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Retraction

Retracted: Statistical Inferences of Burr XII Lifetime Models under Joint Type-1 Competing Risks Samples

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Manipulated or compromised peer review

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] T. A. Abushal, A. A. Soliman, and G. A. Abd-Elmougod, "Statistical Inferences of Burr XII Lifetime Models under Joint Type-1 Competing Risks Samples," *Journal of Mathematics*, vol. 2021, Article ID 9553617, 16 pages, 2021.

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Research Article

Statistical Inferences of Burr XII Lifetime Models under Joint Type-1 Competing Risks Samples

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The problem of statistical inference under joint censoring samples has received considerable attention in the past few years. In this paper, we adopted this problem when units under the test fail with different causes of failure which is known by the competing risks model. The model is formulated under consideration that only two independent causes of failure and the unit are collected from two lines of production and its life distributed with Burr XII lifetime distribution. So, under Type-I joint competing risks samples, we obtained the maximum likelihood (ML) and Bayes estimators. Interval estimation is discussed through asymptotic confidence interval, bootstrap confidence intervals, and Bayes credible interval. The numerical computations which described the quality of theoretical results are discussed in the forms of real data analyzed and Monte Carlo simulation study. Finally, numerical results are discussed and listed through some points as a brief comment.

1. Introduction

The failure times which are obtained from life testing experiments are exposed in complete or censored data. Therefore, the word complete data is used when the failure time of all units under the test is observed but, under some restrictions of time and cost, the failure time of some not all units is observed. Then, we used the word censoring data when the available lifetime data are taken from some units under the test. Censoring scheme can be done under different forms, and the commonly ones are known by Type-I and Type-II censoring schemes (CSs). In Type-I CS, the test has a prefixed time and random number of failure units. However, in Type-II CS, the test time is random and has prefixed number of failure units. Each of Type-I CS and Type-II CS does not allow to remove unit from the test other than the final point. The availability of removed units from the test at any stage is known by progressive censoring scheme (see Balakrishnan and Aggarwala [1]). Under consideration that units of product are taken from different lines of production under the same facility, the joint censoring scheme appeared. Censoring schemes under joint sample are called joint censoring scheme (JCS). Therefore, we combine the joint censoring scheme with Type-I and Type-II censoring schemes to obtain the Type-I and Type-II joint censoring schemes (Type-I and Type-II JCSs).

The product produced from the different lines of production under the same facilities needs some tests to measure the relative merits in a competing duration. In practice, JCSs are applied on random selection taken from lines of production. Different authors exposed to this problem, for early discussion, such as Rao et al. [2] developed the rank order theory under two-sample censoring scheme, Basu [3] presented and discussed the statistics of rank sets from two-sample scheme called Savage statistic, Johnson and Mehrotra [4] used two-sample problem to preset the locally most powerful rank tests under censored data, Bhattacharyya and Johnson [5] applied two-sample censored situation for asymptotic sufficiency and asymptotically most powerful tests, Mehrotra and Bhattacharyya

[6] measured the equality of two exponential distributions testing under Type-II censoring, and Mehrotra and Bhattacharyya [7] discussed under jointly Type-II censored samples the confidence intervals from two exponential distributions. Also, Balakrishnan and Rasouli [8] presented exact likelihood inferences under jointly censoring schemes, Rasouli and Balakrishnan [9] discussed the exact likelihood inference under joint progressive Type-II censoring for two exponential populations, and Shafay et al. [10] discussed the Bayes inference under joint Type-II censored sample for two exponential populations. And, this problem is handled recently by Al-Matrafi and Abd-Elmougod [11], Momenkhan and Abd-Elmougod [12], Mondal and Kundu [13], and Mondal andKundu [14]. The problem of statistical inference under jointly censoring schemes with the competing risks model is recently discussed by Almarashi et al. [15].

Under Type-I JCS, a sample of size N is randomly selected from two lines of production η_1 and η_2 to satisfy that S_1 is selected from the first line η_1 and S_2 is selected from the second line η_2, and the ideal test time τ is given. The sample of size S_1 taken from the line η_1 has $T_1, T_2, \ldots, T_{S_1}$ lifetimes distributed with PDF and CDF given, respectively, by $f_1(\cdot)$ and $F_1(\cdot)$. Also, S_2 from the line η_2 has $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{S_2}$ lifetimes with PDF and CDF given, respectively, by $f_2(\cdot)$ and $F_2(\cdot)$. Under given the test time τ, the ordered life times $\{X_1, X_2, \ldots, X_J\}$, $1 \le J \le N$, obtained from the joint sample $\{T_1, T_2, \ldots, T_{J_1}, \mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{J_2}\}$, $J = J_1 + J_2$, present the Type-I JCS. Therefore, under Type-I JCS, the failure time and the corresponding type of failure (mean from the line η_1 or η_2) are recorded. Hence, the Type-I JCS is given by

$$\mathbf{X} = \{(X_1, v_1), (X_2, v_2), \ldots, (X_J, v_J)\}, \tag{1}$$

where $v_i = 1$ or 0 dependent on the failure from the line η_1 or the line η_2, respectively. Suppose that the integer numbers denoted the number of failure from the line η_1 given by $n_1 = \sum_{i=1}^{J} v_i$ and number of failure from the line η_2 given by $n_2 = \sum_{i=1}^{J} (1 - v_i)$. Hence, the joint likelihood function under $\mathbf{X}$, Type-I JCS, is formulated by

$$L_{1,2,\ldots,J}(\mathbf{X}|\boldsymbol{\omega}) = \frac{S_1!S_2!}{(S_1 - n_1)!(S_2 - n_2)!}\left(\prod_{i=1}^{J} [f_1(x_i)]^{v_i} [f_2(x_i)]^{1-v_i}\right) R_1^{S_1-n_1}(x_J) R_2^{S_2-n_2}(x_J), \tag{2}$$

where $R_i(\cdot)$, $i = 1, 2$, mean the reliability functions and ω presents the parameters vector.

In a real-life testing, commonly the failure times of units/individuals may be reported under different causes of failure which is known by the competing risks model. In this problem, our aim is measuring the risk of one cause of failure with respect to other causes. Early, this problem was discussed under exponential populations by Cox [16] and some properties of the competing risks model by Crowder [17], Balakrishnan and Han [18], Modhesh and Abd-Elmougod [19], and Bakoban and Abd-Elmougod [20]. Recently, the properties of the competing risks model under the accelerated life test model were discussed by Ganguly and Kundu [21], Hanaa and Neveen [22], and Algarn et al. [23]. The competing risk problem under Type-I censoring scheme can be described as follows.

Suppose that N unit is put under life testing experiment and the ideal test τ is given under consideration that only two independent causes of failure exist. The failure time and the corresponding cause of failure are recorded, say $\mathbf{X} = \{(X_1, \delta_1), (X_2, \delta_2), \ldots, (X_J, \delta_J)\}$ and $1 \le J \le N$. The joint likelihood function under competing risks Type-I,$\mathbf{X}$, is formulated by

$$L(\mathbf{X}|\boldsymbol{\omega}) = \frac{n!}{(n-J)!}\left(\prod_{i=1}^{J} [h_1(x_i)]^{\mu(\delta_i=1)} [h_2(x_i)]^{\mu(\delta_i=2)} R_1(x_i) R_2(x_i)\right)\left(R_1(x_J) R_2(x_J)\right)^{(N-J)}, \tag{3}$$

where

$$\mu(\delta_i = l) = \begin{cases} 1, & \delta_i = l, \\ 0, & \delta_i \ne l, \end{cases} \quad l = 1, 2, \tag{4}$$

$$0 < x_1 < x_2 < \cdots < x_J < \infty. \tag{5}$$

Early, the Burr system is introduced as a system that includes twelve types of cumulative distribution functions (see Burr [24]). Also, the Burr system present a variety of density shapes that are applied in different branches of sciences such as chemical engineering, medical and reliability studies, business, and quality control. The Burr XII distribution which is member of this system has different application in life testing models. The random variable X is called Burr XII random variable if it has cumulative distribution function (CDF) given by

$$F(x) = 1 - \left(1 + x^{\beta}\right)^{-\alpha}, \quad x > 0. \tag{6}$$

Burr XII distribution has unimodal or decreasing failure rate function. Also, the shape of failure rate function is not affected by shape parameters α and has unimodal curve when $\beta > 1$. Also, it has decreasing failure rate function when $\beta \le 1$. Therefore, the shape parameter β is more effective in distribution. Different authors discussed Burr XII such as Rodriguez [25], Lee et al. [26], and recently Hassan and Nada [27].

The product coming from different lines of production is tested under the type of testing known by comparative life tests. When population units or individuals fail under different causes of failure, we have joint competing risks' data as an important source of data. Our aims in this paper are building the statistical inferences of Burr XII life populations based on this competing risk Type-I JCS. Then, we give a complete description for the model formulation considering only two independent causes of failure and the unit life distributed with Burr XII lifetime distribution. The collected data observed under this model are used to estimate the model parameters with maximum likelihood estimation for point and corresponding confidence interval. Also, two confidence intervals with bootstrap-p and bootstrap-t are formulated. The Bayes approach is used to construct the point and credible interval estimations. Different tools are used to measure the quality performance of these estimators. The point estimations were measured under mean squared errors (MSEs). And, the interval estimations were measured under interval length (IL) and probability coverage (PC) through the Monte Carlo simulation study. Also, we analyze the real data set to illustrate our purpose.

The paper is planned as follows. Section 2 discusses general assumptions and modeling. Estimation with MLE, point, and asymptotic confidence intervals is presented in Section 3. Bootstrap confidence intervals are discussed in Section 4. Bayes estimation is presented in Section 5. The real example is used and analyzed in Section 6. Assessment and comparing the numerical results with simulation study are presented in Section 7. The brief comments are summarized in Section 8.

2. Model Formulation

Let a sample of size $N = S_1 + S_2$ be selected from two lines η_1 and η_2 (S_1 from η_1 and S_2 from η_2) for a life testing experiment, and the ideal test time τ is proposed. When the experiment is running, the failure time X and the corresponding type υ as well as cause of failure δ are reported. The experiment is continual until τ is observed; then, we can say $(X_i, \upsilon_i, \delta_i)$, $i = 1, 2, \ldots, J$, are observed. Therefore, the random set $\mathbf{X} = \{(X_1, \upsilon_1, \delta_1), (X_2, \upsilon_2, \delta_2), \ldots, (X_J, \upsilon_J, \delta_J)\}$, and $1 \le J \le N$ is called Type-I joint competing risks sample (Type-I JCRS). Therefore, under Type-I JCRS, we have the following assumption:

(1) The number $n_1 = \sum_{i=1}^{J} \upsilon_i$ present number of failure from the line η_1.

(2) The number $n_2 = \sum_{i=1}^{J} (1 - \upsilon_i)$ present number of failure from the line η_2.

(3) The number $m_{1j} = \sum_{i=1}^{J} \upsilon_i * \mu(\delta_i = j)$ present number of failure from the line η_1 and cause j.

