Research Article

Strong Convergence of an Inertial Iterative Algorithm for Generalized Mixed Variational-like Inequality Problem and Bregman Relatively Nonexpansive Mapping in Reflexive Banach Space

Saud Fahad Aldosary, 1 Watcharaporn Cholamjiak 2, Rehan Ali 3 and Mohammad Farid 4

1Department of Mathematics, College of Arts and Sciences, Wadi Al-Dawasir, Prince Sattam Bin Abdulaziz University, Saudi Arabia
2School of Science, University of Phayao, Mae Ka 56000, Phayao, Thailand
3Department of Mathematics, Jamia Millia Islamia, New Delhi 110025, India
4Department of Mathematics, Deanship of Educational Services, Qassim University, Buraidah 51452, Al-Qassim, Saudi Arabia

Correspondence should be addressed to Mohammad Farid; mohdfrd55@gmail.com

Received 17 November 2021; Accepted 9 December 2021; Published 27 December 2021

1.Introduction

Throughout the paper, unless otherwise stated, let X be a reflexive Banach space with X∗ as its dual and C ≠ ∅ be the closed convex subset of X. In this paper, we consider the generalized mixed variational-like inequality problem (in brief, GMVLIP): find u ∈ C such that

\[ G(v, u; u) + b(u, v) - b(u, u) \geq 0, \quad \forall v \in C, \]  

(1)

where \( b: C \times C \rightarrow \mathbb{R} \) and \( G: C \times C \times C \rightarrow \mathbb{R} \) be bifunction and trifunction, respectively, and \( \mathbb{R} \) be the set of real numbers. Sol (GMVLIP equation (1)) stands for the solution of equation (1). If \( b \equiv 0 \), GMVLIP equation (1) is reduced to GVLIP: find \( u \in C \) such that

\[ G(v, u; u) \geq 0, \quad \forall v \in C, \]  

(2)

which is introduced by Preda et al. [1] (see, for instance, [2, 3]).

If we set \( G(v, u; u) = \langle Du + Au, \eta(v, u) \rangle \), where \( D, A: C \rightarrow X \) and \( \eta: C \times C \rightarrow X \), GMVLIP equation (1) is reduced to MVLIP (see for details [4]).

Further, if we set \( G(v, u; u) = \langle Du, \eta(v, u) \rangle \) and \( b \equiv 0 \), GMVLIP equation (1) is reduced to VLIP: find \( u \in C \) such that

\[ \langle Du, \eta(v, u) \rangle \geq 0, \quad \forall v \in C, \]  

(3)

which is presented by Parida et al. [5].

Moreover, if \( \eta(v, u) = v - u \), VLIP is reduced to VIP: find \( u \in C \) such that

\[ \langle Du, v - u \rangle \geq 0, \quad \forall v \in C, \]  

which is introduced by Parida et al. [5].
\[ \langle Du, v - u \rangle \geq 0, \quad \forall v \in C, \]  
which is introduced by Hartmann and Stampacchia [6].

If \( b \geq 0, \ X = \mathbb{R}^n, \) and \( G(v, u; u) = \langle \nabla Du, \eta(v, u) \rangle, \)
where \( \eta \) is continuous and \( D \) is differentiable and \( \eta \)-convex,
GMVLIP equation (1) is reduced to mathematical programming problem as [5]
\[
\min_{u \in C} D(u). \tag{5}
\]

Korpelevich [7] proposed the iterative method for VIP in 1976 on Hilbert space \( H \) as
\[
\begin{align*}
    u_0 & \in C \subseteq H, \\
v_n & = \text{proj}_C(u_n - \sigma Du_n), \\
u_n & = \sigma u_n + (1 - \alpha_n) \text{proj}_C(u_n - \sigma Dv_n), \\
C_n & = \{ z \in C : \|v_n - z\|^2 \leq \|u_n - z\|^2 \}, \\
D_n & = \{ z \in C : \langle u_n - z, u_0 - u_n \rangle \geq 0 \}, \\
u_{n+1} & = \text{proj}_{C \cap D_n} u_0, \ n \geq 0,
\end{align*}
\]  
where \( \sigma \geq 0, \) \( \text{proj}_C \) denotes projection of \( H \) onto \( C, \) and \( D \) is monotone and Lipschitz continuous mapping. This method is called the extragradient iterative method.

Nadezhkina and Takahashi [8] proposed a hybrid extragradient algorithm involving nonexpansive mapping \( T \) on \( C \) and studied the convergence analysis in 2006 as
\[
\begin{align*}
    u_0 & \in C \subseteq H, \\
x_n & = \text{proj}_C(u_n - \sigma Du_n), \\
v_n & = \alpha_n u_n + (1 - \alpha_n) \text{proj}_C(u_n - \sigma Dv_n), \\
C_n & = \{ z \in C : \|v_n - z\|^2 \leq \|u_n - z\|^2 \}, \\
D_n & = \{ z \in C : \langle u_n - z, u_0 - u_n \rangle \geq 0 \}, \\
u_{n+1} & = \text{proj}_{C \cap D_n} u_0, \ n \geq 0,
\end{align*}
\]  
where the idea considered in [8] has been generalized in [9]
from Hilbert to Banach space \( X \) as
\[
\begin{align*}
    u_0 & \in C \subseteq X, \\
x_n & = f^{-1}(a_n j u_n + (1 - a_n) J u_n), \\
v_n & = \text{proj}_C(\alpha_n u_n + (1 - \alpha_n) \text{proj}_C(u_n - \sigma Dv_n)), \\
C_n & = \{ z \in K : \phi(z, v_n) \leq \phi(z, u_n) \}, \\
D_n & = \{ z \in K : \langle u_n - z, J u_0 - J u_n \rangle \geq 0 \}, \\
u_{n+1} & = \bigcap_{z \in D_n} \Omega_z u_0,
\end{align*}
\]  
where \( \Pi_z \) denotes generalized projection of \( X \) onto \( C, \) \( \phi \) is the Lyapunov function such that \( \phi(v, u) = \|v\|^2 - 2 \langle v, ju \rangle + \|u\|^2, \) \( \forall v, u \in X, \) and \( f : X \rightarrow 2^{X^*} \) is the normalized duality mapping with \( f^{-1} \) being its inverse. For further work, see [10–17].

In 1967, an important technique was discovered by Bregman [18] in the light of Bregman distance function. This technique is very useful not only in design and interpretation of the iterative method but also to solve optimization and feasibility problems and to approximate equilibria, fixed point, variational inequalities, etc. (for details [19–22]).

In 2010, Reich and Sabach [23] introduced iterative algorithm on Banach space involving maximal monotone operators. In the light of Bregman projection, there were various iterative algorithms studied by researchers in this field (see, for instance, [19, 24–28]).

In 2008, Maingé [29] developed and studied an inertial Krasnosel’skii–Mann algorithm as
\[
\begin{align*}
    t_n & = u_n + \theta_n (u_n - u_{n-1}), \\
u_{n+1} & = (1 - \alpha_n) t_n + \alpha_n T t_n,
\end{align*}
\]  
For further work, see [30–39].

Inspired by the work in [2, 27, 29], we establish an inertial hybrid iterative algorithm involving Bregman relatively nonexpansive mapping to find a common solution of GMVLIP equation (1) and a fixed-point problem in Banach space. Moreover, we study the convergence analysis for the main result. At last, we list some consequences and computational example to emphasize the efficiency and relevancy of the main result.

2. Preliminaries

Assume \( g : X \rightarrow (-\infty, +\infty) \) is a proper, convex, and lower semicontinuous mapping and \( g^* : X^* \rightarrow (-\infty, +\infty) \) is a Fenchel conjugate of \( g, \) defined as
\[
g^* (u_0) = \sup \{ \langle u_0, u \rangle - g(u) : u \in Y \}, \quad u_0 \in Y^*. \tag{10}
\]

And, for any \( w \in \text{int} (\text{dom} g), \) interior of the domain of \( g \) and \( u \in X, \) the right-hand derivative of \( g \) at \( w \) in the direction \( u \) is
\[
g^0 (w, u) = \lim_{\lambda \rightarrow 0^+} \frac{g(w + \lambda u) - g(w)}{\lambda}. \tag{11}
\]

A mapping \( g \) is called Gateaux differentiable at \( w \) if the above limit exists. So, \( g^0 (w, u) \) agrees with \( \nabla g (w), \) the value of the gradient of \( g \) at \( w. \) It is called Frechet differentiable at \( w, \) if the limit is attained uniformly in \( \|u\| = 1. \) It is called uniformly Frechet differentiable on \( C \subseteq X, \) if the above limit is attained uniformly for \( w \in C \) and \( \|u\| = 1. \)

The mapping \( g \) is called Legendre if the following holds [19]:

(i) \( \text{int} (\text{dom} g) \neq \emptyset, \ g \) is Gateaux differentiable on \( \text{int} (\text{dom} g), \) and \( \text{dom} \nabla g = \text{int} (\text{dom} g) \)

(ii) \( \text{int} (\text{dom} g^* ) \neq \emptyset, \ g^* \) is Gateaux differentiable on \( \text{int} (\text{dom} g^* ), \) and \( \text{dom} \nabla g^* = \text{int} (\text{dom} g^* ) \)

We have the following [19]:

(i) \( g \) be Legendre iff \( g^* \) be Legendre mapping

(ii) \( (\partial g)^{-1} = \partial g^* \)

(iii) \( \nabla g = (\nabla g^*)^{-1}, \ \text{ran} \nabla g = \text{dom} \nabla g^* = \text{int} (\text{dom} g^* ), \) \( \text{ran} \nabla g^* = \text{dom} \nabla g = \text{int} (\text{dom} g) \)

(iv) The mappings \( g \) and \( g^* \) are strictly convex on \( \text{int} (\text{dom} g) \) and \( \text{int} (\text{dom} g^* ) \)

Definition 1 (see [18]). Let \( g : Y \rightarrow (-\infty, +\infty) \) be Gateaux differentiable and convex and \( D_g : \text{dom} g \times \text{int} (\text{dom} g) \rightarrow [0, +\infty) \) such that
is known as Bregman distance with respect to \( g \).

