Research Article

The p -Adic Valuations of Sums of Binomial Coefficients

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In this paper, we prove three supercongruences on sums of binomial coefficients conjectured by Z.-W. Sun. Let p be an odd prime and let $h \in \mathbb{Z}$ with $2h - 1 \equiv 0 \pmod{p}$. For $a \in \mathbb{Z}^+$ and $p^a > 3$, we show that $\sum_{k=0}^{p^a-1} \binom{hp^a-1}{k} \binom{2k}{k} (-h/2)^k \equiv 0 \pmod{p^{a+1}}$. Also, for any $n \in \mathbb{Z}^+$, we have $\gamma_p \left(\sum_{k=0}^{n-1} \binom{hn-1}{k} \binom{2k}{k} (-h/2)^k \right) \geq \gamma_p(n)$, where $\gamma_p(n)$ denotes the p -adic order of n . For any integer $m \equiv 0 \pmod{p}$ and positive integer n , we have $(1/pn) \left(\sum_{k=0}^{pn-1} \binom{pn-1}{k} \binom{2k}{k} (-m)^k \right) - (m(m-4)/p) \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{2k}{k} (-m)^k \in \mathbb{Z}_p$, where $(\cdot)$ is the Legendre symbol and $\mathbb{Z}_p$ is the ring of p -adic integers.

1. Introduction

Let p be an odd prime. In 2006, Pan and Sun [1] proved the congruence

$$\sum_{k=0}^{p-1} \binom{2k}{k+d} \equiv \left(\frac{p-d}{3} \right) \pmod{p}, \quad \text{for } d = 0, \dots, p, \quad (1)$$

via a curious combinatorial identity. For any positive integer a and prime $p \geq 5$, later Sun and Tauraso [2] established the following general result:

$$\sum_{k=0}^{p^a-1} \binom{2k}{k} \equiv \left(\frac{p^a}{3} \right) \pmod{p^2}. \quad (2)$$

Guo [3] conjectured a q -analogue of (2), which was confirmed by Liu and Petrov [4] using a q -analogue of Sun–Zhao congruence on harmonic sums and a q -series identity. Guo and Zudilin [5] also gave the q -generalizations of (2).

Let $A, B \in \mathbb{Z}$. The Lucas sequence $u_n = u_n(A, B)$ ($n \geq 0$) is given by

$$\begin{aligned} u_0 &= 0, \\ u_1 &= 1, \\ u_{n+1} &= Au_n - Bu_{n-1} \quad (n \geq 1). \end{aligned} \quad (3)$$

If p is an odd prime not dividing B , then it is known that $p \mid u_{p-(\Delta/p)}$ (see, e.g., [6]). For a nonzero integer n and a prime p , let $\gamma_p(n)$ denote the p -adic valuation (or p -adic order) of n , i.e., $\gamma_p(n)$ is the largest integer such that $p^{\gamma_p(n)} \mid n$, especially $\gamma_p(0) = +\infty$ and we define $\gamma_p(m/n) = \gamma_p(m) - \gamma_p(n)$ for rational number m/n . For more developments on p -adic valuation, we refer the reader to the papers [7–10].

In 2011, Sun [11] proved that for any nonzero integer m and odd prime p with $p \nmid m$, there holds

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{1}{m^k} \equiv \left(\frac{\Delta}{p} \right) + u_{p-(\Delta/p)}(m-2, 1) \pmod{p^2}, \quad (4)$$

where $\Delta = m(m-4)$. As a common extension of (4), Sun [12] showed that

$$\frac{1}{pn} \left(\sum_{k=0}^{pn-1} \binom{2k}{k} \frac{1}{m^k} - \left(\frac{\Delta}{p} \right) \sum_{k=0}^{n-1} \binom{2k}{k} \frac{1}{m^k} \right) \in \mathbb{Z}_p, \quad (5)$$

and furthermore

$$\begin{aligned} & \frac{1}{n} \left(\sum_{k=0}^{pn-1} \binom{2k}{k} \frac{1}{m^k} - \left(\frac{\Delta}{p} \right) \sum_{k=0}^{n-1} \binom{2k}{k} \frac{1}{m^k} \right) \\ & \equiv \frac{\binom{2n}{n}}{2m^{n-1}} u_{p-(\Delta/p)}(m-2, 1) \pmod{p^2}. \end{aligned} \quad (6)$$