(4) The number $m_{2j} = \sum_{i=1}^{J} (1 - \upsilon_i) * \mu(\delta_i = j)$ present number of failure from the line η_2 and cause j. Hence, the joint likelihood function of Type-I JCRS $\mathbf{X} = \{(X_1, \upsilon_1, \delta_1), (X_2, \upsilon_2, \delta_2), \ldots, (X_J, \upsilon_J, \delta_J)\}$ is formulated by

$$
\begin{aligned}
L_{1,2,\ldots,J}(\mathbf{X}|\boldsymbol{\omega}) \propto \prod_{i=1}^{J} \Big\{ \Big[[h_{11}(x_i)]^{\mu(\delta_i=1)} [h_{12}(t_i)]^{\mu(\delta_i=2)} R_{11}(x_i) R_{12}(x_i) \Big]^{\upsilon_i} \\
\times [h_{21}(x_i)]^{\mu(\delta_i=1)} [h_{22}(t_i)]^{\mu(\delta_i=2)} R_{21}(x_i) R_{22}(x_i) \Big\}^{1-\upsilon_i} \\
\times \big[R_{11}(x_J) R_{12}(x_J)\big]^{S_1 - n_1} \big[R_{21}(x_J) R_{22}(x_J)\big]^{S_2 - n_2},
\end{aligned}
\tag{7}
$$

where $\mu(\delta_i = l)$ is given by (4).

(5) If k defines the unit type, then the observed failure time $x_i = \min\{x_{ik1}, x_{ik2}\}$, $i = 1, 2, \ldots, J$.

(6) The CDF of random variable X_{ikj} of Burr XII lifetime distribution is given by

$$
F_{kj}(x) = 1 - \left(1 + x^{\beta_k}\right)^{-\alpha_{kj}}, \quad x > 0, \beta_k, \alpha_{kj} > 0, k, j = 1, 2. \tag{8}
$$

(7) The minimum value has distribution given by $F_{k1}(\cdot) + F_{k2}(\cdot) - F_{k1}(\cdot) * F_{k2}(\cdot)$. Therefore, the latent failure time is distributed with Burr XII distributions with shape parameters β_k and $\alpha_{k1} + \alpha_{k2}$.

(8) The discrete random variables m_{1j} and m_{2j} have the binomial distributions given by

$$
\begin{aligned}
m_{1j} &\longrightarrow \text{binomial}\left(n_j, \frac{\alpha_{k1}}{\alpha_{k1} + \alpha_{k2}}\right), \\
m_{2j} &\longrightarrow \text{binomial}\left(n_j, \frac{\alpha_{k2}}{\alpha_{k1} + \alpha_{k2}}\right).
\end{aligned}
\tag{9}
$$

3. Maximum Likelihood Estimation

The model parameters in this section are discussed under given Type-I JCRS from Burr XII distribution. The joint likelihood function (7) is reduced to

$$\begin{aligned}L(\boldsymbol{\omega}|\mathbf{X}) \propto \prod_{i=1}^{J}&\left[x_i^{\beta_1-1}\left(1+x_i^{\beta_1}\right)^{-(\alpha_{11}+\alpha_{12}+1)}\right]^{v_i}\left[x_i^{\beta_2-1}\left(1+x_i^{\beta_2}\right)^{-(\alpha_{21}+\alpha_{22}+1)}\right]^{1-v_i}\\ &\times\left(1+x_J^{\beta_1}\right)^{-(S_1-n_1)(\alpha_{11}+\alpha_{12})}\left(1+x_J^{\beta_2}\right)^{-(S_2-n_2)(\alpha_{21}+\alpha_{22})}\\ &\times\beta_1^{n_1}\beta_2^{n_2}\alpha_{11}^{m_{11}}\alpha_{12}^{m_{12}}\alpha_{21}^{m_{21}}\alpha_{22}^{m_{22}}x_J^{\beta_1-1}x_J^{\beta_2-1},\end{aligned} \tag{10}$$

where $\omega = \{\beta_1, \beta_2, \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}\}$ and $\mathbf{X}$ be Type-I JCRS. Function (10) after taking the natural logarithm is reduced to

$$\begin{aligned}\ell(\boldsymbol{\omega}|\mathbf{X}) = {}& n_1\log\beta_1 + n_2\log\beta_2 + m_{11}\log\alpha_{11} + m_{12}\log\alpha_{12} + m_{21}\log\alpha_{21} + m_{22}\log\alpha_{22}\\ &+(\beta_1-1)\sum_{i=1}^{J}v_i\log x_i - (\alpha_{11}+\alpha_{12}+1)\sum_{i=1}^{J}v_i\log\left(1+x_i^{\beta_1}\right)\\ &+(\beta_2-1)\sum_{i=1}^{J}(1-v_i)\log x_i - (\alpha_{21}+\alpha_{22}+1)\sum_{i=1}^{J}(1-v_i)\log\left(1+x_i^{\beta_2}\right)\\ &-(S_1-n_1)(\alpha_{11}+\alpha_{12})\log\left(1+x_J^{\beta_1}\right) - (S_2-n_2)(\alpha_{21}+\alpha_{22})\log\left(1+x_J^{\beta_2}\right).\end{aligned} \tag{11}$$

3.1. Point Estimation. From the log-likelihood function, we obtain the likelihood equations by taking the first partially derivatives respective to the model parameters as follows:

$$\begin{aligned}\frac{\partial\ell(\boldsymbol{\omega}|\mathbf{X})}{\partial\alpha_{1j}} &= \frac{m_{1j}}{\alpha_{1j}} - \sum_{i=1}^{J}v_i\log\left(1+x_i^{\beta_1}\right) - (S_1-n_1)\log\left(1+x_J^{\beta_1}\right) = 0,\\ \frac{\partial\ell(\boldsymbol{\omega}|\mathbf{X})}{\partial\alpha_{2j}} &= \frac{m_{2j}}{\alpha_{2j}} - \sum_{i=1}^{J}(1-v_i)\log\left(1+x_i^{\beta_2}\right) - (S_2-n_2)\log\left(1+x_J^{\beta_2}\right) = 0,\end{aligned} \tag{12}$$

which reduced to

$$\widehat{\alpha}_{1j}(\beta_1) = \frac{m_{1j}}{\sum_{i=1}^{J}v_i\log\left(1+x_i^{\beta_1}\right) + (S_1-n_1)\log\left(1+x_J^{\beta_1}\right)}, \tag{13}$$

$$\widehat{\alpha}_{2j}(\beta_2) = \frac{m_{2j}}{\sum_{i=1}^{J}(1-v_i)\log\left(1+x_i^{\beta_2}\right) + (S_2-n_2)\log\left(1+x_J^{\beta_2}\right)}. \tag{14}$$

And the derivatives with respect to β_k are reduced to the likelihood equations as follows:

$$\frac{\partial\ell(\boldsymbol{\omega}|\mathbf{X})}{\partial\beta_k} = 0, \quad k = 1, 2, \tag{15}$$

which reduced to

$$\begin{aligned}&\frac{n_1}{\beta_1} + \sum_{i=1}^{J}v_i\log x_i - (\alpha_{11}+\alpha_{12}+1)\sum_{i=1}^{J}\frac{v_i x_i^{\beta_1}\log x_i}{1+x_i^{\beta_1}}\\ &\quad - \frac{(S_1-n_1)(\alpha_{11}+\alpha_{12})x_J^{\beta_1}\log x_J}{1+x_J^{\beta_1}} = 0,\end{aligned} \tag{16}$$

$$\begin{aligned}&\frac{n_2}{\beta_2} + \sum_{i=1}^{J}(1-v_i)\log x_i - (\alpha_{21}+\alpha_{22}+1)\sum_{i=1}^{J}\frac{v_i x_i^{\beta_2}\log x_i}{1+x_i^{\beta_2}}\\ &\quad - \frac{(S_2-n_2)(\alpha_{21}+\alpha_{22})x_J^{\beta_2}\log x_J}{1+x_J^{\beta_2}} = 0.\end{aligned} \tag{17}$$

Equations (13) to (17) have shown that the problem of obtaining the ML estimate of model parameters needs to solve two nonlinear equations (16) and (17) to obtain $\widehat{\beta}_k$, $k = 1, 2$. Different iteration methods can be applied such as

Newton–Raphson or fixed point iteration with initial value can be obtained from the profile log-likelihood (11) after replacing the parameters α_{kj} of equations (13) and (14) as follows:

$$\begin{aligned} f(\beta_1,\beta_2|\mathbf{X}) = {} & n_1\log \beta_1 + n_2\log \beta_2 + m_{11}\log \widehat{\alpha}_{11}(\beta_1) + m_{12}\log \widehat{\alpha}_{12}(\beta_1) \\ & + m_{21}\log \widehat{\alpha}_{21}(\beta_2) + m_{22}\log \widehat{\alpha}_{22}(\beta_2) + (\beta_1 - 1)\sum_{i=1}^{J} v_i\log x_i \\ & - (\widehat{\alpha}_{11}(\beta_1) + \widehat{\alpha}_{12}(\beta_1) + 1)\sum_{i=1}^{J} v_i\log\left(1 + x_i^{\beta_1}\right) + (\beta_2 - 1) \\ & \times \sum_{i=1}^{J}(1 - v_i)\log x_i - (\widehat{\alpha}_{21}(\beta_2) + \widehat{\alpha}_{22}(\beta_2) + 1)\sum_{i=1}^{J}(1 - v_i) \\ & \times \log\left(1 + x_i^{\beta_2}\right) - (S_1 - n_1)(\widehat{\alpha}_{11}(\beta_1) + \widehat{\alpha}_{12}(\beta_1))\log\left(1 + x_J^{\beta_1}\right) \\ & - (S_2 - n_2)(\widehat{\alpha}_{21}(\beta_2) + \widehat{\alpha}_{22}(\beta_2))\log\left(1 + x_J^{\beta_2}\right). \end{aligned} \quad (18)$$

Also, the ML estimate of parameters $\widehat{\alpha}_{kj}$ is obtained from (13) and (14) after replacing β_k by $\widehat{\beta}_k$.

Remark 1. The equations from (13) to (17) showed that the conditional estimators of the model parameters depend on the discrete random variable m_{kj}. Hence, the estimate $\widehat{\alpha}_{1j}$ and $\widehat{\alpha}_{2j}$ does not exist for $m_{1j} = 0$ or J and $m_{2j} = 0$ or J, respectively. And, the problem of exact distributions for estimators $\widehat{\alpha}_{1j}$ and $\widehat{\alpha}_{2j}$ is defined as mixture of discrete and continuous distributions, hence as given in Kundu and Joarder [28] is difficult to obtain.

