Research Article

Waring–Goldbach Problem of Even Powers in Short Intervals

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In this paper, we study the average behaviour of the representations of $n = p_1^2 + p_2^4 + p_3^4 + p_4^k$ over short intervals for $k \geq 4$, where p_1, p_2, p_3, p_4 are prime numbers. This improves the previous results.

1. Introduction

Let n and $k_i, i = 1, \dots, r$ be integers; the classical Waring–Goldbach problem considers the representation of n in the form

$$n = p_1^{k_1} + p_2^{k_2} + \dots + p_r^{k_r}. \quad (1)$$

This problem has attracted the attention of many authors [1–11]. For sufficiently large integer N , let $E(k, N)$ be the exceptional set of expressing positive even integers n up to N in the form

$$n = p_1^2 + p_2^4 + p_3^4 + p_4^k. \quad (2)$$

Recently, Feng and Ma [2] established that $E(k, N) \ll N^{1-\theta(k)+\varepsilon}$, here

$$\theta(k) = \begin{cases} \frac{1}{32}, & k = 4, \\ \frac{1}{48}, & k = 5, \\ \frac{1}{64}, & 6 \leq k \leq 8, \\ \frac{1}{48x}, & k \geq 9, \end{cases} \quad (3)$$

where

$$x = \begin{cases} \left\lceil \left(\frac{k}{4} + 1 - \left\lfloor \frac{k}{4} \right\rfloor \right) 2^{\left(\left\lfloor \frac{k}{4} \right\rfloor - 1 \right)} \right\rceil, & 9 \leq k \leq 19, \\ \left\lceil \left(\frac{k}{4} - \frac{1}{2} \left\lfloor \frac{k}{4} \right\rfloor \right) \left(\left\lfloor \frac{k}{4} \right\rfloor + 1 \right) \right\rceil, & k \geq 20. \end{cases} \quad (4)$$

Let

$$r_k(n) = \sum_{n=p_1^2+p_2^4+p_3^4+p_4^k} \log p_1 \log p_2 \log p_3 \log p_4. \quad (5)$$

In our paper, we use the Hardy–Littlewood circle method and some techniques in the works of Languasco and Zaccagnini [3–8] to improve the results of Feng and Ma by studying the average behaviour of $r_k(n)$ over short intervals $[N, N+P]$, $P = o(N)$.

Theorem 1. Let N be a sufficiently large integer and $1 \leq P \leq N, k \geq 4$ be integers. Then, there exists $C = C(\varepsilon) \geq 0$ for every $\varepsilon > 0$, such that

$$\sum_{n=N+1}^{N+P} r_k(n) = \frac{\Gamma(1/2)^2}{4} P N^{1/k} + O_k \left(P N^{1/k} \exp \left(-C \left(\frac{L}{\log L} \right)^{1/3} \right) \right), \quad (6)$$

as $N \rightarrow \infty$, uniformly for $N^{1-5/(6k)+\varepsilon} \leq P \leq N^{1-\varepsilon}$.

Theorem 1 implies that for a sufficiently large integer N , every interval $[N, N + P]$ contains the integers which are sums of one prime square, two prime quartics, and one prime power in short intervals, where $N^{1-5/(6k)+\varepsilon} \leq P \leq N^{1-\varepsilon}$ for $k \geq 4$. This improves Feng and Ma's results obviously. For example, when $k = 4$, the exceptional set $E(4, N) \ll N^{1-5/24+\varepsilon}$ in Theorem 1 is smaller than $N^{1-1/32+\varepsilon}$ in Feng and Ma's results. Comparing the methods in Feng and Ma's proof process, we replace finite sums with infinite series over primes and get a better result.

