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Research Article

Super H-Antimagic Total Covering for Generalized Antiprism and Toroidal Octagonal Map

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Let G be a graph and $H \subseteq G$ be subgraph of G. The graph G is said to be (a,d)-H antimagic total graph if there exists a bijective function $f\colon V(H) \cup E(H) \longrightarrow \{1,2,3,\ldots,|V(H)|+|E(H)|\}$ such that, for all subgraphs isomorphic to H, the total H weights $W(H) = W(H) = \sum_{x\in V(H)} f(x) + \sum_{y\in E(H)} f(y)$ forms an arithmetic sequence $a, a+d, a+2d, \ldots, a+(n-1)d$, where a and d are positive integers and n is the number of subgraphs isomorphic to H. An (a,d)-H antimagic total labeling f is said to be super if the vertex labels are from the set $\{1,2,\ldots,|V(G)|\}$. In this paper, we discuss super (a,d)-C_3-antimagic total labeling for generalized antiprism and a super (a,d)-C_8-antimagic total labeling for toroidal octagonal map.

1. Introduction

All the graphs that we consider in this works are finite, simple, and connected. Let G be a graph with vertex set and edge set denoted by $V(G)$ and $E(G)$, respectively. For the cardinality of vertex set and edge set, we use the notation $|V(G)|$ and $|E(G)|$, respectively. For basic definitions and terminology related to graph theory, the readers can see the book by Gross et al. [1].

A graph labeling is a map f that sends some of the graph elements (vertices or edges or both) to the set of positive integers. If the domain set of f is the set of vertices (edges), then f is called vertex (edge) labeling. If the domain set is $V(G) \cup E(G)$, then f is called total labeling. Let G be a graph and $H_1, H_2, \ldots, H_k$ be subgraphs of G. We say that the graph G has an $H_1, H_2, \ldots, H_k$ covering if each edge of G belongs to at least one of the subgraph H_i, where $1 \leq i \leq k$. If all H_i, $i = 1, 2, \ldots, k$, are isomorphic to a graph H, then such a covering is called H covering of G. Suppose that a graph G admits an H covering. The graph G is called $(a,d)H$ antimagic if there exists a bijective function $f\colon V(H) \cup E(H) \longrightarrow \{1,2,3,\ldots,|V(H)|+|E(H)|\}$ such that, for all subgraphs isomorphic to H, the total H weights,

$$W(H) = W(K) = \sum_{x\in V(K)} f(x) + \sum_{y\in E(K)} f(y), \quad (1)$$

form an arithmetic sequence $a, a+d, a+2d, \ldots, a+(n-1)d$, where a and d are positive integers and n is the number of subgraphs isomorphic to H. An (a,d)-H antimagic total labeling f is said to be super if the vertex labels are from the set $\{1,2,\ldots,|V(G)|\}$. If $d = 0$, then H is called (a,d)-H antimagic.

Kotzig and Rosa [2] and Enomoto et al. [3] introduced the concept of edge-magic and super edge-magic labeling. Gutierrez and Llado [4] first studied the H (super) magic coverings of a graph G. They proved that the cycle C_n and path P_n are P_m super magic for some m. The cycle (super) magic behavior of some classes of connected graphs is studied in Llado et al. [5]. They proved that prisms, windmills, wheels, and books are C_m-magic for some m. Maryati et al. [6] investigated the G-supermagicness of a disjoint union of c copies of a graph G and showed that the disjoint union of any paths is cP_m-supermagic for some c and m. Maryati et al. [7] and Salman et al. [8] proved that certain families of trees are path-supermagic. Ngurah et al. [9] proved that triangles, chains, ladders, wheels, and grids are cycle-supermagic.

Inaya et al. [10] firstly introduced the concept of H-magic decomposition and H-antimagic decomposition. They showed that, for any graceful tree T with n edges, the complete graph K_{2n+1} admits $(a,d)-T$ antimagic decomposition for some a and all even differences $0 \le d \le n+1$. They also proved that if any tree T with n edges admits α labeling, then the complete bipartite graph $K_{n,n}$ admits an $(a,d)-T$ antimagic decomposition for some a and d having same parity as n. The condition on the existence of C_{2k} super magic decomposition of complete n partite graph and its copies were given by Lian [11]. The H-supermagic decomposition of antiprisms is described by Hendy in [12] and the H-supermagic decompositions of the lexicographic product of graphs are discussed by Hendy et al. in [13]. In [14], Hendy et al. examined the existence of super $(a,d)-H$ magic labeling for toroidal grids and toroidal triangulations. Recently, Fenovcikova et al. [15] proved that wheels are cycle antimagic.

In this paper, we discuss the Super (a,d)-C_3-antimagic total labeling for generalized antiprism and a Super (a,d)-C_8-antimagic total labeling for toroidal octagonal map. We proved that the generalized antiprism $\mathbb{A}_r^s$ admits (a,d)-C_3-antimagic total labeling for $d=0,1$ and the toroidal octagonal map O_s^r admits a Super (a,d)-C_8-antimagic total labeling, for $d=1,2,\ldots,7$.