3.2. Interval Estimation. The asymptotic confidence intervals of model parameters depend on the second partial derivative of the log-likelihood function (11) and hence information matrix (see Salah [29]). And, the Fisher information matrix of the model parameters is defined as the minus expectation of the second partial derivatives which is presented as follows:

$$\begin{aligned} \frac{\partial^2 \ell(\boldsymbol{\omega}|\mathbf{X})}{\partial \beta_1^2} &= \frac{-n_1}{\beta_1^2} - (\alpha_{11} + \alpha_{12} + 1)\sum_{i=1}^{J} v_i \frac{x_i^{\beta_1}(\log x_i)^2}{\left(1 + x_i^{\beta_1}\right)^2} \\ & \quad - \frac{(S_1 - n_1)(\alpha_{11} + \alpha_{12}) x_J^{\beta_1}\left(\log x_J\right)^2}{\left(1 + x_J^{\beta_1}\right)^2}, \\ \frac{\partial^2 \ell(\boldsymbol{\omega}|\mathbf{X})}{\partial \beta_2^2} &= \frac{-n_2}{\beta_2^2} - (\alpha_{21} + \alpha_{22} + 1)\sum_{i=1}^{J}(1 - v_i)\frac{x_i^{\beta_2}(\log x_i)^2}{\left(1 + x_i^{\beta_2}\right)^2} \\ & \quad - \frac{(S_1 - n_1)(\alpha_{21} + \alpha_{22}) x_J^{\beta_2}\left(\log x_J\right)^2}{\left(1 + x_J^{\beta_2}\right)^2}, \\ \frac{\partial^2 \ell(\boldsymbol{\omega}|\mathbf{X})}{\partial \alpha_{kj}^2} &= \frac{-m_{kj}}{\alpha_{kj}^2}\bigg|_{k,j=1,2}, \\ \frac{\partial^2 \ell(\boldsymbol{\omega}|\mathbf{X})}{\partial \alpha_{kj}\partial \alpha_{il}} &= 0, \quad \text{For each } kj \neq il, \\ \frac{\partial^2 \ell(\boldsymbol{\omega}|\mathbf{X})}{\partial \beta_1 \partial \alpha_{1j}} &= \frac{\partial^2 \ell(\boldsymbol{\omega}|\mathbf{X})}{\partial \alpha_{1j}\partial \beta_1} = -\sum_{i=1}^{J}\frac{v_i x_i^{\beta_1}\log x_i}{1 + x_i^{\beta_1}} - \frac{(S_1 - n_1)x_J^{\beta_1}\log x_J}{1 + x_J^{\beta_1}}, \quad j = 1,2, \\ \frac{\partial^2 \ell(\boldsymbol{\omega}|\mathbf{X})}{\partial \beta_2 \partial \alpha_{2j}} &= \frac{\partial^2 \ell(\boldsymbol{\omega}|\mathbf{X})}{\partial \alpha_{2j}\partial \beta_2} = -\sum_{i=1}^{J}\frac{(1 - v_i)x_i^{\beta_2}\log x_i}{1 + x_i^{\beta_2}} - \frac{(S_2 - n_2)x_J^{\beta_2}\log x_J}{1 + x_J^{\beta_2}}, \quad j = 1,2, \\ \frac{\partial^2 \ell(\boldsymbol{\omega}|\mathbf{X})}{\partial \beta_1 \partial \alpha_{2j}} &= \frac{\partial^2 \ell(\boldsymbol{\omega}|\mathbf{X})}{\partial \beta_2 \partial \alpha_{1j}} = \frac{\partial^2 \ell(\boldsymbol{\omega}|\mathbf{X})}{\partial \alpha_{2j}\partial \beta_1} = \frac{\partial^2 \ell(\boldsymbol{\omega}|\mathbf{X})}{\partial \alpha_{1j}\partial \beta_2} = \frac{\partial^2 \ell(\boldsymbol{\omega}|\mathbf{X})}{\partial \beta_1 \partial \beta_2} \\ &= \frac{\partial^2 \ell(\boldsymbol{\omega}|\mathbf{X})}{\partial \beta_2 \partial \beta_1} = 0. \end{aligned} \quad (19)$$

Suppose that the fisher information matrix is defined by $\Psi(\beta_1, \beta_2, \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22})$, where

$$\Psi(\beta_1, \beta_2, \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}) = -E\left(\frac{\partial^2 \ell(\boldsymbol{\omega}|\mathbf{X})}{\partial \omega_i \partial \omega\ l}\right), \quad i, l = 1, 2, \ldots, 6, \tag{20}$$

where $\omega = (\beta_1, \beta_2, \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22})$ be the model parameters. Equation (19) has shown that the expectations of the second derivative of the log likelihood function are more serious. Therefore, we applied the approximate information matrix $\widehat{\Psi}_0(\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\alpha}_{11}, \widehat{\alpha}_{12}, \widehat{\alpha}_{21}, \widehat{\alpha}_{22})$ defined by

$$\widehat{\Psi}_0\left(\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\alpha}_{11}, \widehat{\alpha}_{12}, \widehat{\alpha}_{21}, \widehat{\alpha}_{22}\right) = \left.\left(\frac{\partial^2 \ell(\boldsymbol{\omega}|\mathbf{X})}{\partial \omega_i \partial \omega\ l}\right)\right|_{\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\alpha}_{11}, \widehat{\alpha}_{12}, \widehat{\alpha}_{21}, \widehat{\alpha}_{22}}, \quad i, l = 1, 2, \ldots, 6. \tag{21}$$

Therefore, $\widehat{\Psi}_0^{-1}(\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\alpha}_{11}, \widehat{\alpha}_{12}, \widehat{\alpha}_{21}, \widehat{\alpha}_{22})$ exists with nonzero values of the elements of diagonal. Under normal properties of $(\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\alpha}_{11}, \widehat{\alpha}_{12}, \widehat{\alpha}_{21}, \widehat{\alpha}_{22})$, the approximate $(1 - 2\theta)\%$ confidence intervals of the parameters $\beta_1, \beta_2, \alpha_{11}, \alpha_{12}, \alpha_{21}$, and α_{22} are given by

$$\begin{cases} \widehat{\beta}_1 \mp z_\theta \epsilon_{11}, & \widehat{\beta}_2 \mp z_\theta \epsilon_{22}, \\ \widehat{\alpha}_{11} \mp z_\theta \epsilon_{33}, & \widehat{\alpha}_{12} \mp z_\theta \epsilon_{44}, \\ \widehat{\alpha}_{21} \mp z_\theta \epsilon_{55}, & \widehat{\alpha}_{22} \mp z_\theta \epsilon_{66}, \end{cases} \tag{22}$$

where ϵ_{il} is the element of diagonal of the invariance approximate information matrix $\widehat{\Psi}_0^{-1}(\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\alpha}_{11}, \widehat{\alpha}_{12}, \widehat{\alpha}_{21}, \widehat{\alpha}_{22})$ with significant level θ.

4. Bootstrap Confidence Intervals

In this section, we discussed a bootstrap technique in statistical inference problem about parameters estimation. This technique is a commonly resembling method not only in parameter estimation but also used to estimate bias and variance of an estimator or calibrate hypothesis tests. The bootstrap technique is defined in parametric and nonparametric methods (see Davison and Hinkley [30] and Efron and Tibshirani [31]). Therefore, we adopted parametric bootstrap technique to build two different confidence intervals, percentile bootstrap technique, and bootstrap-t technique. For more details, see Efron [32] and Hall [33]. The following algorithms are used to describe the procedure that is used to build different two bootstrap confidence intervals:

(1) Under consideration that the original observed Type-I JCRS $\mathbf{X} = \{(X_1, \upsilon_1, \delta_1), (X_2, \upsilon_2, \delta_2), \ldots, (X_J, \upsilon_J, \delta_J)\}$, the estimates are obtained and given by $\widehat{\omega} = (\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\alpha}_{11}, \widehat{\alpha}_{12}, \widehat{\alpha}_{21}, \widehat{\alpha}_{22})$.

(2) For given $\widehat{\omega}$ and integer values of $N, S_1\ S_2$, and time τ, generate a sample of size S_1 from Burr XII distribution with shape parameters $\widehat{\beta}_1$ and $\widehat{\alpha}_{11} + \widehat{\alpha}_{12}$ and a sample of size S_2 from Burr XII distribution with shape parameters $\widehat{\beta}_2$ and $\widehat{\alpha}_{21} + \widehat{\alpha}_{22}$. The τ-bootstrap Type-I JCRS is obtained from the generated joint sample as a small J satisfies that $X_J < \tau$ denoted by $\mathbf{X} = \{(X_1^*, \upsilon_1, \delta_1), (X_2^*, \upsilon_2, \delta_2), \ldots, (X_J^*, \upsilon_J, \delta_J)\}$.

(3) From Step 2, the two numbers n_1^* and n_2^* (number of failure taken from line η_1 and η_2, respectively) are obtained.

(4) The four numbers m_{1j}^* and m_{2j}^*, $j = 1, 2$, are randomly generated from binomial distribution with size $J - n_{3-k}^*$ and probability $(\widehat{\alpha}_{kj}/(\widehat{\alpha}_{k1} + \widehat{\alpha}_{k2}))$, $k, j = 1, 2$.

(5) The bootstrap estimate sample $\widehat{\omega}^* = (\widehat{\beta}_1^*, \widehat{\beta}_2^*, \widehat{\alpha}_{11}^*, \widehat{\alpha}_{12}^*, \widehat{\alpha}_{21}^*, \widehat{\alpha}_{22}^*)$ is obtained.

(6) Repeat Steps 2 to 5 $\mathbf{M}$ times.

(7) The values $(\widehat{\beta}_1^{[i]*}, \widehat{\beta}_2^{[i]*}, \widehat{\alpha}_{11}^{[i]*}, \widehat{\alpha}_{12}^{[i]*}, \widehat{\alpha}_{21}^{[i]*}, \widehat{\alpha}_{22}^{[i]*})$, $i = 1, 2, \ldots, \mathbf{M}$, are arranged in ascending order to obtain $\widetilde{\omega}^* = (\widehat{\beta}_1^{(i)*}, \widehat{\beta}_2^{(i)*}, \widehat{\alpha}_{11}^{(i)*}, \widehat{\alpha}_{12}^{(i)*}, \widehat{\alpha}_{21}^{(i)*}, \widehat{\alpha}_{22}^{(i)*})$.

4.1. Percentile Bootstrap Confidence Interval (PBCI). Suppose that the ordered sample described by distribution $\Phi(x) = P(\widetilde{\omega}_l^* \le x)$, $l = 1, 2, 3, 4, 5, 6$, be cumulative distribution function of $\widetilde{\omega}_l^*$, where $\widetilde{\omega}_1^*$ mean $\widehat{\beta}_1^*$ and others. So, the point bootstrap estimate is defined by

$$\widehat{\boldsymbol{\omega}}_l^* = \frac{1}{\mathbf{M}} \sum_{i=1}^{\mathbf{M}} \widetilde{\boldsymbol{\omega}}_l^{(i)*}. \tag{23}$$

Also, the $100(1 - 2\theta)\%$ PBCIs are given by

$$\left(\widetilde{\boldsymbol{\omega}}_{l\text{boot}(\theta)}^*, \widetilde{\boldsymbol{\omega}}_{l\text{boot}(1-\theta)}^*\right), \tag{24}$$

where $\widetilde{\omega}_{l\text{boot}}^* = \Phi^{-1}(x)$.

