumption 1. Let g: $X\longrightarrow (-\infty,+\infty]$ be a coercive Legendre function and the resolvent operator $\mathrm{res}^g_{G,b}$: $X\longrightarrow 2^C$ be defined by equation (55). Then, the following holds:*

(i) $\mathrm{res}^g_{G,b}$ is single-valued

(ii) $\mathrm{res}^g_{G,b}$ is Bregman firmly nonexpansive type mapping, that is,

$$\langle \nabla g(\text{res}^g_{G,b}u) - \nabla g(\text{res}^g_{G,b}v), \text{res}^g_{G,b}u - \text{res}^g_{G,b}v\rangle \le \langle \nabla g(u) - \nabla g(v), \text{res}^g_{G,b}u - \text{res}^g_{G,b}y\rangle, \quad \forall u, v \in X \tag{70}$$

(iii) $F(res^g_{G,b}) = Sol(GMVLIP(1))$ *is closed and convex*

(iv)

$$D_g(q, \text{res}^g_{G,b}u) + D_g(\text{res}^g_{G,b}u, u) \le D_g(q, u), \quad \forall q \in F(\text{res}^g_{G,b}). \tag{71}$$

(v) $res^g_{G,b}$ *is Bregman quasi-nonexpansive*

Proof

(i) For $u \in X$, let $w_1, w_2 \in F(\text{res}^g_{G,b})$. Then, $w_1, w_2 \in C$ and hence

$$G(w_2, w_1; w_1) + \langle \nabla g(w_1) - \nabla g(u), w_2 - w_1\rangle + b(w_1, w_2) - b(w_1, w_1) \ge 0. \tag{72}$$

and

$$G(w_1, w_2; w_2) + \langle \nabla g(w_2) - \nabla g(u), w_1 - w_2\rangle + b(w_2, w_1) - b(w_2, w_2). \tag{73}$$

Adding the above two inequalities, we get

$$\begin{aligned} &G(w_2, w_1; w_1) + G(w_1, w_2; w_2) + \langle \nabla g(w_1) - \nabla g(w_2), w_2 - w_1\rangle \\ &+ b(w_1, w_2) - b(w_1, w_1) + b(w_2, w_1) - b(w_2, w_2) \ge 0. \end{aligned} \tag{74}$$

By condition (iii) of Theorem 1, we get

$$\begin{aligned} &-G(w_1, w_2; w_1) + G(w_1, w_2; w_2) + \langle \nabla g(w_1) - \nabla g(w_2), w_2 - w_1\rangle \\ &+ b(w_1, w_2) - b(w_1, w_1) + b(w_2, w_1) - b(w_2, w_2) \ge 0. \end{aligned} \tag{75}$$

As b is skew symmetric and G is a generalized relaxed α-monotone,

$$\begin{aligned} &\alpha(w_2, w_1) - \langle \nabla g(w_1) - \nabla g(w_2), w_2 - w_1\rangle \le 0 \\ &\langle \nabla g(w_1) - \nabla g(w_2), w_2 - w_1\rangle \ge \alpha(w_2, w_1). \end{aligned} \tag{76}$$

By interchanging the position of w_1 and w_2 in equation (76), we get

$$\langle \nabla g(w_2) - \nabla g(w_1), w_1 - w_2\rangle \ge \alpha(w_2, w_1). \tag{77}$$

Adding equations (76) and (77), we have

$$2\langle \nabla g(w_1) - \nabla g(w_2), w_2 - w_1\rangle \ge \{\alpha(w_1, w_2) + \alpha(w_2, w_1)\}. \tag{78}$$

As $\alpha(u, v) + \alpha(v, u) \ge 0$, $\forall v \in C$,

$$\langle \nabla g(w_1) - \nabla g(w_2), w_2 - w_1\rangle \ge 0. \tag{79}$$

This implies that

$$\langle \nabla g(w_2) - \nabla g(w_1), w_2 - w_1\rangle \le 0. \tag{80}$$

As g is convex and Gateaux differentiable,

$$\langle \nabla g(w_2) - \nabla g(w_1), w_2 - w_1\rangle \ge 0. \tag{81}$$

By equations (80) and (81), we have

$$\langle \nabla g(w_2) - \nabla g(w_1), w_2 - w_1\rangle = 0. \tag{82}$$

Since g is a Legendre function, $w_1 = w_2$. Hence, $\mathrm{res}^g_{G,b}$ is single-valued.

(ii) For $u, v \in C$, we obtain

$$G\left(\mathrm{res}^g_{G,b}v, \mathrm{res}^g_{G,b}u; \mathrm{res}^g_{G,b}u\right) + \langle \nabla g\left(\mathrm{res}^g_{G,b}u\right) - \nabla g(u), \mathrm{res}^g_{G,b}v - \mathrm{res}^g_{G,b}u\rangle + b\left(\mathrm{res}^g_{G,b}u, \mathrm{res}^g_{G,b}v\right) - b\left(\mathrm{res}^g_{G,b}u, \mathrm{res}^g_{G,b}u\right) \geq 0 \quad (83)$$

and

$$G\left(\mathrm{res}^g_{G,b}u, \mathrm{res}^g_{G,b}v; \mathrm{res}^g_{G,b}v\right) + \langle \nabla g\left(\mathrm{res}^g_{G,b}v\right) - \nabla g(v), \mathrm{res}^g_{G,b}u - \mathrm{res}^g_{G,b}v\rangle + b\left(\mathrm{res}^g_{G,b}v, \mathrm{res}^g_{G,b}u\right) - b\left(\mathrm{res}^g_{G,b}v, \mathrm{res}^g_{G,b}v\right) \geq 0. \quad (84)$$

Adding the above two inequalities, we have

$$\begin{aligned} &G\left(\mathrm{res}^g_{G,b}v, \mathrm{res}^g_{G,b}u; \mathrm{res}^g_{G,b}u\right) + G\left(\mathrm{res}^g_{G,b}u, \mathrm{res}^g_{G,b}v; \mathrm{res}^g_{G,b}v\right) \\ &+ \langle \nabla g\left(\mathrm{res}^g_{G,b}u\right) - \nabla g(u) - \nabla g\left(\mathrm{res}^g_{G,b}v\right) + \nabla g(v), \mathrm{res}^g_{G,b}v - \mathrm{res}^g_{G,b}u\rangle \\ &+ b\left(\mathrm{res}^g_{G,b}u, \mathrm{res}^g_{G,b}v\right) - b\left(\mathrm{res}^g_{G,b}u, \mathrm{res}^g_{G,b}u\right) + b\left(\mathrm{res}^g_{G,b}v, \mathrm{res}^g_{G,b}u\right) - b\left(\mathrm{res}^g_{G,b}v, \mathrm{res}^g_{G,b}v\right) \geq 0, \end{aligned} \quad (85)$$

which yields by applying the concept of b and G,

$$\begin{aligned} &\langle \nabla g\left(\mathrm{res}^g_{G,b}u\right) - \nabla g(u) - \nabla g\left(\mathrm{res}^g_{G,b}v\right) + \nabla g(v), \mathrm{res}^g_{G,b}v - \mathrm{res}^g_{G,b}u\rangle \\ &\geq -\left\{G\left(\mathrm{res}^g_{G,b}v, \mathrm{res}^g_{G,b}u; \mathrm{res}^g_{G,b}u\right) + G\left(\mathrm{res}^g_{G,b}u, \mathrm{res}^g_{G,b}v; \mathrm{res}^g_{G,b}v\right)\right\} \\ &= G\left(\mathrm{res}^g_{G,b}v, \mathrm{res}^g_{G,b}u; \mathrm{res}^g_{G,b}v\right) - G\left(\mathrm{res}^g_{G,b}v, \mathrm{res}^g_{G,b}u; \mathrm{res}^g_{G,b}u\right) \\ &\geq \alpha\left(\mathrm{res}^g_{G,b}u, \mathrm{res}^g_{G,b}v\right). \end{aligned} \quad (86)$$

In equation (86), interchanging the position of $\mathrm{res}^g_{G,b}u$ and $\mathrm{res}^g_{G,b}v$, we get

$$\langle \nabla g\left(\mathrm{res}^g_{G,b}v\right) - \nabla g(v) - \nabla g\left(\mathrm{res}^g_{G,b}u\right) + \nabla g(u), \mathrm{res}^g_{G,b}v - \mathrm{res}^g_{G,b}u\rangle \geq \alpha\left(\mathrm{res}^g_{G,b}v, \mathrm{res}^g_{G,b}u\right). \quad (87)$$

Adding equations (86) and (87) and using $\alpha(u, v) + \alpha(v, u) \geq 0$, $\forall v \in C$, we get

$$2\langle \nabla g\left(\mathrm{res}^g_{G,b}u\right) - \nabla g(u) - \nabla g\left(\mathrm{res}^g_{G,b}v\right) + \nabla g(v), \mathrm{res}^g_{G,b}v - \mathrm{res}^g_{G,b}u\rangle \geq \left\{\alpha\left(\mathrm{res}^g_{G,b}u, \mathrm{res}^g_{G,b}v\right) + \alpha\left(\mathrm{res}^g_{G,b}v, \mathrm{res}^g_{G,b}u\right)\right\} \geq 0. \quad (88)$$

This implies that

$$\langle \nabla g(\mathrm{res}^g_{G,b}u) - \nabla g(\mathrm{res}^g_{G,b}v), \mathrm{res}^g_{G,b}(u) - \mathrm{res}^g_{G,b}(v)\rangle \leq \langle \nabla g(u) - \nabla g(v), \mathrm{res}^g_{G,b}(u) - \mathrm{res}^g_{G,b}(v)\rangle. \tag{89}$$

This means that $\mathrm{res}^g_{G,b}$ is a Bregman firmly nonexpansive type mapping.