2. Preliminaries and Lemmas

Let N be a sufficiently large integer, $l \geq 1$ be an integer, $z = 1/N - 2\pi i\alpha$, $\alpha \in [-1/2, 1/2]$, $L = \log N$, $e(\alpha) = e^{2\pi i\alpha}$,

$$S_l(\alpha) = \sum_{n=1}^{\infty} e^{-n^l/N} e(n^l \alpha) \Lambda(n), \quad (7)$$

and

$$V_l(\alpha) = \sum_{p=2}^{\infty} e^{-p^l/N} e(p^l \alpha) \log p. \quad (8)$$

We have

$$|z|^{-1} \ll \min(N, |\alpha|^{-1}). \quad (9)$$

We also set

$$U(\alpha, P) = \sum_{m=1}^P e(m\alpha). \quad (10)$$

From Montgomery ([11], p. 39), we can find that

$$|U(\alpha, P)| \leq \min(P, |\alpha|^{-1}). \quad (11)$$

Now, we need some lemmas as follows.

Lemma 1 (see [7], Lemma 3). Let $\mu > 0$, $z = 1/N - 2\pi i\alpha$. Then, we have

$$\int_{-1/2}^{1/2} z^{-\mu} e(-n\alpha) d\alpha = \frac{1}{\Gamma(\mu)} e^{-n/N} n^{\mu-1} + O_{\mu}\left(\frac{1}{n}\right), \quad (12)$$

uniformly for $n \geq 1$.

Lemma 2 (see [7], Lemma 1). $|S_l(\alpha) - V_l(\alpha)| \ll_l N^{1/(2l)}$.

Lemma 3 (see [7], Lemma 4). Let $\varepsilon > 0$ be an arbitrarily small positive constant. Then, we have

$$\int_{-\xi}^{\xi} \left| S_l(\alpha) - \frac{\Gamma(1/l)}{l z^{1/l}} \right|^2 d\alpha \ll_l N^{2/l-1} \exp\left(-c_1 \left(\frac{L}{\log L}\right)^{1/3}\right), \quad (13)$$

uniformly for $0 \leq \xi \leq N^{-1+5/(6l)-\varepsilon}$. Here, $c_1 = c_1(\varepsilon)$ is a positive constant, which does not depend on l .

Lemma 4. Let $\varepsilon > 0$ be an arbitrarily small positive constant and $\tau > 0$ be a positive constant. Then,

$$\begin{aligned} \int_{-\tau}^{\tau} |S_l(\alpha)|^2 d\alpha &\ll_l (\tau N^{1/l} + N^{2/l-1}) L^3, \\ \int_{-\tau}^{\tau} |S_l(\alpha)|^4 d\alpha &\ll_l (\tau N^{2/l} + N^{4/l-1}) N^{\varepsilon}, \\ \int_{-\tau}^{\tau} |V_l(\alpha)|^4 d\alpha &\ll_l (\tau N^{2/l} + N^{4/l-1}) N^{\varepsilon}. \end{aligned} \quad (14)$$

Proof. This is Lemma 3.4 in [1] and Lemma 6 in [7]. $\square$

Lemma 5. Let $\varepsilon > 0$ be a arbitrarily small positive constant, $c \geq 1$ be a fixed positive constant and $N^{-c} \leq \omega \leq N^{2/l-1}$. Let $I(\omega) := [-1/2, -\omega] \cup [\omega, 1/2]$. Then,

$$\begin{aligned} \int_{I(\omega)} |S_l(\alpha)|^2 \frac{d\alpha}{|\alpha|} &\ll_l \frac{N^{2/l-1+\varepsilon}}{\omega} L^3 + N^{1/l} L^4, \\ \int_{I(\omega)} |S_l(\alpha)|^4 \frac{d\alpha}{|\alpha|} &\ll_l \frac{N^{4/l-1+\varepsilon}}{\omega}, \\ \int_{I(\omega)} |V_l(\alpha)|^4 \frac{d\alpha}{|\alpha|} &\ll_l \frac{N^{4/l-1+\varepsilon}}{\omega}. \end{aligned} \quad (15)$$

Proof. This is Lemma 3.5 in [1] and Lemma 7 in [7]. $\square$

Lemma 6. Let c_2 be a fixed positive constant; we have

$$\int_{-1/2}^{1/2} |S_2(\alpha) S_4(\alpha)|^2 d\alpha \ll N L^{c_2}. \quad (16)$$