4.2. Bootstrap-t Confidence Interval (PTCI). From the order sample $\widetilde{\omega}^* = (\widehat{\beta}_1^{(i)*}, \widehat{\beta}_2^{(i)*}, \widehat{\alpha}_{11}^{(i)*}, \widehat{\alpha}_{12}^{(i)*}, \widehat{\alpha}_{21}^{(i)*}, \widehat{\alpha}_{22}^{(i)*})$, we built the order statistics values $\Phi_l^{*\,(1)} < \Phi_l^{*\,(2)} < \cdots < \Phi_l^{*\,(\mathbf{M})}$, where

$$\Phi_l^{*\,[i]} = \frac{\widetilde{\boldsymbol{\omega}}_1^{(i)*} - \widehat{\boldsymbol{\omega}}_1}{\sqrt{\operatorname{var}\left(\widetilde{\boldsymbol{\omega}}_1^{(i)*}\right)}}, \quad i = 1, 2, \ldots, \mathbf{M}, l = 1, 2, 3, 4, 5, 6. \tag{25}$$

The $100(1-2\theta)\%$ PTCIs are given by

$$\left(\widetilde{\boldsymbol{\omega}}^*_{l\text{boot}-t(\theta)}, \widetilde{\boldsymbol{\omega}}^*_{l\text{boot}-t(1-\theta)}\right), \tag{26}$$

where the value $\widetilde{\omega}^*_{l\text{boot}-t}$ is given by

$$\widetilde{\boldsymbol{\omega}}^*_{l\text{boot}-t} = \widehat{\boldsymbol{\omega}}_l^* + \sqrt{\operatorname{Var}(\widehat{\boldsymbol{\omega}}_l)}\Phi^{-1}(x), \tag{27}$$

and $\Phi(x) = P(\widetilde{\omega}_l^* \le x)$ be the cumulative distribution function of $\widetilde{\omega}_l^*$.

5. Bayesian MCMC Estimation

In this section, we adopted Bayesian approach to estimate the model parameters under Type-I JCRS (see Ullah and Aslam [34]). So, we suppose that the prior information available about the parameters are independent Gamma prior distributions. Therefore, for parameters vectors $\omega = (\beta_1, \beta_2, \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22})$, the prior information is defined by

$$P_i^*(\omega_i) = \frac{b_i^{a_i}}{\Gamma(a_i)}\omega_i^{a_i-1}\exp(-b_i\omega_i), \quad \omega_i > 0, (a_i, b_i > 0), i = 1, 2, 3, 4, 5, 6. \tag{28}$$

And the corresponding density is defined by

$$P^*(\boldsymbol{\omega}) \propto \prod_{i=1}^{6} \omega_i^{a_i-1}\exp\left(-b_{i\boldsymbol{\omega}_i}\right). \tag{29}$$

Therefore, the posterior distribution can be formulated by using (10) and (29) as follows:

$$\begin{aligned} P(\omega|\mathbf{X}) \propto\ & \beta_1^{n_1+a_1-1}\beta_2^{n_2+a_1-1}\alpha_{11}^{m_{11}+a_1-1}\alpha_{12}^{m_{12}+a_1-1}\alpha_{21}^{m_{21}+a_1-1}\alpha_{22}^{m_{22}+a_1-1}x_J^{\beta_1-1}x_J^{\beta_2-1}\exp\{-b_1\beta_1 \\ & - b_2\beta_2 - b_3\alpha_{11} - b_4\alpha_{12} - b_5\alpha_{21} - b_6\alpha_{22} + (\beta_1 - 1)\sum_{i=1}^{J} v_i\log x_i \\ & - (\alpha_{11} + \alpha_{12} + 1)\sum_{i=1}^{J} v_i\log\left(1 + x_i^{\beta_1}\right) + (\beta_2 - 1)\sum_{i=1}^{J}(1v_i)\log x_i \\ & - (\alpha_{21} + \alpha_{22} + 1)\sum_{i=1}^{J}(1 - v_i)\log\left(1 + x_i^{\beta_2}\right) - (S_1 - n_1)(\alpha_{11} + \alpha_{12}) \\ & \times\log\left(1 + x_J^{\beta_1}\right) - (S_2 - n_2)(\alpha_{21} + \alpha_{22})\log\left(1 + x_J^{\beta_2}\right)\}. \end{aligned} \tag{30}$$

The full conditional distributions are obtained from the joint posterior distribution (29), as follows:

$$\begin{aligned} P_1(\beta_1|\omega_{-1}, \mathbf{X}) \propto \beta_1^{n_1+a_1-1}\exp\Bigg\{ & -b_1\beta_1 + (\beta_1 - 1)\sum_{i=1}^{J} v_i\log x_i - (\alpha_{11} + \alpha_{12} + 1) \\ & \times\sum_{i=1}^{J} v_i\log\left(1 + x_i^{\beta_1}\right) - (S_1 - n_1)(\alpha_{11} + \alpha_{12})\log\left(1 + x_J^{\beta_1}\right)\Bigg\}, \end{aligned} \tag{31}$$

$$\begin{aligned} P_2(\beta_2|\omega_{-1}, \mathbf{X}) \propto \beta_2^{n_1+a_2-1}\exp\Bigg\{ & -b_2\beta_2 + \beta_2\sum_{i=1}^{J}(1 - v_i)\log x_i - (\alpha_{21} + \alpha_{22} + 1) \\ & \times\sum_{i=1}^{J}(1 - v_i)\log\left(1 + x_i^{\beta_2}\right) - (S_2 - n_2)(\alpha_{21} + \alpha_{22})\log\left(1 + x_J^{\beta_2}\right)\Bigg\}, \end{aligned} \tag{32}$$

and the full conditional distributions of parameters α_{kj} are gamma distributions given as

$$P_3\left(\alpha_{1j}|\omega_{-1},\mathbf{X}\right)\propto \text{Gamma}\left[m_{1j}+a_{j+2},b_{j+2}+\sum_{i=1}^{J}v_i\log\left(1+x_i^{\beta_1}\right)+\left(S_1-n_1\right)\log\left(1+x_J^{\beta_1}\right)\right],\quad j=1,2, \tag{33}$$

$$P_4\left(\alpha_{2j}|\omega_{-1},\mathbf{X}\right)\propto \text{Gamma}\left[m_{2j}+a_{j+4},b_{j+4}+\sum_{i=1}^{J}\left(1-v_i\right)\log\left(1+x_i^{\beta_2}\right)+\left(S_2-n_2\right)\log\left(1+x_J^{\beta_2}\right)\right],\quad j=1,2, \tag{34}$$

where the conditional value $\omega_i|\omega_{-1}$ means that the conditional i-th parameter for given the parameter vector ω without the i-th parameter ω_i. The point and interval estimate of model parameters under MCMC methods depend on the forms of full conditional distributions and the subclass of MCMC that can be applied. Therefore, full conditional distribution given by (31) to (34) has shown that we can use the algorithms of Gibbs and generally Metropolis Hasting (MH) under Gibbs (for more details, see [35]) described in Algorithm 1.

The problem of generation under the MCMC method needs to determine the number of iteration needed to reach stationary distribution (burn-in) which is defined by $\mathbf{M}^*$. Therefore, the point estimate is reduced to

$$\widetilde{\omega}_{lB}=E_P\left(\omega_l|X\right)=\frac{1}{\mathbf{M}-\mathbf{M}^*}\sum_{i=\mathbf{M}^*+1}^{\mathbf{M}}\omega_l^{(i)},\quad l=1,2,3,4,5,6, \tag{35}$$

and the corresponding variance is reduced to

$$\widehat{V}\left(\omega_l|X\right)=\frac{1}{\mathbf{M}-\mathbf{M}^*}\sum_{i=\mathbf{M}^*+1}^{\mathbf{M}}\left(\omega_l^{(i)}-\widetilde{\omega}_{lB}\right)^2. \tag{36}$$

Also, $100\,(1-2\theta)\%$ credible intervals are obtained from ordered vectors given by

$$\left(\omega_{l\theta\,(\mathbf{M}-\mathbf{M}^*)},\omega_{l\,(1-\theta)\,(\mathbf{M}-\mathbf{M}^*)}\right). \tag{37}$$

6. Real Data Analysis

In this section, we analyzed a real data set presented by Hoel [36] to present the failure times and the corresponding cause of failure for two groups of strain male mice under laboratory experiment received a radiation dose of 300r at an age of 5-6 weeks. The life data are presented in Table 1, and let η_1 be considered as the first group which lived in a conventional laboratory environment, but η_2 be the second group lived in a germ-free environment. The data are classified into two causes of failure: thymic lymphoma with reticulum cell sarcoma as the first cause of death (failure) and the second cause is presented by other causes of death (failure); more details are presented by Koley and Kundu [37]. For simplicity, the data are divided by 1000.

Therefore, the observed Type-I JCRS is taken from two lines of production η_1 and η_2 under censoring scheme $N=181, S_1=99, S_2=82$, and $\tau=0.50$ and is reported in Table 2. The data given in Table 2 show that $(n_1,n_2)=(50,30)$, $(m_{11},m_{12},m_{21},m_{22})=(26,24,25,5)$, and $J=80$. Figure 1 shows the joint profile log-likelihood function (18), and the value (2, 2) is a suitable initial value needed in the iteration method. The point estimate under ML, bootstrap, and Bayes estimators for noninformative prior information (mean $a_i=b_i=0.0001$, $i=1,2,3,4,5,6$) is reported in Table 3. And, the corresponding 95% approximate ML, two bootstrap confidence (Bootstrap-p and Bootstrap-t), and credibly intervals are, respectively, reported in Table 4. The generation results of full conditional distribution as a generation from posterior distribution and its convergence for Bayesian approach under MCMC methods are described in Figures 2 to 7 which have shown the quality of posterior generation.

7. Simulation Studies

The proposed model and its theoretical results in section are assessed and compared through the Monte Carlo study. So, we built this study to measure the effect of changing each of random sample size $N=S_1+S_2$, the test time τ, and parameters values. The values of sample size and the corresponding test time used in simulation study are reported in Tables 5 to 8. However, for the parameter values choosing, we used two sets, $\omega=\{2.0,1.2,1.3,1.8,2.0,2.0\}$ and {1.0, 2.0, 3.0, 2.0, 2.5, 1.0}. In our studying, we generate 1000 simulated data sets. The prior parameters are selected to satisfy the property that $E\ (\omega_i)\simeq(a_i/b_i)$ and information presented with two cases noninformative defined by P_0^* and informative prior P_1^*. The informative prior P_1^* is taken to be (a, b) = {(3, 0.8), (2, 1.5), (2, 2), (2, 1), (3, 1.5), (4, 2)} for the first selected parameter values. And the informative prior information for the second selection of the parameters values is (a, b) = {(2, 2), (2, 2), (3, 1.2), (4, 2), (4, 1.5), (1, 1)}. Also, through this problem, mean estimate (ME) and the corresponding mean squared error (MSE) are used to measure the

(1) Put $\xi = 1$ and $\omega^{(0)} = (\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\alpha}_{11}, \widehat{\alpha}_{12}, \widehat{\alpha}_{21}, \widehat{\alpha}_{22})$ as initial values
(2) The parameters α_{kj} are generated from Gamma distributions (32) and (33)
(3) With normal proposal distribution with the accepted rejection method with mean $\beta_k^{(\xi-1)}$ and variances ϵ_i, generate $\beta_i^{(\xi)}$, $i = 1, 2$
(4) Put $\xi = \xi + 1$
(5) Steps 2 to 4 are repeated M times and report the vector $\omega^{(\xi)} = (\beta_1^{(\xi)}, \beta_2^{(\xi)}, \alpha_{11}^{(\xi)}, \alpha_{12}^{(\xi)}, \alpha_{21}^{(\xi)}, \alpha_{22}^{(\xi)})$

ALGORITHM 1: MH under Gibbs algorithms.

