Research Article

Variable Step Size Adams Methods for BSDEs

Qiang Han [ID]

^{1}*Zhongtai Securities Institute for Financial Studies, Shandong University, Jinan, Shandong 250100, China*

Correspondence should be addressed to Qiang Han; 201411272@mail.sdu.edu.cn

For backward stochastic differential equations (BSDEs), we construct variable step size Adams methods by means of Itô–Taylor expansion, and these schemes are nonlinear multistep schemes. It is deduced that the conditions of local truncation errors with respect to Y and Z reach high order. The coefficients in the numerical methods are inferred and bounded under appropriate conditions. A necessary and sufficient condition is given to judge the stability of our numerical schemes. Moreover, the high-order convergence of the schemes is rigorously proved. The numerical illustrations are provided.

1. Introduction

In 1973, Bismut [1] introduced the linear BSDEs. Until 1990, the well-posedness result of nonlinear BSDEs was rigorously proved by Pardoux and Peng [2–4]. After boomingly developed for three past decades, BSDEs become a vital tool to formulate many problems such as mathematical finance [5], partial differential equations [4], actuarial and financial [6], risk measures [7], and finance [8]. However, the theory of nonlinear BSDEs indicates that a majority of nonlinear BSDEs do not have analytical solutions [9]. Thus, the main purpose of this paper is to design a new numerical scheme to solve the following BSDE:

$$Y_t = \Phi(W_T) + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \qquad (1)$$

where $T > 0$ denotes a fixed terminal time and W is a d-dimensional Brownian motion defined on a filtered complete probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P})$; $\Phi(W_T): \mathbb{R}^d \longrightarrow \mathbb{R}^m$ is a given terminal condition of BSDE, and $f(t, y, z): [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \longrightarrow \mathbb{R}^m$ is the generator function. In addition, they satisfy the following.

Assumption 1. $f(t, Y, Z) \in C_b^k, k \in \mathbb{N}^+$, and $\Phi(W_T) \in C_b^1$ where C_b^k are the set of all k-times continuously differential functions with all partial derivatives bounded.

The papers with respect to numerical solutions of BSDEs are unlikely to list exhaustively because there is a vast literature. Therefore, we recommend milestone papers to readers with respect to time-discretization of BSDEs. The paper [10] was the first work of designing efficient algorithms for BSDEs. After that, a modified and implementable numerical scheme was adopted to calculate BSDEs in [11]. In the meantime, the Malliavin calculus and Monte Carlo methods were utilized by [12] to discretize BSDEs. The empirical regression method was constructed by [13] for BSDEs. The papers [14, 15] presented the θ-scheme to discretize BSDEs. The forward Picard iterations method was designed by [16]. The cubature method was used to solve BSDEs in [14, 17]. In [15], authors proposed the BCOS method based on the Fourier cosine series expansions to approximate the solutions of BSDEs. The stochastic grid binding method [18] was introduced to solve BSDEs. The authors in [19] proposed the branching diffusion method for BSDEs, and the branching techniques do not suffer from the curse of dimensionality. A deep learning method was constructed to solve BSDEs in [20, 21]. This method could also overcome the curse of dimensionality and deal with the numerical solutions of BSDEs via the Euler scheme under the condition of minimizing the global loss function. The papers [22, 23] improved the deep learning method via solving the fixed point problem.

If the Euler scheme (explicit, implicit, or generalized) is utilized to discretize BSDEs, the order of discretization error is $1/2$ (see [11, 13, 18, 24]). The θ-schemes [14, 15] are adopted to discretize BSDEs, and the corresponding rate of convergence is 2. To obtain higher-order schemes, the multistep schemes [16, 25–27] are developed to solve BSDEs.

From the above review, the time-discretization of BSDEs can adopt low-order schemes or high-order schemes. Notice that the Euler schemes, the θ-schemes, and the multistep schemes are constant variable step size. And there are a large number of documents about the constant variable step size schemes. This implies that the theory of implementable numerical methods of BSDEs is booming. The variable step size numerical methods play a vital role in the field of numerical methods of stochastic differential equations (see [28, 29]) while they are not seen in the field of numerical theory of BSDEs. Thus, for this motivation, this paper is to provide novel high-order nonlinear discretization schemes called variable step size Adams scheme (14) by utilizing Itô–Taylor expansion. Note that our high-order nonlinear scheme is always explicit with respect to Z. We provide conditions of local truncation errors with respect to Y and Z reaching high order (see Lemmas 3 and 4). Moreover, a sufficient and necessary condition for the stability of our schemes is derived (see Theorem 1). Finally, we derive the convergence of our schemes (see Theorem 3). To the best of our knowledge, this is the first attempt to come up with a variable step size numerical scheme for BSDEs. Note that the developed schemes can be also applied to solve decoupled forward-backward stochastic differential equations, and the forward stochastic differential equation can be approximated by using an appropriate scheme.

The main contributions are as follows. (i) We derive the variable step size Adams scheme for BSDEs by means of Itô–Taylor expansion. And this scheme is a novel high-order nonlinear time-discretization scheme. (ii) The stability and high-order discretization property of our schemes are rigorously proved. Note that we present a sufficient and necessary condition for the stability of our schemes. (iii) The constant variable one-step size schemes [11, 15, 18, 24] and the constant variable multistep size schemes [16, 26, 27] are the particular cases of our variable step size Adams scheme.

An outline of this paper is as follows. In Section 2, we present two lemmas that can be used in the following sections. The variable step size Adams schemes of BSDEs are demonstrated in Section 3. Section 4 shows the stability and convergence of the variable step size Adams scheme. In Section 5, numerical experiments are carried out to illustrate the theoretical consequences. In the end, Section 6 is devoted to the conclusion of this paper.

2. Preliminaries

For readers' convenience, here we present two lemmas which will be utilized in the sequel part.

Lemma 1 (see [3, 4]). *Assume that functions f and Φ are uniformly Lipschitz with respect to (y, z) and $1/2$-Hölder continuous with respect to t. In addition, assume Φ is of class*

$C_b^{2+\kappa}$ *for some $\kappa \in (0, 1)$. Then, the solution (Y_t, Z_t) of the BSDE in (4) can be represented as*

$$Y_t = u(t, X_t), Z_t = \nabla_x u(t, X_t)), t \in [0, T], \qquad (2)$$

where $u \in C_b^{1,2}([0, T] \times \mathbb{R}^d)$ satisfies the parabolic PDE as follows:

$$\mathscr{L}^{(0)} u(t, x) + f(t, x, u(t, x), \nabla_x u(t, x) \sigma(t, x)) = 0, \qquad (3)$$

with the terminal condition $u(T, x) = \Phi(x)$ where $\mathscr{L}^{(0)} = (\partial/\partial t) + (1/2) \sum_{i,j=1}^{d} (\partial^2/\partial x_i \partial x_j)$.

For readers' convenience, we introduce some symbols before providing the lemma. For a multi-index with finite length α, let $\ell(\alpha)$ be the length of a multi-index of α; $\mathscr{A}^\alpha$ is the set of all functions $v: [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ for which $\mathscr{L}^\alpha v$ is well defined and continuous; $\mathscr{A}_b^\alpha$ denotes the subset of all functions $v \in \mathscr{A}^\alpha$ such that the function $\mathscr{L}^\alpha v$ is bounded; for positive integer n, $\mathscr{A}_b^n$ is the set of functions v such that $v^\alpha \in \mathscr{A}_b^\alpha$ for all $\alpha \in \{\alpha | \ell(\alpha) \leq n\} \smallsetminus \{\varnothing\}$.

Lemma 2 (see Proposition 2.2 in [30]). *Let $n \geq 0$. Then, for a function $v \in \mathscr{A}_b^{n+1}$, $\mathbb{E}_t[v(t + h, X_{t+h}] = v_t + h v_t^{(0)} + h^2/2 v_t^{(0,0)} + \cdots + h^n/n! v_t^{(0)_n} + O(h^{n+1})$, where $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathscr{F}_t]$; $v^{(0)} = \mathscr{L}^{(0)} v, v^{(0,0)} = \mathscr{L}^{(0)} \circ \mathscr{L}^{(0)} v, \ldots, v^{(0)_n} = \underbrace{\mathscr{L}^{(0)} \circ \cdots \circ \mathscr{L}^{(0)}}_{n} v$.*

3. Variable Step Size Adams Methods

In this part, we introduce the variable step size Adams schemes of BSDEs in detail.