2. Results on Super (a,d)-C_3-Antimagic Total Covering of Generalized Antiprism $\mathbb{A}_r^s$

An r-sided generalized antiprism $\mathbb{A}_r^s$ is defined as a polyhedron which is composed of s parallel copies of some particular r-sided polygon and connected by an alternating band of triangles. Figure 1 represents the labeled graph of generalized antiprism $\mathbb{A}_r^s$. We denote its vertex set and edge set by $V(\mathbb{A}_r^s)$ and $E(\mathbb{A}_r^s)$, respectively. The vertex set and the edge set of the generalized antiprism $\mathbb{A}_r^s$ can be defined as follows:

$$\begin{aligned} V(\mathbb{A}_r^s) &= \left\{x_i^j, \text{for } 0 \le i \le r-1, 0 \le j \le s-1\right\}, \\ E(\mathbb{A}_r^s) &= \left\{x_i^j x_{i+1}^j, \text{for } 0 \le i \le r-1, 0 \le j \le s-1\right\} \\ &\quad \cup \left\{x_i^j x_i^{j+1}, \text{for } 0 \le i \le r-1, 0 \le j \le s-2\right\} \\ &\quad \cup \left\{x_i^j x_{i+1}^{j+1}, \text{for } 0 \le i \le r-1, 0 \le j \le s-2\right\}. \end{aligned} \tag{2}$$

The generalized antiprism $\mathbb{A}_r^s$ admits a C_3 covering. Let z_i^j and f_i^j be the C_3 cycles which cover $\mathbb{A}_r^s$, where $0 \le i \le r-1$ and $0 \le j \le s-2$. The cycles z_i^j and f_i^j can be defined as

$$\begin{aligned} z_i^j &= x_i^j x_{i+1}^j x_{i+1}^{j+1} x_i^j, \quad \text{for } 0 \le i \le r-1, 0 \le j \le s-2, \\ f_i^j &= x_i^j x_{i+1}^{j+1} x_i^{j+1} x_i^j, \quad \text{for } 0 \le i \le r-1, 0 \le j \le s-2. \end{aligned} \tag{3}$$

It is easy to observe that $|V(\mathbb{A}_r^s)| = rs$ and $|E(\mathbb{A}_r^s)| = 3rs - 2r$. We first give an upper bound for d such that $\mathbb{A}_r^s$ admits a super (a,d)-C_3-antimagic covering.

Theorem 1. *Let $r, s \ge 3$ and $\mathbb{A}_r^s$ be generalized antiprism graph. Then, there is no super (a,d)-C_3-antimagic covering with $d \ge 6$.*

Proof. Suppose that $\mathbb{A}_r^s$ has a super (a,d)-C_3-antimagic covering. Let $f\colon V(\mathbb{A}_r^s) \cup E(\mathbb{A}_r^s) \longrightarrow \{1,2,3,\ldots,4rs-2r\}$ be a super (a,d)-C_3-antimagic covering and $\{a_3, a_3+d, a_3+2d, \ldots, a_3+(2rs-2r-1)d\}$ be the set of C_3 weights. The minimum weight on cycle C_3 is at least $12+3rs$ which is the sum of the smallest vertex labels $(1,2,3)$ and sum of smallest edge labels $(rs+1, rs+2, rs+3)$. Thus,

$$a_3 \ge 12 + 3rs. \tag{4}$$

On the contrary, the maximum possible C_3-weight is the sum of three largest possible vertex labels, namely, $rs-2, rs-1, rs$, and three the largest possible edge labels from the set, $\{4rs-2r-2, 4rs-2r-1, 4rs-2r\}$. Hence, we have

$$a_3 + (2rs-2r-1)d \le 15rs - 6r - 6. \tag{5}$$

From (4) and (5), an upper bound for the parameter d can be obtained as

$$\begin{aligned} d &\le \frac{12rs - 16r - 18}{2rs - 2r - 1}, \\ d &\le 6 - \frac{4r+6}{2rs-2r-1}, \\ d &\le 6. \end{aligned} \tag{6}$$

Thus, we have arrived at the desired result. □

Theorem 2. *Let $r, s \ge 3$; then, the generalized antiprism $\mathbb{A}_r^s$ admits a super $(9rs-3r+4, o)$-C_3-antimagic total covering.*

Proof. Let $\phi\colon V(\mathbb{A}_r^s) \cup E(\mathbb{A}_r^s) \longrightarrow \{1,2,3,\ldots,4rs-2r\}$ be a total labeling of generalized antiprism $\mathbb{A}_r^s$ defined as follows:

$$\begin{aligned} \phi\left(x_i^j\right) &= \{jr + 1 + i, \quad \text{for } 0 \le i \le r-1, 0 \le j \le s-1, \\ \phi\left(x_i^j x_{i+1}^j\right) &= \{(2s-j)r - i, \quad \text{for } 0 \le i \le r-1, 0 \le j \le s-1, \\ \phi\left(x_i^j x_i^{j+1}\right) &= \{(3s-2-j)r + r - i, \quad \text{for } 0 \le i \le r-1, 0 \le j \le s-2, \\ \phi\left(x_i^j x_{i+1}^{j+1}\right) &= \begin{cases} (4s-3-j)r + r - i, & \text{for } 0 \le i \le r-2, 0 \le j \le s-2, \\ (4s-3-j)r + 1, & \text{for } i = r-1, 0 \le j \le s-2. \end{cases} \end{aligned} \tag{7}$$