We notice that the Bregman distance is not a distance in the usual sense of term. Obviously, \( D_g(w,w)=0 \), but \( D_g(w,u)=0 \) may not imply \( w=u \). It holds if \( g \) is the Legendre function. However, \( D_g \) is neither symmetric nor satisfy the triangle inequality. We have the following important properties of \( D_g \) [40] for \( u,u_1,u_2 \in \text{dom} \ g \) and \( w_1,w_2 \in \text{int} \ (\text{dom} \ g) \).

(i) Two-point identity:

\[
D_g(u, w) + D_g(w, u) = \langle \nabla g(u) - \nabla g(w), u - w \rangle, \quad w \in \text{int} \ (\text{dom} \ g), u \in \text{dom} \ g, \tag{12}
\]

(ii) Three-point identity:

\[
D_g(u, w_1) + D_g(w_1, w_2) = \langle \nabla g(w_1) - \nabla g(w_2), w_1 - w_2 \rangle. \tag{13}
\]

(iii) Four-point identity:

\[
D_g(u, w_1) - D_g(u, w_2) - D_g(u_2, w_1) + D_g(u_2, w_2) = \langle \nabla g(w_2) - \nabla g(w_1), u - w_1 \rangle. \tag{14}
\]

Definition 2 (see [23, 25]). Let \( T : C \rightarrow \text{int} \ (\text{dom} \ g) \) be a mapping and \( F(T) = \{u \in C : Tu = u\} \), where \( F(T) \) is the set of fixed points of \( T \). Then, we have the following:

(i) A point \( u_0 \in C \) is called an asymptotic fixed point if \( C \) contains a sequence \( \{u_n\} \) with \( u_n \rightarrow u_0 \) such that \( \lim_{n \rightarrow \infty} \|Tu_n - u_0\| = 0 \). We represent \( \bar{F}(T) \) as the set of asymptotic fixed points of \( T \).

(ii) \( T \) is called Bregman quasi-nonexpansive if \( F(T) \neq \emptyset \); \( D_g(u_0, Tu) \leq D_g(u_0, u) \), \( \forall u \in C, u_0 \in F(T) \). \tag{15}

(iii) \( T \) is called Bregman relatively nonexpansive if

\[
F(T) = \bar{F}(T) \neq \emptyset; D_g(u_0, Tu) \leq D_g(u_0, u), \quad \forall u \in C, u_0 \in F(T). \tag{17}
\]

(iv) \( T \) is called Bregman firmly nonexpansive if \( \forall u_1, u_2 \in C \),

\[
\langle \nabla g(Tu_1) - \nabla g(Tu_2), Tu_1 - Tu_2 \rangle \leq \langle \nabla g(u_1) - \nabla g(u_2), Tu_1 - Tu_2 \rangle, \tag{18}
\]

or, correspondingly,

\[
D_g(Tu_1, Tu_2) + D_g(Tu_2, Tu_1) + D_g(Tu_1, u_1) + D_g(Tu_2, u_2) \leq D_g(Tu_1, u_2) + D_g(Tu_2, u_1). \tag{19}
\]

Example 1 (see [26]). Let \( A : X \rightarrow 2^X \) be a maximal monotone mapping. If \( A^{-1}(0) \neq \emptyset \) and the Legendre function \( g : X \rightarrow (-\infty, +\infty) \) is bounded on bounded subsets of \( X \) and uniformly Frechet differentiable, then the resolvent with respect to \( A \),

\[
\text{res}_A^g(u) = (\nabla g + A)^{-1} \nabla g(u), \tag{20}
\]
is a single-valued, closed, and Bregman relatively non-expansive mapping from $X$ onto $D(A)$ and $F(\text{res}_A) = A^{-1}(0)$.

**Definition 3** (see [18]). Let $g: X \to (-\infty, +\infty]$ be a Gateaux differentiable and convex function. The Bregman projection of $w \in \text{int}(\text{dom} g)$ onto $C \subset \text{int}(\text{dom} g)$ is a unique vector $\text{proj}_C^g w \in C$ with

$$D_g(\text{proj}_C^g w, w) = \inf\{D_g(u, w): u \in C\}. \quad (21)$$

**Remark 1** (see [24]). (i) If $X$ is a smooth Banach space and $g(u) = (1/2)\|u\|^2; \forall u \in X$, then the Bregman projection $\text{proj}_C^g w$ reduces to $\Pi_C^g(u)$, generalized projection (see [41]), and it is defined as

$$\phi(\Pi_C^g(u), u) = \min_{v \in C} \phi(v, u), \quad (22)$$

where $\phi$ is a Lyapunov function. (ii) If $X$ is a Hilbert space and $g(u) = (1/2)\|u\|^2, \forall u \in X$, then $\text{proj}_C^g w$ reduces to the metric projection of $u$ onto $C$.

For all $r > 0$, assume $B_r = \{z \in X: \|z\| \leq r\}$. Then, a map $g: X \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of $X$, if $\rho_r(t) > 0, \forall t > 0$, where $\rho_r: [0, +\infty) \to [0, +\infty)$ is defined as

$$\rho_r(t) = \frac{\alpha g(w) + (1 - \alpha)g(v) - g(\alpha w + (1 - \alpha)v)}{\alpha(1 - \alpha)} \quad (23)$$

$\forall t \geq 0$. The function $\rho_r$ is known as the gauge of uniform convexity of $g$. The function $g$ is also said to be uniformly smooth on bounded subsets of $X$ if $\lim_{t \to 0} (\sigma_r(t)/t) = 0$, for all $r > 0$, where $\sigma_r: [0, +\infty) \to [0, +\infty)$ is defined by

$$\sigma_r(t) = \sup_{w, v \in B_r, \|w - v\| = t} \frac{ag(w + (1 - \alpha)t\nu) + (1 - \alpha)g(w - at\nu) - g(w)}{\alpha(1 - \alpha)} \quad (24)$$

$\forall t \geq 0$. The function $g$ is said to be uniformly convex if the function $\delta_g: [0, +\infty) \to [0, +\infty)$, defined by

$$\delta_g(t) = \sup \left\{ \frac{1}{2}g(w) + \frac{1}{2}g(v) - g\left(\frac{w + v}{2}\right): \|v - w\| = t \right\}, \quad (25)$$

for all $w, v \in B_r$ and $\alpha \in (0, 1)$, where $\rho_r$ is the gauge of uniform convexity of $g$.

**Definition 4** (see [20]). Let $g: X \to (-\infty, +\infty]$ be a Gateaux differentiable and convex function. Then, $g$ is called the following:

(i) Totally convex at $w \in \text{int}(\text{dom} g)$ if its modulus of total convexity at $u$, i.e., the mapping $v_g: \text{int}(\text{dom} g) \times [0, +\infty) \to [0, +\infty)$ such that

$$v_g(w, s) = \inf \{D_g(v, w): v \in \text{dom} g, \|v - w\| = s\}, \quad (26)$$

is positive, for $s > 0$

(ii) Totally convex if it is totally convex at each point of $w \in \text{int}(\text{dom} g)$

(iii) Totally convex on bounded sets if $v_g: \text{int}(\text{dom} g) \times [0, +\infty) \to [0, +\infty)$ such that

$$v_g(B, s) = \inf \{v_g(w, s): w \in B \cap \text{dom} g\}. \quad (27)$$

By [20] (Section 1.3, p.30), we notice that any uniformly convex function is totally convex but the converse is not true. Also, by [21] (Theorem 2.10, p.9), $g$ is totally convex on bounded sets if and only if $g$ is uniformly convex on bounded sets.

**Definition 5** (see [20, 23]). A mapping $g: X \to (-\infty, +\infty]$ is called the following:

(i) Coercive if $\lim_{\|u\| \to +\infty} (g(u)/\|u\|) = +\infty$

(ii) Sequentially consistent if for any $\{u_n\}, \{v_n\} \subseteq X$ with $\{u_n\}$ bounded,
\[
\lim_{n \to \infty} D_g(v_n, u_n) = 0 \Rightarrow \lim_{n \to \infty} \|v_n - u_n\| = 0. \tag{29}
\]

**Lemma 1** (see [21]). Let \( g: X \to (-\infty, +\infty] \) be a convex function with domain at least two points. Then, \( g \) is sequentially consistent iff it is totally convex on bounded sets.

**Lemma 2** (see [42]). Let \( g: X \to (-\infty, +\infty] \) be uniformly Fréchet differentiable and bounded on \( C \subseteq X \), a bounded set. Then, \( g \) is uniformly continuous on \( C \) and \( \forall g \) is uniformly continuous on \( C \) from the strong topology of \( X \) to the strong topology of \( X^* \).

**Lemma 3** (see [23]). Let \( g: X \to (-\infty, +\infty] \) be a Gateaux differentiable and totally convex function. If \( u_0 \in X \) and \( \{D_g(u_n, u_0)\} \) are bounded, then \( \{u_n\} \) is also bounded.