Now, we deduce the variable step size Adams schemes of BSDEs with respect to Y. A discretization $\pi = \{t_0, t_1, \ldots, t_N\}$ of the time interval $[0, T]$ is defined with step size $h_i = t_{i+1} - t_i$ and $h = \max_{0 \leq i \leq N-1} h_i$; then, we can restate the BSDE (1) as follows:

$$Y_{t_i} = Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(s, Y_s, Z_s) ds - \int_{t_i}^{t_{i+1}} Z_s dW_s. \qquad (4)$$

Taking conditional expectations on both sides of (4), we get the result as follows:

$$Y_{t_i} = \mathbb{E}_i[Y_{t_{i+1}}] + \int_{t_i}^{t_{i+1}} \mathbb{E}_i[f(s, Y_s, Z_s)] ds, \qquad (5)$$

where $\mathbb{E}_i[\cdot] = \mathbb{E}[\cdot|\mathscr{F}_{t_i}]$. It is straightforward that the integrand $\mathbb{E}_i[f(s, Y_s, Z_s)]$ in the above equation is a deterministic function of time s. Naturally, we can replace the function $\mathbb{E}_i[f(s, Y_s, Z_s)]$ in (5) by using multistep methods through the support points $\{(t_n, \mathbb{E}_i[f_n]) | n = i, i + 1, \ldots, i + k\}$ where k is a given positive integer and satisfies $0 \leq k < N$, namely,

$$\int_{t_i}^{t_{i+1}} \mathbb{E}_i[f(s, Y_s, Z_s)] ds = h_i \sum_{j=0}^{k} \Gamma_{j,i} \mathbb{E}_i[f(t_{i+j}, Y_{i+j}, Z_{i+j})] + R_i^y, \qquad (6)$$

where coefficients $\Gamma_{j,i}$ depend on h_i for $i = N - 1, \ldots, 0$ and will be given soon; $R_i^y = \int_{t_i}^{t_{i+1}}$

$\left\{\mathbb{E}_i[f(s, Y_s, Z_s)] - \sum_{j=0}^k \Gamma_{j,i}\mathbb{E}_i[f(t_{i+j}, Y_{i+j}, Z_{i+j})]\right\}ds$. Inserting (6) into (5), we obtain

$$Y_{t_i} = \mathbb{E}_i\left[Y_{t_{i+1}}\right] + h_i \sum_{j=0}^k \Gamma_{j,i}\mathbb{E}_i\left[f\left(t_{i+j}, Y_{i+j}, Z_{i+j}\right)\right] + R_i^y. \quad (7)$$

Hence, the time-discretization of Y is, for $i = N - k, N - k - 1, \ldots, 0$,

$$Y_i^\pi = \mathbb{E}_i[Y_{i+1}^\pi] + h_i \sum_{j=0}^k \Gamma_{j,i}\mathbb{E}_i\left[f\left(t_{i+j}, Y_{i+j}^\pi, Z_{i+j}^\pi\right)\right]. \quad (8)$$

In what follows, we demonstrate the expression with respect to Z. Multiplying (4) by $\Delta W_{i,n} := W_{t_{i+n}} - W_{t_i}, n \in \mathbb{N}^+$ and then taking conditional expectation on the derived equation, we obtain

$$0 = \mathbb{E}_i\left[Y_{t_{i+1}}\Delta W_{i,1}^\top\right] + \int_{t_i}^{t_{i+1}} \mathbb{E}_i\left[f(s, Y_s, Z_s)\Delta W_{i,s}^\top\right]ds$$
$$- \int_{t_i}^{t_{i+1}} \mathbb{E}_i[Z_s]ds, \quad (9)$$

where $\Delta W_{i,s} = W_s - W_{t_i}$. Analogously, we approximate the two integral terms on right-hand side of (9) by the manner as calculating Y_{t_i}, namely,

$$\int_{t_i}^{t_{i+1}} \mathbb{E}_i\left[f(s, Y_s, Z_s)\Delta W_{i,s}^\top\right]ds$$
$$= h_i \sum_{j=1}^k \Gamma_{j,i}^*\mathbb{E}_i\left[f\left(t_{i+j}, Y_{i+j}, Z_{i+j}\right)\Delta W_{i,j}^\top\right] + R_{1,i}^z, \quad (10)$$

$$\int_{t_i}^{t_{i+1}} \mathbb{E}_i[Z_s]ds = h_i \sum_{j=0}^k \Gamma_{j,i}^{**}\mathbb{E}_i\left[Z_{t_{i+j}}\right] + R_{2,i}^z, \quad (11)$$

where $R_{1,i}^z = \int_{t_i}^{t_{i+1}} \mathbb{E}_i[f(s, Y_s, Z_s)\Delta W_{i,s}^\top]ds - h_i \sum_{j=1}^k \Gamma_{j,i}^*\mathbb{E}_i[f(t_{i+j}, Y_{i+j}, Z_{i+j})\Delta W_{i,j}^\top]$ and $R_{2,i}^z = \int_{t_i}^{t_{i+1}} \mathbb{E}_i[Z_s]ds - h_i \sum_{j=0}^k \Gamma_{j,i}^{**}\mathbb{E}_i[Z_{t_{i+j}}]$. Plugging (10) and (11) into (9), we deduce

$$0 = \mathbb{E}_i\left[Y_{t_{i+1}}\Delta W_{i,1}^\top\right] + h_i \sum_{j=1}^k \Gamma_{j,i}^*\mathbb{E}_i\left[f\left(t_{i+j}, Y_{i+j}, Z_{i+j}\right)\Delta W_{i,j}^\top\right]$$
$$- h_i \sum_{j=0}^k \Gamma_{j,i}^{**}\mathbb{E}_i\left[Z_{t_{i+j}}\right] + R_i^z, \quad (12)$$

where $R_i^z = R_{1,i}^z - R_{2,i}^z$. Hence, the time-discretization of Z is, for $i = N - k, N - k - 1, \ldots, 0$,

$$\Gamma_{0,i}^{**}\mathbb{E}_i[Z_i^\pi] = \mathbb{E}_i\left[Y_{i+1}^\pi \frac{\Delta W_{i,1}^\top}{h_i}\right] + \sum_{j=1}^k \Gamma_{j,i}^*\mathbb{E}_i\left[f\left(t_{i+j}, Y_{i+j}^\pi, Z_{i+j}^\pi\right)\Delta W_{i,j}^\top\right]$$
$$- \sum_{j=1}^k \Gamma_{j,i}^{**}\mathbb{E}_i\left[Z_{i+j}^\pi\right], \quad (13)$$

Thus, from the two equations (8) and (13), we propose the explicit Adams schemes to solve BSDE (1) as follows.

Giving the initial values $\{(Y_{N-l}^\pi, Y_{N-l}^\pi)\}_{l=0}^{k-1}$, we solve random variables (Y_i^π, Z_i^π) for $i = N - k, N - k - 1, \ldots, 0$ from

$$\begin{cases} Y_i^\pi = \mathbb{E}_i[Y_{i+1}^\pi] + h_i \sum_{j=0}^k \Gamma_{j,i}\mathbb{E}_i\left[f_{i+j}^\pi\right], \\[2ex] 0 = \mathbb{E}_i\left[Y_{i+1}^\pi \frac{\Delta W_{i,1}^\top}{h_i}\right] + \sum_{j=1}^k \Gamma_{j,i}^*\mathbb{E}_i\left[f_{i+j}^\pi \Delta W_{i,j}^\top\right] - \sum_{j=0}^k \Gamma_{j,i}^{**}\mathbb{E}_i\left[Z_{i+j}^\pi\right], \end{cases} \quad (14)$$

where $f_i^\pi = f(t_i, Y_i^\pi, Z_i^\pi)$ for $i = N - 1, N - 2, \ldots, 0$; coefficients $\Gamma_{j,i}, \Gamma_{j,i}^*$, and $\Gamma_{j,i}^{**}$ depend on h_i for $i = N - 1, \ldots, 0$.

Remark 1. If $\Gamma_{0,i} = 0$, the scheme with respect to Y is explicit and $\Gamma_{j,i} = \Gamma_{j,i}^*$ for $j = 1, 2, \ldots, k$. If $\Gamma_{0,i} \neq 0$, the scheme with respect to Y is implicit and $\Gamma_{j,i} = \Gamma_{j,i}^{**}$ for $j = 1, 2, \ldots, k$.

4. Theoretical Analysis

Before showing the stability and convergence analysis of the variable step size Adams scheme (14), we first provide a few lemmas.