Under the labeling ϕ, the weights of 3- cycles z_i^j are

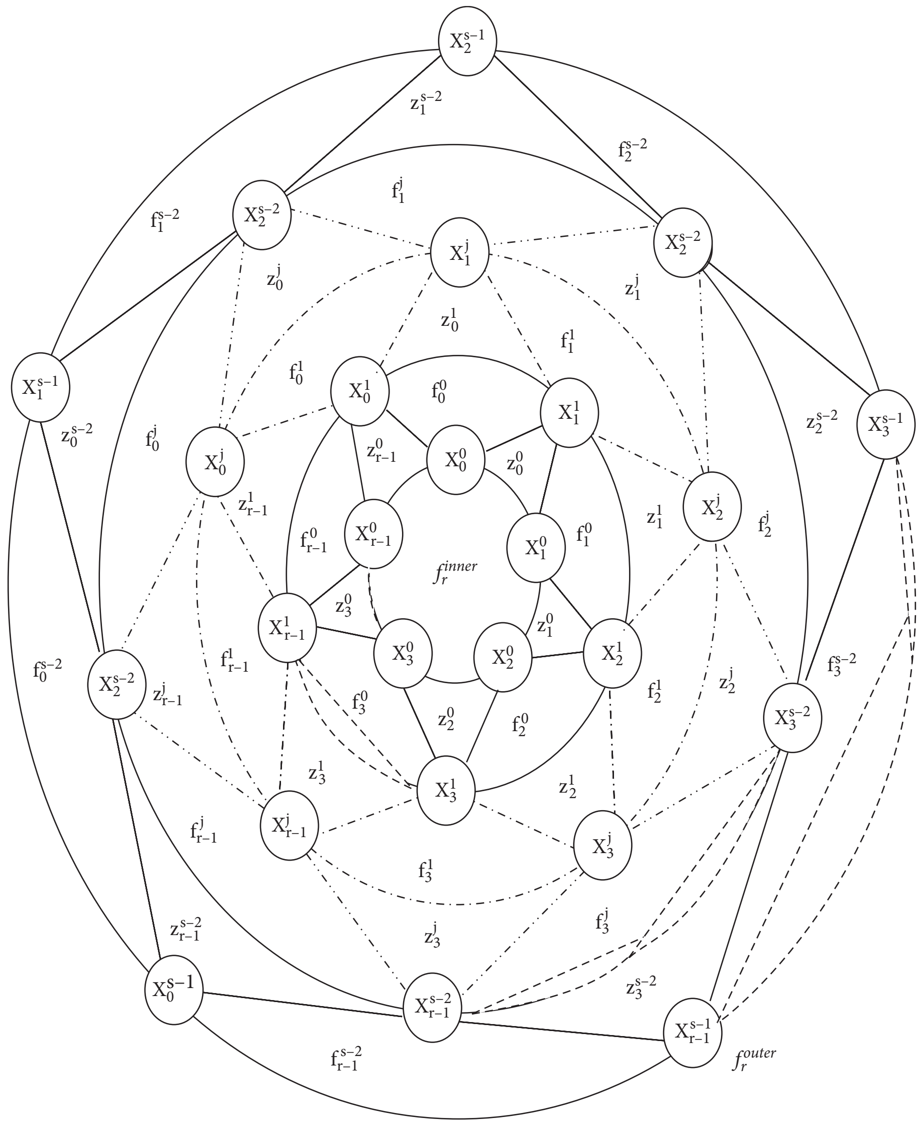

Figure 1: Generalized antiprism $\mathbb{A}_r^s$.

$$W\left(z_i^j\right) = \phi\left(x_i^j\right) + \phi\left(x_{i+1}^j\right) + \phi\left(x_{i+1}^{j+1}\right) + \phi\left(x_i^j x_{i+1}^j\right) + \phi\left(x_{i+1}^j x_{i+1}^{j+1}\right) + \phi\left(x_i^j x_{i+1}^{j+1}\right),$$
$$W\left(z_i^j\right) = \{ 9rs - 3r + 4, \quad \text{for } 0 \le i \le r-1, 0 \le j \le s-2 \tag{8}$$

And, the weights of 3-cycles f_i^j are

$$W\left(f_i^j\right) = \phi\left(x_i^j\right) + \phi\left(x_{i+1}^{j+1}\right) + \phi\left(x_i^{j+1}\right) + \phi\left(x_i^j x_{i+1}^{j+1}\right) + \phi\left(x_{i+1}^{j+1} x_i^{j+1}\right) + \phi\left(x_i^{j+1} x_i^j\right),$$
$$W\left(f_i^j\right) = \{ 9rs - 3r + 4, \quad \text{for } 0 \le i \le r-1, 0 \le j \le s-2. \tag{9}$$