Proof. See the proof of Lemma 5 in [7] and Lemma 4.3 in [12] for $k = 4$. $\square$

3. Proof of Theorem 1

Let $\varepsilon > 0$ and $P \geq A N^{1-5/(6k)+\varepsilon}$ for $k \geq 4$, where $A = N^{2\varepsilon}$. From equation (5), we can get

$$\begin{aligned} &\sum_{n=N+1}^{N+P} e^{-n/N} r_k(n) \\ &= \int_{-1/2}^{1/2} V_2(\alpha) V_4(\alpha)^2 V_k(\alpha) U(-\alpha, P) e(-N\alpha) d\alpha. \end{aligned} \quad (17)$$

We find it also convenient to set

$$E_l(\alpha) = S_l(\alpha) - \frac{\Gamma(1/l)}{l z^{1/l}}. \quad (18)$$

Let $I(A, P) = [-1/2, -A/P] \cup [A/P, 1/2]$; we can get

$$\begin{aligned}
& \sum_{n=N+1}^{N+P} e^{-n/N} r_k(n) \\
&= \int_{-A/P}^{A/P} \frac{\Gamma(1/k)\Gamma(1/2)^2}{4kz^{1+1/k}} U(-\alpha, P) e(-N\alpha) d\alpha \\
&+ \int_{-A/P}^{A/P} \frac{\Gamma(1/k)}{kz^{1/k}} \left(S_2(\alpha) S_4(\alpha)^2 - \left(\frac{\Gamma(1/2)}{2z^{1/2}} \right)^2 \right) U(-\alpha, P) e(-N\alpha) d\alpha \\
&+ \int_{-A/P}^{A/P} E_k(\alpha) S_2(\alpha) S_4(\alpha)^2 U(-\alpha, P) e(-N\alpha) d\alpha \\
&+ \int_{-1/2}^{1/2} V_k(\alpha) (V_2(\alpha) V_4(\alpha)^2 - S_2(\alpha) S_4(\alpha)^2) U(-\alpha, P) e(-N\alpha) d\alpha \\
&+ \int_{-1/2}^{1/2} S_2(\alpha) S_4(\alpha)^2 (V_k(\alpha) - S_k(\alpha)) U(-\alpha, P) e(-N\alpha) d\alpha \\
&+ \int_{I(A,P)} S_k(\alpha) S_2(\alpha) S_4(\alpha)^2 U(-\alpha, P) e(-N\alpha) d\alpha \\
&:= \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 + \mathcal{F}_4 + \mathcal{F}_5 + \mathcal{F}_6.
\end{aligned} \tag{19}$$

Now, we begin to estimate these terms. First, we estimate $\mathcal{F}_1$. By the approximation $e^{-n/N} = e^{-1} + O(PN^{-1})$, Lemma 1, and equations (9) and (11), we can get

$$\begin{aligned}
\mathcal{F}_1 &= \frac{\Gamma(1/k)\Gamma(1/2)^2}{4k\Gamma(1+1/k)} \sum_{n=N+1}^{N+P} n^{1/k} e^{-n/N} + O_k\left(\frac{P}{N}\right) \\
&+ O_k\left(\int_{A/P}^{1/2} \frac{d\alpha}{\alpha^{2+1/k}}\right) \\
&= \frac{\Gamma(1/2)^2}{4e} PN^{1/k} + O_k\left(P^2 N^{1/k-1} + \left(\frac{P}{A}\right)^{1+(1/k)} + N^{(1/k)}\right).
\end{aligned} \tag{20}$$

Next, we estimate $\mathcal{F}_2$.