Lemma 3. *Under Assumption 1, assume that the parameters $\{\Gamma_{j,i}\}_{0 \leq j \leq k}$ satisfy the relation as follows:*

$$\begin{cases} 1 = \sum_{p=0}^k \Gamma_{p,i}, & q = 1, \\[2ex] \frac{1}{q} = \sum_{p=1}^k \frac{\left(\sum_{\ell=0}^{p-1} h_{i+\ell}\right)^{q-1}}{h_i^{q-1}}\Gamma_{p,i}, & q \geq 2. \end{cases} \quad (15)$$

Then,

$$R_i^y = \begin{cases} O(h^k), & \Gamma_{0,i} = 0, \\ O(h^{k+1}), & \Gamma_{0,i} \neq 0. \end{cases} \quad (16)$$

Proof. By Assumption 1, the integrand $\mathbb{E}_t[f(s, Y_s, Z_s)], s > t$, is a continuous function with respect to s (see Theorem 2.2.1 of [31]). Then, by taking derivative with respect to s on

$$\mathbb{E}_t[Y_s] = \mathbb{E}_t[\Phi(W_T)] + \int_s^T \mathbb{E}_t[f(\bar{s}, Y_{\bar{s}}, Z_{\bar{s}})]d\bar{s}, \forall s \in [t, T], \quad (17)$$

we obtain the following reference ordinary differential equation:

$$\frac{d\mathbb{E}_t[Y_s]}{ds} = -\mathbb{E}_t[f(s, Y_s, Z_s)], \quad s \in [t, T]. \quad (18)$$

Thus, from the definition of R_i^y, we have

$$R_i^y = \int_{t_i}^{t_{i+1}} \left\{ \mathbb{E}_i[f(s, Y_s, Z_s)] - \sum_{j=0}^k \Gamma_{j,i} \mathbb{E}_i[f(t_{i+j}, Y_{i+j}, Z_{i+j})] \right\} ds$$

$$= Y_{t_i} - \mathbb{E}_i[Y_{t_{i+1}}] - h_i \sum_{j=0}^k \Gamma_{j,i} \mathbb{E}_i[f(t_{i+j}, Y_{i+j}, Z_{i+j})].$$

$$(19)$$

Substituting (18) into (19) and utilizing Itô–Taylor expansion at (t_i, W_{t_i}), we have

$$R_i^y = \mathbb{E}_i\left[u_{t_i} - u_{t_{i+1}} - h_i \sum_{j=0}^k \Gamma_{j,i} f(t_{i+j}, Y_{i+j}, Z_{i+j}) \right]$$

$$= \mathbb{E}_i\left[u_{t_i} - u_{t_{i+1}} + h_i \sum_{j=1}^k \Gamma_{j,i} u_{t_{i+j}}^{(0)} \right]$$

$$= \mathbb{E}_i\left[h_i u_{t_i}^{(0)}\left(-1 + \Gamma_{1,i} + \Gamma_{2,i} + \cdots + \Gamma_{k,i} \right) \right.$$

$$+ h_i^2 u_{t_i}^{(0,0)}\left(-\frac{1}{2} + \Gamma_{1,i} + \frac{h_i + h_{i+1}}{h_i}\Gamma_{2,i} + \cdots + \frac{\sum_{\ell=0}^{k-1} h_{i+\ell}}{h_i}\Gamma_{k,i} \right)$$

$$+ h_i^3 u_{t_i}^{(0,0,0)}\left(-\frac{1}{6} + \frac{1}{2}\left(\Gamma_{1,i} + \frac{(h_i + h_{i+1})^2}{h_i^2}\Gamma_{2,i} + \cdots + \frac{\left(\sum_{\ell=0}^{k-1} h_{i+\ell}\right)^2}{h_i^2}\Gamma_{k,i} \right) \right)$$

$$+ \cdots$$

$$\left. + h_i^q u_{t_i}^{(0)_q}\left(-\frac{1}{q!} + \frac{1}{(q-1)!}\left(\Gamma_{1,i} + \frac{(h_i + h_{i+1})^{q-1}}{h_i^{q-1}}\Gamma_{2,i} + \cdots + \frac{\left(\sum_{\ell=0}^{k-1} h_{i+\ell}\right)^{q-1}}{h_i^{q-1}}\Gamma_{k,i} \right) \right) + \cdots \right]$$

$$= \mathbb{E}_i\left[h_i u_{t_i}^{(0)}\left(-1 + \sum_{p=0}^k \Gamma_{p,i} \right) + \sum_{q \geq 2} \frac{h^q}{q!} u_{t_i}^{(0)_q}\left(-1 + q \sum_{p=1}^k \left(\frac{\left(\sum_{\ell=0}^{p-1} h_{i+\ell}\right)^{q-1}}{h_i^{q-1}}\Gamma_{p,i} \right) \right) \right],$$

$$(20)$$

where $u_{t_i} = u(t_i, W_{t_i})$. The conclusion is obvious with the help of equation (20). The proof is completed. $\square$

Lemma 4. *Under Assumption 1, assume that the parameters* $\{\Gamma_{j,i}\}_{0 \leq j \leq k}$ *satisfy*

$$\begin{cases} 1 = \sum_{j=0}^k \Gamma_{j,i}^{**}, & 0 \neq \Gamma_{0,i}^{**}, \\ 1 = \sum_{j=1}^k \frac{\left(\sum_{\ell=0}^{j-1} h_{i+\ell}\right)^q}{h_i^q}(q\Gamma_{j,i}^* + \Gamma_{j,i}^{**}), & q \geq 1. \end{cases}$$

$$(21)$$

Then, $R_i^z = O(h^{k+1})$.

Proof. By Assumption 1, the two integrands $\mathbb{E}_t[f(s, Y_s, Z_s)\Delta W_{t,s}^\top]$ and $\mathbb{E}_t[Z_s]$, $s > t$, are continuous function of s (see Theorem 2.2.1 of [31]). Taking derivative with respect to s, we have the ordinary differential equation as follows:

$$\frac{d\mathbb{E}_t[Y_s \Delta W_{t,s}^\top]}{ds} = -\mathbb{E}_t[f(s, X_s, Y_s, Z_s)\Delta W_{t,s}^\top] + \mathbb{E}_t[Z_s],$$

$$s \in [t, T].$$

$$(22)$$

Thus,

$$
\frac{R_i^z}{\Gamma_{0,i}^{**} h_i} = Z_{t_i} - \mathbb{E}_i\left[Y_{t_{i+1}} \frac{\Delta W_{i,1}^{\top}}{\Gamma_{0,i}^{**} h_i} \right] - \sum_{j=1}^{k} \frac{\Gamma_{j,i}^{*}}{\Gamma_{0,i}^{**}} \mathbb{E}_i\left[f\left(t_{i+j}, Y_{i+j}, Z_{i+j}\right) \Delta W_{i,j}^{\top} \right] + \sum_{j=1}^{k} \frac{\Gamma_{j,i}^{**}}{\Gamma_{0,i}^{**}} \mathbb{E}_i\left[Z_{t_{i+j}} \right]
$$

$$
= Z_{t_i} - \mathbb{E}_i\left[Y_{t_{i+1}} \frac{\Delta W_{i,1}^{\top}}{\Gamma_{0,i}^{**} h_i} \right] + \sum_{j=1}^{k} \frac{\Gamma_{j,i}^{*}}{\Gamma_{0,i}^{**}} \mathbb{E}_i\left[\left(Y_{t_{i+j}} \Delta W_{i,j}^{\top} \right)^{(0)} \right] + \sum_{j=1}^{k} \frac{\Gamma_{j,i}^{**} - \Gamma_{j,i}^{*}}{\Gamma_{0,i}^{**}} \mathbb{E}_i\left[Z_{t_{i+j}} \right] \tag{23}
$$

$$
= Z_{t_i} - \frac{1}{\Gamma_{0,i}^{**}} \mathbb{E}_i\left[Z_{t_{i+1}} \right] + \sum_{j=1}^{k} \frac{\Gamma_{j,i}^{*}}{\Gamma_{0,i}^{**}} \mathbb{E}_i\left[\left(t_{i+j} - t_i \right) Z_{t_{i+j}}^{(0)} + \frac{\Gamma_{j,i}^{**}}{\Gamma_{j,i}^{*}} Z_{t_{i+j}} \right],
$$

where the last equality can be verified via relation (2) and integration by parts. Substituting (22) into (23) and utilizing Itô–Taylor expansion at (t_i, W_{t_i}), we have

$$
\frac{R_i^z}{\Gamma_{0,i}^{**} h_i} = \mathbb{E}_i\Bigg[Z_{t_i}\left(1 - \frac{1}{\Gamma_{0,i}^{**}} + \sum_{j=1}^{k} \frac{\Gamma_{j,i}^{**}}{\Gamma_{0,i}^{**}} \right) + h_i Z_{t_i}^{(0)}\left(-\frac{1}{\Gamma_{0,i}^{**}} + \sum_{j=1}^{k} \frac{\sum_{\ell=0}^{j-1} h_{i+\ell}}{\Gamma_{0,i}^{**} h_i}\left(\Gamma_{j,i}^{*} + \Gamma_{j,i}^{**} \right) \right)
$$