Observe that the weights $W(z_i^j)$ and $W(f_i^j)$ of all cycles z_i^j and f_i^j are equal, and therefore, the resulting labeling is super $(9rs-3r+4,0)$-C_3 total labeling. □

Theorem 3. *Let $r,s\geq 3$; then, the generalized antiprism $\mathbb{A}_r^s$ admits a super $(7rs+4,2)$-antimagic total covering.*

Proof. Let $\chi\colon V(\mathbb{A}_r^s)\cup E(\mathbb{A}_r^s)\longrightarrow\{1,2,3,\ldots,4rs-2r\}$ be a total labeling of generalized antiprism $\mathbb{A}_r^s$ defined as follows.

For $j=$ even, the label on vertices x_i^j is defined as

$$\chi(x_i^j)=\begin{cases}1+i, & \text{for } 0\leq i\leq r-1,\ j=0,\\ (j+1)r, & \text{for } i=0,\ 2\leq j\leq s-1,\\ jr+i, & \text{for } 1\leq i\leq r-1,\ 2\leq j\leq s-1.\end{cases}\quad(10)$$

For $j=$ odd, the label on vertices x_i^j is defined as

$$\chi(x_i^j)=\begin{cases}jr+1, & \text{for } i=0,\ 1\leq j\leq s-1,\\ (j+1)r+1-i, & \text{for } 1\leq i\leq r-1,\ 1\leq j\leq s-1.\end{cases}\quad(11)$$

For $j=$ even, the label on edges $(x_i^jx_{i+1}^j)$ is defined as

$$\chi(x_i^jx_{i+1}^j)=\begin{cases}rs+1+i, & \text{for } 0\leq i\leq r-1,\ j=0,\\ rs+(j+1)r, & \text{for } i=0,\ 2\leq j\leq s-1,\\ rs+jr+i, & \text{for } 1\leq i\leq r-1,\ 2\leq j\leq s-1.\end{cases}\quad(12)$$

For $j=$ odd, the label on edges $(x_i^jx_{i+1}^j)$ is defined as

$$\chi(x_i^jx_{i+1}^j)=\begin{cases}rs+jr+1, & \text{for } i=0,\ 1\leq j\leq s-1,\\ rs+(j+1)r+1-i, & \text{for } 1\leq i\leq r-1,\ 1\leq j\leq s-1.\end{cases}\quad(13)$$

The label on edges $(x_i^jx_i^{j+1})$ is defined as

$$\chi(x_i^jx_i^{j+1})=\begin{cases}(3s-2)r+1+i, & \text{for } 0\leq i\leq r-1,\ j=0,\\ (3s-1-j)r, & \text{for } i=0,\ 1\leq j\leq s-1,\\ (3s-2-j)r+i, & \text{for } 1\leq i\leq r-1,\ 1\leq j\leq s-1.\end{cases}\quad(14)$$

And, the label on edges $(x_i^jx_{i+1}^{j+1})$ is defined as

$$\chi(x_i^jx_{i+1}^{j+1})=3rs+jr-i,\quad \text{for } 0\leq i\leq r-1,\ 0\leq j\leq s-2.\quad(15)$$

Under the labeling χ, the weights of 3-cycle z_i^j are

$$W(z_i^j)=\chi(x_i^j)+\chi(x_{i+1}^j)+\chi(x_{i+1}^{j+1})+\chi(x_i^jx_{i+1}^j)+\chi(x_{i+1}^jx_{i+1}^{j+1})+\chi(x_i^jx_{i+1}^{j+1}).\quad(16)$$

For $j=$ even, we have

$$W(z_i^j)=\begin{cases}7rs+8+2i, & \text{for } 0\leq i\leq r-2,\ j=0,\\ 7rs+4, & \text{for } i=r-1,\ j=0,\\ 7rs+4jr+2r+2, & \text{for } i=0,\ 2\leq j\leq s-2,\\ 7rs+4jr+2+2i, & \text{for } 1\leq i\leq r-1,\ 2\leq j\leq s-2.\end{cases}\quad(17)$$

For $j=$ odd, we have

$$W(z_i^j)=\begin{cases}7rs+4jr+4, & \text{for } i=0,\ 1\leq j\leq s-2,\\ 7rs+4jr+2r+4-2i, & \text{for } 1\leq i\leq r-1,\ 1\leq j\leq s-2.\end{cases}\quad(18)$$

The weight of 3-cycle f_i^j are

$$W(f_i^j)=\chi(x_i^j)+\chi(x_{i+1}^{j+1})+\chi(x_i^{j+1})+\chi(x_i^jx_{i+1}^{j+1})+\chi(x_{i+1}^{j+1}x_i^{j+1})+\chi(x_i^{j+1}x_i^j).\quad(19)$$