$$\begin{aligned}
& \int_{-A/P}^{A/P} \frac{\Gamma(1/k)}{kz^{1/k}} \left(S_2(\alpha) S_4(\alpha)^2 - \left(\frac{\Gamma(1/2)}{2z^{1/2}} \right)^2 \right) U(-\alpha, P) e(-N\alpha) d\alpha \\
&= \int_{-A/P}^{A/P} \frac{\Gamma(1/k)}{kz^{1/k}} \left(S_2(\alpha)^2 - \frac{\Gamma(1/2)^2}{4z} \right) U(-\alpha, P) e(-N\alpha) d\alpha \\
&+ \int_{-A/P}^{A/P} \frac{\Gamma(1/k)}{kz^{1/k}} (S_2(\alpha) S_4(\alpha)^2 - S_2(\alpha)^2) U(-\alpha, P) e(-N\alpha) d\alpha \\
&= E_1 + E_2.
\end{aligned} \tag{21}$$

From $S_2(\alpha) \ll N^{1/2}$ and $f^2 - g^2 = 2f(f-g) - (f-g)^2$, we can get

$$\begin{aligned}
E_1 &\ll_k N^{1/2} \int_{-A/P}^{A/P} |E_2(\alpha)| \frac{|U(-\alpha, P)|}{|z|^{1/k}} d\alpha \\
&+ \int_{-A/P}^{A/P} |E_2(\alpha)^2| \frac{|U(-\alpha, P)|}{|z|^{1/k}} d\alpha \\
&:= E_{11} + E_{12}.
\end{aligned} \tag{22}$$

By Lemma 3 and equation (11), for every $\varepsilon > 0$, then there exists $c_1 = c_1(\varepsilon) > 0$ such that

$$\begin{aligned}
E_{12} &\ll_k PN^{1/k} \left(\int_{-A/P}^{A/P} |E_2(\alpha)|^2 d\alpha \right) \\
&\ll_k PN^{1/k} \exp\left(-c_1 \left(\frac{L}{\log L}\right)^{1/3}\right),
\end{aligned} \tag{23}$$

provided that $P \geq AN^{1-5/(6k)+\varepsilon}$ for $k \geq 4$. By Lemma 3, equation (11), and the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}
\mathcal{E}_{11} &\ll_k N^{1/2} \left(\int_{-A/P}^{A/P} |E_2(\alpha)|^2 d\alpha \right)^{1/2} P \left(\int_{-A/P}^{A/P} \frac{d\alpha}{|z|^{2/k}} \right)^{1/2} \\
&\ll_k PN^{1/2} \exp\left(-\frac{c_1}{2} \left(\frac{L}{\log L}\right)^{1/3}\right) \\
&\cdot \left(\int_{-1/N}^{1/N} N^{2/k} d\alpha + \int_{1/N}^{A/P} \frac{d\alpha}{|\alpha|^{2/k}} \right)^{1/2} \\
&\ll_k PN^{1/k} \exp\left(-\frac{c_1}{2} \left(\frac{L}{\log L}\right)^{1/3}\right),
\end{aligned} \tag{24}$$

provided that $P \geq AN^{1-5/(6k)+\varepsilon}$ for $k \geq 4$. Hence, by equations (23) and (24), for every $\varepsilon > 0$, then there exists $c_1 = c_1(\varepsilon) > 0$ such that

$$E_1 \ll_k PN^{1/k} \exp\left(-\frac{c_1}{2}\left(\frac{L}{\log L}\right)^{1/3}\right), \quad (25)$$

provided that $P \geq AN^{1-5/(6k)+\varepsilon}$ for $k \geq 4$.

$$\begin{aligned} E_2 &= \int_{-A/P}^{A/P} \frac{\Gamma(1/k)}{kz^{1/k}} (S_2(\alpha)S_4(\alpha)^2 - S_2(\alpha)^2) \\ &\quad \cdot U(-\alpha, P)e(-N\alpha)d\alpha \\ &= E_{21} + E_{22}. \end{aligned} \quad (26)$$