$$
+ \frac{h_i^2}{2} Z_{t_i}^{(0,0)}\left(-\frac{1}{\Gamma_{0,i}^{**}} + \sum_{j=1}^{k} \frac{\left(\sum_{\ell=0}^{j-1} h_{i+\ell} \right)^2}{\Gamma_{0,i}^{**} h_i^2}\left(2\Gamma_{j,i}^{*} + \Gamma_{j,i}^{**} \right) \right) + \frac{h_i^3}{6} Z_{t_i}^{(0,0,0)}\left(-\frac{1}{\Gamma_{0,i}^{**}} + \sum_{j=1}^{k} \frac{\left(\sum_{\ell=0}^{j-1} h_{i+\ell} \right)^3}{\Gamma_{0,i}^{**} h_i^3}\left(3\Gamma_{j,i}^{*} + \Gamma_{j,i}^{**} \right) \right)
$$

$$
+ \cdots + \frac{h_i^q}{q!} Z_{t_i}^{(0)_q}\left(-\frac{1}{\Gamma_{0,i}^{**}} + \sum_{j=1}^{k} \frac{\left(\sum_{\ell=0}^{j-1} h_{i+\ell} \right)^q}{\Gamma_{0,i}^{**} h_i^q}\left(q\Gamma_{j,i}^{*} + \Gamma_{j,i}^{**} \right) \right) + \cdots \Bigg] \tag{24}
$$

$$
= \mathbb{E}_i\Bigg[Z_{t_i}\left(1 - \frac{1}{\Gamma_{0,i}^{**}} + \sum_{j=1}^{k} \frac{\Gamma_{j,i}}{\Gamma_{0,i}^{**}} \right) + \sum_{q \ge 1} \frac{h_i^q}{q!} Z_{t_i}^{(0)_q}\left(-\frac{1}{\Gamma_{0,i}^{**}} + \sum_{j=1}^{k} \frac{\left(\sum_{\ell=0}^{j-1} h_{i+\ell} \right)^q}{\Gamma_{0,i}^{**} h_i^q}\left(q\Gamma_{j,i}^{*} + \Gamma_{j,i}^{**} \right) \right) \Bigg].
$$

The conclusion is obvious with the help of equation (24). The proof is completed.

In what follows, we list the numerical expressions with respect to Y and Z by utilizing Lemmas 3 and 4.

(1) If $\Gamma_{0,i} = 0$, the numerical schemes of Y for $k = 1, 2, 3$ with respect to time are

$$
k = 1, \quad Y_i^{\pi} = \mathbb{E}_i\left[Y_{i+1}^{\pi} \right] + h_i \mathbb{E}_i\left[f_{i+1}^{\pi} \right],
$$

$$
k = 2, \quad Y_i^{\pi} = \mathbb{E}_i\left[Y_{i+1}^{\pi} \right] + h_i \mathbb{E}_i\left[f_{i+1}^{\pi} + \frac{h_i}{2 h_{i+1}}\left(f_{i+1}^{\pi} - f_{i+2}^{\pi} \right) \right],
$$

$$
k = 3, \quad Y_i^{\pi} = \mathbb{E}_i\left[Y_{i+1}^{\pi} \right] + h_i \mathbb{E}_i\Bigg[f_{i+1}^{\pi} + \frac{h_i}{2 h_{i+1}}\left(f_{i+1}^{\pi} - f_{i+2}^{\pi} \right) + \frac{2 h_i + 3 h_{i+1}}{6\left(h_i + h_{i+1} \right)}\left(\frac{h_i\left(h_i + h_{i+1} \right)}{\left(h_{i+1} + h_{i+2} \right) h_{i+1}} f_{i+1}^{\pi} \right.
$$

$$
\left. - \frac{h_i\left(h_i + h_{i+1} \right)}{h_{i+1} h_{i+2}} f_{i+2}^{\pi} + \frac{h_i\left(h_i + h_{i+1} \right)}{\left(h_{i+1} + h_{i+2} \right) h_{i+2}} f_{i+3}^{\pi} \right) \Bigg]. \tag{25}
$$

(2) If $\Gamma_{0,i} \neq 0$, the numerical schemes of Y for $k = 0, 1, 2$ with respect to time are

$$k = 0, Y_i^\pi = \mathbb{E}_i[Y_{i+1}^\pi] + h_i f_i^\pi,$$

$$k = 1, Y_i^\pi = \mathbb{E}_i[Y_{i+1}^\pi] + \frac{h_i}{2}\mathbb{E}_i[f_i^\pi + f_{i+1}^\pi],$$

$$k = 2, Y_i^\pi = \mathbb{E}_i[Y_{i+1}^\pi] + h_i\mathbb{E}_i\left[\frac{2h_i + 3h_{i+1}}{6(h_i + h_{i+1})}f_i^\pi + \frac{h_i + 3h_{i+1}}{6h_{i+1}}f_{i+1}^\pi - \frac{h_i^2}{6(h_i + h_{i+1})h_{i+1}}f_{i+2}^\pi\right].$$

$$(26)$$

(3) The numerical expressions with respect to Z are provided for $k = 0, 1, 2$, namely,

$$k = 0, Z_i^\pi = \mathbb{E}_i\left[Y_{i+1}^\pi \frac{\Delta W_{i,1}^\top}{h_i}\right],$$

$$k = 1, \frac{1}{2}Z_i^\pi = \mathbb{E}_i\left[Y_{i+1}^\pi \frac{\Delta W_{i,1}^\top}{h_i}\right] + \mathbb{E}_i[f_{i+1}^\pi \Delta W_{i,1}^\top] - \frac{1}{2}\mathbb{E}_i[Z_{i+1}^\pi],$$

$$k = 2, \frac{2h_i + 3h_{i+1}}{6(h_i + h_{i+1})}Z_i^\pi = \mathbb{E}_i\left[Y_{i+1}^\pi \frac{\Delta W_{i,1}^\top}{h_i}\right] + \mathbb{E}_i\left[f_{i+1}^\pi \Delta W_{i,1}^\top + \frac{h_i}{2h_{i+1}}\left(f_{i+1}^\pi \Delta W_{i,1}^\top - f_{i+2}^\pi \Delta W_{i,2}^\top\right)\right]$$

$$- \mathbb{E}_i\left[\frac{4h_i + 3h_{i+1}}{6(h_i + h_{i+1})}Z_{i+1}^\pi + \frac{h_i^2}{6h_{i+1}(h_i + h_{i+1})}\left(Z_{i+1}^\pi - Z_{i+2}^\pi\right)\right].$$

$$(27)$$

Lemma 5. *If constraints (15) and (21) and $(h_i/h_{i+j}) \leq M, M \in (0, +\infty)$ are satisfied, then the coefficients $\Gamma_{j,i}$, $\Gamma_{j,i}^*$, and $\Gamma_{j,i}^{**}$ in scheme (14) are bounded.*

Proof. From Lemma 3, we know that the coefficients $\Gamma_{j,i}$ are composed of products and sums of h_i/h_{i+j} for $j = 1, 2, \ldots, k$. Under the condition $h_i/h_{i+j} \leq M$, it is clear that the coefficients $\Gamma_{j,i}$ are bounded. Analogously, we derive that coefficients $\Gamma_{j,i}^*$ and $\Gamma_{j,i}^{**}$ are bounded.

In what follows, two definitions are introduced to serve the stability of scheme (14). □

Definition 1. The characteristic polynomials of (14) are given by

$$P_y(\zeta) = \zeta - 1, \qquad (28)$$

$$P_z(\zeta) = \zeta^{k-1} - \sum_{j=0}^k \Gamma_{j,i}\zeta^{k-j}. \qquad (29)$$

and Equation (14) is said to fulfil Dahlquist's root condition if

(i) The roots of $P_y(\zeta)$ and $P_z(\zeta)$ lie on or within the unit circle

(ii) The roots on the unit circle are simple

Definition 2. Let $(Y_i^\pi, Z_i^\pi), i = 0, 1, \ldots, N - k$, be the time-discretization approximate solution given by (14) and $(\overline{Y}_i^\pi, \overline{Z}_i^\pi)$ be the solution of its perturbed form (see (31)). Then, scheme (14) is said to be $\mathbb{L}_2$-stable if

$$\max_{0 \leq i \leq N-k} \mathbb{E}\left[\left|Y_i^\pi - \overline{Y}_i^\pi\right|^2 + \left|Z_i^\pi - \overline{Z}_i^\pi\right|^2\right]$$

$$\leq C\left(\max_{N-k+1 \leq \ell \leq N} \mathbb{E}\left[\left|Y_i^\pi - \overline{Y}_i^\pi\right|^2 + \left|Z_i^\pi - \overline{Z}_i^\pi\right|^2\right]\right.$$