Let p be an odd prime and let m be an integer with $p \nmid m$. One can easily get the following formula:

$$\sum_{k=0}^{p-1} \binom{p-1}{k} (-1)^k \binom{2k}{k} / m^k \equiv \sum_{k=0}^{p-1} \binom{2k}{k} \frac{1}{m^k} \pmod{p}, \quad (7)$$

since for any $k \in \{1, 2, \dots, p-1\}$, we get $\binom{p-1}{k} \equiv (-1)^k \pmod{p}$. It looks like the left-hand side of (7) has some connection with the right-hand side. Motivated by (4) and (7), Sun [13] determined the sum $\left(\sum_{k=0}^{p^a-1} \binom{hp^a-1}{k} \binom{2k}{k} / m^k \right)$ modulo p^2 , where h is a p -adic integer and $m \in \mathbb{Z}$ with $p \nmid m$. For example, if $h \equiv 0 \pmod{p}$ and $((2h \equiv 1 \pmod{p})$ or $p^a > 3$), then

$$\begin{aligned} & \sum_{k=0}^{p^a-1} \binom{hp^a-1}{k} \binom{2k}{k} \left(\frac{h}{2} \right)^k \\ & \equiv \left(\frac{1-2h}{p^a} \right) \left(1 + h \left(\left(4 - \frac{2}{h} \right)^{p-1} - 1 \right) \right) \pmod{p^2}. \end{aligned} \quad (8)$$

It is interesting to consider whether there exists the supercongruence as (8) modulo the higher powers of p in the case $2h-1 \equiv 0 \pmod{p}$ and $p^a > 3$. Sun [13] managed to investigate the above case and made the following conjecture. The first aim of this paper is to prove the conjectured results.

Theorem 1. Let p be an odd prime and let $h \in \mathbb{Z}$ with $2h-1 \equiv 0 \pmod{p}$. If $a \in \mathbb{Z}^+$ and $p^a > 3$, then

$$\sum_{k=0}^{p^a-1} \binom{hp^a-1}{k} \binom{2k}{k} \left(\frac{h}{2} \right)^k \equiv 0 \pmod{p^{a+1}}. \quad (9)$$

Also, for any $n \in \mathbb{Z}^+$, we have

$$\nu_p \left(\sum_{k=0}^{n-1} \binom{hn-1}{k} \binom{2k}{k} \left(\frac{h}{2} \right)^k \right) \geq \nu_p(n). \quad (10)$$

On the other hand, based on (5) and (7), Sun [12] conjectured the corresponding result with $p \nmid m$. The second aim of this paper is to show the following result.

Theorem 2. Let p be an odd prime and let $\Delta = m(m-4)$. For any integer $m \equiv 0 \pmod{p}$ and positive integer n , we have

$$\begin{aligned} & \frac{1}{pn} \left(\sum_{k=0}^{pn-1} \binom{pn-1}{k} \binom{2k}{k} \frac{1}{-m^k} - \left(\frac{\Delta}{p} \right) \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{2k}{k} \frac{1}{(-m)^k} \right) \\ & \in \mathbb{Z}_p. \end{aligned} \quad (11)$$

The remainder of the paper is organized as follows. In the next section, we give some lemmas. The proofs of Theorems 1 and 2 will be given in Section 3.

2. Some Lemmas

In the following section, for an assertion A , we adopt the notation:

$$[A] = \begin{cases} 1, & \text{if } A \text{ holds,} \\ 0, & \text{otherwise.} \end{cases} \quad (12)$$

We know that $[m=n]$ coincides with the Kronecker symbol $\delta_{m,n}$.

Lemma 1. Let n, k, α be positive integers and p be a prime. Then,

$$\binom{p^\alpha n - 1}{k} \equiv \binom{p^{\alpha-1} n - 1}{\lfloor \frac{k}{p} \rfloor} (-1)^{k - [k/p]} \left(1 - np^\alpha \sum_{\substack{j=1 \\ p \nmid j}}^k \frac{1}{j} \right) \pmod{p^{2\alpha}}. \quad (13)$$

Proof. Note that

$$\begin{aligned} \binom{p^\alpha n - 1}{k} &= \prod_{j=1}^k \frac{p^\alpha n - j}{j} = \binom{p^{\alpha-1} n - 1}{\lfloor \frac{k}{p} \rfloor} \prod_{\substack{j=1 \\ p \nmid j}}^k \frac{p^\alpha n - j}{j} \\ &\equiv \binom{p^{\alpha-1} n - 1}{\lfloor \frac{k}{p} \rfloor} (-1)^{k - [k/p]} \left(1 - \sum_{\substack{j=1 \\ p \nmid j}}^k \frac{p^\alpha n}{j} \right) \pmod{p^{2\alpha}}. \end{aligned} \quad (14)$$

This proves (13). The congruence (13) is a result of Beukers [[14], Lemma 2]. $\square$

Lemma 2. *Let p be an odd prime. Then, for any integers a, b and positive integers r, s , we have*

$$\frac{\binom{p^r a}{p^s b}}{\binom{p^{r-1} a}{p^{s-1} b}} \equiv 1 \pmod{p^{r+s+\min\{r,s\}-\delta_{p,3}}}. \quad (15)$$

This lemma is a well-known congruence due to Osburn et al., see, e.g., [[15], (19)].

The following curious result is due to Sun [16].