By Lemma 4, equation (11), and $S_k(\alpha) \ll_k N^{1/k}$, we have

$$\begin{aligned} E_{21} &\ll_k N^{1/k+1/2} \left(\int_{-A/P}^{A/P} |S_4(\alpha)|^2 |U(-\alpha, P)| d\alpha \right) \\ &\ll_k N^{1/k+1/2} \left(\int_{-A/P}^{A/P} |S_4(\alpha)|^4 d\alpha \right)^{1/2} \\ &\quad \cdot \left(\int_{-1/P}^{1/P} P^2 d\alpha + \int_{1/P}^{A/P} |\alpha|^{-2} d\alpha \right)^{1/2} \\ &\ll_k N^{1/k+1/2} \left(\frac{A}{P} N^{1/2+\varepsilon} \right)^{1/2} \left(P + \frac{P}{A} \right)^{1/2} \\ &\ll_k A^{1/2} N^{1/k+3/4+\varepsilon}, \end{aligned} \quad (27)$$

and

$$\begin{aligned} E_{22} &\ll_k N^{1/k} \int_{-A/P}^{A/P} |S_2(\alpha)|^2 |U(-\alpha, P)| d\alpha \\ &\ll_k PN^{1/k} \left(\int_{-A/P}^{A/P} |S_2(\alpha)|^4 d\alpha \right)^{1/2} \left(\int_{-A/P}^{A/P} d\alpha \right)^{1/2} \\ &\ll_k PN^{1/k} \left(\frac{A}{P} N^{1+\varepsilon} \right)^{1/2} \left(\frac{A}{P} \right)^{1/2} \\ &\ll_k AN^{1/k+1/2+\varepsilon}. \end{aligned} \quad (28)$$

Hence, by equations (27) and (28), we have

$$\begin{aligned} E_2 &\ll_k AN^{1/k+1/2+\varepsilon} + A^{1/2} N^{1/k+3/4+\varepsilon} \\ &\ll_k PN^{1/k} \exp\left(-\frac{c_1}{2}\left(\frac{L}{\log L}\right)^{1/3}\right), \end{aligned} \quad (29)$$

provided that $P \geq AN^{1-5/(6k)+\varepsilon}$ for $k \geq 4$. Thus, from equations (25) and (29), for every $\varepsilon > 0$, there exists $c_1 = c_1(\varepsilon) > 0$ such that

$$\mathcal{F}_2 \ll_k PN^{1/k} \exp\left(-\frac{c_1}{2}\left(\frac{L}{\log L}\right)^{1/3}\right), \quad (30)$$

provided that $P \geq AN^{1-5/(6k)+\varepsilon}$ for $k \geq 4$

Now, we estimate $\mathcal{F}_3$. By equations (11) and (19), Lemmas 3 and 6, and the Cauchy-Schwarz inequality, we obtain that for every $\varepsilon > 0$, there exists $c_1 = c_1(\varepsilon) > 0$ such that

$$\begin{aligned} \mathcal{F}_3 &\ll_k P \left(\int_{-1/2}^{1/2} |S_2(\alpha)S_4(\alpha)|^2 d\alpha \right)^{1/2} \left(\int_{-A/P}^{A/P} |E_k(\alpha)|^2 d\alpha \right)^{1/2} \\ &\ll_k PN^{1/2} L^{c_2/2} \left(\int_{-A/P}^{A/P} |E_k(\alpha)|^2 d\alpha \right)^{1/2} \\ &\ll_k PN^{1/k} \exp\left(-\frac{c_1}{2}\left(\frac{L}{\log L}\right)^{1/3}\right), \end{aligned} \quad (31)$$

provided that $P \geq AN^{1-5/(6k)+\varepsilon}$ for $k \geq 4$.