$$\left. + \sum_{\ell=0}^{N-k} \mathbb{E}\left[|\varepsilon_\ell^Y|^2 + |\varepsilon_\ell^Z|^2\right]\right), \qquad (30)$$

where C denotes a constant which changes from line to line; $(\overline{Y}_i^\pi, \overline{Z}_i^\pi)$ satisfies a perturbed form of (14) for $i = N - k, N - k - 1, \ldots, 0$:

$$
\begin{cases}
\overline{Y}_i^\pi = \mathbb{E}_i\left[\overline{Y}_{i+1}^\pi\right] + h_i \sum_{j=0}^{k} \Gamma_{j,i} \mathbb{E}_i\left[\overline{f}_{i+j}^\pi\right] + \varepsilon_i^Y, \\[2mm]
0 = \dfrac{1}{h_i} \mathbb{E}_i\left[\overline{Y}_{i+1}^\pi \Delta W_{i,1}^\top\right] + \sum_{j=1}^{k} \Gamma_{j,i}^* \mathbb{E}_i\left[\overline{f}_{i+j}^\pi \Delta W_{i,j}^\top\right] - \sum_{j=0}^{k} \Gamma_{j,i}^{**} \mathbb{E}_i\left[\overline{Z}_{i+j}^\pi\right] + \varepsilon_i^Z,
\end{cases}
\tag{31}
$$

where $\overline{f}_i^\pi = f(t_i, \overline{Y}_i^\pi, \overline{Z}_i^\pi)$; sequences ε_i^Y and ε_i^Z which belong to $\mathbb{L}_2(\mathscr{F}_i)$ are random variables.

The following theorem is devoted to analyze the stability of scheme (14).

Theorem 1. *Suppose Assumption 1 and the condition of Lemma 5 hold. Then, the variable step size explicit Adams methods (that is, the coefficient $\Gamma_{0,i} = 0$) is numerically stable if and only if its characteristic polynomial (29) satisfies Dahlquist's root condition.*

Proof (sufficiency). Let $\Delta Y_i = Y_i^\pi - \overline{Y}_i^\pi, \Delta Z_i = Z_i^\pi - \overline{Z}_i^\pi$, and $\Delta f_i = f(t_i, Y_i^\pi, Z_i^\pi) - f(t_i, \overline{Y}_i^\pi, \overline{Z}_i^\pi)$ for $i = N - k, N - k - 1, \ldots, 0$. We complete the proof of the theorem in three steps.

Step 1. From (14) and (31) with respect to Y, one obtains

$$
\Delta Y_i = \mathbb{E}_i\left[\Delta Y_{i+1} + h_i \sum_{j=1}^{k} \Gamma_{j,i} \Delta f_{i+j}\right] - \varepsilon_i^Y.
\tag{32}
$$

Furthermore,

$$
\begin{aligned}
\left|\Delta Y_i\right| &\leq \left|\mathbb{E}_i[\Delta Y_{i+1}]\right| + h_i \sum_{j=1}^{k} \left|\Gamma_{j,i} \mathbb{E}_i\left[\Delta f_{i+j}\right]\right| + \left|\varepsilon_i^Y\right| \pi, \\[2mm]
&\leq \left|\mathbb{E}_i[\Delta Y_{i+1}]\right| + h_i L_f \sum_{j=1}^{k} \left|\Gamma_{j,i}\right| \mathbb{E}_i\left[\left|\Delta Y_{i+j}\right| + \left|\Delta Z_{i+j}\right|\right] \\[2mm]
&\quad + \left|\varepsilon_i^Y\right|,
\end{aligned}
\tag{33}
$$

where L_f denotes the Lipschitz constant. Squaring equation (33), then from the inequalities $(\sum_{i=1}^{n} a_i)^2 \leq n \sum_{i=1}^{n} a_i^2$ and $(a + b)^2 \leq \delta(1 + \delta)a^2 + (1 + 1/\delta)b^2$, $\delta > 0$, one deduces

$$
\begin{aligned}
\left|\Delta Y_i\right|^2 &\leq \left|\mathbb{E}_i[\Delta Y_{i+1}]\right|^2 + \frac{3 + \sqrt{5}}{2}\left(h_i L_f \sum_{j=1}^{k} \left|\Gamma_{j,i}\right| \mathbb{E}_i\left[\left|\Delta Y_{i+j}\right| + \left|\Delta Z_{i+j}\right|\right] + \left|\varepsilon_i^Y\right|\right)^2 \\[2mm]
&\leq \left|\mathbb{E}_i[\Delta Y_{i+1}]\right|^2 + \frac{3 + \sqrt{5}}{2}(2k + 1)L_f^2 h_i^2 \max_{1 \leq j \leq k}\left|\Gamma_{j,i}\right|^2 \sum_{j=1}^{k} \mathbb{E}_i\left[\left|\Delta Y_{i+j}\right|^2 + \left|\Delta Z_{i+j}\right|^2\right] \\[2mm]
&\quad + \frac{3 + \sqrt{5}}{2}(2k + 1)\left|\varepsilon_i^Y\right|^2.
\end{aligned}
\tag{34}
$$

Summing over the above inequality from i to $N - k$, we have

$$
\begin{aligned}
\left|\Delta Y_i\right|^2 &\leq \left|\mathbb{E}_i[\Delta Y_{N-k+1}]\right|^2 + \frac{3 + \sqrt{5}}{2}(2k + 1)L_f^2 h^2 \max_{1 \leq j \leq k}\left|\Gamma_{j,i}\right|^2 \sum_{\ell=i}^{N-k} \sum_{j=1}^{k} \mathbb{E}_i\left[\left|\Delta Y_{\ell+j}\right|^2 + \left|\Delta Z_{\ell+j}\right|^2\right] \\[2mm]
&\quad + \frac{3 + \sqrt{5}}{2}(2k + 1) \sum_{\ell=i}^{N-k} \left|\varepsilon_\ell^Y\right|^2.
\end{aligned}
\tag{35}
$$

Step 2. Subtracting (31) from (14) with respect to Z, we obtain

$$\Delta Z_i = \frac{1}{\Gamma_{0,i}^{**} h_i} \mathbb{E}_i \big[\Delta Y_{i+1} \Delta W_{i,1}^\top \big] + \sum_{j=1}^{k} \frac{\Gamma_{j,i}^*}{\Gamma_{0,i}^{**}} \mathbb{E}_i \big[\Delta f_{i+j} \Delta W_{i,j}^\top \big]$$

$$- \sum_{j=1}^{k} \frac{\Gamma_{j,i}^{**}}{\Gamma_{0,i}^{**}} \mathbb{E}_i \big[\Delta Z_{i+j} \big] - \frac{\varepsilon_i^Z}{\Gamma_{0,i}^{**}}. \tag{36}$$

By (2) and (3), one can verify that

$$\mathbb{E}_i \left[Y_{t_{i+1}} \frac{\Delta W_{i,1}^\top}{h_i} \right] = \frac{1}{h_i \sqrt{2\pi h_i}} \int_{-\infty}^{+\infty} u(t_{i+1}, x+v) v e^{-v^2/2h_i} dv$$

$$= \frac{1}{\sqrt{2\pi h_i}} \int_{-\infty}^{+\infty} \frac{\partial u}{\partial x}(t_{i+1}, x+v) e^{-v^2/2h_i} dv$$

$$= \mathbb{E}_i \big[Z_{t_{i+1}} \big]. \tag{37}$$

Plugging (37) into (36), we have

$$\Delta Z_i = \frac{1-\Gamma_{1,i}^{**}}{\Gamma_{0,i}^{**}} \mathbb{E}_i \big[\Delta Z_{i+1} \big] - \sum_{j=2}^{k} \frac{\Gamma_{j,i}^{**}}{\Gamma_{0,i}^{**}} \mathbb{E}_i \big[\Delta Z_{i+j} \big]$$

$$+ \sum_{j=1}^{k} \frac{\Gamma_{j,i}^*}{\Gamma_{0,i}^{**}} \mathbb{E}_i \big[\Delta f_{i+j} \Delta W_{i,j}^\top \big] - \frac{\varepsilon_i^Z}{\Gamma_{0,i}^{**}}. \tag{38}$$