**Lemma 4** (see [21]). Let \( g: X \to (-\infty, +\infty] \) be a Gateaux differentiable and totally convex function. If \( v \in C \) is the Bregman projection of \( w \) onto \( C \) with respect to \( g \), i.e., \( v = \text{proj}_C(w) \), then \( v \) is the unique solution of the variational inequality:

\[
\langle \nabla g(w) - \nabla g(v), v - u \rangle \geq 0, \quad \forall u \in C \tag{30}
\]

(iii) The vector \( v \) is the unique solution of the inequality:

\[
D_g(u, v) + D_g(v, w) \leq D_g(u, w), \quad \forall u, v \in C \tag{31}
\]

**Lemma 5** (see [25]). Let \( g: X \to (-\infty, +\infty] \) be Legendre and \( T: C \to C \) be Bregman quasi nonexpansive mapping with respect to \( g \). Then, \( F(T) \) is closed and convex.

**Lemma 6** (see [23]). Let \( g: X \to (-\infty, +\infty] \) be Gateaux differentiable and totally convex function, \( u_0 \in X \), and \( C \subseteq X \), a nonempty closed convex set. Suppose that \( \{u_n\} \) is bounded and any weak subsequential limit of \( \{u_n\} \) belongs to \( C \). If \( D_g(u_n, u_0) \leq D_g(\text{proj}_C(u_0), u_0) \), then \( \{u_n\} \) strongly converges to \( \text{proj}_C(u_0) \).

**Lemma 7** (see [43]). Let \( C \) be a nonempty subset of a Hausdorff topological vector space \( X^* \) and let \( f: C \to 2^X \) be a KKM mapping. If \( f(v) \) is closed in \( X^* \) for all \( v \in C \) and compact for some \( v \in C \), then \( \bigcap_{v \in C} f(v) \neq \emptyset \).

**Definition 6** (see [1]). A function \( G: C \times C \times C \to \mathbb{R} \) is said to be generalized relaxed \( \alpha \)-monotone if for any \( u, v \in C \), we have

\[
G(v, u; v) - G(v, u; u) \geq \alpha(u, v), \tag{32}
\]

where

\[
\lim_{t \to 0} \frac{\alpha(u, tv + (1-t)u)}{t} = 0. \tag{33}
\]

**Remark 3**

(i) If \( G(v, u; w) = \langle Aw, \eta(v, u) \rangle \), where \( \eta: C \times C \to X \), we say that the mapping \( A \) is a generalized \( \eta \)-\( \alpha \) monotone.

(ii) In Definition 6, let \( G(v, u; w) = \langle Aw, \eta(v, u) \rangle \) and \( \alpha(u, v) = \beta(v - u) \), where \( \beta: C \to R \) with \( \beta(tw) = t^p\beta(w) \), for \( t > 0 \) and \( p > 1 \), then we say that \( A \) is called a relaxed \( \eta \)-\( \alpha \) monotone mapping.

(iii) In case (ii), if \( \eta(v, u) = v - u \) for all \( u, v \in C \), then Definition 6 reduces to \( \langle Av - Au, v - u \rangle \geq \beta(v - u) \) for all \( u, v \in C \) and \( A \) is called a relaxed \( \alpha \)-monotone mapping.

**Example 2.** Consider \( X = X^* \), \( C = (-\infty, \infty) \), and

\[
G(v, u; w) = \begin{cases} 
-cw((v - u), v < u, \\
-cw(v - u), v \geq u, 
\end{cases} \tag{34}
\]

where \( c > 0 \) is a constant. Thus, \( G \) is generalized relaxed \( \alpha \)-monotone with

\[
\alpha(u, v) = \begin{cases} 
-c(v - u)^2, v < u, \\
c(v - u)^2, v \geq u. 
\end{cases} \tag{35}
\]

**Assumption 1.** Let \( b: C \times C \to \mathbb{R} \) satisfy the following:

(i) \( b \) is skew-symmetric, i.e., \( b(u, u) - b(u, v) - b(v, u) + b(v, v) \geq 0 \), \( \forall u, v \in C \)

(ii) \( b \) is convex in the second argument

(iii) \( b \) is continuous

**3. Existence of Solutions and Resolvent Operator**

For \( w \in C \), assume the auxiliary problems (in short, AP) related to GMVLIP equation (I): find \( u \in C \) such that

\[
G(v, u; u) + \langle \nabla g(u) - \nabla g(u), v - u \rangle + b(u, v) - b(u, u) \geq 0, \quad \forall v \in C. \tag{36}
\]
and find $u \in C$ such that

$$G(v, u; v) + \langle \nabla g(u) - \nabla g(w), v - u \rangle + b(u, v) - b(u, u) \geq \alpha(u, v), \quad \forall v \in C. \quad (37)$$

We have the Minty-type lemma as follows.

Lemma 8. Let $g: X \to (-\infty, +\infty]$ be Gateaux differentiable and coercive function, and let $b: C \times C \to \mathbb{R}$ satisfy Assumption 1 (ii). Assume $G: C \times C \times C \to \mathbb{R}$ with the following cases:

(i) $G(v, u; \cdot)$ is hemicontinuous
(ii) $G(\cdot, u; w)$ is convex

Then, AP equation (36) and AP equation (37) are equivalent.

Proof. Let $u \in C$ be a solution of AP equation (36) and by the concept of $G$, we obtain

$$G(v, u; v) + \langle \nabla g(u) - \nabla g(w), v - u \rangle + b(u, v) - b(u, u) \geq \alpha(u, v), \quad \forall v \in C. \quad (38)$$

Conversely, let $u \in C$ be a solution of AP equation (37). Then,

$$G(v, u; v) + \langle \nabla g(u) - \nabla g(w), v - u \rangle + b(u, v) - b(u, u) \geq \alpha(u, v), \quad \forall v \in C. \quad (39)$$

For any $v \in C$, let $v_t = tv + (1-t)u$, $t \in (0, 1]$, and we get $v_t \in C$. By equation (39), we have

$$G(v_t, u; v_t) + \langle \nabla g(u) - \nabla g(w), v_t - u \rangle + b(u, v_t) - b(u, u) \geq \alpha(u, v_t). \quad (40)$$

Using conditions (ii) and (iii), we obtain

$$G(v_t, u; v_t) \leq t\psi(v, u; v_t) + (1-t)\psi(u, u; v_t) = t\psi(v, u; v_t). \quad (41)$$

By Assumption 1 (ii), we have

$$tG(v, u; v_t) + t\langle \nabla g(u) - \nabla g(w), v - u \rangle + tb(u, v) + (1-t)b(u, u) - b(u, u) \geq \alpha(u, v_t). \quad (42)$$

Using equations (40)–(43), we have

$$tG(v, u; v_t) + t\langle \nabla g(u) - \nabla g(w), v - u \rangle + tb(u, v) - tb(u, u) \geq \alpha(u, v_t). \quad (44)$$

Hence,

$$G(v, u; v_t) + \langle \nabla g(u) - \nabla g(w), v - u \rangle + b(u, v) - b(u, u) \geq \frac{\alpha(u, v_t)}{t}. \quad (45)$$
Let \( t \rightarrow 0 \), and by condition (i), we obtain
\[
G(v, u; u) + \langle \nabla g(u) - \nabla g(w), v - u \rangle + b(u, v) - b(u, u) \geq 0.
\] (46)

Thus, \( u \in C \) be a solution of AP equation (95). \( \square \)

**Theorem 1.** Let \( g: X \rightarrow (-\infty, +\infty] \) be a Gateaux differentiable and coercive function, \( b: C \times C \rightarrow \mathbb{R} \) satisfy Assumption 1 (ii)-(iii), and \( \alpha: C \times C \rightarrow \mathbb{R} \) be a bifunction. Consider \( G: C \times C \times C \rightarrow \mathbb{R} \) and for any \( u, v, w \in C \), assume the following:

\[
F_w(v) = \{ u \in C: G(v, u; u) + \langle \nabla g(u) - \nabla g(w), v - u \rangle + b(u, v) - b(u, u) \geq 0 \}, \quad \forall v \in C,
\] (47)

and

\[
G_w(v) = \{ u \in C: G(v, u; v) + \langle \nabla g(u) - \nabla g(w), v - u \rangle + b(u, v) - b(u, u) \geq \alpha(u, v) \}, \quad \forall v \in C.
\] (48)

Obviously, \( \Pi \in C \) solves AP equation (36) if and only if \( \Pi \in \bigcap_{v \in C} F_w(v) \). Hence, \( \bigcap_{v \in C} F_w(v) \neq \emptyset \). Next, we prove that \( F_w \) is a KKM mapping. On the contrary, let \( F_w \) be not a KKM mapping; then, \( \exists \{v_1, v_2, \ldots, v_m\} \subset C \) such that

\[
\alpha(v_1, v_2, \ldots, v_m) \notin \bigcup_{i=1}^{m} F_w(v_i); \text{ this means there exists a } u_0 \in \alpha[v_1, v_2, \ldots, v_m], u_0 = \sum_{i=1}^{m} t_i v_i \text{ where } t_i \geq 0, \quad i = 1, 2, \ldots, m, \sum_{i=1}^{m} t_i = 1, \text{ but } u_0 \notin \bigcup_{i=1}^{m} F_w(v_i). \text{ Then,}
\]

\[
G(v_1, u_0; u_0) + \langle \nabla g(u_0) - \nabla g(w), v_1 - u_0 \rangle + b(u_0, v_1) - b(u_0, u_0) < 0.
\] (49)

By Theorem 1 (ii)-(iii), we get

\[
0 = G(u_0, u_0; u_0) + \langle \nabla g(u_0) - \nabla g(w), u_0 - u_0 \rangle + b(u_0, u_0) - b(u_0, u_0)
\leq \sum_{i=1}^{m} t_i G(v_i, u_0; u_0) + \sum_{i=1}^{m} t_i \langle \nabla g(u_0) - \nabla g(w), v_i - u_0 \rangle + \sum_{i=1}^{m} t_i b(u_0, v_i) - \sum_{i=1}^{m} t_i b(u_0, u_0)
= \sum_{i=1}^{m} t_i [G(v_i, u_0; u_0) + \langle \nabla g(u_0) - \nabla g(w), v_i - u_0 \rangle + b(u_0, v_i) - b(u_0, u_0)]
< 0,
\] (50)

which is a contradiction. Thus, \( F_w \) is a KKM mapping.