Next, we estimate $\mathcal{F}_4$.

$$\begin{aligned} \mathcal{F}_4 &= \int_{-1/2}^{1/2} V_k(\alpha) (V_2(\alpha)V_4(\alpha)^2 - S_2(\alpha)S_4(\alpha)^2) U(-\alpha, P)e(-N\alpha) d\alpha \\ &= \int_{-1/2}^{1/2} V_k(\alpha)V_4(\alpha)^2 (V_2(\alpha) - S_2(\alpha)) U(-\alpha, P)e(-N\alpha) d\alpha \\ &\quad + \int_{-1/2}^{1/2} V_k(\alpha)S_2(\alpha) (V_4(\alpha)^2 - S_4(\alpha)^2) U(-\alpha, P)e(-N\alpha) d\alpha \\ &:= \mathcal{U}_1 + \mathcal{U}_2. \end{aligned} \quad (32)$$

By Lemma 2 and $V_k(\alpha) \ll_k N^{1/k}$, we obtain that

$$\begin{aligned}
\mathcal{U}_2 &= \int_{-1/2}^{1/2} V_k(\alpha)(V_4(\alpha) - S_4(\alpha))S_2(\alpha)(V_4(\alpha) + S_4(\alpha))U(-\alpha, P)e(-N\alpha) d\alpha \\
&\ll_k N^{1/k+1/8} \int_{-1/2}^{1/2} (S_2(\alpha)V_4(\alpha) + S_2(\alpha)S_4(\alpha))U(-\alpha, P)e(-N\alpha) d\alpha \\
&:= N^{1/k+1/8}(\mathcal{J}_1 + \mathcal{J}_2).
\end{aligned} \tag{33}$$

By Lemmas 4 and 5, equation (11), and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\mathcal{J}_1 &= \int_{-1/2}^{1/2} S_2(\alpha)V_4(\alpha)U(-\alpha, P)e(-N\alpha) d\alpha \\
&\ll P \int_{-1/P}^{1/P} |S_2(\alpha)| |V_4(\alpha)| d\alpha + \int_{I(1,P)} \frac{|S_2(\alpha)| |V_4(\alpha)|}{|\alpha|} d\alpha \\
&\ll P \left(\int_{-1/P}^{1/P} |S_2(\alpha)|^2 d\alpha \right)^{1/2} \left(\int_{-1/P}^{1/P} |V_4(\alpha)|^2 d\alpha \right)^{1/2} \\
&\quad + \left(\int_{I(1,P)} \frac{|S_2(\alpha)|^2}{|\alpha|} d\alpha \right)^{1/2} \left(\int_{I(1,P)} \frac{|V_4(\alpha)|^2}{|\alpha|} d\alpha \right)^{1/2} \\
&\ll P \left(\frac{1}{P} N^{1/2} + 1 \right)^{1/2} L^{3/2} \left(\int_{-1/P}^{1/P} |V_4(\alpha)|^4 d\alpha \right)^{1/4} \left(\int_{-1/P}^{1/P} d\alpha \right)^{1/4} \\
&\quad + (N^{1/2} L^4 + PL^3)^{1/2} \left(\int_{I(1,P)} \frac{|V_4(\alpha)|^4}{|\alpha|} d\alpha \right)^{1/4} \left(\int_{I(1,P)} \frac{1}{|\alpha|} d\alpha \right)^{1/4} \\
&\ll P^{3/4} N^{\varepsilon/4} L^{3/2} + N^{1/4+\varepsilon/4} P^{1/4} L^{9/4} \\
&\ll P^{3/4} N^{\varepsilon/4} L^{3/2},
\end{aligned} \tag{34}$$

provided that $P \geq AN^{1-5/(6k)+\varepsilon}$ for $k \geq 4$.

Similarly, we also have

$$\begin{aligned}
\mathcal{J}_2 &= \int_{-1/2}^{1/2} S_2(\alpha)S_4(\alpha)U(-\alpha, P)e(-N\alpha) d\alpha \\
&\ll P \int_{-1/P}^{1/P} |S_2(\alpha)| |S_4(\alpha)| d\alpha + \int_{I(1,P)} \frac{|S_2(\alpha)| |S_4(\alpha)|}{|\alpha|} d\alpha \\
&\ll P \left(\int_{-1/P}^{1/P} |S_2(\alpha)|^2 d\alpha \right)^{1/2} \left(\int_{-1/P}^{1/P} |S_4(\alpha)|^2 d\alpha \right)^{1/2} \\
&\quad + \left(\int_{I(1,P)} \frac{|S_2(\alpha)|^2}{|\alpha|} d\alpha \right)^{1/2} \left(\int_{I(1,P)} \frac{|S_4(\alpha)|^2}{|\alpha|} d\alpha \right)^{1/2} \\
&\ll P^{3/4} N^{\varepsilon/4} L^{3/2},
\end{aligned} \tag{35}$$