(iii) Let $u \in F(\mathrm{res}^g_{G,b})$; then,

$$\begin{aligned} u \in F(\mathrm{res}^g_{G,b}) &\Leftrightarrow u = \mathrm{res}^g_{G,b}u \\ &\Leftrightarrow G(v,u;u) + \langle \nabla g(u) - \nabla g(u), v-u\rangle + b(u,v) - b(u,u) \geq 0, \quad \forall v \in C \\ &\Leftrightarrow G(v,u;u) + b(u,v) - b(u,u), \quad \forall v \in C \\ &\Leftrightarrow u \in \mathrm{Sol}(\mathrm{GMVLIP}(1)). \end{aligned} \tag{90}$$

Furthermore, Since $\mathrm{res}^g_{G,b}$ is a Bregman firmly nonexpansive type mapping, in ([42], Lemma 1.3.1), $F(\mathrm{res}^g_{G,b})$ is a closed and convex subset of C. Therefore, by equation (90), we get that $\mathrm{Sol}(\mathrm{GMVLIP}(1)) = F(\mathrm{res}^g_{G,b})$ is closed and convex.

(iv) Now, we show that $\mathrm{res}^g_{G,b}$ is Bregman quasi-nonexpansive mapping.

For $u, v \in C$, from (b), we have

$$\langle \nabla g(\mathrm{res}^g_{G,b}u) - \nabla g(\mathrm{res}^g_{G,b}v), \mathrm{res}^g_{G,b}(u) - \mathrm{res}^g_{G,b}(v)\rangle \leq \langle \nabla g(u) - \nabla g(v), \mathrm{res}^g_{G,b}(u) - \mathrm{res}^g_{G,b}(v)\rangle. \tag{91}$$

Moreover, we have

$$\begin{aligned} D_g(\mathrm{res}^g_{G,b}(u), \mathrm{res}^g_{G,b}(v)) + D_g(\mathrm{res}^g_{G,b}(v), \mathrm{res}^g_{G,b}(u)) &\leq D_g(\mathrm{res}^g_{G,b}(u), v) - D_g(\mathrm{res}^g_{G,b}(u), u) \\ &\quad + D_g(\mathrm{res}^g_{G,b}(v), u) - D_g(\mathrm{res}^g_{G,b}(v), v). \end{aligned} \tag{92}$$

Taking $v = w \in F(\mathrm{res}^g_{G,b})$, we see that

$$D_g(\mathrm{res}^g_{G,b}(u), w) + D_g(w, \mathrm{res}^g_{G,b}(u)) \leq D_g(\mathrm{res}^g_{G,b}(u), w) - D_g(\mathrm{res}^g_{G,b}(u), u) + D_g(w,u) - D_g(w,w). \tag{93}$$

Hence,

$$D_g(w, \mathrm{res}^g_{G,b}(u)) + D_g(\mathrm{res}^g_{G,b}(u), u) \leq D_g(w,u). \tag{94}$$

(v) Equation (94) implies that $\mathrm{res}^g_{G,b}$ is Bregman quasi-nonexpansive mapping. □

4. Main Result

We developed the strong convergence algorithm for the inertial iterative method to find the common solution of GMVLIP equation (1) and fixed-point problem of a Bregman relatively nonexpansive mapping in reflexive Banach space.

Iterative Algorithm 1. Let the sequences $\{x_n\}$ and $\{z_n\}$ be generated by the iterative algorithm:

$$\left.\begin{aligned} & x_0, x_{-1} \in C, \\ & u_n = x_n + \theta_n(x_n - x_{n-1}), \\ & v_n = \nabla g^*(\alpha_n \nabla g(u_n) + (1-\alpha_n)\nabla g(Tu_n)), \\ & w_n = \nabla g^*(\beta_n \nabla g(Tu_n) + (1-\beta_n)\nabla g(v_n)), \\ & z_n = \mathrm{res}^g_{G,b} w_n, \\ & C_n = \{z \in C: D_g(z,z_n) \leq D_g(z,u_n)\}, \\ & Q_n = \{z \in C: \langle \nabla g(x_0) - \nabla g(x_n), z - x_n\rangle \leq 0\}, \\ & x_{n+1} = \mathrm{proj}^g_{C_n \cap Q_n} x_0, \quad \mathrm{forall}\, n \geq 0, \end{aligned}\right\} \tag{95}$$

where $\{\theta_n\} \subseteq (0,1)$ and $\{\alpha_n\}, \{\beta_n\} \subseteq [0,1]$.

Theorem 2. *Let $C \subseteq X$ with $C \subseteq int(domg)$, where $g: X \longrightarrow (-\infty, +\infty]$ be a coercive Legendre function which is bounded, uniformly Frechet differentiable, and totally convex on bounded subsets of X. Let $G: C \times C \times C \longrightarrow \mathbb{R}$*

satisfy all conditions of Theorem 1 with continuous $G(y,\cdot;y)$, *and* $b\colon C\times C\longrightarrow\mathbb{R}$ *satisfies Assumption 1, respectively. Let* $T\colon C\longrightarrow C$ *be a Bregman relatively nonexpansive mapping. Let* $\Omega = Sol(GMVLIP(1))\cap F(T)\neq\varnothing$. *Let* $\{x_n\},\{z_n\}$ *be generated by Iterative 1 and* $\{\theta_n\}\subseteq(0,1)$, $\{\alpha_n\},\{\beta_n\}\subseteq[0,1]$ *with* $\lim_{n\longrightarrow\infty}\alpha_n = 0$. *Then,* $\{x_n\}$ *converges strongly to* $proj_\Omega^g x_0$.

Proof. For convenience, we divide its proof into several steps as in the following. □

Step 1. Ω and $C_n\cap Q_n$ are closed and convex, $\forall n\geq 0$.

By Lemmas 5 and 9, Ω is a closed and convex, and therefore, $\text{proj}_\Omega^g x_0$ is well defined.

Obviously, Q_n is closed and convex. Furthermore, we prove that C_n is closed and convex, $\forall n\geq 0$. We can easily show that C_n is closed and convex, $\forall n$. Thus, $C_n\cap Q_n$ is closed and convex, $\forall n\geq 0$.

Step 2. $\Omega\subset C_n\cap Q_n$, $\forall n\geq 0$, and $\{x_n\}$ is well defined.

Let $p\in\Omega$; then,

$$\begin{aligned}D_g(p,z_n) &= D_g\left(p,\text{res}_{G,b}^g w_n\right)\\ &\leq D_g(p,w_n)\\ &= D_g\left(p,\nabla g^*\left(\beta_n\nabla g(Tu_n)+(1-\beta_n)\nabla g(v_n)\right)\right)\\ &\leq \beta_n D_g(p,u_n)+(1-\beta_n)D_g(p,v_n),\end{aligned}\tag{96}$$

and

$$\begin{aligned}D_g(p,v_n) &= D_g\left(p,\nabla g^*\left(\alpha_n\nabla g(u_n)+(1-\alpha_n)\nabla g(Tu_n)\right)\right)\\ &\leq \alpha_n D_g(p,u_n)+(1-\alpha_n)D_g(p,u_n)\\ &= D_g(p,u_n).\end{aligned}\tag{97}$$

Substituting equation (97) into equation (96), we have

$$D_g(p,z_n)\leq D_g(p,u_n).\tag{98}$$

Thus, $p\in C_n$. Therefore, $\Omega\subset C_n$, $\forall n\geq 0$. Furthermore, by induction, we show that $\Omega\subset C_n\cap Q_n$, $n\geq 0$. As $Q_0 = C$, $\Omega\subset C_0\cap Q_0$. Suppose that $\Omega\subset C_m\cap Q_m$, for some $m>0$. Then, $\exists x_{m+1}\in C_m\cap Q_m$ such that $x_{m+1} = \text{proj}_{C_m\cap Q_m}^g x_0$. From the definition of x_{m+1}, we get $\langle\nabla g(x_0)-\nabla g(x_{m+1}), x_{m+1}-z\rangle\geq 0$, $\forall z\in C_k\cap Q_m$. Since $\Omega\subset C_m\cap Q_m$, we have

$$\langle\nabla g(x_0)-\nabla g(x_{m+1}), p-x_{m+1}\rangle\leq 0,\quad\forall p\in\Omega,\tag{99}$$

which implies $p\in Q_{m+1}$. Hence, $\Omega\subset C_{m+1}\cap Q_{m+1}$ implies $\Omega\subset C_n\cap Q_n$, $\forall n\geq 0$, and thus, $x_{n+1} = \text{proj}_{C_n\cap Q_n}^g x_0$ is well defined, $\forall n\geq 0$. Hence, $\{x_n\}$ is well defined.