TABLE 1: Time-to-failure of male mice which received a radiation dose at age 5-6 weeks.

η_1	Thymic lymphoma	159	189	191	198	200	207	220	235	245	250	256
		261	265	266	280	343	356	383	403	414	428	432
		317	318	399	495	525	536	549	552	554	557	558
	Reticulum cell sarcoma	571	586	594	596	605	612	621	628	631	636	643
		647	648	649	661	663	666	670	695	697	700	705
		712	713	738	748	753						
	Other cases	40	42	51	62	163	179	206	222	228	249	252
		282	324	333	341	366	385	407	420	431	441	461
		462	482	517	517	524	564	567	586	619	620	621
		622	647	651	686	761	763					
η_2	Thymic lymphoma	158	192	193	194	195	202	212	215	229	230	237
		240	244	247	259	300	301	321	337	415	434	444
		485	496	529	537	624	707	800				
	Reticulum cell sarcoma	430	590	606	638	655	679	691	693	696	747	752
		760	778	821	986							
	Other cases	136	246	255	376	421	565	616	617	652	655	658
		660	662	675	681	734	736	737	757	769	777	800
		807	825	855	857	864	868	870	873	882	895	910
		934	942	1015	1019							

TABLE 2: Type-I JCRS from heal data with $\tau = 0.5$.

Data	0.040	0.042	0.051	0.062	0.136	0.158	0.159	0.163	0.179	0.189
	0.191	0.192	0.193	0.194	0.195	0.198	0.200	0.202	0.206	0.207
	0.212	0.215	0.220	0.222	0.228	0.229	0.230	0.235	0.237	0.240
	0.244	0.245	0.246	0.247	0.249	0.250	0.252	0.255	0.256	0.259
	0.261	0.265	0.266	0.280	0.282	0.300	0.301	0.317	0.318	0.321
	0.324	0.333	0.337	0.341	0.343	0.356	0.366	0.376	0.383	0.385
	0.399	0.403	0.407	0.414	0.415	0.42	0.421	0.428	0.430	0.431
	0.432	0.434	0.441	0.444	0.461	0.462	0.482	0.485	0.495	0.496
(η_1 or η_2)	1	1	1	1	0	0	1	1	1	1
	1	0	0	0	0	1	1	0	1	1
	0	0	1	1	1	0	0	1	0	0
	0	1	0	0	1	1	1	0	1	0
	1	1	1	1	1	0	0	1	1	0
	1	1	0	1	1	1	1	0	1	1
	1	1	1	1	0	1	0	1	0	1
	1	0	1	0	1	1	1	0	1	0
(δ_1 or δ_2)	2	2	2	2	2	1	1	2	2	1
	1	1	1	1	1	1	1	1	2	1
	1	1	1	2	2	1	1	1	1	1
	1	1	2	1	2	1	2	2	1	1
	1	1	1	1	2	1	1	1	1	1
	2	2	1	2	1	1	2	2	1	2
	1	1	2	1	1	2	2	1	1	2
	1	1	2	1	2	2	2	1	1	1

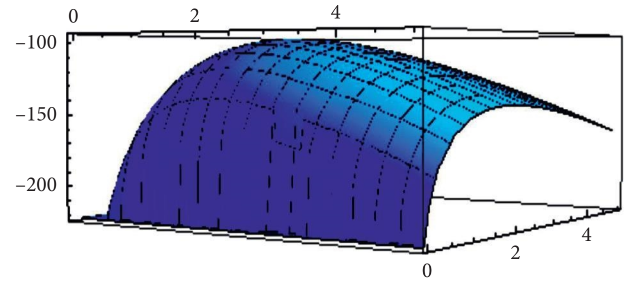

Figure 1: The profile loglikelihood of α_1 and α_2.

Table 3: The point ML, bootstrap, and Bayes estimate.

Method	β_1	β_2	α_{11}	α_{12}	α_{21}	α_{22}
$(\cdot)_{ML}$	1.864 9	1.947 3	1.495 7	1.380 6	1.691 1	0.338 2
$(\cdot)_{Boot}$	2.001 3	1.854 1	1.625 4	1.564 2	1.745 2	0.625 4
$(\cdot)_{B-MCMC}$	1.751 4	1.821 7	1.303 3	1.202 5	1.456 4	0.291 0

Table 4: 95% ML, bootstrap, and Bayes interval estimate.

Pa.	ACI	Length	Boo-p		Boot-t		CI	Length
β_1	(1.577, 2.153)	0.577	(1.474, 2.65)	1.180	(1.452, 2.274)	0.822	(1.343, 2.232)	0.889
β_2	(1.665, 2.064)	0.397	(1.212, 2.845)	1.633	(1.275, 2.414)	1.138	(1.273 7, 2.448)	1.174
α_{11}	(1.066, 1.926)	0.860	(0.422, 2.854)	2.433	(0.748, 2.184)	1.436	(0.754, 2.147)	1.393
α_{12}	(0.955, 1.806)	0.851	(0.425, 2.321)	1.396	(0.692, 2.066)	1.374	(0.675, 1.969)	1.294 06
α_{21}	(1.298, 2.084)	0.786	(0.215, 2.965)	1.896	(0.746, 2.624)	1.878	(0.757, 2.521)	1.765
α_{22}	(0.062, 0.615)	0.553	(0.001, 0.966)	0.964	(0.020, 0.659)	0.639	(0.085, 0.645)	0.559

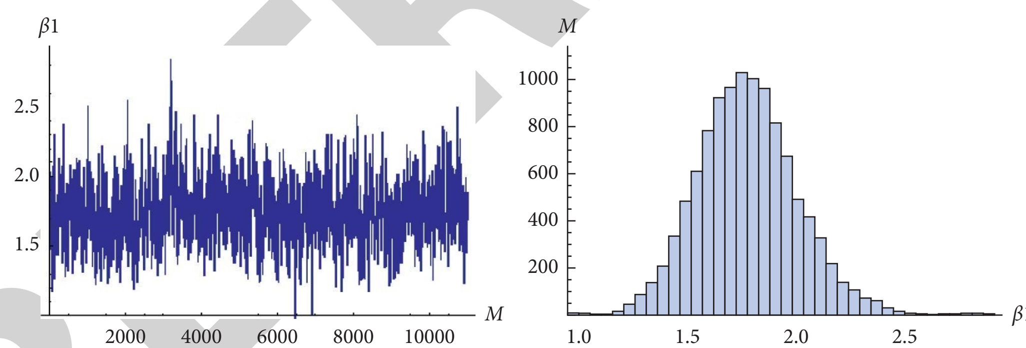

Figure 2: Simulation MCMC generated number/histogram of the parameter α_1.

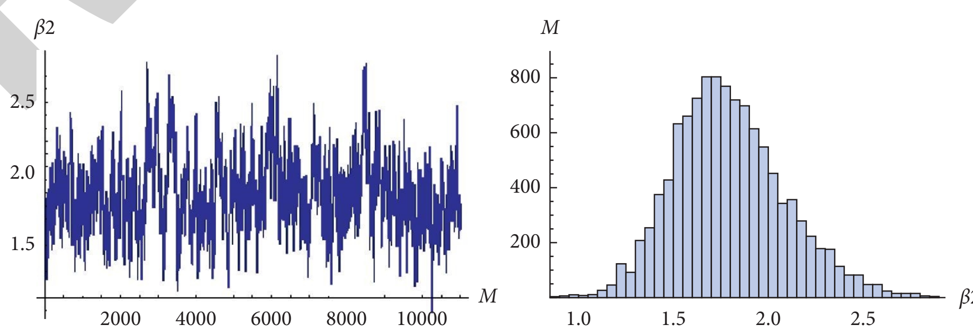

Figure 3: Simulation MCMC generated number/histogram of the parameter α_2.

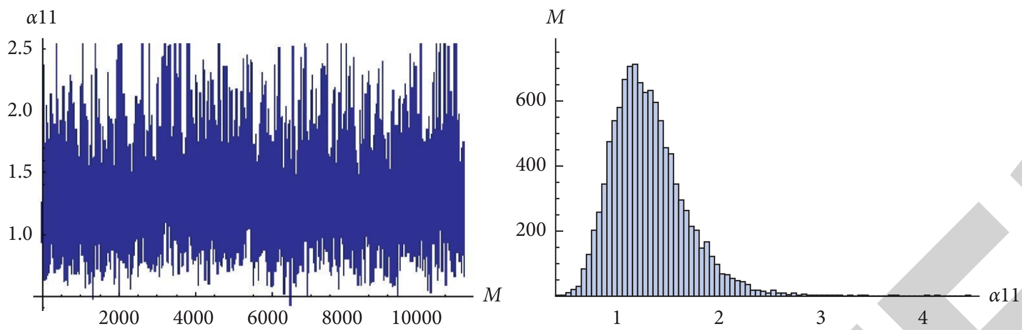

Figure 4: Simulation MCMC generated number/histogram of the parameter β_{11}.

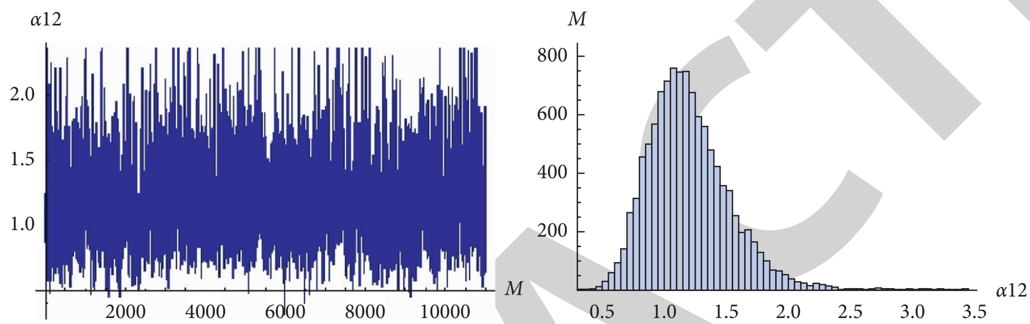

Figure 5: Simulation MCMC generated number/histogram of the parameter β_{12}.