Lemma 3 (see [16], Theorem 1)). *Let $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. Suppose that p is an odd prime dividing $m-4$. Then,*

$$\begin{aligned} \nu_p \left(\sum_{k=0}^{n-1} \binom{2k}{k} \frac{1}{m^k} \right) &\geq \nu_p(n), \\ \nu_p \left(\sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \binom{2k}{k} \frac{1}{m^k} \right) &\geq \nu_p(n). \end{aligned} \quad (16)$$

Furthermore,

$$\begin{aligned} \frac{1}{n} \sum_{k=0}^{n-1} \binom{2k}{k} \frac{1}{m^k} &\equiv \frac{\binom{2n-1}{n-1}}{4^{n-1}} + \delta_{p,3} [3|n] \frac{m-4}{3} \\ &\cdot \left(\frac{\frac{2n}{3^{\nu_3(n)}} - 1}{\frac{n}{3^{\nu_3(n)}} - 1} \right) \pmod{p^{\nu_p(m-4)}}, \end{aligned} \quad (17)$$

and also

$$\frac{1}{n} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{\binom{2k}{k}}{m^k} \equiv \frac{C_{n-1}}{4^{n-1}} \pmod{p^{\nu_p(m-4)-\delta_{p,3}}}, \quad (18)$$

where C_k denotes the Catalan number $1/(k+1) \binom{2k}{k} = \binom{2k}{k} - \binom{2k}{k+1}$. Thus, for $a \in \mathbb{Z}^+$, we have

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \binom{2k}{k} \frac{1}{m^k} \equiv 1 + \delta_{p,3} \frac{m-4}{3} \equiv \frac{m-1}{3} \pmod{p}. \quad (19)$$

Lemma 4. *Let p be an odd prime and let $h \in \mathbb{Z}$ with $2h-1 \equiv 0 \pmod{p}$. Let l, α be nonnegative integers. If $p \geq 5$, then we have*

$$\sum_{[k/p^\alpha]=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \equiv \frac{p^\alpha}{4^l} \binom{2l}{l} \pmod{p^{\alpha+1}}. \quad (20)$$

If $p=3$, then

$$\sum_{[k/p^\alpha]=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \equiv 0 \pmod{p^\alpha}. \quad (21)$$

If $p=3$ and $l \equiv 1 \pmod{3}$, then

$$\begin{aligned} \sum_{[k/p^\alpha]=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k &\equiv \frac{p^\alpha}{4^l} \binom{2l}{l} + 2[\alpha \geq 1] p^{\alpha-1} \\ &\cdot \frac{(1-2h)}{h} \binom{2l}{l} \pmod{p^{\alpha+1}}. \end{aligned} \quad (22)$$

Proof. Observe that

$$\begin{aligned} \sum_{[k/p^\alpha]=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k &= \sum_{k=0}^{p^\alpha l + p^\alpha - 1} \binom{2k}{k} \left(\frac{h}{2}\right)^k \\ &- \sum_{k=0}^{p^\alpha l - 1} \binom{2k}{k} \left(\frac{h}{2}\right)^k. \end{aligned} \quad (23)$$

Since $(2/h) - 4 = 2(1-2h)/h \equiv 0 \pmod{p}$ and $l \equiv 1 \pmod{3}$, by (17), we obtain that

$$\begin{aligned}
& \sum_{\lfloor h/p^\alpha \rfloor = l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \\
& \equiv (p^\alpha l + p^\alpha) \left(\frac{\binom{(2l+2)p^\alpha}{(l+1)p^\alpha}}{4^{p^\alpha l + p^\alpha - 1} 2} + \delta_{p,3} [3 | (p^\alpha l + p^\alpha)] \frac{2(1-2h)}{3h} \left(\frac{\frac{2(p^\alpha l + p^\alpha)}{3^{\nu_3(p^\alpha l + p^\alpha)} - 1} - 1 \right) \right) \\
& \quad - p^\alpha l \left(\frac{\binom{2p^\alpha l}{p^\alpha l}}{4^{p^\alpha l - 1} 2} + \delta_{p,3} [3 | (p^\alpha l)] \frac{2(1-2h)}{3h} \left(\frac{\frac{2p^\alpha l}{3^{\nu_3(p^\alpha l)} - 1} - 1 \right) \right) \pmod{p^{\alpha+1}}.
\end{aligned} \tag{24}$$

If $p \geq 5$, by (15) and (24), we get

$$\begin{aligned}
& \sum_{\lfloor k/p^\alpha \rfloor = l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \\
& \equiv p^\alpha (l+1) \frac{\binom{2l+2}{l+1}}{4^l 2} - p^\alpha l \frac{\binom{2l}{l}}{4^{l-1} 2} = p^\alpha (2l+1) \frac{\binom{2l}{l}}{4^l} - 2p^\alpha l \frac{\binom{2l}{l}}{4^l} = p^\alpha \frac{\binom{2l}{l}}{4^l} \pmod{p^{\alpha+1}}.
\end{aligned} \tag{25}$$

Thus, (20) is proved. The congruence (21) is easily deduced from (24).