provided that $P \geq AN^{1-5/(6k)+\varepsilon}$ for $k \geq 4$.

Hence, by equations (34) and (35), we have

$$\mathcal{U}_2 \ll_k P^{3/4} N^{1/k+1/8+\varepsilon}, \tag{36}$$

provided that $P \geq AN^{1-5/(6k)+\varepsilon}$ for $k \geq 4$.

By Lemma 2, equation (11), and $V_k(\alpha) \ll_k N^{1/k}$, we have

$$\begin{aligned}
\mathcal{U}_1 &= \int_{-1/2}^{1/2} V_k(\alpha)V_4(\alpha)^2(V_2(\alpha) - S_2(\alpha))U(-\alpha, P)e(-N\alpha) d\alpha \\
&\ll_k N^{1/k+1/4} \left(\int_{-1/2}^{1/2} |V_4(\alpha)|^2 |U(-\alpha, P)| d\alpha \right) \\
&\ll_k N^{1/k+1/4} \left(P \int_{-1/P}^{1/P} |V_4(\alpha)|^2 d\alpha + \int_{I(1,P)} \frac{|V_4(\alpha)|^2}{|\alpha|} d\alpha \right) \\
&:= N^{1/k+1/4}(\mathcal{M}_1 + \mathcal{M}_2).
\end{aligned} \tag{37}$$

By Lemmas 4 and 5 and the Cauchy–Schwarz inequality, we have

$$\mathcal{M}_1 \ll P \left(\int_{-1/P}^{1/P} |V_4(\alpha)|^4 d\alpha \right)^{1/2} \left(\int_{-1/P}^{1/P} d\alpha \right)^{1/2} \ll P^{1/2} N^{\varepsilon/2}, \quad (38)$$

and

$$\mathcal{M}_2 \ll \left(\int_{I(1,P)} \frac{|V_k(\alpha)|^4}{|\alpha|} d\alpha \right)^{1/2} \left(\int_{I(1,P)} \frac{1}{|\alpha|} d\alpha \right)^{1/2} \ll P^{1/2} N^{\varepsilon}, \quad (39)$$

provided that $P \geq AN^{1-5/(6k)+\varepsilon}$ for $k \geq 4$. Then, we have

$$\mathcal{U}_1 \ll_k P^{1/2} N^{1/k+1/4+\varepsilon}, \quad (40)$$

provided that $P \geq AN^{1-5/(6k)+\varepsilon}$ for $k \geq 4$.

Hence, by equations (36) and (40), we have

$$\begin{aligned} \mathcal{F}_4 &\ll_k P^{1/2} N^{1/k+1/4+\varepsilon} + P^{3/4} N^{1/k+1/8+\varepsilon} \\ &\ll_k P N^{1/k} \exp\left(-\frac{c_1}{2} \left(\frac{L}{\log L}\right)^{1/3}\right), \end{aligned} \quad (41)$$

provided that $P \geq AN^{1-5/(6k)+\varepsilon}$ for $k \geq 4$.