Step 3. The sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{z_n\}$, and $\{w_n\}$ are bounded.

Using the concept of Q_n, we get $x_n = \text{proj}_{Q_n}^g x_0$. By $x_n = \text{proj}_{Q_n}^g x_0$ and Lemma 10 (iii), we obtain

$$\begin{aligned}D_g(x_n,x_0) &= D_g\left(\text{proj}_{Q_n}^g x_0,x_0\right)\\ &\leq D_g(u,x_0)-D_g\left(u,\text{proj}_{Q_n}^g x_0\right)\leq D_g(u,x_0),\\ &\quad\forall u\in\Omega\subset Q_n.\end{aligned}\tag{100}$$

This implies that $\{D_g(x_n,x_0)\}$ is bounded, and hence, $\{x_n\}$ is bounded by Lemma 3.

Now,

$$\begin{aligned}D_g(p,x_n) &= D_g\left(p,\text{proj}_{C_{n-1}\cap Q_{n-1}}^g x_0\right)\\ &\leq D_g(p,x_0)-D_g(x_n,x_0),\end{aligned}\tag{101}$$

which implies that $\{D_g(p,x_n)\}$ is bounded. Using $D_g(p,Tx_n)\leq D_g(p,x_n)$, $\forall p\in\Omega$, $\{Tx_n\}$ is bounded. Therefore, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, and $\{z_n\}$ are bounded.

Step 4. $\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\| = 0$; $\lim_{n\longrightarrow\infty}\|x_n-u_n\| = 0$; $\lim_{n\longrightarrow\infty}\|z_n-u_n\| = 0$; $\lim_{n\longrightarrow\infty}\|z_n-w_n\| = 0$; $\lim_{n\longrightarrow\infty}\|u_n-w_n\| = 0$, and $\lim_{n\longrightarrow\infty}\|u_n-Tu_n\| = 0$.

Since $x_{n+1} = \text{proj}_{C_n\cap Q_n}^g x_0\in Q_n$ and $x_n\in\text{proj}_{Q_n}^g x_0$, we get

$$D_g(x_n,x_0)\leq D_g(x_{n+1},x_0),\quad\forall n\geq 0,\tag{102}$$

which implies $\{D_g(x_n,x_0)\}$ is nondecreasing. By boundedness of $\{D_g(x_n,x_0)\}$, $\lim_{n\longrightarrow\infty}D_g(x_n,x_0)$ exists and is finite. Furthermore,

$$\begin{aligned}D_g(x_{n+1},x_n) &= D_g\left(x_{n+1},\text{proj}_{Q_n}^g x_0\right)\\ &\leq D_g(x_{n+1},x_0)-D_g\left(\text{proj}_{Q_n}^g x_0,x_0\right)\\ &= D_g(x_{n+1},x_0)-D_g(x_n,x_0),\end{aligned}\tag{103}$$

which yields

$$\lim_{n\longrightarrow\infty}D_g(x_{n+1},x_n) = 0.\tag{104}$$

Using Lemma 1,

$$\lim_{n\longrightarrow\infty}\left\|x_{n+1}-x_n\right\| = 0.\tag{105}$$

From the definition of u_n, $\|u_n-x_n\| = \|\theta_n(x_n-x_{n-1})\|\leq\|x_n-x_{n-1}\|$, which implies by equation (105) that

$$\lim_{n\longrightarrow\infty}\left\|u_n-x_n\right\| = 0.\tag{106}$$

Since

$$\left\|u_n-x_{n+1}\right\|\leq\left\|u_n-x_n\right\|+\left\|x_n-x_{n+1}\right\|,\tag{107}$$

it follows from equations (105) and (106) that

$$\lim_{n\longrightarrow\infty}\left\|u_n-x_{n+1}\right\| = 0.\tag{108}$$

Using Lemma 2 because g is uniformly Frechet differentiable, we get

$$\lim_{n\longrightarrow\infty}\left|g(u_n)-g(x_{n+1})\right| = 0\tag{109}$$

and

$$\lim_{n\longrightarrow\infty}\|\nabla g(u_n)-\nabla g(x_{n+1})\|=0. \tag{110}$$

By the concept of D_g, we get

$$D_g(x_{n+1},u_n)=g(x_{n+1})-g(u_n)-\langle\nabla g(u_n),x_{n+1}-u_n\rangle. \tag{111}$$

∇g is bounded on the bounded subset of X because g is bounded on X. Since g is uniformly Frechet differentiable, it is uniformly continuous on bounded subsets. Hence, by equations (108), (109), and (111),

$$\lim_{n\longrightarrow\infty}D_g(x_{n+1},u_n)=0. \tag{112}$$

As $x_{n+1}=\text{proj}^g_{C_n\cap Q_n}x_0\in C_n$, we have

$$D_g(x_{n+1},z_n)\le D_g(x_{n+1},u_n), \tag{113}$$

and hence, by equations (112) and (113),

$$\lim_{n\longrightarrow\infty}D_g(x_{n+1},z_n)=0. \tag{114}$$

Thanks to Lemma 1,

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-z_n\|=0. \tag{115}$$

Taking into account

$$\|z_n-u_n\|\le\|z_n-x_{n+1}\|+\|x_{n+1}-u_n\|, \tag{116}$$

by equations (108) and (115), we get

$$\lim_{n\longrightarrow\infty}\|z_n-u_n\|=0. \tag{117}$$

By Lemma 2,

$$\lim_{n\longrightarrow\infty}|g(z_n)-g(u_n)|=0 \tag{118}$$

and

$$\lim_{n\longrightarrow\infty}\|\nabla g(z_n)-\nabla g(u_n)\|=0. \tag{119}$$

Next, we estimate

$$\begin{aligned}D_g(p,u_n)-D_g(p,z_n)&=g(p)-g(u_n)-\langle\nabla g(u_n),p-u_n\rangle\\&\quad-g(p)+g(z_n)+\langle\nabla g(z_n),p-z_n\rangle\\&=g(z_n)-g(u_n)+\langle\nabla g(z_n),p-z_n\rangle-\langle\nabla g(u_n),p-u_n\rangle\\&=g(z_n)-g(u_n)+\langle\nabla g(z_n),u_n-z_n\rangle\\&\quad+\langle\nabla g(z_n)-\nabla g(u_n),p-u_n\rangle.\end{aligned} \tag{120}$$

Since $\{z_n\}$, $\{u_n\}$, $\{\nabla g(z_n)\}$, and $\{\nabla g(u_n)\}$ are bounded and by equations (117)–(120), we get

$$\lim_{n\longrightarrow\infty}|D_g(p,u_n)-D_g(p,z_n)|=0. \tag{121}$$

Furthermore, it follows from Lemma 9 (v) that

$$\begin{aligned}D_g(z_n,w_n)&\le D_g(p,w_n)-D_g(p,z_n)\\&\le D_g(p,\nabla g^*(\beta_n\nabla g(Tu_n)+(1-\beta_n)\nabla g(v_n)))-D_g(p,z_n)\\&\le\beta_nD_g(p,Tu_n)+(1-\beta_n)D_g(p,u_n)-D_g(p,z_n)\\&\le D_g(p,u_n)-D_g(p,z_n).\end{aligned} \tag{122}$$