Finally, we will prove (22). The congruence (22) is trivial when $\alpha = 0$. Now we may assume $\alpha \geq 1$. With the help of

(15), (17), and (24), for any nonnegative integer l with $l \equiv 1 \pmod{3}$, we have

$$\begin{aligned}
& \sum_{\lfloor k/3^\alpha \rfloor = l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \\
& \equiv 3^\alpha (l+1) \left(\frac{\binom{2l+2}{l+1}}{4^l 2} + \frac{2(1-2h)}{3h} \binom{2l+1}{l+1} \right) - 3^\alpha l \left(\frac{\binom{2l}{l}}{4^{l-1} 2} + \frac{1-2h}{3h} \binom{2l}{l} \right) \\
& \equiv \frac{3^\alpha}{4^l} \binom{2l}{l} + 2 \cdot 3^{\alpha-1} \frac{(1-2h) \binom{2l}{l}}{h} \pmod{3^{\alpha+1}}.
\end{aligned} \tag{26}$$

This concludes the proof. $\square$

Lemma 5. Let p be an odd prime and l, s be nonnegative integers. Let $\Delta = m(m-4)$. For any integer $m \equiv 0 \pmod{p}$, we have

$$\begin{aligned} & \sum_{k=p^s l}^{p^s l + p^s - 1} \binom{2k}{k} \frac{1}{m^k} - \left(\frac{\Delta}{p}\right) \sum_{k=p^{s-1} l}^{p^{s-1} l + p^{s-1} - 1} \binom{2k}{k} \frac{1}{m^k} \\ & \equiv p^{s-1} \frac{\binom{2l}{l}}{m^l} \left(\left(2 - \frac{m}{2}\right)l + 1 \right) u_{p-\left(\frac{\Delta}{p}\right)}^{(m-2, 1) \pmod{p^{s+1}}}. \end{aligned} \quad (27)$$

Proof. The proof is very similar to (20). Clearly,

$$\sum_{k=p^s l}^{p^s l + p^s - 1} \binom{2k}{k} \frac{1}{m^k} = \sum_{k=0}^{p^s l + p^s - 1} \binom{2k}{k} \frac{1}{m^k} - \sum_{k=0}^{p^s l - 1} \binom{2k}{k} \frac{1}{m^k}. \quad (28)$$

Substituting $n = p^{s-1}(l+1)$ and $n = p^{s-1}l$ in (6), we obtain (27). $\square$

Lemma 6. Let p be an odd prime and $h \in \mathbb{Z}$ with $2h-1 \equiv 0 \pmod{p}$. Let a be a positive integer. Then,

$$\sum_{k=0}^{p^a-1} \binom{2k}{k} \left(\frac{h}{2}\right)^k \sum_{j=1, p \nmid j}^k \frac{1}{j} \equiv \begin{cases} 0 \pmod{p}, & \text{if } a \geq 2, \\ 2 \pmod{p}, & \text{if } a = 1. \end{cases} \quad (29)$$

Proof. Note that

$$\begin{aligned} & \sum_{k=0}^{p^a-1} \binom{2k}{k} \left(\frac{h}{2}\right)^k \sum_{j=1, p \nmid j}^k \frac{1}{j} = \sum_{s=0}^{p^{a-1}-1} \sum_{t=0}^{p-1} \binom{2(ps+t)}{ps+t} \left(\frac{h}{2}\right)^{ps+t} \sum_{j=1, p \nmid j}^{ps+t} \frac{1}{j}, \\ & \sum_{j=1, p \nmid j}^{ps+t} \frac{1}{j} = \sum_{j=1, p \nmid j}^{ps-1} \frac{1}{j} + \sum_{j=ps+1}^{ps+t} \frac{1}{j} = \sum_{j=1, p \nmid j}^{ps-1} \frac{1}{ps-j} + \sum_{j=1}^t \frac{1}{ps+j} \equiv \sum_{j=1}^t \frac{1}{j} \pmod{p}. \end{aligned} \quad (30)$$

With the help of Lucas' theorem (cf. [17], p. 44)), it follows that

$$\begin{aligned} & \sum_{k=0}^{p^a-1} \binom{2k}{k} \left(\frac{h}{2}\right)^k \sum_{j=1, p \nmid j}^k \frac{1}{j} \equiv \sum_{s=0}^{p^{a-1}-1} \sum_{t=1}^{(p-1)/2} \binom{2s}{s} \binom{2t}{t} \left(\frac{h}{2}\right)^{s+t} \sum_{j=1}^t \frac{1}{j} \\ & + \sum_{s=0}^{p^{a-1}-1} \sum_{t=(p+1)/2}^{p-1} \binom{2s+1}{s} \binom{2t-p}{t} \left(\frac{h}{2}\right)^{s+t} \sum_{j=1}^t \frac{1}{j} \\ & \equiv \sum_{s=0}^{p^{a-1}-1} \binom{2s}{s} \left(\frac{h}{2}\right)^s \sum_{t=1}^{(p-1)/2} \binom{2t}{t} \left(\frac{h}{2}\right)^t \sum_{j=1}^t \frac{1}{j} \pmod{p}. \end{aligned} \quad (31)$$

Since $(2/h) - 4 \equiv 0 \pmod{p}$, in view of (16), then

$$\sum_{s=0}^{p^{a-1}-1} \binom{2s}{s} \left(\frac{h}{2}\right)^s \equiv 0 \pmod{p^{a-1}}. \quad (32)$$

The congruence (29) with $a \geq 2$ holds by (31) and (32).