We rearrange the k-step recursion to a one-step recursion as follows:

$$\mathbb{E}_i [\mathscr{Z}_i] = \mathbb{E}_i [A\mathscr{Z}_{i+1} + F_i + R_i], \tag{39}$$

where

$$\mathscr{Z}_i = \begin{pmatrix} \Delta Z_i \\ \Delta Z_{i+1} \\ \vdots \\ \Delta Z_{i+k-1} \end{pmatrix}, A = \begin{pmatrix} \frac{1-\Gamma_{1,i}^{**}}{\Gamma_{0,i}^{**}} & -\frac{\Gamma_{2,i}^{**}}{\Gamma_{0,i}^{**}} & \cdots & -\frac{\Gamma_{k,i}^{**}}{\Gamma_{0,i}^{**}} \\ 1 & 0 & & \\ & & \ddots & \ddots \\ & & & 1 & 0 \end{pmatrix}, F_i = \begin{pmatrix} \sum_{j=1}^{k} \frac{\Gamma_{j,i}^*}{\Gamma_{0,i}^{**}} \mathbb{E}_i \big[\Delta f_{i+j} \Delta W_{i,j}^\top \big] \\ 0 \\ \vdots \\ 0 \end{pmatrix}, R_i = \begin{pmatrix} -\frac{\varepsilon_i^Z}{\Gamma_{0,i}^{**}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{40}$$

To ensure the stability of the k-step scheme, the norm of the matrix A in equation (39) is not more than 1 (see Chapter III.4, Lemma 4.4 in [32]). This can be satisfied if the eigenvalues $eig(A)$ of the matrix A make $|eig(A)| \leq 1$ and the eigenvalues are simple if $|eig(A)| = 1$. In addition, the eigenvalues of A satisfy the root condition by Definition 1. By Dahlquist's root condition, it is possible that there exists a nonsingular matrix $\mathscr{D}$ such that $\|\mathscr{D}^{-1} A \mathscr{D}\|_2 \leq 1$ where $\|\cdot\|_2$ denotes the spectral matrix norm induced by Euclidian vector norm in $\mathbb{R}^{k \times m \times d}$. Hence, we can choose a scalar product for $\overline{A}, \widetilde{A} \in \mathbb{R}^{k \times m \times d}$ as $\langle \overline{A}, \widetilde{A} \rangle_* := \langle \mathscr{D}^{-1} \overline{A}, \mathscr{D}^{-1} \widetilde{A} \rangle = \overline{A}^\top (\mathscr{D}^{-1})^\top \mathscr{D}^{-1} \widetilde{A}$. And we have $|\overline{A}|_*^2 := \langle \overline{A}, \overline{A} \rangle_*$ with the induced vector norm on $\mathbb{R}^{k \times m \times d}$. Let $\|A\|_* = \|\mathscr{D}^{-1} A \mathscr{D}\|_2$ be the induced matrix norm.

Owing to the norm equivalence, we know that there exist positive constants c_1 and c_2 such that

$$c_1 |\overline{A}|_2^2 \leq |\overline{A}|_*^2 \leq c_2 |\overline{A}|_2^2, \quad \forall \overline{A} \in \mathbb{R}^{k \times m \times d}, \tag{41}$$

where $|\overline{A}|_2^2 = \sum_{j=1,2,\ldots,m} |a_j|^2$ for $\overline{A} = (a_1^\top, \ldots, a_m^\top)^\top$. Applying $|\cdot|_*$ to equation (39), we have

$$|\mathbb{E}_i[\mathscr{Z}_i]|_* = |\mathbb{E}_i[A\mathscr{Z}_{i+1} + F_i + R_i]|_*$$

$$= \|A\|_* |\mathbb{E}_i[\mathscr{Z}_{i+1}]|_* + |\mathbb{E}_i[F_i]|_* + |\mathbb{E}_i[R_i]|_*$$

$$\leq |\mathbb{E}_i[\mathscr{Z}_{i+1}]|_* + \mathbb{E}_i[F_i]|_* + |\mathbb{E}_i[R_i]|_*. \tag{42}$$

Squaring equation (42), then from the inequalities $(\sum_{i=1}^{n} a_i)^2 \le n \sum_{i=1}^{n} a_i^2$ and $(a+b)^2 \le \delta(1+\delta)a^2 + (1+1/\delta)b^2$, $\delta > 0$, one deduces

$$
\begin{aligned}
\left|\mathbb{E}_i[\mathscr{Z}_i]\right|_*^2 &\le \left|\mathbb{E}_i[\mathscr{Z}_{i+1}]\right|_*^2 + \frac{3+\sqrt{5}}{2}\left(\left|\mathbb{E}_i[F_i]\right|_* + \left|\mathbb{E}_i[R_i]\right|_*\right)^2 \\
&\le \left|\mathbb{E}_i[\mathscr{Z}_{i+1}]\right|_*^2 + (k+1)c_2\frac{3+\sqrt{5}}{2}\sum_{j=1}^{k}\left|\mathbb{E}_i\left[\frac{\Gamma_{j,i}^*}{\Gamma_{0,i}^{**}}\Delta f_{i+j}\Delta W_{i,j}^{\top}\right]\right|^2 + \frac{(k+1)c_2 3+\sqrt{5}/2}{\left(\Gamma_{0,i}^{**}\right)^2}\left|\mathbb{E}_i[\varepsilon_i^Z]\right|^2.
\end{aligned}
\tag{43}
$$

By the Lipschitz condition of f with respect to (y,z), (43) can be restated as

$$
\begin{aligned}
\left|\mathbb{E}_i[\mathscr{Z}_i]\right|_*^2 &\le \left|\mathbb{E}_i[\mathscr{Z}_{i+1}]\right|_*^2 + (k+1)c_2\frac{3+\sqrt{5}}{2}L_f^2\sum_{j=1}^{k}\left(\frac{\Gamma_{j,i}^*}{\Gamma_{0,i}^{**}}\right)^2\left|\mathbb{E}_i\left[\left(\left|\Delta Y_{i+j}\right| + \left|\Delta Z_{i+j}\right|\right)\Delta W_{i,j}^{\top}\right]\right|^2 \\
&\quad + \frac{(k+1)c_2 3+\sqrt{5}/2}{\left(\Gamma_{0,i}^{**}\right)^2}\left|\mathbb{E}_i[\varepsilon_i^Z]\right|^2.
\end{aligned}
\tag{44}
$$

By the Cauchy–Schwarz inequality, we have the following estimates:

$$
\begin{aligned}
\left|\mathbb{E}_i[\mathscr{Z}_i]\right|_*^2 &\le \left|\mathbb{E}_i[\mathscr{Z}_{i+1}]\right|_*^2 + \frac{3+\sqrt{5}}{2}k(k+1)c_2L_f^2 h\frac{\max_{1\le j\le k}\left|\Gamma_{j,i}^*\right|^2}{\left(\Gamma_{0,i}^{**}\right)^2}\sum_{j=1}^{k}\mathbb{E}_i\left[\left|\Delta Y_{i+j}\right|^2 + \left|\Delta Z_{i+j}\right|^2\right] \\
&\quad + \frac{(k+1)c_2 3+\sqrt{5}/2}{\left(\Gamma_{0,i}^{**}\right)^2}\left|\mathbb{E}_i[\varepsilon_i^Z]\right|^2.
\end{aligned}
\tag{45}
$$

Summing over the above inequality from i to $N-k$, we have

$$
\begin{aligned}
\left|\mathbb{E}_i[\mathscr{Z}_i]\right|_*^2 &\le \left|\mathbb{E}_i[\mathscr{Z}_{N-k+1}]\right|_*^2 + \frac{3+\sqrt{5}}{2}k(k+1)c_2L_f^2 h\frac{\max_{1\le j\le k}\left|\Gamma_{j,i}^*\right|^2}{\left(\Gamma_{0,i}^{**}\right)^2}\sum_{\ell=i}^{N-k}\sum_{j=1}^{k}\mathbb{E}_i\left[\left|\Delta Y_{\ell+j}\right|^2 + \left|\Delta Z_{\ell+j}\right|^2\right] \\
&\quad + \frac{(k+1)c_2 3+\sqrt{5}/2}{\left(\Gamma_{0,i}^{**}\right)^2}\sum_{\ell=i}^{N-k}\left|\mathbb{E}_i[\varepsilon_\ell^Z]\right|^2 \\
&\le \left|\mathbb{E}_i[\mathscr{Z}_{N-k+1}]\right|_*^2 + \frac{3+\sqrt{5}}{2}k(k+1)\frac{c_2}{c_1}L_f^2 h\frac{\max_{1\le j\le k}\left|\Gamma_{j,i}^*\right|^2}{\left(\Gamma_{0,i}^{**}\right)^2}\left(\sum_{\ell=i}^{N-k}\sum_{j=1}^{k}c_1\mathbb{E}_i\left[\left|\Delta Y_{\ell+j}\right|^2\right] + \sum_{\ell=i+1}^{N-k+1}\mathbb{E}_i\left[\left|\mathscr{Z}_\ell\right|_*^2\right]\right) \\
&\quad + \frac{(k+1)c_2 3+\sqrt{5}/2}{\left(\Gamma_{0,i}^{**}\right)^2}\sum_{\ell=i}^{N-k}\left|\left[\mathbb{E}_i\varepsilon_\ell^Z\right]\right|^2.
\end{aligned}
\tag{46}
$$