Since $\{D_g(p,u_n)\}$ and $\{D_g(p,z_n)\}$ are bounded, by equations (121) and (122),

$$\lim_{n\longrightarrow\infty}D_g(z_n,w_n)=0, \tag{123}$$

and hence,

$$\lim_{n\longrightarrow\infty}\|z_n-w_n\|=0. \tag{124}$$

From equations (117) and (124), we get

$$\lim_{n\longrightarrow\infty}\|u_n-w_n\|=0. \tag{125}$$

By uniform Frechet differentiable of g, Lemma 2, and equations (124) and (125), we have

$$\lim_{n\longrightarrow\infty}\|\nabla g(z_n)-\nabla g(w_n)\|=0, \tag{126}$$

$$\lim_{n\longrightarrow\infty}\|\nabla g(u_n)-\nabla g(w_n)\|=0. \tag{127}$$

Note that

$$\begin{aligned}\|\nabla g(u_n)-\nabla g(w_n)\| &= \|\nabla g(u_n)-\nabla g(\nabla g^*(\beta_n\nabla g(Tu_n)+(1-\beta_n)\nabla g(v_n)))\|\\ &= \|\nabla g(u_n)-\beta_n\nabla g(Tu_n)-(1-\beta_n)\nabla g(v_n))\|\\ &= \|\beta_n(\nabla g(u_n)-\nabla g(Tu_n))+(1-\beta_n)(\nabla g(u_n)-\nabla g(v_n))\|\\ &= \|\beta_n(\nabla g(u_n)-\nabla g(Tu_n))+(1-\beta_n)(\nabla g(u_n)-\nabla g(\nabla g^*(\alpha_n\nabla g(u_n)+(1-\alpha_n)\nabla g(Tu_n))))\|\\ &= \|\beta_n(\nabla g(u_n)-\nabla g(Tu_n))+(1-\beta_n)(1-\alpha_n)(\nabla g(u_n)-\nabla g(Tu_n))\|\\ &= [1-\alpha_n(1-\beta_n)]\|\nabla g(u_n)-\nabla g(Tu_n))\|.\end{aligned} \tag{128}$$

By equations (127) and (128) and $\lim_{n\longrightarrow\infty}\alpha_n=0$, we get

$$\lim_{n\longrightarrow\infty}\|\nabla g(u_n)-\nabla g(Tu_n)\|=0. \tag{129}$$

Moreover, we have from equation (129) that

$$\lim_{n\longrightarrow\infty}\|u_n-Tu_n\|=0. \tag{130}$$

Step 5. $\overline{x}\in\Omega$.

First, we prove that $\overline{x}\in F(T)$. As $\{x_n\}$ is bounded, $\exists$ a subsequence $\{x_{n_k}\}\subseteq\{x_n\}$ such that $x_{n_k}\rightharpoonup\overline{x}\in C$ as $k\longrightarrow\infty$. By equations (106), (117), (124), and (125), $\{x_n\}$, $\{u_n\}$, $\{w_n\}$, and $\{z_n\}$ have the same asymptotic behaviour and thus $\exists$ subsequences $\{u_{n_k}\}$ of $\{u_n\}$, $\{w_{n_k}\}$ of $\{w_n\}$, and $\{z_{n_k}\}$ of $\{z_n\}$ such that $u_{n_k}\rightharpoonup\overline{x}$, $w_{n_k}\rightharpoonup\overline{x}$, and $z_{n_k}\rightharpoonup\overline{x}$ as $k\longrightarrow\infty$. Using $u_{n_k}\rightharpoonup\overline{x}$ and equation (130), we get

$$\lim_{k\longrightarrow\infty}\|u_{n_k}-Tu_{n_k}\|=0. \tag{131}$$

By the concept of T, $\overline{x}\in\widehat{F}(T)=F(T)$.

Next, prove that $\overline{x}\in\mathrm{Sol}(\mathrm{GMVLIP}(1))$. As $z_n=\mathrm{res}^g_{G,b}w_n$, we have

$$G(v,z_{n_k};z_{n_k})+\langle\nabla g(z_{n_k})-\nabla g(w_{n_k}),v-z_{n_k}\rangle+b(v,z_{n_k})-b(z_{n_k},z_{n_k})\geq 0,\quad\forall v\in C. \tag{132}$$

Using generalized relaxed α-monotonicity of G, we have

$$\begin{aligned}\langle\nabla g(z_{n_k})-\nabla g(w_{n_k}),v-z_{n_k}\rangle &\geq -G(v,z_{n_k};z_{n_k})-b(v,z_{n_k})+b(z_{n_k},z_{n_k}),\quad\forall v\in C,\\ &\geq\alpha(z_{n_k},v)-G(v,z_{n_k};v)-b(v,z_{n_k})+b(z_{n_k},z_{n_k}).\end{aligned} \tag{133}$$

Using the concept of G, b, equation (126), and $k\longrightarrow\infty$ in equation (133), we obtain

$$\alpha(\overline{x},v)-G(v,\overline{x};v)+b(\overline{x},\overline{x})-b(\overline{x},v)\leq 0,\quad\text{for all } v\in C. \tag{134}$$

For $t\in(0,1)$ and $v\in C$, let $v_t=tv+(1-t)\overline{x}$. Since $v_t\in C$, we have

$$\alpha_i(\overline{x},v_t)-G(v_t,\overline{x};v_t)+b(\overline{x},\overline{x})-b(\overline{x},v_t)\leq 0, \tag{135}$$

which implies that

$$\begin{aligned}\alpha(\overline{x},v_t) &\leq G(v_t,\overline{x};v_t)-b(\overline{x},\overline{x})+b(\overline{x},v_t)\\ &\leq tG(v,\overline{x};v_t)+(1-t)G(\overline{x},\overline{x};v_t)-b(\overline{x},\overline{x})+tb(\overline{x},v)+(1-t)b(\overline{x},\overline{x})\\ &\leq t[G(v,\overline{x};v_t)+b(\overline{x},v)-b(\overline{x},\overline{x})].\end{aligned} \tag{136}$$

Since $G(v,\overline{x};\cdot)$ is hemicontinuous, we have

$$\lim_{t\longrightarrow 0}\{G(v,\overline{x};v_t)+b(\overline{x},v)-b(\overline{x},\overline{x})\}\geq\lim_{t\longrightarrow 0}\frac{\alpha(\overline{x},v_t)}{t}, \tag{137}$$

which implies

$$G(v,\overline{x};\overline{x})+b(\overline{x},v)-b(\overline{x},\overline{x})\geq 0. \tag{138}$$

Hence, $\overline{x}\in\mathrm{Sol}(\mathrm{GMVLIP}(1))$. Thus, $\overline{x}\in\Omega$.

Step 6. We prove that $x_n\longrightarrow\overline{x}=\mathrm{proj}^g_\Omega x_0$.

Proof of Step 6. Let $\tilde{u} = \text{proj}^g_\Omega x_0$. As $\{x_n\}$ is weakly convergent, $x_{n+1} = \text{proj}^g_\Omega x_0$ and $\text{proj}^g_\Omega x_0 \in \Omega \subset C_n \cap Q_n$. By equation (100), we have

$$D_g(x_{n+1}, x_0) \le D_g(\text{proj}^g_\Omega x_0, x_0). \quad (139)$$

Using Lemma 6, $\{x_n\}$ is strongly convergent to $\tilde{u} = \text{proj}^g_\Omega x_0$. Hence, by the uniqueness of the limit, $\{x_n\}$ converges strongly to $\overline{x} = \text{proj}^g_\Omega x_0$. □

5. Consequences

Finally, we get the following consequences of Theorem 2.

Corollary 1. *Let $C \subseteq X$ with $C \subseteq int(domg)$, where $g: X \longrightarrow (-\infty, +\infty]$ be a coercive Legendre function which is bounded, uniformly Frechet differentiable, and totally convex on bounded subsets of X. Let $G: C \times C \times C \longrightarrow \mathbb{R}$ satisfy conditions (i), (ii), and (iii) of Theorem 1 and G be monotone, i.e.,*

$$G(y, x; y) - G(y, x; x) \ge 0, \quad \text{for any } x, y \in C. \quad (140)$$

Let $b: C \times C \longrightarrow \mathbb{R}$ satisfy Assumption 1, and Let $T: C \longrightarrow C$ be a Bregman relatively nonexpansive mapping. Let $\Omega = Sol(GMVLIP(1)) \cap F(T) \ne \varnothing$. Let $\{x_n\}, \{z_n\}$ be generated by Iterative 1 and $\{\theta_n\} \subseteq (0, 1)$ and $\{\alpha_n\}, \{\beta_n\} \subseteq [0, 1]$ with $\lim_{n\longrightarrow\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $proj^g_\Omega x_0$.

Moreover, if GMVLIP equation (1)= C and by the concept of Example 1 for $A: X \longrightarrow 2^{X^*}$, we have the maximal monotone operator.