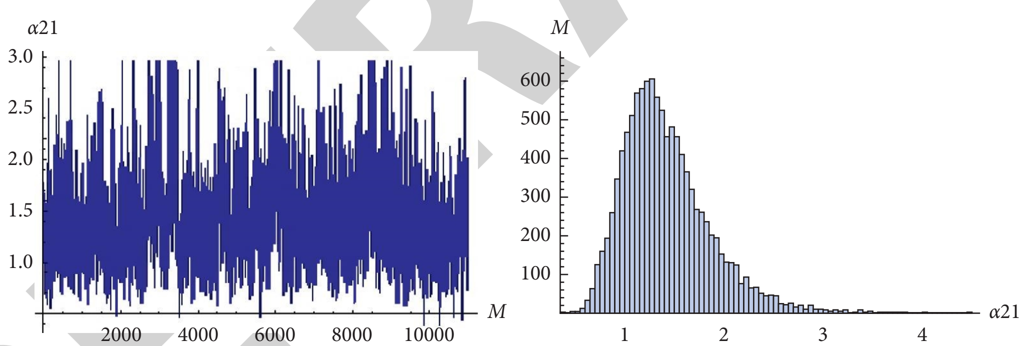

Figure 6: Simulation MCMC generated number/histogram of the parameter β_{21}.

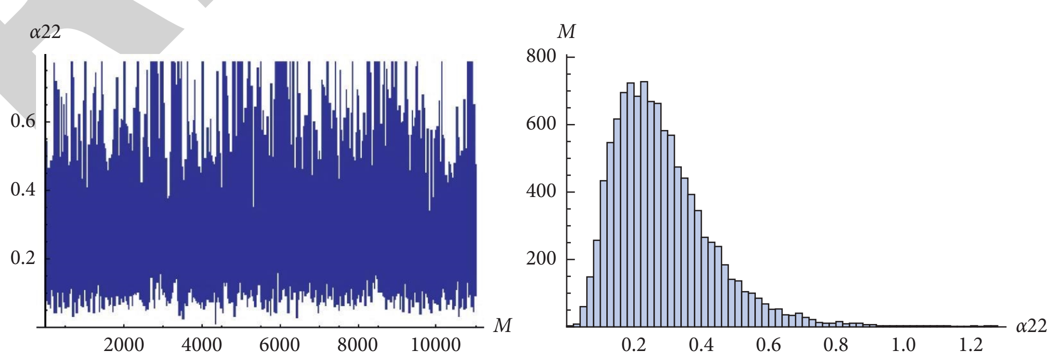

Figure 7: Simulation MCMC generated number/histogram of the parameter β_{21}.

(1) Two samples of size S_1 and S_2 are generated form Burr XII distribution with parameters β_k and $\alpha_{k1} + \alpha_{k2}$, $k = 1, 2$, respectively. Hence, the joint sample of size $N = S_1 + S_2$ is generated.
(2) For given τ, the Type-I JCRS and its size J are determined.
(3) The integers n_1 and n_2 are computed from the Type-I JCRS.
(4) The random integers m_{kj} are generated from binomial distributions.
(5) Steps 1 to 4 are repeated 1000 times to obtain 1000 Type-I JCRS.
(6) The MLE, bootstrap, and Bayes point and intervals estimates are computed for each sample.
(7) The values of each ME, MSEs, MILs, and PCs are computed and reported in Tables 5–8.

ALGORITHM 2: General steps used to generate Type-I joint competing risk samples and the corresponding estimate (see Almarashi et al. [15]).

TABLE 5: The ME and MSEs of ML, boot, and Bayes methods under $\omega = \{2.0, 1.2, 1.3, 1.8, 2.0, 2.0\}$.

(τ, S_1, S_2)		β_1		β_2		α_{11}		α_{12}		α_{21}		α_{22}	
		ME	MSE	ME	MSE	ME	MSE	ME	MSE	ME	MSE	ME	MSE
(0.2, 30, 30)	ML	2.542	0.454	1.423	0.423	1.524	0.421	2.352	0.632	2.457	0.489	2.397	0.500
	Boot	2.577	0.481	1.625	0.488	1.599	0.502	2.387	0.689	2.499	0.532	2.421	0.552
	Bayes P_0	2.537	0.452	1.414	0.411	1.517	0.411	2.341	0.618	2.442	0.481	2.387	0.492
	Bayes P_1	2.425	0.355	1.317	0.318	1.427	0.357	2.240	0.518	2.314	0.392	2.301	0.380
(0.2, 50, 30)	ML	2.521	0.428	1.419	0.442	1.481	0.395	2.329	0.601	2.449	0.483	2.390	0.492
	Boot	2.555	0.452	1.627	0.493	1.548	0.481	2.365	0.661	2.481	0.528	2.417	0.553
	Bayes P_0	2.519	0.424	1.418	0.421	1.478	0.392	2.318	0.585	2.445	0.484	2.379	0.487
	Bayes P_1	2.407	0.321	1.313	0.313	1.403	0.331	2.215	0.481	2.307	0.387	2.303	0.371
(0.2, 40, 60)	ML	2.525	0.431	1.379	0.404	1.488	0.391	2.325	0.598	2.407	0.452	2.361	0.459
	Boot	2.551	0.449	1.600	0.458	1.539	0.483	2.354	0.663	2.438	0.490	2.275	0.511
	Bayes P_0	2.522	0.418	1.378	0.381	1.479	0.388	2.321	0.579	2.409	0.449	2.341	0.451
	Bayes P_1	2.403	0.317	1.281	0.279	1.297	0.327	2.219	0.480	2.278	0.351	2.269	0.339
(0.2, 80, 80)	ML	2.500	0.400	1.362	0.292	1.429	0.358	2.300	0.563	2.298	0.447	2.354	0.451
	Boot	2.512	0.411	1.591	0.441	1.512	0.448	2.318	0.618	2.430	0.483	2.270	0.501
	Bayes P_0	2.477	0.281	1.371	0.373	1.441	0.339	2.292	0.543	2.397	0.432	2.337	0.444
	Bayes P_1	2.275	0.282	1.275	0.270	1.263	0.300	2.189	0.439	2.267	0.338	2.254	0.328
(0.8, 30, 30)	ML	2.507	0.407	1.371	0.296	1.441	0.365	2.308	0.571	2.301	0.452	2.362	0.457
	Boot	2.518	0.418	1.598	0.447	1.518	0.457	2.315	0.624	2.447	0.489	2.279	0.514
	Bayes P_0	2.481	0.293	1.377	0.375	1.453	0.351	2.299	0.555	2.405	0.438	2.347	0.457
	Bayes P_1	2.279	0.285	1.281	0.277	1.269	0.308	2.161	0.449	2.278	0.345	2.263	0.341
(0.8, 50, 30)	ML	2.481	0.380	1.374	0.292	1.407	0.328	2.274	0.538	2.307	0.450	2.360	0.449
	Boot	2.489	0.385	1.593	0.441	1.475	0.414	2.281	0.571	2.441	0.491	2.271	0.510
	Bayes P_0	2.455	0.261	1.372	0.370	1.411	0.304	2.254	0.514	2.401	0.433	2.342	0.458
	Bayes P_1	2.244	0.285	1.284	0.279	1.219	0.172	2.147	0.404	2.281	0.342	2.260	0.338
(0.8, 40, 60)	ML	2.487	0.385	1.331	0.263	1.411	0.331	2.274	0.541	2.279	0.418	2.328	0.411
	Boot	2.493	0.387	1.479	0.404	1.479	0.422	2.289	0.578	2.408	0.458	2.237	0.471
	Bayes P_0	2.459	0.264	1.338	0.318	1.418	0.314	2.260	0.525	2.369	0.404	2.315	0.411
	Bayes P_1	2.248	0.289	1.224	0.232	1.221	0.183	2.151	0.413	2.234	0.300	2.218	0.300
(0.8, 80, 80)	ML	2.415	0.341	1.311	0.245	1.390	0.290	2.215	0.502	2.255	0.402	2.302	0.292
	Boot	2.425	0.359	1.452	0.381	1.462	0.382	2.227	0.514	2.401	0.441	2.218	0.449
	Bayes P_0	2.411	0.241	1.314	0.301	1.390	0.271	2.211	0.500	2.354	0.292	2.301	0.395
	Bayes P_1	2.207	0.242	1.200	0.214	1.200	0.144	2.114	0.351 3	2.218	0.271	2.202	0.281

TABLE 6: The MILs and CPs of ML, boot, and Bayes methods under $\omega = \{2.0, 1.2, 1.3, 1.8, 2.0, 2.0\}$.

(τ, S_1, S_2)		β_1		β_2		α_{11}		α_{12}		α_{21}		α_{22}	
		MIL	CP	MIL	CP	MIL	CP	MIL	CP	MIL	CP	MIL	CP
(0.2, 30, 30)	ML	4.142	0.85	3.211	0.87	3.512	0.88	4.215	0.89	5.413	0.88	5.124	0.88
	Boot	4.314	0.88	3.415	0.88	3.688	0.89	4.389	0.89	5.598	0.89	5.311	0.88
	Bayes P_0	4.101	0.89	3.178	0.89	3.481	0.90	4.182	0.90	5.389	0.89	5.047	0.90
	Bayes P_1	3.245	0.91	3.001	0.90	3.214	0.91	4.007	0.90	5.217	0.90	4.874	0.91

Table 6: Continued.