Now suppose that $a = 1$. In fact, for $k = 0, \dots, (p-1)/2$, we clearly have

$$\binom{2k}{k} = \binom{\frac{1}{2}}{k} (-4)^k \equiv \binom{\frac{p-1}{2}}{k} (-4)^k \pmod{p}. \quad (33)$$

Observing that $2h \equiv 1 \pmod{p}$, we obtain

$$\begin{aligned} \sum_{t=1}^{(p-1)/2} \binom{2t}{t} \left(\frac{h}{2}\right)^t \sum_{j=1}^t \frac{1}{j} &\equiv \sum_{t=1}^{(p-1)/2} \binom{\frac{p-1}{2}}{t} (-2h)^t \sum_{j=1}^t \frac{1}{j} \\ &\equiv \sum_{j=1}^{(p-1)/2} \frac{(-1)^j}{j} \sum_{t=j}^{(p-1)/2} \binom{\frac{p-1}{2}}{t} \binom{-1}{t-j} \pmod{p}. \end{aligned} \quad (34)$$

Recall the Chu–Vandermonde identity (see, e.g., [18], p. 169])

$$\sum_{k=0}^n \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n}. \quad (35)$$

Thus,

$$\begin{aligned} \sum_{j=1}^{(p-1)/2} \frac{(-1)^j}{j} \sum_{t=j}^{(p-1)/2} \binom{\frac{p-1}{2}}{t} \binom{-1}{t-j} &= \sum_{j=1}^{(p-1)/2} \frac{(-1)^j}{j} \binom{\frac{p-3}{2}}{j-1} \\ &= \frac{2}{p-1} \sum_{j=1}^{(p-1)/2} \binom{\frac{p-1}{2}}{j} (-1)^j = \frac{-2}{p-1} \equiv 2 \pmod{p}. \end{aligned} \quad (36)$$

Therefore, (29) with $a = 1$ is proved by (31), (34), and (36).

By the above, we have completed the proof of Lemma 6. $\square$

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. We first prove (10). Let $v_p(n) = a$. (10) is evidently trivial when $a = 0$. Next, we suppose that $a \geq 1$. With the help of (13), we have

$$\begin{aligned}
\sum_{k=0}^{n-1} \binom{hn-1}{k} \binom{2k}{k} \left(\frac{h}{2}\right)^k &\equiv \sum_{k=0}^{n-1} \binom{\frac{n}{p}h-1}{\lfloor \frac{k}{p} \rfloor} (-1)^{k-\lfloor k/p \rfloor} \binom{2k}{k} \left(\frac{h}{2}\right)^k \\
&= \sum_{l=0}^{(n/p)-1} \binom{\frac{n}{p}h-1}{l} (-1)^l \sum_{\lfloor k/p \rfloor=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \pmod{p^a}.
\end{aligned} \tag{37}$$

By (20) and (21), for any odd prime p , we get

Repeating the above process $a-1$ times, we obtain that

$$\sum_{\lfloor k/p \rfloor=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \equiv 0 \pmod{p}. \tag{38}$$

$$\begin{aligned}
&\sum_{l=0}^{(n/p)-1} \binom{\frac{n}{p}h-1}{l} (-1)^l \sum_{\lfloor k/p \rfloor=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \\
&\equiv \sum_{m=0}^{(n/p^2)-1} \binom{\frac{n}{p^2}h-1}{m} (-1)^m \sum_{\lfloor l/p \rfloor=m} \sum_{\lfloor k/p \rfloor=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \text{ (by (2.1))} \\
&\equiv \sum_{l=0}^{(n/p^a)-1} \binom{\frac{n}{p^a}h-1}{l} (-1)^l \sum_{\lfloor k/p^a \rfloor=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \equiv 0 \pmod{p^a}.
\end{aligned} \tag{39}$$

Let us turn to (9). We assume that $p \geq 5$. In view of (13), we obtain

$$\begin{aligned}
&\sum_{k=0}^{p^a-1} \binom{hp^a-1}{k} \binom{2k}{k} \left(\frac{h}{2}\right)^k \\
&\equiv \sum_{k=0}^{p^a-1} \binom{hp^{a-1}-1}{\lfloor \frac{k}{p} \rfloor} (-1)^{k-\lfloor k/p \rfloor} \left(1 - hp^a \sum_{j=1, p \nmid j}^k \frac{1}{j}\right) \binom{2k}{k} \left(\frac{h}{2}\right)^k \\
&= \sum_{l=0}^{p^{a-1}-1} \left(\binom{hp^{a-1}-1}{l} (-1)^l - 1 \right) \sum_{\lfloor k/p \rfloor=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \left(1 - hp^a \sum_{j=1, p \nmid j}^k \frac{1}{j}\right) \\
&\quad + \sum_{l=0}^{p^{a-1}-1} \sum_{\lfloor k/p \rfloor=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \left(1 - hp^a \sum_{j=1, p \nmid j}^k \frac{1}{j}\right) \pmod{p^{a+1}}.
\end{aligned} \tag{40}$$