Next, we prove that \( F_w(v) \subset G_w(v), \forall v \in C \). Let \( u \in F_w(v) \), for any \( v \in C \); then,

\[
G(v, u; u) + \langle \nabla g(u) - \nabla g(w), v - u \rangle + b(u, v) - b(u, u) \geq 0.
\]

Using the concept of \( G \), we obtain

\[
G(v, u; v) + \langle \nabla g(u) - \nabla g(w), v - u \rangle + b(u, v) - b(u, u)
\geq G(v, u; u) + \langle \nabla g(u) - \nabla g(w), v - u \rangle + b(u, v) - b(u, u) + \alpha(u, v)
\geq \alpha(u, v).
\] (52)
Thus, $F_w(\nu) \subset G_w(\nu), \forall \nu \in C$, which yields that $G_w(\nu)$ is a KKM mapping.

Let $\{u_n\}$ be any sequence in $G_w(\nu)$ with $u_n \to u$ as $n \to \infty$. Then,

\[ G(\nu, u_n; \nu) + \langle \nabla g(u_n) - \nabla g(w), v - u_n \rangle + b(u_n, v) - b(u_n, u) \geq \alpha(u, v), \quad (53) \]

which yields that $G(\nu; u; v) + \langle \nabla g(u) - \nabla g(w), v - u \rangle + b(u, v) - b(u, u) \geq \alpha(u, v).$ Thus, $u \in G_w(\nu)$ and $G_w(\nu)$ are the closed subset of $C, \forall \nu \in C$. As $C$ is closed convex and bounded subset in $X$, it is weakly compact. Thus, $G_w(\nu)$ is also compact. By Lemmas 7 and 10, we have $\cap_{\nu \in \mathcal{C}} F_w(\nu) = \cap_{\nu \in \mathcal{C}} G_w(\nu) \neq \emptyset$. Therefore, AP equation (56) has a solution.

The resolvent of $G: C \times C \times C \to \mathbb{R}$ with respect to $b$ is the operator $\text{res}_{G,b}^\alpha: X \to 2^C$, defined as follows:

\[ \text{res}_{G,b}^\alpha(u) = \{ w \in C: G(v, w; w) + \langle \nabla g(w) - \nabla g(u), v - w \rangle + b(w, v) - b(w, u) \geq 0, \forall v \in C \}, \quad \forall u \in X. \quad (55) \]

\[ h(u, v) = g(v) - g(u) - \langle \xi, v - u \rangle \quad (57) \]
satisfies

\[ \lim_{\|u - v\| \to \infty} \frac{h(u, v)}{\|u - v\|} = -\infty, \quad (58) \]

for each fixed $v \in C$. By Theorem 1 in [44], equation (56) holds. Now, we show that equation (56) yields

\[ G(v, u; u) + b(u, v) - b(u, u) + \langle \nabla g(u), v - u \rangle - \langle \xi, v - u \rangle \geq 0, \quad (59) \]

for any $v \in C$. Assume $v_t = tv + (1 - t)u$ and $t \in (0, 1]$; we get $v_t \in C$. By equation (59) and the concept of $G$, we get

\[ G(v_t, u; v_t) + b(u, v_t) - b(u, u) + \langle \nabla g(u), v_t - u \rangle - \langle \xi, v_t - u \rangle \geq \alpha(u, v_t). \quad (60) \]

\[ G(tv + (1 - t)u, u; v_t) + b(u, tv + (1 - t)u) - b(u, u) + g(tv + (1 - t)u) - g(v) - \langle \xi, tv + (1 - t)u - u \rangle \geq \alpha(u, v_t), \forall v \in C. \quad (61) \]

Since
\[ g(tv + (1-t)u) - g(v) \leq \langle \nabla g(tv + (1-t)u), tv + (1-t)u - v \rangle, \] 
\( t \geq 0 \) \quad \text{(62)}

we get from equation (61), Theorem 1 (ii), and Assumption 1 (ii) that

\[ tG(v; u; v_I) + (1-t)G(u; u; u_I) + tb(u, v) + (1-t)b(u, u) - b(u, u_I) + \langle \nabla g(tv + (1-t)u), tv + (1-t)u - v \rangle - \langle \xi, tv + (1-t)u - u \rangle \geq a(u, v_I), \quad \forall \xi \in C. \] 
\( t \geq 0 \) \quad \text{(63)}

From Lemma 10 (iii), we have

\[ tG(v; u; v_I) + tb(u, v) - tb(u, u) + \langle \nabla g(tv + (1-t)u), t(v - u) \rangle - \langle \xi, t(v - u) \rangle \geq a(u, v_I) \] 
\( t \geq 0 \) \quad \text{(64)}

and

\[ t[G(v; u; v_I) + b(u, v) - b(u, u) + \langle \nabla g(tv + (1-t)u), (v - u) \rangle - \langle \xi, (v - u) \rangle] \geq a(u, v_I). \] 
\( t \geq 0 \) \quad \text{(65)}

Therefore,

\[ G(v; u; v_I) + b(u, v) - b(u, u) + \langle \nabla g(tv + (1-t)u), (v - u) \rangle - \langle \xi, (v - u) \rangle \geq \frac{a(u, v_I)}{t}, \quad \forall \xi \in C. \] 
\( t \geq 0 \) \quad \text{(66)}

As \( g \) is a Gateaux differentiable function, \( \nabla g \) is norm-to-weak * continuous. Taking \( t \rightarrow 0 \), we have

\[ G(v; u; u_I) + b(u, v) - b(u, u) + \langle \nabla g(u), (v - u) \rangle - \langle \xi, (v - u) \rangle \geq 0, \quad \forall \xi \in C. \] 
\( t \geq 0 \) \quad \text{(67)}

Thus, for any \( u \in X \), let \( \xi = \nabla g(u) \); we have \( \xi \in C \) such that

\[ G(v; u; u_I) + b(u, v) - b(u, u) + \langle \nabla g(u), (v - u) \rangle - \langle \nabla g(u), (v - u) \rangle \geq 0, \quad \forall \xi \in C, \] 
\( t \geq 0 \) \quad \text{(68)}

i.e.,

\[ G(v; u; u_I) + b(u, v) - b(u, u) + \langle \nabla g(u) - \nabla g(u), (v - u) \rangle \geq 0, \quad \forall \xi \in C, \] 
\( t \geq 0 \) \quad \text{(69)}

that is, \( u \in \operatorname{res}_{G,b}^g(u) \). Hence, \( \operatorname{dom}(\operatorname{res}_{G,b}^g) = X \). \( \square \)

Lemma 10. Let \( G: C \times C \times C \rightarrow \mathbb{R} \) satisfy all conditions of Theorem 1, and let \( b: C \times C \rightarrow \mathbb{R} \) satisfy Assumption 1. Let \( g: X \rightarrow (-\infty, +\infty] \) be a coercive Legendre function and the \( \text{resolvent operator } \operatorname{res}_{G,b}^g: X \rightarrow 2^C \) be defined by equation (55). Then, the following holds:

(i) \( \operatorname{res}_{G,b}^g \) is single-valued
(ii) \( \operatorname{res}_{G,b}^g \) is Bregman firmly nonexpansive type mapping, that is,
\[ \langle \nabla g(\text{res}_{G,b}^0 u) - \nabla g(\text{res}_{G,b}^0 v), \text{res}_{G,b}^0 u - \text{res}_{G,b}^0 v \rangle \leq \langle \nabla g(u) - \nabla g(v), \text{res}_{G,b}^0 u - \text{res}_{G,b}^0 v \rangle, \quad \forall u, v \in X \]  

(iii) \( F(\text{res}_{G,b}^0) = \text{Sol}(\text{GMVLIP}(1)) \) is closed and convex

(iv) \( D_g(q, \text{res}_{G,b}^0 u) + D_g(\text{res}_{G,b}^0 u, u) \leq D_g(q, u), \quad \forall q \in F(\text{res}_{G,b}^0). \)

(v) \( \text{res}_{G,b}^0 \) is Bregman quasi-nonexpansive

\[ G(w_2, w_1; w_1) + \langle \nabla g(w_1) - \nabla g(u), w_2 - w_1 \rangle + b(w_1, w_2) - b(w_1, w_1) \geq 0. \]  

and

\[ G(w_1, w_2; w_2) + \langle \nabla g(w_2) - \nabla g(u), w_1 - w_2 \rangle + b(w_2, w_1) - b(w_2, w_2). \]

Adding the above two inequalities, we get

\[ G(w_2, w_1; w_1) + G(w_1, w_2; w_2) + \langle \nabla g(w_1) - \nabla g(w_2), w_2 - w_1 \rangle + b(w_1, w_2) - b(w_1, w_1) + b(w_2, w_1) - b(w_2, w_2) \geq 0. \]

By condition (iii) of Theorem 1, we get

\[ -G(w_1, w_2; w_1) + G(w_1, w_2; w_2) + \langle \nabla g(w_1) - \nabla g(w_2), w_2 - w_1 \rangle + b(w_1, w_2) - b(w_1, w_1) + b(w_2, w_1) - b(w_2, w_2) \geq 0. \]

As \( b \) is skew symmetric and \( G \) is a generalized relaxed \( \alpha \)-monotone,

\[ \alpha(w_2, w_1) - \langle \nabla g(w_1) - \nabla g(w_2), w_1 - w_2 \rangle \leq 0 \]  

\[ \langle \nabla g(w_1) - \nabla g(w_2), w_2 - w_1 \rangle \geq \alpha(w_2, w_1). \]  

By interchanging the position of \( w_1 \) and \( w_2 \) in equation (76), we get

\[ \langle \nabla g(w_2) - \nabla g(w_1), w_1 - w_2 \rangle \geq \alpha(w_2, w_1). \]  

Adding equations (76) and (77), we have

\[ 2\langle \nabla g(w_1) - \nabla g(w_2), w_2 - w_1 \rangle \geq \{\alpha(w_1, w_2) + \alpha(w_2, w_1)\}. \]  

As \( \alpha(u, v) + \alpha(v, u) \geq 0, \forall v \in C, \)

\[ \langle \nabla g(w_1) - \nabla g(w_2), w_2 - w_1 \rangle \geq 0. \]  

This implies that

\[ \langle \nabla g(w_2) - \nabla g(w_1), w_1 - w_2 \rangle \leq 0. \]

As \( g \) is convex and Gateaux differentiable,

\[ \langle \nabla g(w_2) - \nabla g(w_1), w_1 - w_2 \rangle \geq 0. \]

By equations (80) and (81), we have

\[ \langle \nabla g(w_2) - \nabla g(w_1), w_2 - w_1 \rangle = 0. \]  

\[ \]
Since \( g \) is a Legendre function, \( w_1 = w_2 \). Hence, \( \text{res}_{G,b}^g \) is single-valued.