(τ, S_1, S_2)		β_1		β_2		α_{11}		α_{12}		α_{21}		α_{22}	
		MIL	CP	MIL	CP	MIL	CP	MIL	CP	MIL	CP	MIL	CP
(0.2, 50, 30)	ML	3.850	0.90	2.901	0.89	3.012	0.90	3.841	0.90	5.211	0.90	5.109	0.90
	Boot	3.950	0.90	3.080	0.90	3.130	0.90	4.007	0.89	5.417	0.89	5.214	0.89
	Bayes P_0	3.811	0.91	2.920	0.90	3.011	0.92	3.700	0.92	5.198	0.89	5.001	0.90
	Bayes P_1	3.124	0.93	2.710	0.91	2.910	0.96	3.507	0.94	4.987	0.92	4.780	0.94
(0.2, 40, 60)	ML	3.899	0.89	2.725	0.89	3.025	0.90	3.871	0.91	5.003	0.90	4.952	0.91
	Boot	3.981	0.90	2.895	0.91	3.142	0.91	4.019	0.89	5.274	0.92	5.001	0.91
	Bayes P_0	3.861	0.92	2.701	0.90	3.019	0.92	3.722	0.90	5.000	0.91	4.890	0.92
	Bayes P_1	3.190	0.91	2.503	0.93	2.927	0.90	3.531	0.91	4.711	0.93	4.520	0.91
(0.2, 80, 80)	ML	3.601	0.90	2.610	0.91	2.911	0.91	3.590	0.91	4.802	0.91	4.815	0.91
	Boot	3.690	0.92	2.781	0.94	3.000	0.93	3.811	0.90	5.003	0.92	5.912	0.91
	Bayes P_0	3.570	0.92	2.590	0.90	2.890	0.90	3.530	0.92	4.779	0.90	4.715	0.91
	Bayes P_1	2.854	0.93	2.401	0.94	2.711	0.94	3.224	0.93	4.490	0.92	4.412	0.95
(0.8, 30, 30)	ML	3.654	0.89	2.680	0.90	2.978	0.91	3.610	0.89	4.875	0.90	4.864	0.90
	Boot	3.697	0.90	2.775	0.90	3.069	0.90	3.882	0.90	5.069	0.92	5.949	0.90
	Bayes P_0	3.630	0.90	2.640	0.91	2.945	0.90	3.591	0.90	4.819	0.92	4.760	0.93
	Bayes P_1	2.915	0.90	2.501	0.92	2.774	0.91	3.305	0.92	4.500	0.92	4.445	0.92
(0.8, 50, 30)	ML	3.418	0.91	2.671	0.90	2.760	0.91	3.401	0.91	4.879	0.89	4.871	0.90
	Boot	3.498	0.90	2.754	0.93	2.879	0.93	3.670	0.92	5.085	0.91	5.939	0.90
	Bayes P_0	3.401	0.91	2.621	0.91	2.847	0.92	3.402	0.94	4.804	0.91	4.748	0.91
	Bayes P_1	2.721	0.92	2.491	0.90	2.576	0.94	3.115	0.94	4.503	0.94	4.445	0.93
(0.8, 40, 60)	ML	3.441	0.89	2.451	0.91	2.772	0.91	3.424	0.90	4.610	0.96	4.623	0.90
	Boot	3.514	0.90	2.524	0.97	2.881	0.90	3.679	0.91	4.850	0.92	5.790	0.91
	Bayes P_0	3.422	0.92	2.405	0.92	2.842	0.91	3.414	0.90	4.579	0.94	4.624	0.91
	Bayes P_1	2.738	0.92	2.213	0.92	2.495	0.92	3.008	0.91	4.280	0.95	4.215	0.94
(0.8, 80, 80)	ML	3.150	0.93	2.178	0.92	2.684	0.92	3.314	0.92	4.390	0.90	4.398	0.93
	Boot	3.241	0.92	2.290	0.90	2.701	0.92	3.450	0.91	4.512	0.93	5.588	0.90
	Bayes P_0	3.110	0.93	2.154	0.94	2.629	0.92	3.375	0.93	4.370	0.94	4.401	0.93
	Bayes P_1	2.415	0.95	2.001	0.95	2.478	0.95	3.000	0.93	4.003	0.93	4.005	0.96

Table 7: The ME and MSEs of ML, boot, and Bayes methods under $\omega = \{1.0, 2.0, 3.0, 2.0, 2.5, 1.0\}$.

(τ, S_1, S_2)		β_1		β_2		α_{11}		α_{12}		α_{21}		α_{22}	
		ME	MSE	ME	MSE	ME	MSE	ME	MSE	ME	MSE	ME	MSE
(0.1, 30, 30)	ML	1.234	0.234	2.421	0.421	3.342	0.645	2.324	0.402	2.842	0.495	1.321	0.255
	Boot	1.335	0.238	2.542	0.426	3.452	0.648	2.426	0.405	2.890	0.498	1.435	0.291
	Bayes P_0	1.229	0.233	2.418	0.420	3.302	0.644	2.314	0.400	2.834	0.493	1.313	0.243
	Bayes P_1	1.198	0.224	2.370	0.411	3.201	0.634	2.280	0.390	2.790	0.484	1.280	0.242
(0.1, 50, 30)	ML	1.217	0.217	2.408	0.403	3.264	0.627	2.305	0.387	2.829	0.477	1.304	0.237
	Boot	1.319	0.221	2.528	0.409	3.401	0.621	2.407	0.391	2.875	0.481	1.421	0.278
	Bayes P_0	1.208	0.215	2.404	0.401	3.255	0.627	2.293	0.385	2.817	0.472	1.300	0.225
	Bayes P_1	1.181	0.209	2.367	0.382	3.185	0.613	2.261	0.369	2.777	0.468	1.266	0.222
(0.1, 40, 60)	ML	1.223	0.222	2.365	0.367	3.272	0.631	2.314	0.395	2.780	0.441	1.285	0.209
	Boot	1.325	0.227	2.451	0.371	3.415	0.627	2.412	0.397	2.831	0.449	1.370	0.251
	Bayes P_0	1.214	0.218	2.361	0.362	3.267	0.635	2.299	0.388	2.777	0.447	1.275	0.201
	Bayes P_1	1.193	0.213	2.322	0.348	3.192	0.621	2.274	0.374	2.731	0.415	1.214	0.191
(0.1, 80, 80)	ML	1.175	0.187	2.314	0.328	3.231	0.561	2.272	0.351	2.735	0.407	1.251	0.172
	Boot	1.286	0.192	2.407	0.347	3.370	0.588	2.350	0.362	2.800	0.415	1.361	0.199
	Bayes P_0	1.269	0.179	2.321	0.327	3.229	0.564	2.264	0.347	2.722	0.401	1.254	0.162
	Bayes P_1	1.144	0.152	2.279	0.301	3.151	0.541	2.217	0.322	2.680	0.362	1.182	0.127
(0.5, 30, 30)	ML	1.211	0.214	2.407	0.401	3.324	0.615	2.285	0.284	2.815	0.476	1.309	0.241
	Boot	1.310	0.217	2.531	0.409	3.428	0.619	2.404	0.292	2.867	0.481	1.424	0.279
	Bayes P_0	1.203	0.212	2.400	0.397	3.287	0.617	2.289	0.271	2.807	0.482	1.304	0.231
	Bayes P_1	1.175	0.205	2.356	0.394	3.180	0.601	2.260	0.362	2.766	0.461	1.271	0.228

Table 7: Continued.

(τ, S_1, S_2)		β_1		β_2		α_{11}		α_{12}		α_{21}		α_{22}	
		ME	MSE	ME	MSE	ME	MSE	ME	MSE	ME	MSE	ME	MSE
(0.5, 50, 30)	ML	1.200	0.200	2.397	0.385	3.244	0.600	2.284	0.365	2.801	0.455	1.292	0.222
	Boot	1.302	0.204	2.510	0.397	3.381	0.593	2.391	0.373	2.849	0.463	1.409	0.266
	Bayes P_0	1.185	0.197	2.389	0.387	3.237	0.601	2.273	0.367	2.790	0.454	1.291	0.210
	Bayes P_1	1.160	0.192	2.351	0.362	3.161	0.586	2.244	0.351	2.751	0.449	1.252	0.214
(0.5, 40, 60)	ML	1.207	0.201	2.347	0.349	3.251	0.603	2.292	0.376	2.751	0.424	1.252	0.189
	Boot	1.303	0.204	2.438	0.355	3.400	0.601	2.394	0.380	2.802	0.430	1.358	0.238
	Bayes P_0	1.191	0.200	2.347	0.341	3.244	0.607	2.281	0.369	2.748	0.418	1.263	0.187
	Bayes P_1	1.181	0.192	2.307	0.329	3.175	0.590	2.255	0.359	2.703	0.400	1.200	0.177
(0.5, 80, 80)	ML	1.162	0.170	2.300	0.309	3.214	0.534	2.249	0.335	2.708	0.389	1.239	0.160
	Boot	1.271	0.173	2.394	0.328	3.352	0.559	2.329	0.347	2.771	0.401	1.350	0.184
	Bayes P_0	1.252	0.158	2.302	0.307	3.207	0.537	2.241	0.322	2.700	0.381	1.241	0.151
	Bayes P_1	1.129	0.134	2.262	0.282	3.133	0.515	2.200	0.301	2.653	0.341	1.170	0.114

Table 8: The MILs and CPs of ML, boot, and Bayes methods under $\omega = \{1.0, 2.0, 3.0, 2.0, 2.5, 1.0\}$.

(τ, S_1, S_2)		β_1		β_2		α_{11}		α_{12}		α_{21}		α_{22}	
		MIL	CP	MIL	CP	MIL	CP	MIL	CP	MIL	CP	MIL	CP
(0.1, 30, 30)	ML	2.514	0.88	4.521	0.89	5.985	0.87	3.985	0.90	5.234	0.90	2.451	0.90
	Boot	2.674	0.89	4.654	0.89	6.124	0.89	4.231	0.89	5.385	0.90	2.562	0.91
	Bayes P_0	2.485	0.90	4.492	0.90	5.941	0.89	3.937	0.90	5.201	0.90	2.414	0.90
	Bayes P_1	2.350	0.90	4.320	0.91	5.752	0.91	3.813	0.90	5.025	0.90	2.285	0.91
(0.1, 50, 30)	ML	2.465	0.90	4.472	0.90	5.944	0.90	3.941	0.90	5.192	0.91	2.400	0.90
	Boot	2.627	0.91	4.601	0.89	6.051	0.91	4.188	0.90	5.341	0.90	2.523	0.90
	Bayes P_0	2.441	0.90	4.451	0.91	5.900	0.91	3.900	0.92	5.154	0.91	2.366	0.94
	Bayes P_1	2.304	0.93	4.275	0.92	5.707	0.92	3.762	0.92	4.955	0.93	2.241	0.92
(0.1, 40, 60)	ML	2.454	0.90	4.439	0.90	5.936	0.90	3.939	0.90	5.155	0.91	2.360	0.92
	Boot	2.631	0.89	4.569	0.92	6.061	0.91	4.175	0.92	5.311	0.93	2.481	0.91
	Bayes P_0	2.447	0.90	4.418	0.91	5.903	0.92	3.909	0.90	5.119	0.91	2.328	0.93
	Bayes P_1	2.314	0.90	4.245	0.94	5.715	0.91	3.769	0.93	4.921	0.94	2.207	0.96
(0.1, 80, 80)	ML	2.407	0.90	4.401	0.92	5.902	0.91	3.903	0.92	5.111	0.94	2.360	0.92
	Boot	2.582	0.92	4.515	0.93	6.024	0.92	4.132	0.92	5.271	0.93	2.481	0.90
	Bayes P_0	2.400	0.94	4.375	0.91	5.871	0.92	3.861	0.92	5.062	0.93	2.328	0.94
	Bayes P_1	2.267	0.95	4.208	0.96	5.674	0.94	3.715	0.94	4.874	0.94	2.187	0.95
(0.5, 30, 30)	ML	2.485	0.89	4.502	0.89	5.955	0.89	3.961	0.91	5.205	0.91	2.436	0.92
	Boot	2.641	0.90	4.635	0.89	6.101	0.89	4.209	0.90	5.344	0.91	2.544	0.91
	Bayes P_0	2.449	0.90	4.474	0.91	5.915	0.90	3.915	0.90	5.145	0.90	2.400	0.92
	Bayes P_1	2.324	0.90	4.301	0.91	5.727	0.91	3.800	0.91	5.000	0.92	2.271	0.91
(0.5, 50, 30)	ML	2.441	0.91	4.456	0.90	5.919	0.90	3.918	0.92	5.151	0.91	2.378	0.90
	Boot	2.600	0.91	4.580	0.90	6.024	0.92	4.161	0.92	5.302	0.92	2.502	0.92
	Bayes P_0	2.415	0.91	4.433	0.91	5.875	0.91	3.882	0.92	5.115	0.91	2.341	0.92
	Bayes P_1	2.274	0.93	4.257	0.93	5.691	0.92	3.744	0.92	4.912	0.92	2.221	0.92
(0.5, 40, 60)	ML	2.427	0.91	4.418	0.90	5.914	0.92	3.925	0.92	5.114	0.91	2.339	0.92
	Boot	2.607	0.90	4.550	0.92	6.032	0.91	4.151	0.92	5.311	0.93	2.460	0.94
	Bayes P_0	2.421	0.90	4.401	0.91	5.274	0.93	3.892	0.90	5.066	0.92	2.309	0.93
	Bayes P_1	2.288	0.92	4.227	0.93	5.894	0.91	3.751	0.93	4.869	0.94	2.191	0.96
(0.5, 80, 80)	ML	2.381	0.93	4.384	0.92	5.877	0.92	3.877	0.93	5.070	0.92	2.339	0.92
	Boot	2.555	0.92	4.500	0.92	5.982	0.94	4.114	0.92	5.239	0.93	2.455	0.94
	Bayes P_0	2.274	0.92	4.362	0.91	5.850	0.92	3.851	0.92	5.030	0.95	2.307	0.94
	Bayes P_1	2.231	0.95	4.189	0.94	5.644	0.95	3.703	0.92	4.835	0.94	2.179	0.94