Step 3. Adding (35) to (46), we obtain

$$\left|\Delta Y_i\right|^2 + \left|\mathbb{E}_i\left[\mathscr{Z}_i\right]\right|_*^2 \leq C\left(h + h^2\right) \sum_{\ell=i+1}^{N-k+1} \mathbb{E}_i\left[\left|\Delta Y_\ell\right|^2 + \left|\mathscr{Z}_\ell\right|_*^2\right] + C \sum_{\ell=N-k+1}^{N} \mathbb{E}_i\left[\left|\Delta Y_\ell\right|^2 + \left|Z_\ell\right|^2\right]$$

$$+ \frac{3 + \sqrt{5}}{2}(2k+1) \sum_{\ell=i}^{N-k} \left|\varepsilon_i^Y\right|^2 + \frac{(k+1)c_2 3 + \sqrt{5}/2}{\left(\Gamma_{0,i}^{**}\right)^2} \sum_{\ell=i}^{N-k} \left|\mathbb{E}_i\left[\varepsilon_\ell^Z\right]\right|^2. \tag{47}$$

From the discrete Gronwall inequality, we have

$$\left|\Delta Y_i\right|^2 + \left|\mathbb{E}_i\left[\mathscr{Z}_i\right]\right|_*^2 \leq C\left(\max_{i+1 \leq \ell \leq N-k+1} kh\mathbb{E}_i\left[\left|\Delta Y_\ell\right|^2 + \left|\mathscr{Z}_\ell\right|_*^2\right] + \sum_{\ell=N-k+1}^{N} \mathbb{E}_i\left[\left|\Delta Y_\ell\right|^2 + \left|Z_\ell\right|^2\right] + \sum_{\ell=i}^{N-k} \mathbb{E}_i\left[\left|\varepsilon_\ell^Y\right|^2 + \left|\varepsilon_\ell^Z\right|^2\right]\right). \tag{48}$$

Moreover,

$$\max_{0 \leq i \leq N-k} \mathbb{E}\left[\left|\Delta Y_i\right|^2 + \left|\Delta Z_i\right|^2\right] \leq C\left(\max_{N-k+1 \leq \ell \leq N} \mathbb{E}\left[\left|\Delta Y_\ell\right|^2 + \left|\Delta Z_\ell\right|^2\right] + \sum_{\ell=0}^{N-k} \mathbb{E}\left[\left|\varepsilon_\ell^Y\right|^2 + \left|\varepsilon_\ell^Z\right|^2\right]\right). \tag{49}$$

Necessity. The proof is analogous to ordinary differential equations (see Theorem 6.3.3 in [33]). So, we omit it. □

Theorem 2. *Suppose Assumption 1 and the condition of Lemma 5 hold. Then, the variable step size implicit Adams methods (that is, the coefficient $\Gamma_{0,i} \neq 0$) is numerically stable if and only if its characteristic polynomial (29) satisfies Dahlquist's root condition.*

Proof. The proof of this theorem is analogous to that of Theorem 1. Thus, we omit it here.

Next, the convergence property of scheme (14) is given in the theorem as below. □

Theorem 3. *Suppose Assumption 1 and the condition of Lemma 5 hold. Let (Y_{t_i}, Z_{t_i}) and (Y_i^π, Z_i^π) be solutions of the BSDE in (1) and solutions of the variable step size Adams methods (14), respectively. The terminal values satisfy $\mathbb{E}[sup_{N-k<i\leq N}|Y_i^\pi - Y_{t_i}|^2 + |Z_i^\pi - Z_{t_i}|^2]^{1/2} \leq Ch^k$. Then, as h is small enough,*

$$\mathbb{E}\left[\sup_{0 \leq i \leq N-k}\left|Y_i^\pi - Y_{t_i}\right|^2 + \left|Z_i^\pi - Z_{t_i}\right|^2\right]^{\frac{1}{2}} \leq Ch^k, \tag{50}$$

where C is a constant changing from line to line.

Proof. The proof is obvious with the help of Theorem 1 and Lemmas 3 and 4. □

5. Numerical Experiments

In this section, we demonstrate the theory results of scheme (14) via numerical examples. First, we choose a method to approximate the conditional mathematical expectations numerically. Among the popular methods, we focus on the least squares Monte Carlo (LSMC) method (see [13, 27]). Next, we review how to approximate mathematical expectations by the LSMC method.

For $i \in \{0, 1, \ldots, N\}$, let $\mathscr{C}_i := \left\{W^{(i,\widetilde{m})} : \widetilde{m} = 1, 2, \ldots, \widetilde{M} \in \mathbb{N}^+\right\}$ be independently generated copies of W_{t_i}. The empirical probability measure of the $\mathscr{C}_i$-simulations is denoted by $\nu_{i,\widetilde{M}} = 1/\widetilde{M} \sum_{\widetilde{m}=1}^{\widetilde{M}} \delta_{(W_{i+1}^{(i,\widetilde{m})}, \ldots, W_{i+k}^{(i,\widetilde{m})})}$ where δ_x is the Dirac measure on x. Denote by $\mathscr{K}_{Y,i}$ and $\mathscr{K}_{Z,i}$ vector spaces of functions, i.e., $\mathscr{K}_{Y,i} := span\left\{p_{Y,i}^{(1)}, p_{Y,i}^{(2)}, \ldots, p_{Y,i}^{(K_{Y,i})}\right\}$ where $K_{Y,i}$ is a nonnegative integer; $p_{Y,i}^{(\mathbf{k})} : \mathbb{R}^d \longrightarrow \mathbb{R}^m$ such that $\mathbb{E}[|p_{Y,i}^{(\mathbf{k})}(W_i)|^2] < +\infty$; $\mathscr{K}_{Z,i} := \left\{p_{Z,i}^{(1)}(\cdot), p_{Z,i}^{(2)}(\cdot), \ldots, p_{Z,i}^{(K_{Z,i})}(\cdot)\right\}$ where $K_{Z,i}$ is a positive integer; $p_{Z,i}^{(\mathbf{k})} : \mathbb{R}^d \longrightarrow \mathbb{R}^{m \times d}$ such that $\mathbb{E}[|p_{Z,i}^{(\mathbf{k})}(X_i)|^2] < +\infty$.

From ordinary least squares, numerical solutions $Y_i^{\pi,(\widetilde{M})}$ and $Z_i^{\pi,(M)}$ are obtained by the following manner:

$$Y_i^{\pi,(\widetilde{M})} = \arg \inf_{\phi \in \mathscr{K}_{Y,i}} \frac{1}{\widetilde{M}} \sum_{m=1}^{\widetilde{M}} \left| \phi\left(W_i^{(\widetilde{m})}\right) - S_{Y,i} \right|^2, \qquad (51)$$

$$Z_i^{\pi,(\widetilde{M})} = \arg \inf_{\phi \in \mathscr{K}_{Z,i}} \frac{1}{\widetilde{M}} \sum_{m=1}^{\widetilde{M}} \left| \phi\left(W_i^{(\widetilde{m})}\right) - S_{Z,i} \right|^2, \qquad (52)$$

where $S_{Z,i} = Y_{i+1}^{\pi,(\widetilde{M})} \Delta W_{i,i+1}^{\top} / \Gamma_{0,i}^{**} h_i + \sum_{j=1}^{k} \Gamma_{j,i}^{*} / \Gamma_{0,i}^{**} f(t_{i+j}, Y_{i+j}^{\pi,(\widetilde{M})}, Z_{i+j}^{\pi,(\widetilde{M})}) \Delta W_{i,i+j}^{\top} - \sum_{j=1}^{k} \Gamma_{j,i}^{**} / \Gamma_{0,i}^{**} Z_{i+j}^{\pi,(\widetilde{M})}$ and $S_{Y,i} = Y_{i+1}^{\pi,(\widetilde{M})} + h_i \sum_{j=0}^{k} \Gamma_{j,i} f(t_{i+j}, Y_{i+j}^{\pi,(\widetilde{M})}, Z_{i+j}^{\pi,(\widetilde{M})})$.