(ii) For \( u, v \in C \), we obtain

\[
\begin{align*}
G(\text{res}_{G,b}^g v, \text{res}_{G,b}^g u; \text{res}_{G,b}^g u) + &\langle \nabla g(\text{res}_{G,b}^g u) - \nabla g(u), \text{res}_{G,b}^g v - \text{res}_{G,b}^g u \rangle \\
+ b(\text{res}_{G,b}^g u, \text{res}_{G,b}^g v) - b(\text{res}_{G,b}^g u, \text{res}_{G,b}^g u) &\geq 0
\end{align*}
\]

and

\[
\begin{align*}
G(\text{res}_{G,b}^g u, \text{res}_{G,b}^g v; \text{res}_{G,b}^g v) + &\langle \nabla g(\text{res}_{G,b}^g v) - \nabla g(v), \text{res}_{G,b}^g u - \text{res}_{G,b}^g v \rangle \\
+ b(\text{res}_{G,b}^g v, \text{res}_{G,b}^g u) - b(\text{res}_{G,b}^g v, \text{res}_{G,b}^g v) &\geq 0.
\end{align*}
\]

Adding the above two inequalities, we have

\[
\begin{align*}
G(\text{res}_{G,b}^g v, \text{res}_{G,b}^g u; \text{res}_{G,b}^g u) + &\langle \nabla g(\text{res}_{G,b}^g u) - \nabla g(u), \text{res}_{G,b}^g v - \text{res}_{G,b}^g u \rangle \\
+ b(\text{res}_{G,b}^g u, \text{res}_{G,b}^g v) - b(\text{res}_{G,b}^g u, \text{res}_{G,b}^g u) + &\langle \nabla g(\text{res}_{G,b}^g v) - \nabla g(v), \text{res}_{G,b}^g u - \text{res}_{G,b}^g v \rangle \\
+ b(\text{res}_{G,b}^g v, \text{res}_{G,b}^g u) - b(\text{res}_{G,b}^g v, \text{res}_{G,b}^g v) &\geq 0,
\end{align*}
\]

which yields by applying the concept of \( b \) and \( G \),

\[
\langle \nabla g(\text{res}_{G,b}^g u) - \nabla g(u), \text{res}_{G,b}^g v - \text{res}_{G,b}^g u \rangle \geq -\{G(\text{res}_{G,b}^g v, \text{res}_{G,b}^g u; \text{res}_{G,b}^g u) + G(\text{res}_{G,b}^g u, \text{res}_{G,b}^g v; \text{res}_{G,b}^g v)\}
\]

\[
= G(\text{res}_{G,b}^g v, \text{res}_{G,b}^g u; \text{res}_{G,b}^g u) - G(\text{res}_{G,b}^g v, \text{res}_{G,b}^g u; \text{res}_{G,b}^g u)
\]

\[
\geq a(\text{res}_{G,b}^g u, \text{res}_{G,b}^g v).
\]

In equation (86), interchanging the position of \( \text{res}_{G,b}^g u \) and \( \text{res}_{G,b}^g v \), we get

\[
\langle \nabla g(\text{res}_{G,b}^g v) - \nabla g(v), \text{res}_{G,b}^g u - \text{res}_{G,b}^g v \rangle \geq a(\text{res}_{G,b}^g v, \text{res}_{G,b}^g u).
\]

Adding equations (86) and (87) and using \( a(u, v) + a(v, u) \geq 0, \forall v \in C \), we get

\[
2\langle \nabla g(\text{res}_{G,b}^g u) - \nabla g(u), \text{res}_{G,b}^g v - \text{res}_{G,b}^g u \rangle \geq \{a(\text{res}_{G,b}^g u, \text{res}_{G,b}^g v) + a(\text{res}_{G,b}^g v, \text{res}_{G,b}^g u)\} \geq 0.
\]

This implies that...
\[ \langle \nabla g (\text{res}^g_{G,b} u) - \nabla g (\text{res}^g_{G,b} v), \text{res}^g_{G,b} (u) - \text{res}^g_{G,b} (v) \rangle \leq \langle \nabla g (u) - \nabla g (v), \text{res}^g_{G,b} (u) - \text{res}^g_{G,b} (v) \rangle \quad (89) \]

This means that \( \text{res}^g_{G,b} \) is a Bregman firmly nonexpansive type mapping.

\[ \begin{align*}
&u \in F(\text{res}^g_{G,b}) \iff u = \text{res}^g_{G,b} u \\
&\iff G(v, u) + \langle \nabla g(u) - \nabla g(u), v - u \rangle + b(u, v) - b(u, u) \geq 0, \quad \forall v \in C \\
&\iff G(v, u) + b(u, v) - b(u, u), \quad \forall v \in C \\
&\iff u \in \text{Sol}(\text{GMVLIP}(1)).
\end{align*} \quad (90) \]

Furthermore, Since \( \text{res}^g_{G,b} \) is a Bregman firmly nonexpansive type mapping, in ([42], Lemma 1.3.1), \( F(\text{res}^g_{G,b}) \) is a closed and convex subset of \( C \). Therefore, by equation (90), we get that \( \text{Sol}(\text{GMVLIP}(1)) = F(\text{res}^g_{G,b}) \) is closed and convex.

\[ \langle \nabla g (\text{res}^g_{G,b} u) - \nabla g (\text{res}^g_{G,b} v), \text{res}^g_{G,b} (u) - \text{res}^g_{G,b} (v) \rangle \leq \langle \nabla g (u) - \nabla g (v), \text{res}^g_{G,b} (u) - \text{res}^g_{G,b} (v) \rangle \quad (91) \]

Moreover, we have

\[ \begin{align*}
D_g (\text{res}^g_{G,b} (u), \text{res}^g_{G,b} (v)) + D_g (\text{res}^g_{G,b} (v), \text{res}^g_{G,b} (u)) &\leq D_g (\text{res}^g_{G,b} (u), v) - D_g (\text{res}^g_{G,b} (u), u) \\
&+ D_g (\text{res}^g_{G,b} (v), u) - D_g (\text{res}^g_{G,b} (v), v).
\end{align*} \quad (92) \]

Taking \( v = w \in F(\text{res}^g_{G,b}) \), we see that

\[ D_g (\text{res}^g_{G,b} (u), w) + D_g (w, \text{res}^g_{G,b} (u)) \leq D_g (\text{res}^g_{G,b} (u), w) - D_g (\text{res}^g_{G,b} (u), u) + D_g (w, u) - D_g (w, w). \quad (93) \]

Hence,

\[ D_g (w, \text{res}^g_{G,b} (u)) + D_g (\text{res}^g_{G,b} (u), w) \leq D_g (w, u). \quad (94) \]

Equation (94) implies that \( \text{res}^g_{G,b} \) is Bregman quasi-nonexpansive mapping. \( \square \)

4. Main Result

We developed the strong convergence algorithm for the inertial iterative method to find the common solution of GMVLIP equation (1) and fixed-point problem of a Bregman relatively nonexpansive mapping in reflexive Banach space.

Iterative Algorithm 1. Let the sequences \( \{x_n\} \) and \( \{z_n\} \) be generated by the iterative algorithm:

\[ x_n = \alpha_n x_n + \beta_n (x_{n-1} - x_n), \]
\[ \nu_n = \nabla g (a_n \nu (u_n) + (1 - a_n) \nabla g (Tu_n)), \]
\[ w_n = \nabla g (\beta_n \nu (Tu_n) + (1 - \beta_n) \nabla g (v_n)), \]
\[ z_n = \text{res}^g_{G,b} w_n, \]
\[ C_n = \{ z \in C : D_g (z, z_n) \leq D_g (z, u_n) \}, \]
\[ Q_n = \{ z \in C : \langle \nabla g (x_n) - \nabla g (x), z - x_n \rangle \geq 0 \}, \]
\[ x_{n+1} = \text{proj}_{C \cap Q_n}^{\text{dom}g} x_n, \quad \text{for all } n \geq 0, \]

where \( |\theta_n| \leq (0, 1) \) and \( \{a_n\}, \{\beta_n\} \subseteq [0, 1] \).

Theorem 2. Let \( C \subseteq X \) with \( C \subseteq \text{int}(\text{dom}g) \), where \( g : X \rightarrow (-\infty, +\infty] \) be a coercive Legendre function which is bounded, uniformly Frechet differentiable, and totally convex on bounded subsets of \( X \). Let \( G : C \times C \times C \rightarrow \mathbb{R} \)
satisfy all conditions of Theorem 1 with continuous $G(y, \cdot; y)$, and b. $C \times C \to \mathbb{R}$ satisfies Assumption 1, respectively. Let $T: \Omega \to C$ be a Bregman relatively nonexpansive mapping.