For any positive integer a , we have

$$\binom{hp^{a-1}-1}{l}(-1)^l - 1 \equiv 0 \pmod{p}. \quad (41)$$

Therefore,

$$\begin{aligned} & \sum_{k=0}^{p^a-1} \binom{hp^a-1}{k} \binom{2k}{k} \left(\frac{h}{2}\right)^k \\ & \equiv \sum_{l=0}^{p^{a-1}-1} \binom{hp^{a-1}-1}{l} (-1)^l \sum_{\lfloor k/p \rfloor=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \end{aligned} \quad (42)$$

$$- hp^a \sum_{l=0}^{p^{a-1}-1} \sum_{\lfloor k/p \rfloor=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \sum_{j=1, p \nmid j}^k \frac{1}{j} \pmod{p^{a+1}}.$$

With the help of Lemma 6, for $a \geq 2$, we get

$$\begin{aligned} & \sum_{l=0}^{p^{a-1}-1} \sum_{\lfloor k/p \rfloor=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \sum_{j=1, p \nmid j}^k \frac{1}{j} \\ & = \sum_{k=0}^{p^a-1} \binom{2k}{k} \left(\frac{h}{2}\right)^k \sum_{j=1, p \nmid j}^k \frac{1}{j} \equiv 0 \pmod{p}. \end{aligned} \quad (43)$$

If $a = 1$, by (20), (29), and (42), then

$$\begin{aligned} & \sum_{k=0}^{p^a-1} \binom{hp^a-1}{k} \binom{2k}{k} \left(\frac{h}{2}\right)^k \\ & \equiv \sum_{k=0}^{p-1} \binom{2k}{k} \left(\frac{h}{2}\right)^k - 2hp \equiv p(1-2h) \equiv 0 \pmod{p^2}. \end{aligned} \quad (44)$$

The congruence (9) holds with $a = 1$ and $p \geq 5$. If $a \geq 2$, combining (42) and (43), we have modulo p^{a+1} ,

$$\begin{aligned} & \sum_{k=0}^{p^a-1} \binom{hp^a-1}{k} \binom{2k}{k} \left(\frac{h}{2}\right)^k \equiv \sum_{l=0}^{p^{a-1}-1} \binom{hp^{a-1}-1}{l} (-1)^l \sum_{\lfloor k/p \rfloor=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \\ & \equiv \sum_{l=0}^{p^{a-s}-1} \binom{hp^{a-s}-1}{l} (-1)^l \sum_{\lfloor k/p^s \rfloor=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k. \end{aligned} \quad (45)$$

Repeating this process $s - 1$ times, we have

$$\begin{aligned} & \sum_{l=0}^{p^{a-s}-1} \binom{hp^{a-s}-1}{l} (-1)^l \sum_{\lfloor k/p^s \rfloor=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \\ & \equiv \sum_{m=0}^{p^{a-s-1}-1} \binom{hp^{a-s-1}-1}{m} (-1)^m \sum_{\lfloor l/p \rfloor=m} \sum_{\lfloor k/p^s \rfloor=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \\ & \quad - hp^{a-s} \sum_{m=0}^{p^{a-s-1}-1} \sum_{\lfloor l/p \rfloor=m} \sum_{j=1, p \nmid j}^l \frac{1}{j} \sum_{\lfloor k/p^s \rfloor=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k. \end{aligned} \quad (46)$$

By (20) and (29), we get

$$\begin{aligned}
hp^{a-s} \sum_{l=1}^{p^{a-s}-1} \sum_{j=1, p \nmid j}^l \frac{1}{j} \sum_{[k/p^s]=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k &\equiv hp^a \sum_{l=1}^{p^{a-s}-1} \frac{\binom{2l}{l}}{4^l} \sum_{j=1, p \nmid j}^l \frac{1}{j} \\
&\equiv hp^a \sum_{l=1}^{p^{a-s}-1} \binom{2l}{l} \left(\frac{h}{2}\right)^l \sum_{j=1, p \nmid j}^l \frac{1}{j} \\
&\equiv \begin{cases} 2hp^a \pmod{p^{a+1}}, & \text{if } s = a - 1, \\ 0 \pmod{p^{a+1}}, & \text{if } s < a - 1. \end{cases}
\end{aligned} \tag{47}$$