To obtain the numerical solutions of BSDE (1.1), we have to determine the parameters in the numerical scheme (14). The coefficients $\Gamma_{j,i}$, $\Gamma_{j,i}^{*}$, and $\Gamma_{j,i}^{**}$ are obtained by means of Lemmas 3 and 4 as the value of k is given. In what follows, we discuss the numerical schemes according to the following two cases with the nonuniform time grid $t_i = T(1 - (1 - i/N)^2)$:

$$\begin{cases} Y_i^{\pi} = \mathbb{E}_i[Y_{i+1}^{\pi}] + h_i \mathbb{E}_i[f_{i+1}^{\pi}], \\ Z_i^{\pi} = \mathbb{E}_i\left[Y_{i+1}^{\pi} \frac{\Delta W_{i,1}^{\top}}{h_i} \right], \end{cases} \qquad (53)$$

$$\begin{cases} Y_i^{\pi} = \mathbb{E}_i[Y_{i+1}^{\pi}] + h_i \mathbb{E}_i\left[f_{i+1}^{\pi} + \frac{h_i}{2h_{i+1}} (f_{i+1}^{\pi} - f_{i+2}^{\pi}) \right], \\ \frac{1}{2} Z_i^{\pi} = \mathbb{E}_i\left[Y_{i+1}^{\pi} \frac{\Delta W_{i,1}^{\top}}{h_i} \right] + \mathbb{E}_i[f_{i+1}^{\pi} \Delta W_{i,1}^{\top}] - \frac{1}{2} \mathbb{E}_i[Z_{i+1}^{\pi}]. \end{cases} \qquad (54)$$

To compare the performance of scheme (53) and scheme (54), we also provide the corresponding constant variable step numerical schemes, namely,

$$\begin{cases} Y_i^{\pi} = \mathbb{E}_i[Y_{i+1}^{\pi}] + h \mathbb{E}_i[f_{i+1}^{\pi}], \\ Z_i^{\pi} = \mathbb{E}_i\left[Y_{i+1}^{\pi} \frac{\Delta W_{i,1}^{\top}}{h} \right], \end{cases} \qquad (55)$$

$$\begin{cases} Y_i^{\pi} = \mathbb{E}_i[Y_{i+1}^{\pi}] + h \mathbb{E}_i\left[\frac{3}{2} f_{i+1}^{\pi} - \frac{1}{2} f_{i+2}^{\pi} \right], \\ \frac{1}{2} Z_i^{\pi} = \mathbb{E}_i\left[Y_{i+1}^{\pi} \frac{\Delta W_{i,1}^{\top}}{h} \right] + \mathbb{E}_i[f_{i+1}^{\pi} \Delta W_{i,1}^{\top}] - \frac{1}{2} \mathbb{E}_i[Z_{i+1}^{\pi}], \end{cases} \qquad (56)$$

where the constant step size $h = T/N$. Before we apply our scheme (14) to the following BSDEs, we first introduce the notations. Let N be the number of time points; $\widetilde{M}$ represents the number of simulation paths. The basis functions which are spanned by polynomials whose degree is less than or equal to the order of discretization error are applied to compute the value of $Y_i^{\pi,(M)}$. The error of both the numerical solution and the exact solution of the BSDE (1) at the time $t = 0$ is denoted by $|Y_0 - Y_0^{\pi,(M)}|$.

and

Example 1. Consider the BSDE as follows:

$$\begin{aligned} Y_t = 1 + \eta &+ \sin\left(\tau \mathbf{1}_d^{\top} W_T\right) \\ &+ \int_t^T \min\left\{ 1, \left(Y_s - \eta - 1 - \frac{\sin\left(\tau \mathbf{1}_d^{\top} W_s\right)}{\exp\left(\tau^2 d(T-t)/2\right)} \right)^2 \right\} ds \\ &- \int_t^T Z_s dW_s, \end{aligned} \qquad (57)$$

which appears in [26] and is used to illustrate the variance reduction problem with closed-form solutions. Here, $\eta > 0$; $\tau > 0$; $\mathbf{1}_d$ is a d-dimensional vector with components all 1. Now, the solution to the above BSDE is

$$Y_t = 1 + \eta + \frac{\sin\left(\tau \mathbf{1}_d^{\top} W_t\right)}{\exp\left(\tau^2 d(T-t)/2\right)}. \qquad (58)$$

Take $T = 1, \eta = 0.6, \tau = 1/\sqrt{d}, d = 9$, and $\widetilde{M} = 100000$.

Figure 1 presents the relationship of the absolute error between the numerical solution and the exact solution of BSDE (57) with respect to Y at time 0 and the number of time points via the variable step size scheme (53), the variable step size scheme (54), the constant variable step size scheme (55), and the constant variable step size scheme (56). To be specific, Figure 1 implies that (i) on the whole, the error of Y becomes smaller as N gets bigger in scheme (53), scheme (54), scheme (55), and scheme (56); (ii) if the the absolute error $|Y_0 - Y_0^{\pi,(M)}|\pi$ is fixed, the number of steps of the constant variable step size schemes is bigger than that of the variable step size schemes; and (iii) the absolute error with respect to Y obtained by schemes (54) and (56) is smaller than the absolute error of Y obtained by schemes (53) and (55).

Example 2. We consider the BSDE as follows:

$$Y_t = \frac{\exp\left(T + \sum_{k=1}^{d} W_k\right)}{1 + \exp\left(T + \sum_{k=1}^{d} W_k\right)} + \int_t^T \left(\sum_{k=1}^{d} Z_{k,s} \right) \left(Y_s - \frac{2+d}{d} \right) ds - \int_t^T Z_s dW_s. \qquad (59)$$

Therefore, the exact solutions of BSDE (59) can be represented in the following form:

$$Y_{t_i} = \frac{\exp\left(t_i + \sum_{k=1}^{d} W_k\right)}{1 + \exp\left(t_i + \sum_{k=1}^{d} W_k\right)}. \qquad (60)$$
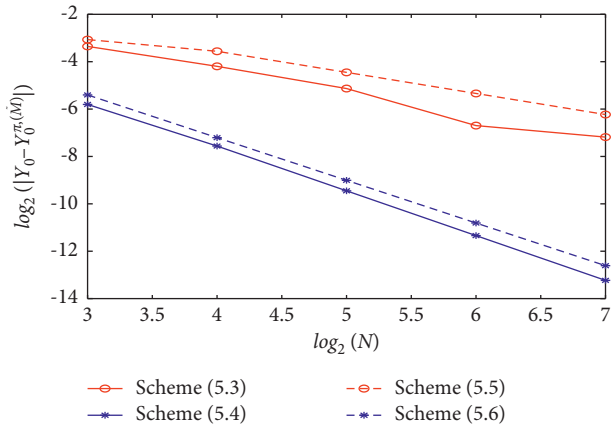
FIGURE 1: $\text{Log}_2 (|Y_0 - Y_0^{\pi,\widetilde{M}}|)$ versus $\log_2 (N)$.
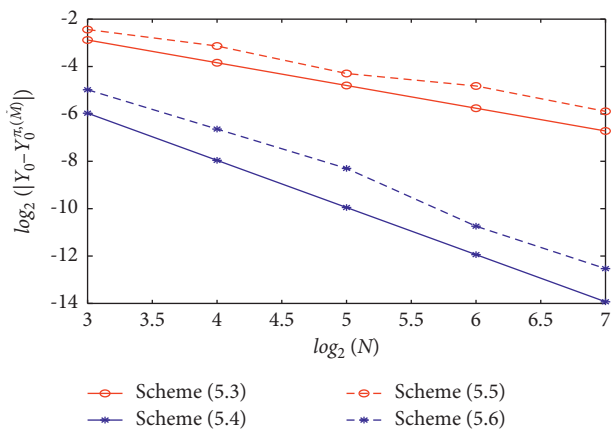

FIGURE 2: $\text{Log}_2 (|Y_0 - Y_0^{\pi,\widetilde{M}}|)$ versus $\log_2 (N)$.

Take $T = 1, d = 5$, and $\widetilde{M} = 100000$.

Figure 2 presents the relationship of the absolute error between the numerical solution and the exact solution of BSDE (59) with respect to Y at time 0 and the number of time points via the variable step size scheme (53), the variable step size scheme (54), the constant variable step size scheme (55), and the constant variable step size scheme (56). Figure 2 implies that the variable step size schemes possess almost the same convergence rates as the constant variable step size scheme, but the number of steps of the variable step size schemes is smaller than that of the constant variable step size schemes.

6. Conclusions

We design in this article the variable step size Adams schemes to calculate the solution of BSDEs. To enrich the numerical schemes with high-order convergence, we infer the conditions with respect to the coefficients in the numerical method by means of Itô–Taylor expansion. We then provide a sufficient and necessary condition for the stability of our variable step size Adams schemes via Dahlquist's root condition. The convergence of the proposed scheme is analyzed too. We illustrate the theory results of our numerical algorithms with two examples. Finally, note that the constant variable one-step size schemes [11, 15, 18, 24] and the constant variable multistep size schemes [16, 26, 27] are the particular cases of our variable step size Adams schemes.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

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