For $j=$ even, we have

$$W(f_i^j)=\begin{cases}7rs+2r+4, & \text{for } i=0,\ j=0,\\ 7rs+4r+4-2i, & \text{for } 1\leq i\leq r-1,\ j=0,\\ 7rs+4jr+4r+2-2i, & \text{for } 0\leq i\leq r-1,\ 2\leq j\leq s-2.\end{cases}\quad(20)$$

For $j=$ odd, we have

$$W(f_i^j)=\begin{cases}7rs+4jr+4r+2, & \text{for } i=0,\ 1\leq j\leq s-2,\\ 7rs+4jr+2r+2+2i, & \text{for } 1\leq i\leq r-1,\ 1\leq j\leq s-2.\end{cases}\quad(21)$$

Observe that the weights $W(z_i^j)$ and $W(f_i^j)$ form an arithmetic progression with common difference 2 starting from $7rs+4, 7rs+6$ and ending at $11rs-4r+2$. This implies that the defined labeling is a super $(7rs+4,2)$-C_3-antimagic total covering. □

3. Results on Super (a,d)-C_8-Antimagic Total Covering of Toroidal Octagonal Planner Map O_s^r

A planar octagonal map is a graph obtained by joining octagons and squares in such a way that they cover the plane. To obtain the toroidal octagonal map, we apply torus identification on octagonal planner map. We denote the toroidal octagonal map with r rows and s column of octagons by O_s^r, where $s,r\geq 2$. The planar representation of O_s^r is depicted in Figure 2. The vertex set $V(O_s^r)$ and the edge set $E(O_s^r)$ of octagonal planner map O_s^r can be defined as follows:

$$\begin{aligned}V(O_s^r)&=\{u_i^j,v_i^j,w_i^j,x_i^j;0\leq i\leq r-1 \text{ and } 0\leq j\leq s-1\},\\ E(O_s^r)&=\{u_i^jv_i^j,w_i^jx_i^j;0\leq i\leq r-1 \text{ and } 0\leq j\leq s-1\}\\ &\cup\{w_i^ju_i^{j-1};1\leq i\leq s-1 \text{ and } 0\leq j\leq r-1\}\\ &\cup\{w_i^0u_i^{s-1};0\leq i\leq r-1\}\\ &\cup\{v_i^jw_{i+1}^{j+1};0\leq i\leq r-1 \text{ and } 0\leq j\leq s-2\}\\ &\cup\{v_i^{n-1}w_{i+1}^0;0\leq i\leq r-1\}\\ &\cup\{v_i^jx_{i+1}^j;0\leq i\leq r-1 \text{ and } 0\leq j\leq s-1\}\\ &\cup\{u_i^jx_i^j;0\leq i\leq r-1 \text{ and } 0\leq j\leq s-1\}.\end{aligned}\quad(22)$$

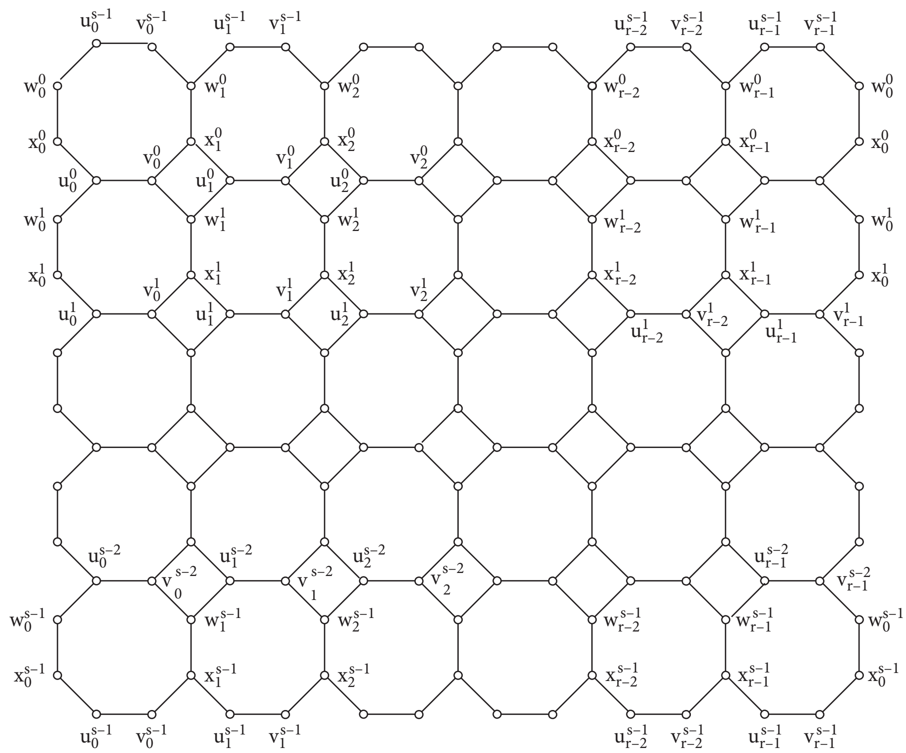

FIGURE 2: Toroidal octagonal map identification O_s^r

From the above sets, we have $|V(O_s^r)| = 4rs$ and $|E(O_s^r)| = 6rs$. We can cover the toroidal octagonal map O_s^r by the 8-sided cycles $C_{8,i}^j$. For $0 \le j \le s-1$ and $0 \le i \le r-1$, the vertex set and edge set of 8-sided cycles $C_{8,i}^j$ can be defined as