Let $\Omega = \text{Sol}({\text{GMVLIP}}(1)) \cap F(T) \neq \emptyset$. Let $\{x_n\}, \{z_n\}$ be generated by Iterative 1 and $\{\theta_n\} \subseteq (0, 1), \{\alpha_n\}, \{\beta_n\} \subseteq [0, 1]$ with $\lim_{n \to \infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $\text{proj}_{\Omega}^g x_0$.

**Proof.** For convenience, we divide its proof into several steps as in the following.

**Step 1.** $\Omega$ and $C_n \cap Q_n$ are closed and convex, $\forall n \geq 0$.

By Lemmas 5 and 9, $\Omega$ is a closed and convex, and therefore, $\text{proj}_{\Omega}^g x_0$ is well defined.

Obviously, $Q_n$ is closed and convex. Furthermore, we prove that $C_n$ is closed and convex, $\forall n \geq 0$. We can easily show that $C_n$ is closed and convex, $\forall n$. Thus, $C_n \cap Q_n$ is closed and convex, $\forall n \geq 0$.

**Step 2.** $\Omega \subseteq C_n \cap Q_n$, $\forall n \geq 0$, and $\{x_n\}$ is well defined.

Let $p \in \Omega$; then,

$$D_g(p, z_n) = D_g(p, \text{res}_g^\alpha, w_n) \\
\leq D_g(p, u_n) \\
= D_g(p, \nabla g^\ast (\beta_n \nabla g(Tu_n) + (1 - \beta_n) \nabla g(v_n))) \\
\leq \beta_n D_g(p, u_n) + (1 - \beta_n) D_g(p, v_n),$$

and

$$D_g(p, v_n) = D_g(p, \nabla g^\ast (\alpha_n \nabla g(u_n) + (1 - \alpha_n) \nabla g(Tu_n))) \\
\leq \alpha_n D_g(p, u_n) + (1 - \alpha_n) D_g(p, u_n) \\
= D_g(p, u_n).$$

(96)

Substituting equation (97) into equation (96), we have

$$D_g(p, z_n) \leq D_g(p, u_n).$$

(98)

Thus, $p \in C_n$. Therefore, $\Omega \subseteq C_n$, $\forall n \geq 0$. Furthermore, by induction, we show that $\Omega \subseteq C_n \cap Q_n$, $n \geq 0$. As $Q_0 = C$, $\Omega \subseteq C_0 \cap Q_0$. Suppose that $\Omega \subseteq C_m \cap Q_m$, for some $m \geq 0$. Then, $\exists x_{m+1} \in C_m \cap Q_m$ such that $x_{m+1} = \text{proj}_{C_m \cap Q_m}^g x_0$. From the definition of $x_{m+1}$, we get $\langle \nabla g(x_0) - \nabla g(x_{m+1}), x_{m+1} - z \rangle \geq 0, \forall z \in C \cap Q_m$. Since $\Omega \subseteq C_m \cap Q_m$, we have

$$\langle \nabla g(x_0) - \nabla g(x_{m+1}), p - x_{m+1} \rangle \leq 0, \quad \forall p \in \Omega,$$

(99)

which implies $p \in Q_{m+1}$. Hence, $\Omega \subseteq C_{m+1} \cap Q_{m+1}$ implies $\Omega \subseteq C_n \cap Q_n$, $\forall n \geq 0$, and thus, $x_n = \text{proj}_{C_n \cap Q_n}^g x_0$ is well defined, $\forall n \geq 0$. Hence, $\{x_n\}$ is well defined.

**Step 3.** The sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{z_n\}$, and $\{w_n\}$ are bounded.

Using the concept of $Q_n$, we get $x_n = \text{proj}_{Q_n}^g x_0$. By $x_n = \text{proj}_{Q_n}^g x_0$ and Lemma 10 (iii), we obtain

$$D_g(x_n, x_0) = D_g(\text{proj}_{Q_n}^g x_0, x_0) \\
\leq D_g(u, x_0) - D_g(u, \text{proj}_{Q_n}^g x_0) \leq D_g(u, x_0),$$

$$\forall u \in \Omega \subseteq Q_n.$$

(100)

This implies that $D_g(x_n, x_0)$ is bounded, and hence, $\{x_n\}$ is bounded by Lemma 3.

Now,

$$D_g(p, x_n) = D_g(\text{proj}_{C_n \cap Q_n}^g x_0, x_0) \\
\leq D_g(p, x_n) - D_g(x_n, x_0),$$

(101)

which implies that $\{D_g(p, x_n)\}$ is bounded. Using $D_g(p, x_n) \leq D_g(p, T x_n) \subseteq D_g(p, x_p) \forall p \in \Omega \subseteq \{T x_n\}$ is bounded. Therefore, $\{u_n\}, \{v_n\}, \{w_n\}$, and $\{z_n\}$ are bounded.

**Step 4.** $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$; $\lim_{n \to \infty} \|x_n - u_n\| = 0$; $\lim_{n \to \infty} (\|z_n - w_n\| = 0$ and $\lim_{n \to \infty} \|u_n - w_n\| = 0$; $\lim_{n \to \infty} \|T u_n - T u_n\| = 0$.

Since $x_{n+1} = \text{proj}_{C_n \cap Q_n}^g x_0$ and $x_n \in \text{proj}_{Q_n}^g x_0$, we get

$$D_g(x_n, x_0) \leq D_g(x_{n+1}, x_0),$$

$$\forall n \geq 0,$$

which implies $\{D_g(x_n, x_0)\}$ is nondecreasing. By boundedness of $\{D_g(x_n, x_0)\}$, $\lim_{n \to \infty} D_g(x_n, x_0)$ exists and is finite. Furthermore,

$$D_g(x_{n+1}, x_n) = D_g(x_{n+1}, \text{proj}_{Q_n}^g x_0) \\
\leq D_g(x_{n+1}, x_n) - D_g(\text{proj}_{Q_n}^g x_0, x_0)$$

(103)

which yields

$$\lim_{n \to \infty} D_g(x_{n+1}, x_n) = 0.$$

(104)

Using Lemma 1,

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.$$

(105)

From the definition of $u_n$, $\|u_n - x_n\| = \|\theta_n (x_n - x_{n-1})\| \leq \|x_n - x_{n-1}\|$, which implies by equation (105) that

$$\lim_{n \to \infty} \|u_n - x_n\| = 0.$$

(106)

Since

$$\|u_{n+1} - x_n\| \leq \|u_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|,$$

(107)

it follows from equations (105) and (106) that

$$\lim_{n \to \infty} \|u_{n+1} - x_{n+1}\| = 0.$$

(108)

Using Lemma 2 because $g$ is uniformly Frechet differentiable, we get

$$\lim_{n \to \infty} \|g(u_n) - g(x_{n+1})\| = 0$$

(109)

and
\[ \lim_{n \to \infty} \| \nabla g(u_n) - \nabla g(x_{m+1}) \| = 0. \quad (110) \]

By the concept of \( D_g \), we get
\[ D_g(x_{n+1}, u_n) = g(x_{n+1}) - g(u_n) - \langle \nabla g(u_n), x_{n+1} - u_n \rangle. \quad (111) \]

\( \nabla g \) is bounded on the bounded subset of \( X \) because \( g \) is bounded on \( X \). Since \( g \) is uniformly Frechet differentiable, it is uniformly continuous on bounded subsets. Hence, by equations (108), (109), and (111),
\[ \lim_{n \to \infty} D_g(x_{n+1}, u_n) = 0. \quad (112) \]

As \( x_{n+1} = \text{proj}_{C_n} x_n \), we have
\[ D_g(x_{n+1}, z_n) \leq D_g(x_{n+1}, u_n), \quad (113) \]

and hence, by equations (112) and (113),
\[ \lim_{n \to \infty} D_g(x_{n+1}, z_n) = 0. \quad (114) \]

Thanks to Lemma 1,
\[ \lim_{n \to \infty} \| x_{n+1} - z_n \| = 0. \quad (115) \]

Taking into account
\[ \| z_n - u_n \| \leq \| z_n - x_{n+1} \| + \| x_{n+1} - u_n \|. \]

by equations (108) and (115), we get
\[ \lim_{n \to \infty} \| z_n - u_n \| = 0. \quad (116) \]

By Lemma 2,
\[ \lim_{n \to \infty} \| g(z_n) - g(u_n) \| = 0 \]

and
\[ \lim_{n \to \infty} \| \nabla g(z_n) - \nabla g(u_n) \| = 0. \quad (119) \]

Next, we estimate
\[ \lim_{n \to \infty} \| D_g(p, u_n) - D_g(p, z_n) \| = 0. \quad (121) \]

Furthermore, it follows from Lemma 9 (v) that
\[ \lim_{n \to \infty} \| u_n - w_n \| = 0. \quad (125) \]

By uniform Frechet differentiable of \( g \), Lemma 2, and equations (124) and (125), we have
\[ \lim_{n \to \infty} \| \nabla g(z_n) - \nabla g(u_n) \| = 0, \quad (126) \]

\[ \lim_{n \to \infty} \| \nabla g(u_n) - \nabla g(w_n) \| = 0. \quad (127) \]
\[\|\nabla g(u_n) - \nabla g(w_n)\| = \|\nabla g(u_n) - \nabla g(\nabla^* (\beta_n \nabla g(Tu_n) + (1 - \beta_n) \nabla g(v_n)))\|\]
\[= \|\nabla g(u_n) - \beta_n \nabla g(Tu_n) - (1 - \beta_n) \nabla g(v_n)\|\]
\[= \|\beta_n \nabla g(u_n) - \nabla g(Tu_n) + (1 - \beta_n) \nabla g(u_n) - \nabla g(v_n)\|\]
\[= \|\beta_n \nabla g(u_n) - \nabla g(Tu_n) + (1 - \beta_n) (1 - \alpha_n) \nabla g(u_n) - \nabla g(Tu_n)\|\]
\[= \|1 - \alpha_n (1 - \beta_n)\| \|\nabla g(u_n) - \nabla g(Tu_n)\|.\]  