point estimate. And, mean interval length (MIL) and probability coverage (PC) are used to measure interval estimate. The Monte Carlo study is done with respect to Algorithm 2.

8. Conclusions

Recently, the joint censoring scheme is more widely used in a comparative life testing specially for products coming from different lines of production. The problem of comparative life testes under different causes of failure has been discussed recently under the joint censoring scheme of competing risks exponential lifetime model by Almarashi et al. [15]. In this paper, we adopted this problem when units or individual is distributed with Burr XII distributions. The unknown model parameters are estimated with classical methods (ML and bootstrap) and Bayes method with noninformative and informative prior. Numerical computation is exposed with real data analysis and Monto Carlo simulation study to assess and discuss the developed results. The numerical result discusses changing of sample size, test time, and available information. Therefore, we observed the following points:

(1) The proposed model under Type-I JCRS serves well for all choice of censoring schemes and parameters choices

(2) The Bayes estimation under noninformative prior P_0 is more close to maximum likelihood estimation

(3) The informative priors P_1 serve better than noninformative prior and maximum likelihood estimations

(4) The increasing effect of sample size $S_1 + S_2$ reduces the MSE and MIL

(5) The large value of test time τ serves well than small value of τ

Data Availability

The used data are the real data set presented by Hoel (1972) in [36].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] N. Balakrishnan and R. Aggarwala, *Progressive Censoring-Theory, Methods, and Applications*, Birkhäuser, Basel, Switzerland, 2000.

[2] U. V. R. Rao, I. R. Savage, and M. Sobel, "Contributions to the theory of rank order statistics: the two-sample censored case," *The Annals of Mathematical Statistics*, vol. 31, no. 2, pp. 415–426, 1960.

[3] A. P. Basu, "On a generalized savage statistic with applications to life testing," *The Annals of Mathematical Statistics*, vol. 39, no. 5, pp. 1591–1604, 1968.

[4] R. A. Johnson and K. G. Mehrotra, "Locally most powerful rank tests for the two-sample problem with censored data," *The Annals of Mathematical Statistics*, vol. 43, no. 3, pp. 823–831, 1972.

[5] K. G. Mehrotra and R. A. Johnson, "Asymptotic sufficiency and asymptotically most powerful tests for the two sample censored situation," *Annals of Statistics*, vol. 4, pp. 589–596, 1976.

[6] G. K. Bhattacharyya and K. G. Mehrotra, "On testing equality of two exponential distributions under combined type-IIcensoring," *Journal of the American Statistical Association*, vol. 76, no. 376, pp. 886–894, 1981.

[7] K. G. Mehrotra and G. K. Bhattacharyya, "Confidence intervals with jointly type-II censored samples from two exponential distributions," *Journal of the American Statistical Association*, vol. 77, no. 378, pp. 441–446, 1982.

[8] N. Balakrishnan and A. Rasouli, "Exact likelihood inference for two exponential populations under joint type-II censoring," *Computational Statistics & Data Analysis*, vol. 52, no. 5, pp. 2725–2738, 2008.

[9] A. Rasouli and N. Balakrishnan, "Exact likelihood inference for two exponential populations under joint progressive type-II censoring," *Communications in Statistics -Theory and Methods*, vol. 39, no. 12, pp. 2172–2191, 2010.

[10] A. R. Shafay, N. Balakrishnan, and Y. Abdel-Aty, "Bayesian inference based on a jointly type-II censored sample from two exponential populations," *Journal of Statistical Computation and Simulation*, vol. 84, no. 11, pp. 2427–2440, 2014.

[11] B. N. Al-Matrafi and G. A. Abd-Elmougod, "Statistical Inferences with jointly type-II censored samples from two rayleigh distributions," *Global Journal of Pure and Applied Mathematics*, vol. 13, pp. 8361–8372, 2017.

[12] F. A. Momenkhan and G. A. Abd-Elmougod, "Stimations in partially step-stress accelerate life tests with jointly type-II censored samples from rayleigh distributions," *Transylvanian Review*, vol. 28, pp. 7609–7616, 2018.

[13] S. Mondal and D. Kundu, "Bayesian inference for weibull distribution under the balanced joint type-II progressive censoring scheme," *American Journal of Mathematical and Management Sciences*, vol. 39, no. 1, pp. 56–74, 2019.

[14] S. Mondal and D. Kundu, "Inferences of weibull parameters under balance two sample type-II progressive censoring scheme," *Quality and Reliability Engineering International*, vol. 36, no. 1, pp. 1–17, 2020.

[15] A. M. Almarashi, A. Algarni, A. M. Daghistani, G. A. Abd-Elmougod, S. Abdel-Khalek, and M. Z. Raqab, "Inferences for joint hybrid progressive censored exponential lifetimes under competing risk model," *Mathematical Problems in Engineering*, vol. 2021, Article ID 3380467, 12 pages, 2021.

[16] D. R. Cox, "The analysis of exponentially distributed lifetimes with two types of failures," *Journal of the Royal Statistical Society: Series B (Methodological)*, vol. 21, no. 2, pp. 411–421, 1959.

[17] M. J. Crowder, *Classical Competing Risks*, Chapman and Hall, London, UK, 2001.

[18] N. Balakrishnan and D. Han, "Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under type-II censoring," *Journal of*

Statistical Planning and Inference, vol. 138, no. 12, pp. 4172–4186, 2008.

[19] A. A. Modhesh and G. A. Abd-Elmougod, “Analysis of progressive first-failure-censoring for non-normal model using competing risks data,” *American Journal of Theoretical and Applied Statistics*, vol. 4, no. 6, pp. 610–618, 2015.

[20] R. A. Bakoban and G. A. Abd-Elmougod, “MCMC in analysis of progressively first failure censored competing risks data for gompertz model,” *Journal of Computational and Theoretical Nanoscience*, vol. 13, no. 10, pp. 6662–6670, 2016.

[21] A. Ganguly and D. Kundu, “Analysis of simple step-stress model in presence of competing risks,” *Journal of Statistical Computation and Simulation*, vol. 86, no. 10, pp. 1989–2006, 2016.

[22] H. H. Abu-Zinadah and N. Sayed-Ahmed, “Competing risks model with partially step-stress accelerate life tests in analyses lifetime chen data under type-II censoring scheme,” *Open Physics*, vol. 17, no. 1, pp. 192–199, 2019.

[23] A. Algarn, A. M. Almarashi, G. A. Abd-Elmougod, and Z. A. Abo-Eleneen, “Partially constant stress accelerate life tests model in analyses lifetime competing risks with a bathtub shape lifetime distribution in presence of type-I censoring,” *Transylvanian Review, to Apper*, vol. 27, no. 40, pp. 1–15, 2019.

[24] I. W. Burr, “Cumulative frequency functions,” *The Annals of Mathematical Statistics*, vol. 13, no. 2, pp. 215–232, 1942.

[25] R. N. Rodriguez, “A guide to the burr type XII distributions,” *Biometrika*, vol. 64, no. 1, pp. 129–134, 1977.

[26] W. C. Lee, J. W. Wu, and C. W. Hong, “Assessing the lifetime performance index of products from progressively type II right censored data using burr XII model,” *Mathematics and Computers in Simulation*, vol. 79, no. 7, pp. 2167–2179, 2009.

[27] H. M. Aljohani and N. M. Alfar, “Estimations with step-stress partially accelerated life tests for competing risks burr XII lifetime model under type-II censored data,” *Alexandria Engineering Journal*, vol. 59, no. 3, pp. 1171–1180, 2020.

[28] D. Kundu and A. Joarder, “Analysis of type-II progressively hybrid censored data,” *Computational Statistics & Data Analysis*, vol. 50, no. 10, pp. 2509–2528, 2006.

[29] M. M. Salah, “On progressive type-II censored samples from alpha power exponential distribution,” *Journal of Mathematics*, vol. 2020, Article ID 2584184, 8 pages, 2020.

[30] A. C. Davison and D. V. Hinkley, *Bootstrap Methods and Their Applications*, Cambridge University Press, Cambridge, UK, 2nd edition, 1997.

[31] B. Efron and R. J. Tibshirani, *An Introduction to the Bootstrap*, New York Chapman and Hall, New York, NY, USA, 1993.

[32] B. Efron, “The jackknife, the bootstrap and other resampling plans,” in *CBMS-NSF Regional Conference Series in Applied Mathematics*SIAM, Phiadelphia, PA, USA, 1982.

[33] P. Hall, “Theoretical comparison of bootstrap confidence intervals,” *Annals of Statistics*, vol. 16, pp. 927–953, 1988.

[34] K. Ullah and M. Aslam, “Bayesian analysis of the weibull paired comparison model using numerical approximation,” *Journal of Mathematics*, vol. 2020, Article ID 6628379, 6 pages, 2020.

[35] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, “Equation of state calculations by fast computing machines,” *The Journal of Chemical Physics*, vol. 21, no. 6, pp. 1087–1092, 1953.

[36] D. G. Hoel, “A representation of mortality data by competing risks,” *Biometrics*, vol. 28, no. 2, pp. 475–488, 1972.

[37] A. Koley and D. Kundu, “On generalized progressive hybrid censoring in presence of competing risks,” *Metrika*, vol. 80, no. 4, pp. 401–426, 2017.