Now, we estimate $\mathcal{F}_5$. By Lemmas 4 and 5, equation (11), and a partial integration, we have

$$\begin{aligned} \mathcal{F}_5 &\ll_k P N^{1/2k+1/2} \left(\int_{-1/P}^{1/P} |S_4(\alpha)|^2 d\alpha \right) \\ &\quad + N^{1/2k+1/2} \left(\int_{I(1,P)} \frac{|S_4(\alpha)|^2}{|\alpha|} d\alpha \right) \\ &\ll_k P N^{1/2k+1/2} \left(\frac{N^{1/4}}{P} + N^{-1/2} \right) L^3 + (P N^{\varepsilon}) N^{1/2k+1/2} \\ &\ll_k P N^{1/k} \exp\left(-\frac{c_1}{2} \left(\frac{L}{\log L}\right)^{1/3}\right), \end{aligned} \quad (42)$$

provided that $P \geq AN^{1-5/(6k)+\varepsilon}$ for $k \geq 4$.

Finally, we estimate $\mathcal{F}_6$. Clearly, by Lemma 5, equation (11), and Cauchy-Schwarz inequality, we have

$$\begin{aligned} \mathcal{F}_6 &\ll_k N^{1/k} \left(\int_{I(A,P)} \frac{|S_2(\alpha)|^2}{|\alpha|} d\alpha \right)^{1/2} \left(\int_{I(A,P)} \frac{|S_4(\alpha)|^4}{|\alpha|} d\alpha \right)^{1/2} \\ &\ll_k N^{1/k} \left(N^{1/2} L^4 + \frac{P}{A} L^3 \right)^{1/2} \left(N^{\varepsilon} \frac{P}{A} \right)^{1/2} \\ &\ll_k N^{1/k} \left(N^{1/2+\varepsilon} L^4 + \left(\frac{P}{A} \right)^2 N^{\varepsilon} L^3 \right)^{1/2} \\ &\ll_k \frac{P}{A} N^{1/k+\varepsilon/2} L^{2/3} \\ &\ll_k P N^{1/k} \exp\left(-\frac{c_1}{2} \left(\frac{L}{\log L}\right)^{1/3}\right), \end{aligned} \quad (43)$$

provided that $P \geq AN^{1-5/(6k)+\varepsilon}$ for $k \geq 4$.

Completion of the proof. Let $k \geq 4$. By equations (19)–(37) and (40)–(43), we obtain that, for every $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$, such that

$$\begin{aligned} \sum_{n=N+1}^{N+P} e^{-n/N} r_k(n) &= \frac{\Gamma(1/4)^2}{4e} P N^{1/k} \\ &\quad + O_k \left(P N^{1/k} \exp \left(-C \left(\frac{L}{\log L} \right)^{1/3} \right) \right), \end{aligned} \quad (44)$$

provided that $P \geq AN^{1-5/(6k)+\varepsilon}$ with $A = N^{2\varepsilon}$. From $e^{-n/N} = e^{-1} + O(PN^{-1})$, for $n \in [N+1, N+P]$, $1 \leq P \leq N$ and every $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$, such that

$$\begin{aligned} e^{-1} \sum_{n=N+1}^{N+P} r_k(n) &= \frac{\Gamma(1/2)^2}{4e} P N^{1/k} \\ &\quad + O_k \left(P N^{1/k} \exp \left(-C \left(\frac{L}{\log L} \right)^{1/3} \right) \right) \\ &\quad + O_k \left(\frac{P}{N} \sum_{n=N+1}^{N+P} r_k(n) \right), \end{aligned} \quad (45)$$

provided that $N^{1-5/6k+\varepsilon} \leq P \leq N^{1-\varepsilon}$ for $k \geq 4$. Using $e \leq e^{n/N} \leq e^2$ and equation (44), the last error term is $\ll_k P^2 N^{1/k-1}$. Thus, we have

$$\begin{aligned} \sum_{n=N+1}^{N+P} r_k(n) &= \frac{\Gamma(1/2)^2}{4} P N^{1/k} \\ &\quad + O_k \left(P N^{1/k} \exp \left(-C \left(\frac{L}{\log L} \right)^{1/3} \right) \right), \end{aligned} \quad (46)$$

uniformly for $N^{1-5/6k+\varepsilon} \leq P \leq N^{1-\varepsilon}$. Now, Theorem 1 follows.

Data Availability

The data used to support this study are available upon request from the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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