Corollary 2. *Let $C \subseteq X$ with $C \subseteq int(domg)$, where $g: X \longrightarrow (-\infty, +\infty]$ be a coercive Legendre function which is bounded, uniformly Frechet differentiable, and totally convex on bounded subsets of X. Let $A: X \longrightarrow 2^{X^*}$ be a maximal monotone operator with $A^{-1}(0) \ne \varnothing$. Let $\{x_n\}, \{z_n\} \subseteq C$ generated by*

$$\left.\begin{aligned}
&x_0, x_{-1} \in C,\\
&u_n = x_n + \alpha_n(x_n - x_{n-1}),\\
&v_n = \nabla g^*\left(\alpha_n \nabla g(u_n) + (1-\alpha_n)\nabla g(\text{res}^g_A u_n)\right),\\
&z_n = \nabla g^*\left(\beta_n \nabla g(\text{res}^g_A u_n) + (1-\beta_n)\nabla g(v_n)\right),\\
&C_n = \left\{z \in C: D_g(z, z_n) \le D_g(z, u_n)\right\},\\
&Q_n = \left\{z \in C: \langle \nabla g(x_0) - \nabla g(x_n), z - x_n\rangle \le 0\right\},\\
&x_{n+1} = \text{proj}^g_{C_n \cap Q_n} x_0, \forall n \ge 0,
\end{aligned}\right\} \quad (141)$$

where $\{\theta_n\} \subseteq (0, 1)$ and $\{\alpha_n\}, \{\beta_n\} \subseteq [0, 1]$ with $\lim_{n\longrightarrow\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $proj_{A^{-1}(0)} x_0$.

Remark 4. If $g(x) = (1/2)\|x\|^2$, $\forall x \in X$, then Theorem 2 is reduced to the strong convergence theorem for finding the common solution of GMVLIP equation (1) and fixed-point problem of a relatively nonexpansive mapping in reflexive Banach space.

6. Numerical Example

Finally, to support our main theorem, we now give an example in infinitely dimensional spaces $L_2[0, 1]$ such that $\|\cdot\|$ is L_2-norm defined by $\|x\| = \sqrt{\int_0^1 |x(t)|^2 \mathrm{d}t}$ where $x(t) \in L_2[0, 1]$.

Example 3. Let $X = L_2[0, 1]$ and $C = \{x(t) \in L_2[0, 1]: \int_0^1 tx(t)\,\mathrm{d}t \le 2\}$. Define mappings as follows:

(i) Coercive Legendre function $g: X \longrightarrow (-\infty, +\infty]$ by $g(x) = (1/2)\|x\|^2$, $\forall x \in X$

(ii) ∀x,y,z∈C, Function $G: C \times C \times C \longrightarrow \mathbb{R}$ by $G(x, y, z) = (1/2)(\|y\|^2 - \|x\|^2)$, with $\alpha: C \times C \longrightarrow \mathbb{R}$ such that $\alpha(x, y) = 0$, $\forall x, y \in C$

(iii) Bifunction $b: C \times C \longrightarrow \mathbb{R}$ by $b(x, y) = -\langle x, y\rangle$, $\forall x, y \in C$

(iv) Bregman relatively nonexpansive mapping $T: C \longrightarrow C$ with respect to g by $Tx = (x/2)$, $\forall x \in C$

It is obvious that $G: C \times C \times C \longrightarrow \mathbb{R}$ satisfies all conditions of Theorem 1 with continuous $G(y, \cdot; y)$ and $b: C \times C \longrightarrow \mathbb{R}$ satisfies Assumption 1, respectively. On the other hand, we consider

Table 1: Numerical results of the difference ε_n.

ε_n		$(1/n+1)$	$(1/2n+1)$	$(1/n^2+1)$	$(1/2n^2+1)$	$(1/n^3+1)$
$x_{-1} = (\sin(t)/2), x_0 = \sin(t)$	No. of iter. CPU time (s)	9 7.59932	15 12.22748	19 14.71024	20 15.57306	20 15.66219
$x_{-1} = (2\sin(t) - t/2), x_0 = 2\sin(t) - t$	No. of iter. CPU time (s)	10 8.75820	15 12.24971	20 15.65068	20 15.78607	21 17.30471
$x_{-1} = t + \log^2(t+1), x_0 = \log(t+1)$	No. of iter. CPU time (s)	11 9.06217	15 10.15574	21 16.53084	20 15.81738	22 17.65972

Table 2: Numerical results of the difference θ.

θ		0.1	0.3	0.5	0.7	0.9
$x_{-1} = (\sin(t)/2), x_0 = \sin(t)$	No. of iter. CPU time (s)	9 7.75878	9 7.50740	9 7.67907	9 7.59864	9 7.60107
$x_{-1} = (2\sin(t) - t/2), x_0 = 2\sin(t) - t$	No. of iter. CPU time (s)	10 8.53362	10 8.82150	10 8.62202	10 8.82075	10 8.64536
$x_{-1} = t + \log^2(t+1), x_0 = \log(t+1)$	No. of iter. CPU time (s)	11 9.61967	11 9.06217	11 9.56274	11 9.47570	11 9.10264

Table 3: Numerical results of the difference α_n.

α_n		$(1/2n+1)$	$(1/10n+1)$	$(1/100n+1)$	$(1/2n^2+1)$	$(1/10n^2+1)$
$x_{-1} = \sin(t)/2, x_0 = \sin(t)$	No. of iter. CPU time (s)	9 7.53828	6 5.63066	5 4.78461	7 6.19290	5 4.80899
$x_{-1} = (2\sin(t) - t/2), x_0 = 2\sin(t) - t$	No. of iter. CPU time (s)	10 8.51165	6 5.84207	5 5.10883	7 6.47365	6 5.94383
$x_{-1} = t + \log^2(t+1), x_0 = \log(t+1)$	No. of iter. CPU time (s)	11 8.87843	6 5.52223	5 4.95105	7 6.23286	6 5.59795

Table 4: Numerical results of the difference β_n.

β_n		$(1/2n+1)$	$(1/10n+1)$	$(1/100n+1)$	$(1/2n^2+1)$	$(1/10n^2+1)$
$x_{-1} = (\sin(t)/2), x_0 = \sin(t)$	No. of iter. CPU time (s)	5 4.80889	5 4.75128	5 4.79156	5 4.75109	5 4.76763
$x_{-1} = (2\sin(t) - t/2), x_0 = 2\sin(t) - t$	No. of iter. CPU time (s)	5 4.97311	5 5.09668	5 5.03153	5 4.99385	5 4.98385
$x_{-1} = t + \log^2(t+1), x_0 = \log(t+1)$	No. of iter. CPU time (s)	5 5.16031	5 4.97200	5 4.83550	5 5.49581	5 5.47782

$$\begin{aligned}
u \in res_{G,b}^{g}(w) &\Longleftrightarrow G(u,y,y) + \langle \nabla g(u) - \nabla g(w), y-u\rangle + b(u,y) - b(u,u) \geq 0, \quad \forall y \in C \\
&\Longleftrightarrow \frac{1}{2}\left(\|y\|^2 - \|u\|^2\right) + \langle u-w, y-u\rangle - \langle u,y\rangle + \langle u,u\rangle \geq 0, \quad \forall y \in C \\
&\Longleftrightarrow \frac{1}{2}\left(\|y\|^2 - \|u\|^2\right) - \langle w, y-u\rangle \geq 0, \quad \forall y \in C \\
&\Longleftrightarrow \frac{1}{2}\left(\|y\|^2 - \|u\|^2\right) - \langle w, y-w\rangle + \langle w, u-w\rangle \geq 0, \quad \forall y \in C \\
&\Longleftrightarrow \frac{1}{2}\left(\|u\|^2 - \|w\|^2\right) - \langle w, u-w\rangle \leq \frac{1}{2}\left(\|y\|^2 - \|w\|^2\right) - \langle w, y-w\rangle, \quad \forall y \in C \\
&\Longleftrightarrow D_g(u,w) \leq D_g(y,w), \quad \forall y \in C \\
&\Longleftrightarrow u = Proj_C^g(w).
\end{aligned} \tag{142}$$

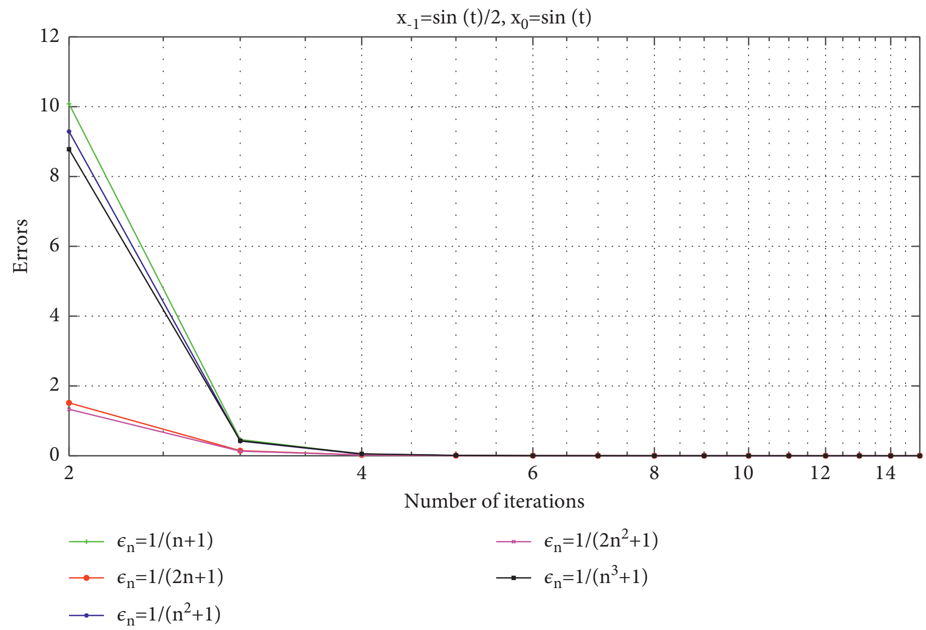

Figure 1: The Cauchy error plotting number of iterations for different parameters ε_n.