At last, we only need to think about the case $s = a - 1$. From (45)–(47), we have

$$\begin{aligned}
\sum_{k=0}^{p^a-1} \binom{hp^a-1}{k} \binom{2k}{k} \left(\frac{h}{2}\right)^k &\equiv \sum_{k=0}^{p^a-1} \binom{2k}{k} \left(\frac{h}{2}\right)^k - 2hp^a \\
&\equiv (1-2h)p^a \equiv 0 \pmod{p^{a+1}}.
\end{aligned} \tag{48}$$

So, we obtain (9) with $a \geq 2$ and $p \geq 5$. When $a \geq 2$ and $p = 3$, the proof of (9) is very similar to the case $p \geq 5$, only requiring a few additional discussions. Without loss of generality, it suffices to study the following sum in (46). Note that

$$\begin{aligned}
h3^{a-s} \sum_{m=0}^{3^{a-s-1}-1} \sum_{[k/3^s]=m} \sum_{j=1, 3 \nmid j}^l \frac{1}{j} \sum_{[k/3^s]=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \\
= h3^{a-s} \sum_{m=0}^{3^{a-s-1}-1} \sum_{l=3m}^{3m+2} \sum_{j=1, 3 \nmid j}^l \frac{1}{j} \sum_{[k/3^s]=l} \left\{ \binom{2k}{k} \left(\frac{h}{2}\right)^k \right. \\
\equiv h3^{a-s} \sum_{m=0}^{3^{a-s-1}-1} \left(\sum_{j=1, 3 \nmid j}^{3m+1} \frac{1}{j} \sum_{[k/3^s]=3m+1} \left\{ \binom{2k}{k} \left(\frac{h}{2}\right)^k \right\} + \sum_{j=1, 3 \nmid j}^{3m+2} \frac{1}{j} \sum_{[k/3^s]=3m+2} \left\{ \binom{2k}{k} \left(\frac{h}{2}\right)^k \right\} \right) \pmod{3^{a+1}}, \\
\sum_{j=1, 3 \nmid j}^{3m+1} \frac{1}{j} = \sum_{j=1, 3 \nmid j}^{3m-1} \frac{1}{j} + \frac{1}{3m+1} \equiv 1 \pmod{3}, \\
\sum_{j=1, 3 \nmid j}^{3m+2} \frac{1}{j} \equiv 1 + \frac{1}{2} = \frac{3}{2} \equiv 0 \pmod{3}.
\end{aligned} \tag{49}$$

With the help of (22), we obtain

$$\begin{aligned}
& h3^{a-s} \sum_{m=0}^{3^{a-s-1}-1} \sum_{[l/3]=m}^l \sum_{j=1, 3 \nmid j}^l \frac{1}{j} \sum_{[k/3^s]=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \\
& \equiv h3^{a-s} \sum_{m=0}^{3^{a-s-1}-1} \sum_{[k/3^s]=3m+1} \binom{2k}{k} \left(\frac{h}{2}\right)^k \\
& \equiv h3^{a-s} \sum_{m=0}^{3^{a-s-1}-1} \left(\frac{3^s}{4^{3m+1}} \binom{6m+2}{3m+1} + 2 \cdot 3^{s-1} \frac{1-2h}{h} \binom{6m+2}{3m+1} \right) \pmod{3^{a+1}}.
\end{aligned} \tag{50}$$

In light of Lucas' theorem,

$$\frac{1}{4^{3m+1}} \binom{6m+2}{3m+1} = \frac{2(6m+1)}{4^{3m+1}(3m+1)} \binom{6m}{3m} \equiv 2 \binom{2m}{m} \pmod{3}. \tag{51}$$

Recall that Strauss et al. [19] proved that for any positive integer n ,

$$v_3 \left(\sum_{k=0}^{n-1} \binom{2k}{k} \right) = 2v_3(n) + v_3 \left(\binom{2n}{n} \right). \tag{52}$$

Guo and Zeng [20] deduced a partial q -analogue of (52). Combining (50)–(52), we have modulo $(\text{mod } 3^{a+1})$,

$$\begin{aligned}
& h3^{a-s} \sum_{m=0}^{3^{a-s-1}-1} \sum_{[l/3]=m}^l \sum_{j=1, 3 \nmid j}^l \frac{1}{j} \sum_{[k/3^s]=l} \binom{2k}{k} \left(\frac{h}{2}\right)^k \\
& \equiv \frac{2(2-h)3^a}{3} \sum_{m=0}^{3^{a-s-1}-1} \binom{2m}{m} \equiv \begin{cases} 0 \pmod{3^{a+1}}, & \text{if } s \leq a-2, \\ \frac{2(2-h)3^a}{3} \pmod{3^{a+1}}, & \text{if } s = a-1. \end{cases}
\end{aligned} \tag{53}$$