$$\begin{aligned}
V\left(C_{8,i}^j\right) &= \left\{w_i^j, u_i^{j-1}, v_i^{j-1}, w_{i+1}^j, x_{i+1}^j, v_i^j, u_i^j, x_i^j; 0 \le i \le r-1, 1 \le j \le s-1\right\},\\
E\left(C_{8,i}^j\right) &= \left\{w_i^j u_i^{j-1}, u_i^{j-1} v_i^{j-1}, v_i^{j-1} w_{i+1}^j, w_{i+1}^j x_{i+1}^j, v_i^j x_{i+1}^j, u_i^j v_i^j, x_i^j u_i^j, x_i^j w_i^j; 0 \le i \le r-1, 1 \le j \le s-1\right\},\\
V\left(C_{8,i}^0\right) &= \left\{w_i^0, u_i^{s-1}, v_i^{s-1}, w_{i+1}^0, x_{i+1}^0, v_i^0, u_i^0, x_i^0; 0 \le i \le r-1,\right\},\\
E\left(C_{8,i}^0\right) &= \left\{w_i^0 u_i^{s-1}, u_i^{s-1} v_i^{s-1}, v_i^{s-1} w_{i+1}^0, w_{i+1}^0 x_{i+1}^0, v_i^0 x_{i+1}^0, u_i^0 v_i^0, x_i^0 u_i^0, x_i^0 w_i^0; 0 \le i \le s-1\right\}.
\end{aligned} \tag{23}$$

We start by giving an upper bound for d such that O_s^r admits a super (a,d)-C_8-antimagic covering.

Theorem 4. *Suppose O_s^r admits a super (a,d)-C_8-antimagic covering; then, $d \le 80$.*

Proof. Suppose O_s^r admits a super (a,d)-C_8-antimagic covering. Then, the weight on cycle C_8 is atleast

$$\sum_{i=1}^{8} i + \sum_{i=1}^{8} (4rs + i) = 32rs + 72, \tag{24}$$

and the largest weight of C_8 is atmost

$$\sum_{i=1}^{8} (4rs + 1 - i) + \sum_{i=1}^{8} (10rs + 1 - i) = 112rs - 56. \tag{25}$$

Thus, we have

$$\begin{aligned}
&a + (rs-1)d \le 112rs - 56,\\
&(rs-1)d \le 112rs - 56 - 32rs - 72,\\
&d \le \frac{80rs - 128}{rs - 1},\\
&d \le 80.
\end{aligned} \tag{26}$$

□

In the next two theorems, we show that toroidal octagonal map O_s^r admits a super (a,d)-C_8-antimagic covering for $d = 1, 2, \ldots 7$.

Theorem 5. *Let $r, s \ge 2$; then, the toroidal octagonal map O_s^r is super (a,d)-C_8-antimagic for $d \in \{1,3,5,7\}$.*

Proof. Define a total labeling $\varphi_d\colon V(O_s^r)\cup E(O_s^r)\longrightarrow\{1,2,3,\ldots,|V(O_s^r)|+|E(O_s^r)|\}$, where $d\in\{1,3,5,7\}$ as follows:

$$\begin{aligned}
\varphi_d\left(u_i^j\right) &= jr+1+i, \quad 0\le i\le r-1, 0\le j\le s-1,\\
\varphi_d\left(v_i^j\right) &= rs+jr+1+i, \quad 0\le i\le r-1, 0\le j\le s-1,\\
\varphi_d\left(x_i^j\right) &= 3rs+(s-1-j)r+r-i, \quad 0\le i\le r-1, 0\le j\le s-1,\\
\varphi_d\left(w_i^j\right) &= 2rs+jr+1+i, \quad 0\le i\le r-1, 0\le j\le s-1,\\
\varphi_d\left(u_i^jv_i^j\right) &= 4rs+(s-1-j)r+r-i, \quad 0\le i\le r-1, 0\le j\le s-1,\\
\varphi\left(x_i^jw_i^j\right) &= 5mn++2jm+1+2i, \quad 0\le i\le r-1, 0\le j\le s-1,\\
\varphi_1\left(v_i^jx_{i+1}^j\right) &= \varphi_3\left(v_i^jx_{i+1}^j\right)=8rs+(s-1-j)r+r-i, \quad 0\le i\le r-1, 0\le j\le s-1,\\
\varphi_5\left(v_i^jx_{i+1}^j\right) &= \varphi_7\left(v_i^jx_{i+1}^j\right)=8rs+jr+1+i, \quad 0\le i\le r-1, 0\le j\le s-1,\\
\varphi_d\left(u_i^jw_i^{j+1}\right) &= 5rs+2(s-1-j)r+2r-2i, \quad 0\le i\le r-1, 0\le j\le s-1,\\
\varphi_1\left(v_i^jw_i^{j+1}\right) &= 7rs+(s-1-j)r+r-i, \quad 0\le i\le r-1, 0\le j\le s-1,\\
\varphi_3\left(v_i^jw_i^{j+1}\right) &= \varphi_5\left(v_i^jw_i^{j+1}\right)=\varphi_7\left(v_i^jw_i^{j+1}\right)=7rs+rj+1+i, \quad 0\le i\le r-1, 0\le j\le s-1,\\
\varphi_1\left(x_i^ju_i^j\right) &= \varphi_3\left(x_i^ju_i^j\right)=\varphi_5\left(x_i^ju_i^j\right)=9rs+(s-1-j)r+r-i, \quad 0\le i\le r-1, 0\le j\le s-1,\\
\varphi_7\left(x_i^ju_i^j\right) &= 9rs+jr+1+i, \quad 0\le i\le r-1, 0\le j\le s-1.
\end{aligned} \tag{27}$$