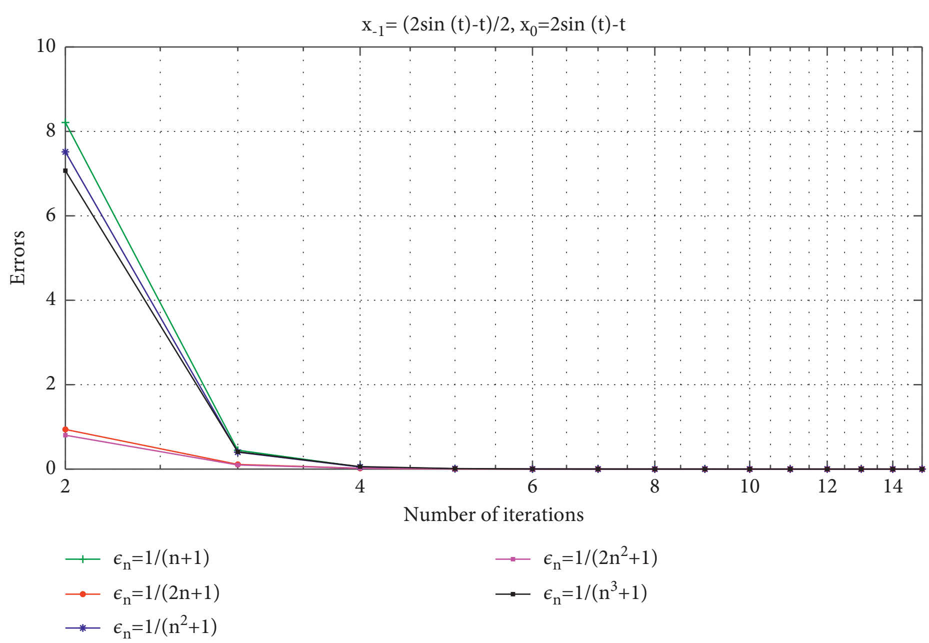

Figure 2: The Cauchy error plotting number of iterations for different parameters ε_n.

For the experiments in this section, we use the Cauchy error $\|x_{n+1} - x_n\|^2 < 10^{-5}$ for the stopping criterion. We will start with the initialization x_{-1} and x_0 in two cases. We split considering all of the performances of our algorithm in four cases by considering all of the parameters that have an effect on the convergence of the algorithm.

Case 1. We start computation by comparison of the algorithm with different parameters ϵ_n where

$$\theta_n = \begin{cases} \min\left\{\dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}, \theta\right\}, & \text{if } n \leq N, \\ \epsilon_n, & \text{otherwise,} \end{cases} \tag{143}$$

where N is the number of iterations that we want to stop, $\lim_{n\longrightarrow\infty}\epsilon_n = 0$, and $\theta \in (0,1)$. We choose $\theta = 0.3$, $\alpha_n = (1/2n+1)$, and $\beta_n = \alpha_n$. Then, the results are presented in Table 1.

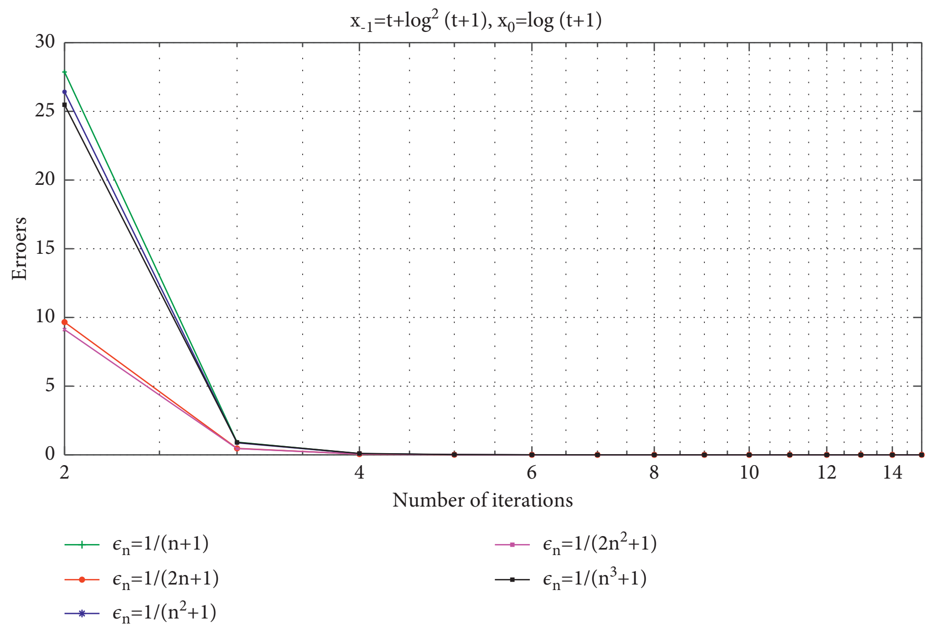

Figure 3: The Cauchy error plotting number of iterations for different parameters ε_n.

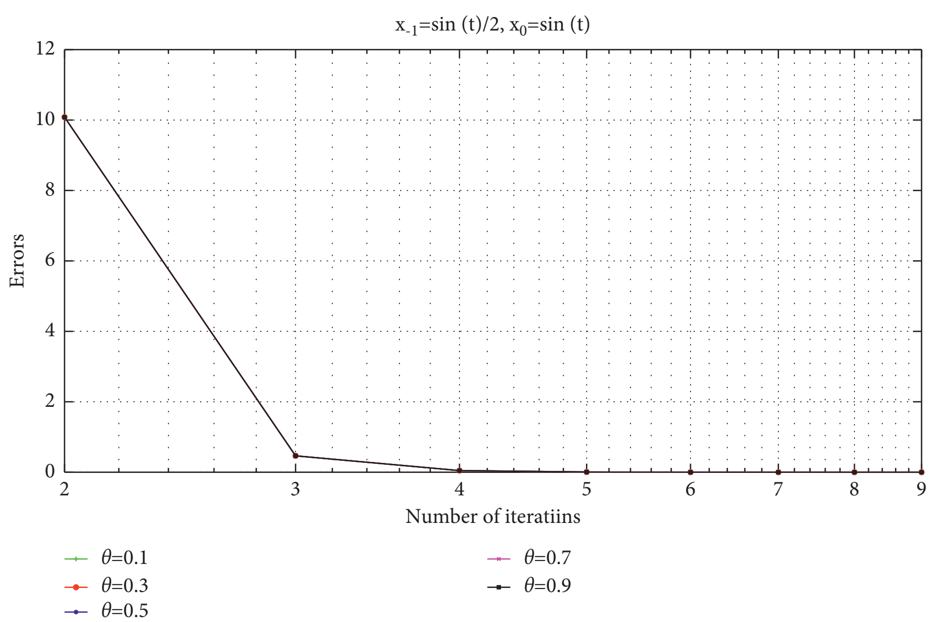

Figure 4: The Cauchy error plotting number of iterations for different parameters θ.

Case 2. We compare the performance of the algorithm with different parameters θ by setting $\varepsilon_n = (1/n+1)$, $\alpha_n = (1/2n+1)$, and $\beta_n = \alpha_n$. Then, the results are presented in Table 2.

Case 3. We compare the performance of the algorithm with different parameters α_n by setting $\varepsilon_n = (1/n+1)$, $\beta_n = \alpha_n$, and $\theta = 0.3$ for the initialization $x_{-1} = (\sin(t)/2), x_0 = \sin(t)$ and $x_{-1} = t + \log^2(t+1), x_0 = \log(t+1)$ and $\theta = 0.1$ for the initialization $x_{-1} = (2\sin(t) - t/2), x_0 = 2\sin(t) - t$. Then, the results are presented in Table 3.

Case 4. We compare the performance of the algorithm with different parameters β_n by setting $\varepsilon_n = (1/n+1)$, $\alpha_n = (1/100n+1)$, and $\theta = 0.3$ for the initialization $x_{-1} = (\sin(t)/2), x_0 = \sin(t)$ and $x_{-1} = t + \log^2(t+1), x_0 = \log(t+1)$ and $\theta = 0.1$ for the initialization $x_{-1} = (2\sin(t) - t/2), x_0 = 2\sin(t) - t$. Then, the results are presented in Table 4.