Since $2h \equiv 1 \equiv -2 \pmod{3}$, then $h \equiv 2 \pmod{3}$. From (45), (46), and (53),

$$\begin{aligned}
& \sum_{k=0}^{3^a-1} \binom{h3^a-1}{k} \binom{2k}{k} \left(\frac{h}{2}\right)^k \equiv \sum_{k=0}^{3^a-1} \binom{2k}{k} \left(\frac{h}{2}\right)^k - \frac{2(2-h)3^a}{3} \\
& \equiv \frac{(2-h)3^a}{3h} - \frac{2(2-h)3^a}{3} \text{ (by (2.5))} \\
& = \frac{(1-2h)(2-h)3^a}{3h} \equiv 0 \pmod{3^{a+1}}.
\end{aligned} \tag{54}$$

Proof of Theorem 1 is now complete. $\square$

Proof of Theorem 2. Let $n = p^{a-1}d$ with $p \nmid d$ and $a \geq 1$. The proof of (11) is very similar to (10). With the help of (13), for any integer $s \in [0, a-1]$ and nonnegative integer k , now we have

$$(-1)^k \binom{p^{a-s}d-1}{k} - (-1)^{[k/p]} \binom{p^{a-s-1}d-1}{\lfloor \frac{k}{p} \rfloor} \in p^{a-s}d\mathbb{Z}_p. \tag{55}$$

Therefore,

$$\begin{aligned}
& \frac{1}{d} \left(\sum_{k=0}^{p^{a-1}d-1} \binom{p^{a-1}d-1}{k} \binom{2k}{k} \frac{1}{(-m)^k} - \left(\frac{\Delta}{p} \right) \sum_{k=0}^{p^{a-1}d-1} \binom{p^{a-1}d-1}{k} \binom{2k}{k} \frac{1}{(-m)^k} \right) \\
& \equiv \frac{1}{d} \sum_{l=0}^{p^{a-1}d-1} \binom{p^{a-1}d-1}{l} (-1)^l \left(\sum_{[k/p]=l} \binom{2k}{k} \frac{1}{m^k} - \left(\frac{\Delta}{p} \right) \frac{\binom{2l}{l}}{m^l} \right) \pmod{p^a}.
\end{aligned} \tag{56}$$

In light of Lemma 5, we have

$$\begin{aligned}
& \frac{1}{d} \sum_{l=0}^{p^{a-1}d-1} \binom{p^{a-1}d-1}{l} (-1)^l \left(\sum_{[k/p]=l} \binom{2k}{k} \frac{1}{m^k} - \left(\frac{\Delta}{p} \right) \frac{\binom{2l}{l}}{m^l} \right) \\
& = \frac{1}{d} \sum_{l=0}^{p^{a-1}d-1} \left(\binom{p^{a-1}d-1}{l} (-1)^l - \binom{p^{a-2}d-1}{\lfloor \frac{l}{p} \rfloor} (-1)^{\lfloor l/p \rfloor} \right) \left(\sum_{[k/p]=l} \binom{2k}{k} \frac{1}{m^k} - \left(\frac{\Delta}{p} \right) \frac{\binom{2l}{l}}{m^l} \right) \\
& \quad + \frac{1}{d} \sum_{l=0}^{p^{a-2}d-1} \binom{p^{a-2}d-1}{l} (-1)^l \left(\sum_{k=p^2l}^{p^2l+p^2-1} \binom{2k}{k} \frac{1}{m^k} - \left(\frac{\Delta}{p} \right) \sum_{k=pl}^{pl+p-1} \frac{\binom{2l}{l}}{m^l} \right).
\end{aligned} \tag{57}$$

By Lemma 5 and (55), we get

$$\begin{aligned}
& \frac{1}{d} \sum_{l=0}^{p^{a-1}d-1} \binom{p^{a-1}d-1}{l} (-1)^l \left(\sum_{[k/p]=l} \binom{2k}{k} \frac{1}{m^k} - \left(\frac{\Delta}{p} \right) \frac{\binom{2l}{l}}{m^l} \right) \\
& \equiv \frac{1}{d} \sum_{l=0}^{p^{a-2}d-1} \binom{p^{a-2}d-1}{l} (-1)^l \left(\sum_{k=p^2l}^{p^2l+p^2-1} \binom{2k}{k} \frac{1}{m^k} - \left(\frac{\Delta}{p} \right) \sum_{k=pl}^{pl+p-1} \frac{\binom{2l}{l}}{m^l} \right) \\
& \equiv \frac{1}{d} \sum_{l=0}^{d-1} \binom{d-1}{l} (-1)^l \left(\sum_{k=p^al}^{p^al+p^a-1} \binom{2k}{k} \frac{1}{m^k} - \left(\frac{\Delta}{p} \right) \sum_{k=p^{a-1}l}^{p^{a-1}l+p^{a-1}-1} \frac{\binom{2l}{l}}{m^l} \right) \equiv 0 \pmod{p^a},
\end{aligned} \tag{58}$$

where the last result comes from Lemma 5.

In view of the above, we have completed the Proof of Theorem 2. $\square$

Data Availability

No data were used to support this study.

Disclosure

An earlier version of this article has been presented as preprint on arXiv which can be accessed from the following link: <https://arxiv.org/abs/1911.00005>.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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