By equations (127) and (128) and \(\lim_{n \to \infty} \alpha_n = 0\), we get
\[\lim_{n \to \infty} \|\nabla g(u_n) - \nabla g(Tu_n)\| = 0.\]  

Moreover, we have from equation (129) that
\[\lim_{n \to \infty} \|u_n - Tu_n\| = 0.\]

Step 5. \(\overline{x} \in \Omega\).  
First, we prove that \(\overline{x} \in F(T)\). As \(\{x_n\}\) is bounded, \(\exists\) a subsequence \(\{x_{n_k}\} \subseteq \{x_n\}\) such that \(x_{n_k} \to \overline{x} \in C\) as \(k \to \infty\).

\[G(v, z_{n_k}; z_{n_k}) + \langle \nabla g(z_{n_k}) - \nabla g(w_{n_k}), v - z_{n_k} \rangle + b(v, z_{n_k}) - b(z_{n_k}, z_{n_k}) \geq 0, \quad \forall v \in C.\]  

Using generalized relaxed \(\alpha\)-monotonicity of \(G\), we have
\[\langle \nabla g(z_{n_k}) - \nabla g(w_{n_k}), v - z_{n_k} \rangle \geq -G(v, z_{n_k}; z_{n_k}) - b(v, z_{n_k}) + b(z_{n_k}, z_{n_k}), \quad \forall v \in C,\]
\[\geq \alpha(z_{n_k}, v) - G(v, z_{n_k}; v) - b(v, z_{n_k}) + b(z_{n_k}, z_{n_k}).\]  

Using the concept of \(G, b\), equation (126), and \(k \to \infty\) in equation (133), we obtain
\[\alpha(\overline{x}, v) = G(v, \overline{x}; v) + b(\overline{x}, \overline{x}) - b(\overline{x}, v) \leq 0, \quad \forall v \in C.\]  

For \(t \in (0, 1)\) and \(v \in C\), let \(v_t = tv + (1 - t)\overline{x}\). Since \(v_t \in C\), we have
\[\alpha_t(\overline{x}, v_t) = G(v_t, \overline{x}; v_t) + b(\overline{x}, \overline{x}) - b(\overline{x}, v_t) \leq 0,\]
which implies that
\[\alpha(\overline{x}, v) \leq G(v, \overline{x}; v) + b(\overline{x}, \overline{x}) + b(\overline{x}, v) \]
\[\leq tG(v, \overline{x}; v) + (1 - t)G(\overline{x}, \overline{x}; v) - b(\overline{x}, v) + tb(\overline{x}, v) + (1 - t)b(\overline{x}, \overline{x})\]
\[\leq t[ G(v, \overline{x}; v) + b(\overline{x}, v) - b(\overline{x}, \overline{x})].\]  

Since \(G(v, \overline{x}; v)\) is hemi-continuous, we have
\[\lim_{t \to 0} \{G(v, \overline{x}; v) + b(\overline{x}, v) - b(\overline{x}, \overline{x})\} \geq \lim_{t \to 0} \frac{\alpha(\overline{x}, v)}{t},\]
which implies
\[G(v, \overline{x}; v) + b(\overline{x}, v) - b(\overline{x}, \overline{x}) \geq 0.\]

Hence, \(\overline{x} \in \text{Sol}(\text{GMVLIP} (1)).\) Thus, \(\overline{x} \in \Omega\).

Step 6. We prove that \(x_n \to \overline{x} = \text{proj}_{\Omega}^\beta x_0\).
Remark 4. Let \( u = \text{proj}_{\Omega}^{\theta} x_0 \). As \( \{x_n\} \) is weakly convergent, \( x_{n+1} = \text{proj}_{\Omega}^{\theta} x_0 \), and \( \text{proj}_{\Omega}^{\theta} x_0 \in \Omega \cap Q_n \). By equation (100), we have
\[
D_g(x_{n+1}, x_0) \leq D_g(\text{proj}_{\Omega}^{\theta} x_0, x_0). \tag{139}
\]
Using Lemma 6, \( \{x_n\} \) is strongly convergent to \( u = \text{proj}_{\Omega}^{\theta} x_0 \). Hence, by the uniqueness of the limit, \( \{x_n\} \) converges strongly to \( x = \text{proj}_{\Omega}^{\theta} x_0 \). \( \square \)

5. Consequences

Finally, we get the following consequences of Theorem 2.

Corollary 1. Let \( C \subseteq X \) with \( C \subseteq \text{int}(\text{dom} g) \), where \( g : X \rightarrow (-\infty, +\infty] \) be a coercive Legendre function which is bounded, uniformly Frechet differentiable, and totally convex on bounded subsets of \( X \). Let \( G : C \times C \times C \rightarrow \mathbb{R} \) satisfy conditions (i), (ii), and (iii) of Theorem 1 and \( G \) be monotone, i.e.,
\[
\begin{align*}
x_0, x_1 \in C, \\
u_n &= x_n + \alpha_n (x_n - x_{n-1}), \\
v_n &= \nabla g^* (\alpha_n \nabla g(u_n) + (1 - \alpha_n) \nabla g(\text{res}_{\Omega}^{\theta} u_n)), \\
z_n &= \nabla g^* (\beta_n \nabla g(\text{res}_{\Omega}^{\theta} u_n) + (1 - \beta_n) \nabla g(v_n)), \\
C_n &= \{z \in C : D_g(z, z_n) \leq D_g(z, u_n)\}, \\
Q_n &= \{z \in C : \langle \nabla g(x_0) - \nabla g(x_n), z - x_n \rangle \leq 0\}, \\
x_{n+1} &= \text{proj}_{C \cap Q_n}^{\theta} x_0, \forall n \geq 0,
\end{align*}
\]  
(141)
where \( [\theta_n] \subseteq (0, 1) \) and \( [\alpha_n], [\beta_n] \subseteq [0, 1] \) with \( \lim \alpha_n = 0 \). Then, \( \{x_n\} \) converges strongly to \( \text{proj}_{A^{-1}(0)}^{\theta} x_0 \).

Remark 4. If \( g(x) = (1/2)\|x\|^2 \), \( \forall x \in X \), then Theorem 2 is reduced to the strong convergence theorem for finding the common solution of GMVLIP equation (1) and fixed-point problem of a relatively nonexpansive mapping in reflexive Banach space.

6. Numerical Example

Finally, to support our main theorem, we now give an example in infinitely dimensional spaces \( L_1 \) such that \( \| \cdot \| \) is \( L_2 \)-norm defined by \( \|x\| = \sqrt{\int_0^1 |x(t)|^2 \, dt} \) when \( x(t) \in L_2[0, 1] \).

Example 3. Let \( X = L_1[0, 1] \) and \( C = \{x(t) \in L_2[0, 1] : \int_0^1 |x(t)| \, dt \leq 2\} \). Define mappings as follows:

(i) Coercive Legendre function \( g : X \rightarrow (-\infty, +\infty] \) by \( g(x) = (1/2)\|x\|^2 \), \( \forall x \in X \)
(ii) \( \forall x, y, z \in C \), Function \( G : C \times C \times C \rightarrow \mathbb{R} \) by \( G(x, y, z) = (1/2)(\|y\|^2 - \|z\|^2) \), with \( a : C \times C \rightarrow \mathbb{R} \) such that \( a(x, y) = 0, \forall x, y \in C \)
(iii) Bifunction \( b : C \times C \rightarrow \mathbb{R} \) by \( b(x, y) = -(x, y), \forall x, y \in C \)
(iv) Bregman relatively nonexpansive mapping \( T : C \rightarrow C \) with respect to \( g \) by \( Tx = (x/2), \forall x \in C \)

It is obvious that \( G \), \( a \), and \( b \) satisfies all conditions of Theorem 1 with continuous \( G(y, \cdot ; y) \) and \( b : C \times C \rightarrow \mathbb{R} \) satisfies Assumption 1, respectively. On the other hand, we consider
Table 1: Numerical results of the difference $e_n$.