The total labeling φ_d labels the vertices $u_i^j, v_i^j, w_i^j, x_i^j$ from the set $\{1,2,\ldots,4rs\}$ and the edges from the set $\{4rs+1,4rs+2,\ldots,10rs\}$. For $0\ge i\ge r-1$ and $0\ge j\ge s-1$, the weight of cycles $C_{8,i}^j$ under φ_d is

$$\begin{aligned}
W_d\left(C_{8,i}^j\right) &= \varphi_d\left(u_i^{j-1}\right)+\varphi_d\left(v_i^{j-1}\right)+\varphi_d\left(u_i^{j-1}v_i^{j-1}\right)+\varphi_d\left(w_{i+1}^j\right)+\varphi_d\left(w_i^j\right)+\varphi_d\left(w_{i+1}^jv_i^{j-1}\right)\\
&\quad+\varphi_d\left(x_{i+1}^j\right)+\varphi_d\left(x_{i+1}^jw_{i+1}^j\right)+\varphi_d\left(v_i^j\right)+\varphi_d\left(v_i^jx_{i+1}^j\right)+\varphi_d\left(u_i^j\right)\\
&\quad+\varphi_d\left(u_i^jv_i^j\right)+\varphi_d\left(x_i^j\right)+\varphi_d\left(x_i^ju_i^j\right)+\varphi_d\left(x_i^jw_i^j\right)+\varphi_d\left(w_i^ju_i^{j-1}\right),\\
W_d\left(C_{8,i}^j\right) &= \begin{cases} 68rs+2r+10+jr+i, & \text{for } d=1,\\ 67rs+r+11+3jr+3i, & \text{for } d=3,\\ 66rs+r+12+5jr+5i, & \text{for } d=5,\\ 65rs+13+7jr+7i, & \text{for } d=7.\end{cases}
\end{aligned} \tag{28}$$

For the case $d=1$, we have weights' set $\{68rs+2r+10, 68rs+2r+11,\ldots,69rs+2r+9\}$; similarly, for cases $d=3,5,7$, we get the weights from the sets $\{67rs+r+11, 67rs+2r+12,\ldots,70rs+r+8\}$, $\{66rs+r+12, 66rs+r+17,\ldots,71rs+r+7\}$, and $\{65rs+r+13, 65rs+r+20,\ldots,72rs+r+5\}$, respectively. Hence, the weights of cycles $C_{8,i}^j$ form an arithmetic sequence with difference 1, 3, 5, and 7. □

Theorem 6. *Let $r, s \geq 2$; then, the toroidal map O_s^r is super (a,d)-C_8-antimagic for $d \in \{2,4,6\}$.*

Proof. Let $d \in \{2,4,6\}$ and $0 \leq i \leq r-1, 0 \leq j \leq s-1$. We define a total labeling ϕ_d of O_s^r as follows:

$$
\begin{aligned}
\phi_d\left(u_i^j\right) &= jr + 1 + i, \quad 0 \leq i \leq r-1, 0 \leq j \leq s-1,\\
\phi_d\left(v_i^j\right) &= rs + jr + 1 + i, \quad 0 \leq i \leq r-1, 0 \leq j \leq s-1,\\
\phi_d\left(x_i^j\right) &= 3rs + (s-1-j)r + r - i, \quad 0 \leq i \leq r-1, 0 \leq j \leq s-1,\\
\phi_d\left(w_i^j\right) &= 2rs + jr + 1 + i, \quad 0 \leq i \leq r-1, 0 \leq j \leq s-1,\\
\phi_d\left(u_i^j v_i^j\right) &= 8rs + (s-1-j)r + r - i, \quad 0 \leq i \leq r-1, 0 \leq j \leq s-1,\\
\phi_2\left(x_i^j w_i^j\right) &= \varphi_4\left(x_i^j w_i^j\right) = 9rs + (s-1-j)r + r - i, \quad 0 \leq i \leq r-1, 0 \leq j \leq s-1,\\
\phi_6\left(x_i^j w_i^j\right) &= 9rs + jr + 1 + i, \quad 0 \leq i \leq r-1, 0 \leq j \leq s-1,\\
\phi_2\left(v_i^j x_{i+1}^j\right) &= \varphi_4\left(v_i^j x_{i+1}^j\right) = 6rs + jr + 1 + i, \quad 0 \leq i \leq r-1, 0 \leq j \leq s-1,\\
\phi_6\left(v_i^j x_{i+1}^j\right) &= 6rs + (s-1-j)r + r - i, \quad 0 \leq i \leq r-1, 0 \leq j \leq s-1,\\
\phi_d\left(u_i^j w_i^{j+1}\right) &= 4rs + jr + 1 + i, \quad 0 \leq i \leq r-1, 0 \leq j \leq s-1,\\
\phi_d\left(v_i^j w_i^{j+1}\right) &= 5rs + rj + 1 + i, \quad 0 \leq i \leq r-1, 0 \leq j \leq s-1,\\
\phi_2\left(x_i^j u_i^j\right) &= 7rs + (s-1-j)r + r - i, \quad 0 \leq i \leq r-1, 0 \leq j \leq s-1,\\
\phi_4\left(x_i^j u_i^j\right) &= \phi_6\left(x_i^j u_i^j\right) = 7rs + jr + 1 + i, \quad 0 \leq i \leq r-1, 0 \leq j \leq s-1.
\end{aligned}
\tag{29}
$$

The total labeling ϕ_d labels the vertices $u_i^j, v_i^j, w_i^j, x_i^j$ from the set $\{1,2,\ldots,4rs\}$ and edges from the set $\{4rs+1, 4rs+2, \ldots, 10rs\}$. This show that φ_d is a bijection from set $V(O_s^r) \cup E(O_s^r)$ to set $\{1,2,\ldots,10rs\}$. For $1 \geq i \geq l$ and $i \geq j \geq k$, the weights of $C_{8,i}^j$ under the labeling ϕ_d are

$$
\begin{aligned}
W_d\left(C_{8,i}^j\right) = {} & \phi_d\left(u_i^{j-1}\right) + \phi_d\left(v_i^{j-1}\right) + \phi_d\left(u_i^{j-1} v_i^{j-1}\right) + \phi_d\left(w_{i+1}^j\right) + \phi_d\left(w_i^j\right) + \phi_d\left(w_{i+1}^j v_i^{j-1}\right)\\
& + \phi_d\left(x_{i+1}^j\right) + \phi_d\left(x_{i+1}^j w_{i+1}^j\right) + \phi_d\left(v_i^j\right) + \phi_d\left(v_i^j x_{i+1}^j\right) + \phi_d\left(u_i^j\right)\\
& + \phi_d\left(u_i^j v_i^j\right) + \phi_d\left(x_i^j\right) + \phi_d\left(x_i^j u_i^j\right) + \phi_d\left(x_i^j w_i^j\right) + \phi_d\left(w_i^j u_i^{j-1}\right),\\
W_d\left(C_{8,i}^j\right) = {} & \begin{cases} 75rs - 4r + 8 + 2jr + 2i, & \text{for } d = 2,\\ 74rs - 4r + 9 + 4jr + 4i, & \text{for } d = 4,\\ 73rs - 4r + 12 + 6jr + 6i, & \text{for } d = 6. \end{cases}
\end{aligned}
\tag{30}
$$

For the case $d = 2$, we have weights from the set $\{75rs - 4r + 8, 75rs - 4r + 10, \ldots, 77rs - 4r + 6\}$. Similarly, for cases $d = 4, 6$, we get weights from the sets $\{74rs - 4r + 9, 74rs - 4r + 13, \ldots, 78rs - 4r + 5\}$ and $\{73rs - 4r + 12, 73rs - 4r + 18, \ldots, 79rs - 4r + 6\}$, respectively. This showed that weights of the cycles $C_{8,i}^j$ form an arithmetic sequence with difference 2, 4, and 6. □

4. Conclusion

In the present paper first, we constructed an upper bound for the parameter d for super (a,d)-C_3-antimagic covering. Secondly, we examined the existence of super (a,d)-C_3-antimagic labeling of generalized antiprism $\mathbb{A}_r^s$. We showed that, for $r, s \geq 3$ the generalized antiprism A_r^s had

(a, d)-C_3-antimagic covering for $d \in \{0, 2\}$. Thirdly, we constructed an upper bound for the parameter d for super (a, d)-C_8-antimagic covering. Finally, we examined the existence of super (a, d)-C_8-antimagic labeling of torodial map O_s^r. We showed that, for $m, n \geq 2$, the torodial octagonal map O_s^r had (a, d)-C_8-antimagic covering for $d \in \{1, 2, 3, 4, 5, 6, 7\}$. We conclude the paper with the following open problems.

Open problem 1: find other possible bound for parameter d under (a, d)-C_3-antimagic total covering and the corresponding remaining labeling of d for generalized antiprism $\mathbb{A}_r^s$

Open problem 2: find other possible bound for parameter d under (a, d)-C_8-antimagic total covering and the corresponding remaining labeling of d for torodial octagonal map O_s^r

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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