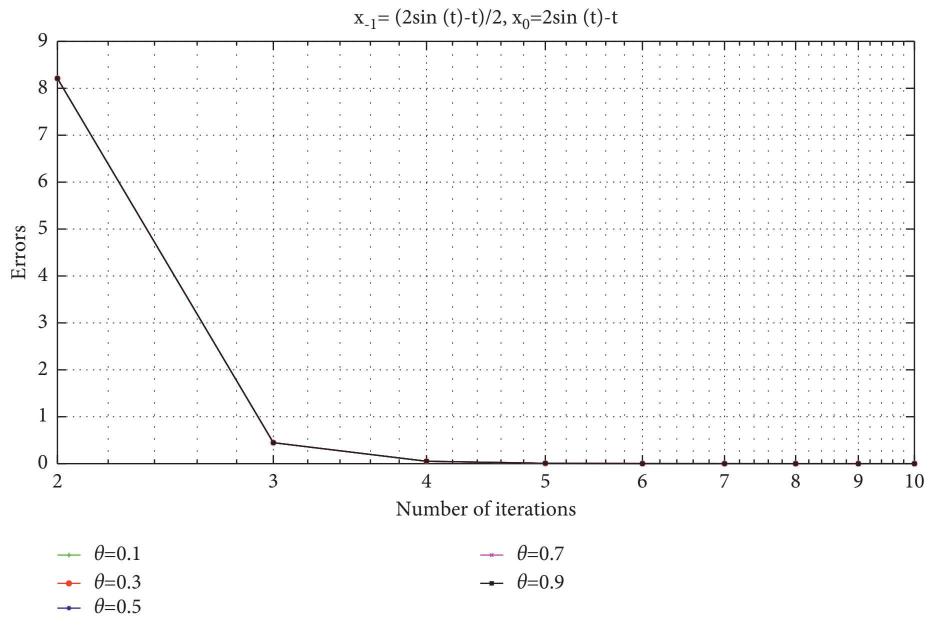

FIGURE 5: The Cauchy error plotting number of iterations for different parameters θ.

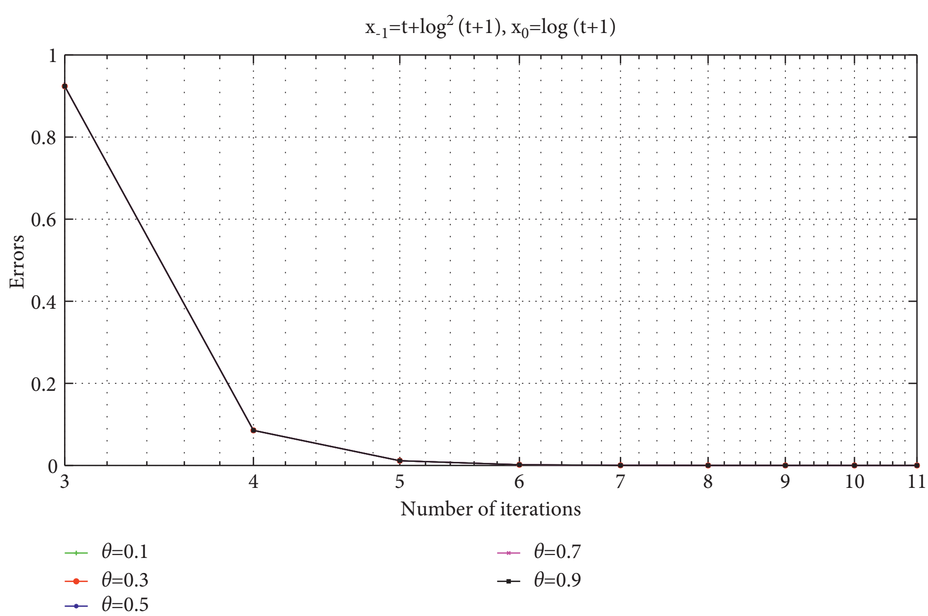

FIGURE 6: The Cauchy error plotting number of iterations for different parameters θ.

From Tables 1–4 and Figures 1–12, we noticed that in all the above 4 cases, choosing $\theta = 0.3$, $\varepsilon_n = (1/n+1)$, $\alpha_n = (1/100n+1)$, and $\beta_n = (1/2n^2+1)$ yields the best results for the initialization $x_{-1} = (\sin(t)/2), x_0 = \sin(t)$. Choosing $\theta = 0.1$, $\varepsilon_n = (1/n+1)$, $\alpha_n = (1/100n+1)$, and $\beta_n = (1/2n+1)$ yields the best results for the initialization $x_{-1} = (2\sin(t) - t/2), x_0 = 2\sin(t) - t$, and choosing $\theta = 0.3$, $\varepsilon_n = (1/n+1)$, $\alpha_n = (1/100n+1)$, and

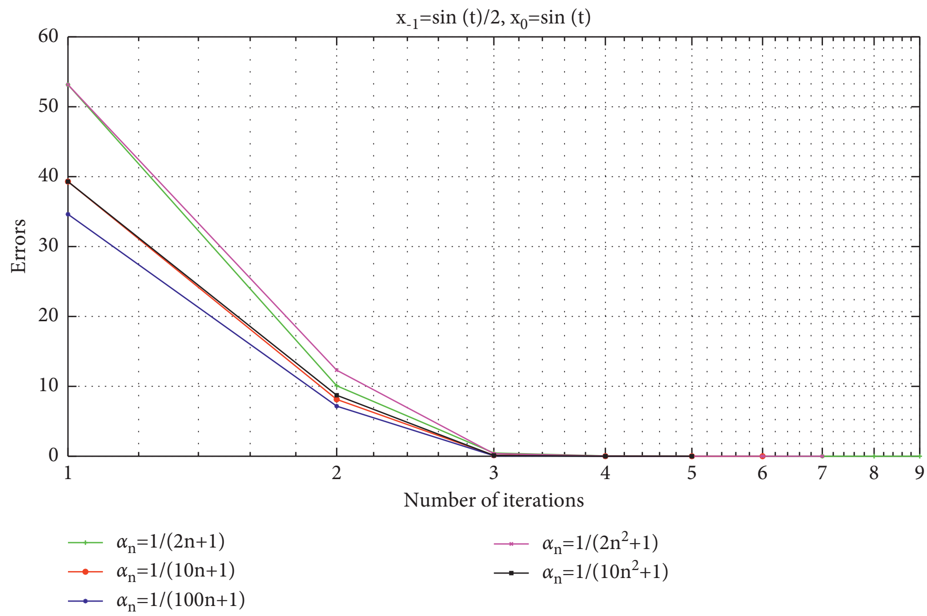

Figure 7: The Cauchy error plotting number of iterations for different parameters α_n.

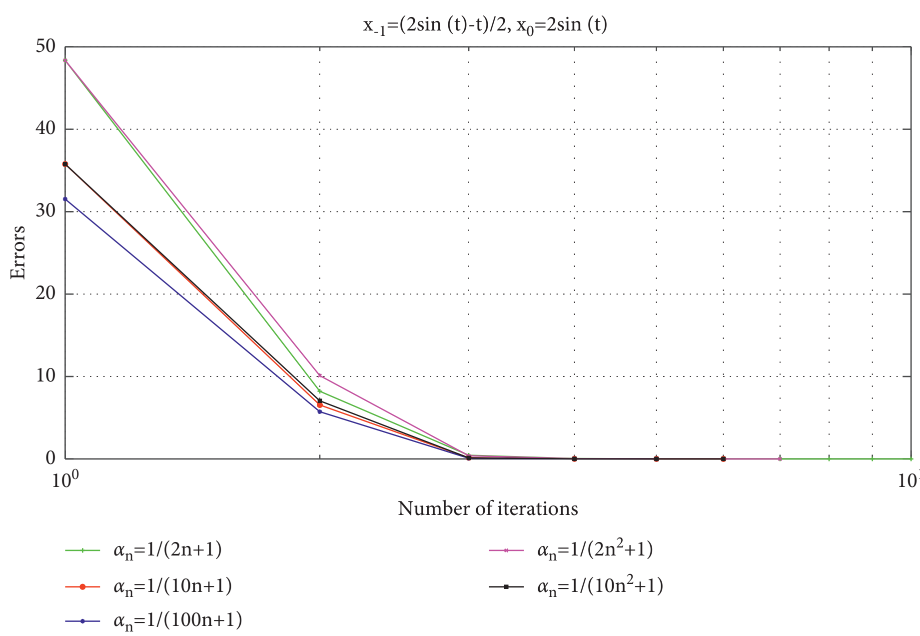

Figure 8: The Cauchy error plotting number of iterations for different parameters α_n.