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>$e_n$</th>
<th>No. of iter.</th>
<th>CPU time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0 = (\sin(t)/2), x_0 = \sin(t)$</td>
<td>$1/n + 1$</td>
<td>9</td>
<td>7.59932</td>
</tr>
<tr>
<td></td>
<td></td>
<td>15</td>
<td>12.22748</td>
</tr>
<tr>
<td></td>
<td></td>
<td>19</td>
<td>14.71024</td>
</tr>
<tr>
<td></td>
<td></td>
<td>20</td>
<td>15.57306</td>
</tr>
<tr>
<td></td>
<td></td>
<td>20</td>
<td>15.66219</td>
</tr>
<tr>
<td>$x_1 = (2 \sin(t) - t/2), x_0 = 2 \sin(t) - t$</td>
<td>$1/2n + 1$</td>
<td>10</td>
<td>8.75820</td>
</tr>
<tr>
<td></td>
<td></td>
<td>15</td>
<td>12.24971</td>
</tr>
<tr>
<td></td>
<td></td>
<td>20</td>
<td>15.65068</td>
</tr>
<tr>
<td></td>
<td></td>
<td>20</td>
<td>17.30471</td>
</tr>
<tr>
<td>$x_2 = t + \log^2(t + 1), x_0 = \log(t + 1)$</td>
<td>$1/n^2 + 1$</td>
<td>11</td>
<td>9.06217</td>
</tr>
<tr>
<td></td>
<td></td>
<td>15</td>
<td>10.15574</td>
</tr>
<tr>
<td></td>
<td></td>
<td>21</td>
<td>15.81738</td>
</tr>
<tr>
<td></td>
<td></td>
<td>22</td>
<td>17.65972</td>
</tr>
</tbody>
</table>

Table 2: Numerical results of the difference $\theta$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\beta_n$</th>
<th>No. of iter.</th>
<th>CPU time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0 = (\sin(t)/2), x_0 = \sin(t)$</td>
<td>$1/10n + 1$</td>
<td>9</td>
<td>7.57878</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9</td>
<td>7.50740</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9</td>
<td>7.67907</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9</td>
<td>7.59864</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9</td>
<td>7.60107</td>
</tr>
<tr>
<td>$x_1 = (2 \sin(t) - t/2), x_0 = 2 \sin(t) - t$</td>
<td>$1/100n + 1$</td>
<td>10</td>
<td>8.53362</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>8.82150</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>8.62202</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>8.82075</td>
</tr>
<tr>
<td>$x_2 = t + \log^2(t + 1), x_0 = \log(t + 1)$</td>
<td>$1/n^2 + 1$</td>
<td>11</td>
<td>9.61967</td>
</tr>
<tr>
<td></td>
<td></td>
<td>11</td>
<td>9.06217</td>
</tr>
<tr>
<td></td>
<td></td>
<td>11</td>
<td>9.56274</td>
</tr>
<tr>
<td></td>
<td></td>
<td>11</td>
<td>9.47570</td>
</tr>
</tbody>
</table>

Table 3: Numerical results of the difference $\alpha_n$.

<table>
<thead>
<tr>
<th>$\alpha_n$</th>
<th>$\beta_n$</th>
<th>No. of iter.</th>
<th>CPU time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0 = (\sin(t)/2), x_0 = \sin(t)$</td>
<td>$1/10n^2 + 1$</td>
<td>9</td>
<td>7.53828</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>5.63066</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>4.78461</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7</td>
<td>6.19290</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>4.80899</td>
</tr>
<tr>
<td>$x_1 = (2 \sin(t) - t/2), x_0 = 2 \sin(t) - t$</td>
<td>$1/100n^2 + 1$</td>
<td>10</td>
<td>8.51165</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>5.84207</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>5.10883</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7</td>
<td>6.47365</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>5.94383</td>
</tr>
<tr>
<td>$x_2 = t + \log^2(t + 1), x_0 = \log(t + 1)$</td>
<td>$1/n^2 + 1$</td>
<td>11</td>
<td>8.87843</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>5.52223</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>4.95105</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7</td>
<td>6.23286</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>5.59795</td>
</tr>
</tbody>
</table>

Table 4: Numerical results of the difference $\beta_n$.

<table>
<thead>
<tr>
<th>$\beta_n$</th>
<th>$\gamma_n$</th>
<th>No. of iter.</th>
<th>CPU time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0 = (\sin(t)/2), x_0 = \sin(t)$</td>
<td>$1/10n + 1$</td>
<td>5</td>
<td>4.80889</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>4.75128</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>4.79156</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>4.76109</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>4.76763</td>
</tr>
<tr>
<td>$x_1 = (2 \sin(t) - t/2), x_0 = 2 \sin(t) - t$</td>
<td>$1/100n + 1$</td>
<td>5</td>
<td>4.97311</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>5.09668</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>5.03153</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>4.99385</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>4.98385</td>
</tr>
<tr>
<td>$x_2 = t + \log^2(t + 1), x_0 = \log(t + 1)$</td>
<td>$1/n^2 + 1$</td>
<td>5</td>
<td>5.16031</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>4.97200</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>4.83550</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>5.49581</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>5.47782</td>
</tr>
</tbody>
</table>

$u \in \text{res}_{G,b}^\beta(w) \iff G(u, y, y) + \langle \nabla g(u) - \nabla g(w), y - u \rangle + b(u, y) - b(u, u) \geq 0, \quad \forall y \in C$

\[
\iff \frac{1}{2} \left( \|y\|^2 - \|w\|^2 \right) + \langle y - u, y - u \rangle - \langle u, y \rangle + \langle u, u \rangle \geq 0, \quad \forall y \in C
\]

\[
\iff \frac{1}{2} \left( \|y\|^2 - \|u\|^2 \right) - \langle u, y - u \rangle \geq 0, \quad \forall y \in C
\]

\[
\iff \frac{1}{2} \left( \|y\|^2 - \|u\|^2 \right) - \langle u, y - w \rangle + \langle w, u - w \rangle \geq 0, \quad \forall y \in C
\]

\[
\iff D_g(u, w) \leq D_g(y, w), \quad \forall y \in C
\]

\[
\iff u = \text{Proj}_{C}^\beta(w).
\]
For the experiments in this section, we use the Cauchy error \( \|x_n - x_{n-1}\|^2 < 10^{-5} \) for the stopping criterion. We will start with the initialization \( x_{-1} \) and \( x_0 \) in two cases. We split considering all of the performances of our algorithm in four cases by considering all of the parameters that have an effect on the convergence of the algorithm.

**Case 1.** We start computation by comparison of the algorithm with different parameters \( \epsilon_n \), where

\[
\theta_n = \begin{cases} 
\min \left\{ \frac{\epsilon_n}{\|x_n - x_{n-1}\|}, \theta \right\}, & \text{if } n \leq N, \\
\epsilon_n, & \text{otherwise},
\end{cases} 
\]

(143)

where \( N \) is the number of iterations that we want to stop, \( \lim_{n \to \infty} \epsilon_n = 0 \), and \( \theta \in (0, 1) \). We choose \( \theta = 0.3 \), \( \alpha_n = (1/2n + 1) \), and \( \beta_n = \alpha_n \). Then, the results are presented in Table 1.
Case 2. We compare the performance of the algorithm with different parameters $\theta$ by setting $\epsilon_n \approx \frac{1}{n+1}$, $\alpha_n \approx \frac{1}{2n+1}$, and $\beta_n = \alpha_n$. Then, the results are presented in Table 2.

Case 3. We compare the performance of the algorithm with different parameters $\alpha_n$ by setting $\epsilon_n \approx \frac{1}{n+1}$, $\beta_n = \alpha_n$, and $\theta = 0.3$ for the initialization $x_{-1} = \sin(t)/2, x_0 = \sin(t)$ and $x_{-1} = t + \log^2(t + 1), x_0 = \log(t + 1)$ and $\theta = 0.1$ for the initialization $x_{-1} = (2 \sin(t) - t)/2, x_0 = 2 \sin(t) - t$. Then, the results are presented in Table 3.

Case 4. We compare the performance of the algorithm with different parameters $\beta_n$ by setting $\epsilon_n \approx \frac{1}{n+1}$, $\alpha_n \approx \frac{1}{100n + 1}$, and $\theta = 0.3$ for the initialization $x_{-1} = (1/n + 1), x_0 = \sin(t)/2$, and $x_{-1} = t + \log^2(t + 1), x_0 = \log(t + 1)$ and $\theta = 0.1$ for the initialization $x_{-1} = (2 \sin(t) - t)/2, x_0 = 2 \sin(t) - t$. Then, the results are presented in Table 4.
From Tables 1–4 and Figures 1–12, we noticed that in all the above 4 cases, choosing $\theta = 0.3$, $\varepsilon_n = (1/n + 1)$, $\alpha_n = (1/100n + 1)$, and $\beta_n = (1/2n^2 + 1)$ yields the best results for the initialization $x_{-1} = (\sin(t)/2), x_0 = \sin(t)$. Choosing $\theta = 0.1$, $\varepsilon_n = (1/n + 1)$, $\alpha_n = (1/100n + 1)$, and $\beta_n = (1/2n + 1)$ yields the best results for the initialization $x_{-1} = (2 \sin(t) - t/2), x_0 = 2 \sin(t) - t$, and choosing $\theta = 0.3$, $\varepsilon_n = (1/n + 1)$, $\alpha_n = (1/100n + 1)$, and $\beta_n = (1/2n + 1)$ yields the best results for the initialization $x_{-1} = (\sin(t)/2), x_0 = \sin(t)$. 

![Figure 5: The Cauchy error plotting number of iterations for different parameters $\theta$.](image1)

![Figure 6: The Cauchy error plotting number of iterations for different parameters $\theta$.](image2)
Figure 7: The Cauchy error plotting number of iterations for different parameters $\alpha_n$.

Figure 8: The Cauchy error plotting number of iterations for different parameters $\alpha_n$. 
\[ x_1 = t + \log^2(t+1), \quad x_0 = \log(t+1) \]

Figure 9: The Cauchy error plotting number of iterations for different parameters \( a_n \).

\[ x_1 = \frac{\sin(t)}{2}, \quad x_0 = \sin(t) \]

Figure 10: The Cauchy error plotting number of iterations for different parameters \( \beta_n \).
\( \beta_n = \frac{1}{100n+1} \) yields the best results for the initialization \( x_{-1} = t + \log^2 (t + 1) \), \( x_0 = \log (t + 1) \).

### 7. Conclusion

In this paper, we established an inertial hybrid iterative algorithm involving Bregman relatively nonexpansive mapping to find a common solution of GMVLIP equation (1) and FPP in Banach space. Moreover, we study the convergence analysis for the main result. At last, we list some consequences and computational example to emphasize the efficiency and relevancy of the main result. From the theoretical and application point of view, the inertial method via Bregman relatively nonexpansive mapping has a great importance on data analysis and some imaging problems. The inertial method has been studied by various researchers due to its importance (see for details [19, 24–28, 30, 31, 33–36, 39]).
References