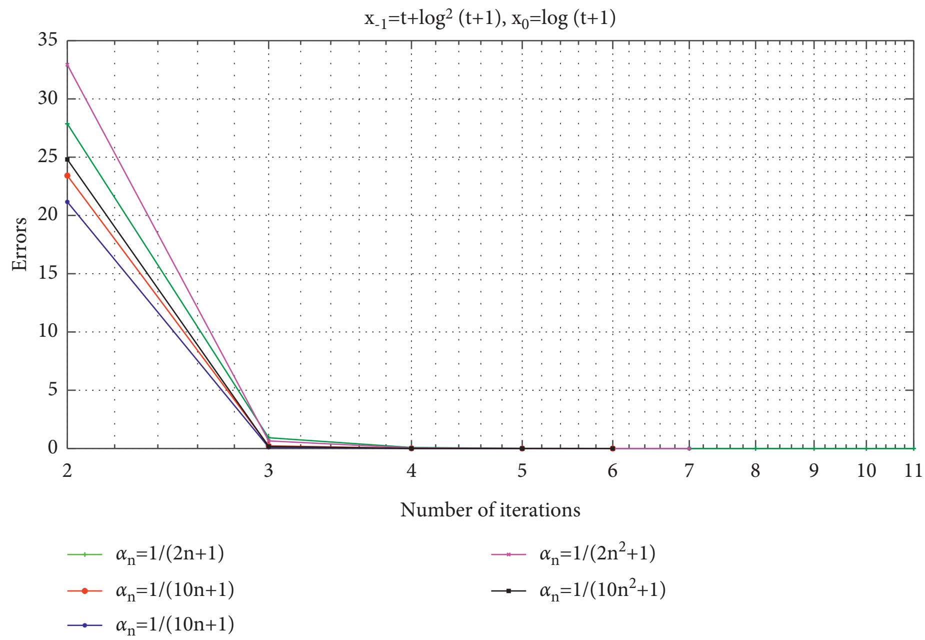

FIGURE 9: The Cauchy error plotting number of iterations for different parameters α_n.

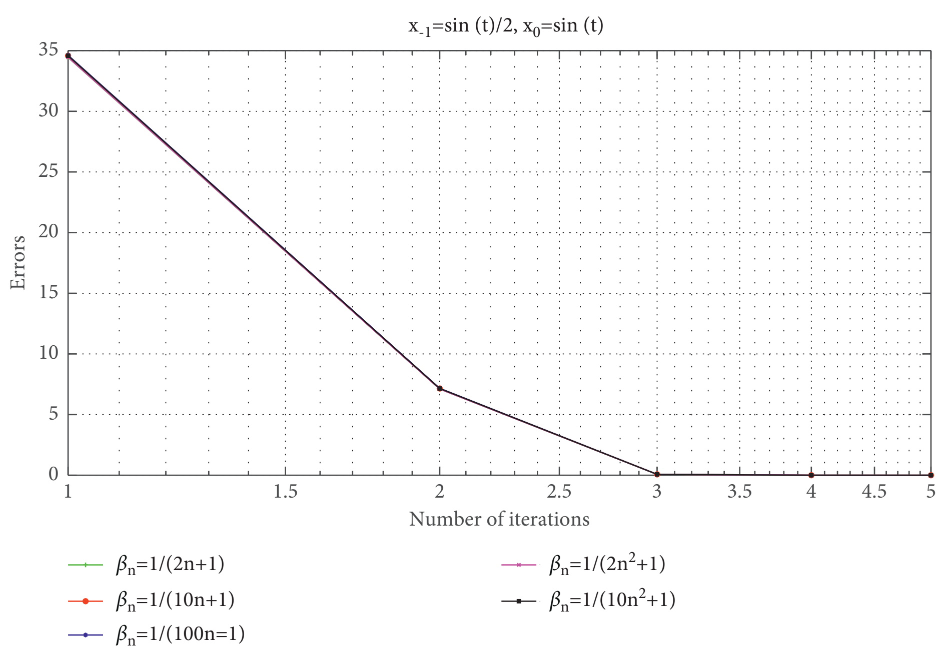

FIGURE 10: The Cauchy error plotting number of iterations for different parameters β_n.

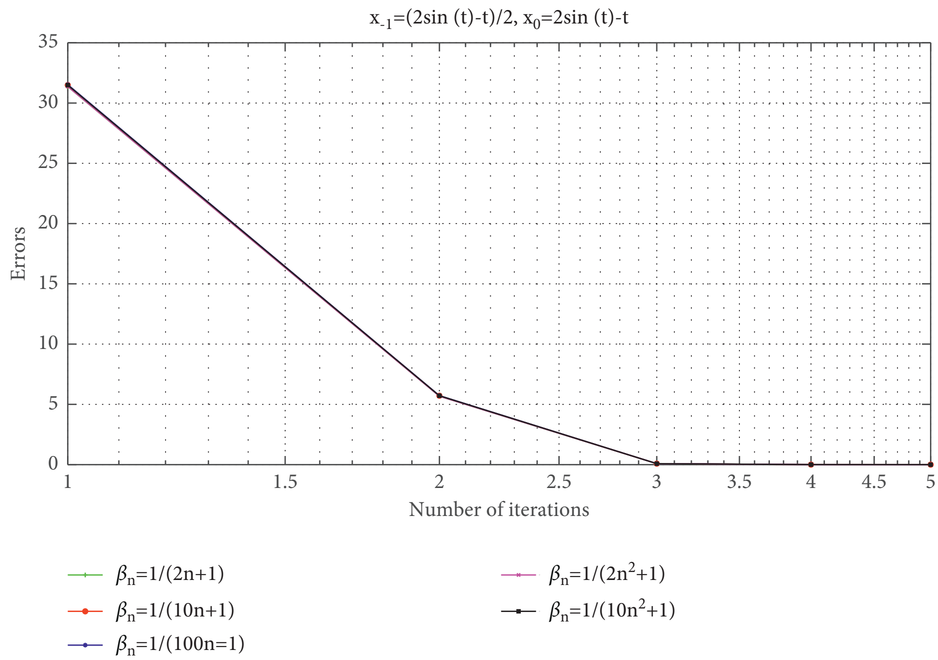

Figure 11: The Cauchy error plotting number of iterations for different parameters β_n.

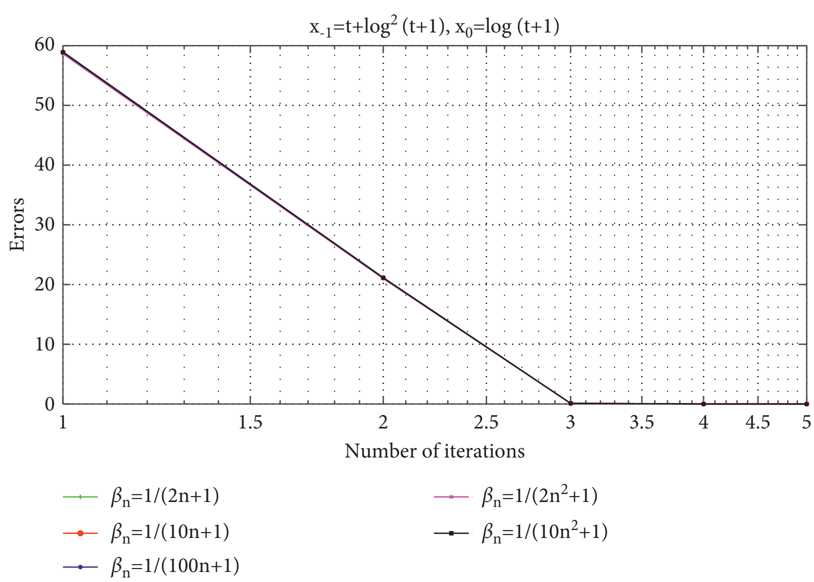

Figure 12: The Cauchy error plotting number of iterations for different parameters β_n.

$\beta_n = (1/100n + 1)$ yields the best results for the initialization $x_{-1} = t + \log^2(t + 1), x_0 = \log(t + 1)$.

7. Conclusion

In this paper, we established an inertial hybrid iterative algorithm involving Bregman relatively nonexpansive mapping to find a common solution of GMVLIP equation (1) and FPP in Banach space. Moreover, we study the convergence analysis for the main result. At last, we list some consequences and computational example to emphasize the efficiency and relevancy of the main result. From the theoretical and application point of view, the inertial method via Bregman relatively nonexpansive mapping has a great importance on data analysis and some imaging problems. The inertial method has been studied by various researchers due to its importance (see for details [19, 24–28, 30, 31, 33–36, 39]).

Abbreviations.

GMVLIP: Generalized mixed variational-like inequality problem
GVLIP: General variational-like inequality problem
MVLIP: Mixed variational-like inequality problem
VLIP: Variational-like inequality problem
VIP: Variational inequality problem
FPP: Fixed-point problem.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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