Research Article

An Implicit Algorithm for Finding a Fixed Point of a Q-Nonexpansive Mapping in Locally Convex Spaces

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In this paper, the notion of the q -duality mappings in locally convex spaces is introduced. An implicit method for finding a fixed point of a Q -nonexpansive mapping is provided. Finally, the convergence of the proposed implicit scheme is investigated. Some examples in order to illustrate of the main results are presented.

1. Introduction

Let C be a nonempty closed and convex subset of a Banach space E and E^* be the dual space of E . Let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^* . The normalized duality mapping $J: E \longrightarrow E^*$ is defined by

$$J(x) = \{f \in E^*: \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \quad (1)$$

for all $x \in E$. It is well known that if J is a single-valued mapping, then it is norm-weak* continuous (see, for more details, Theorem 2.6.10 in [1]). In this paper, the duality mapping of locally convex space is studied and denoted by J_q for the seminorm q .

Suppose that Q is a family of seminorms on a locally convex space E which determines the topology of E that will be denoted by τ_Q . Let C be a nonempty closed and convex subset of E . A mapping $T: C \longrightarrow C$ is called Q -nonexpansive if $q(Tx - Ty) \leq q(x - y)$, for all $x, y \in C$ and $q \in Q$. Also, the function $f: E \longrightarrow E$ is Q -contraction if there exists $\beta \in [0, 1)$ such that $q(f(x) - f(y)) \leq \beta q(x - y)$, for all $x, y \in C$ and $q \in Q$.

The methods of iterations, which find a number of real applications, are popular for problems in sciences and engineering; see, e.g., [2–6]. Recently, various iterative

algorithms have been investigated for fixed points of the nonlinear mapping, especially, for the mappings of nonexpansive type; see, e.g., [7–11] and the references therein.

Also, the general implicit algorithm for finding a fixed point of a Q -nonexpansive mapping is given by

$$z_n = \varepsilon_n f z_n + (1 - \varepsilon_n) T z_n, \quad (n = 1, 2, \dots), \quad (2)$$

where f is a Q -contraction and T is a Q -nonexpansive mapping. On the contrary, it is proven that there exists a sunny Q -nonexpansive retraction P of C onto $\text{Fix}(T)$ and $x \in C$ such that the sequence $\{z_n\}$ converges to Px with respect to the topology τ_Q .

Moreover, some notions and concepts in locally convex spaces are recalled. The Hahn–Banach theorem and Banach contraction principle will be generalized to locally convex spaces. The results of this article can be viewed as implicit and locally convex space version of some results given in [1, 12–17].

2. Methods and Preliminaries

Suppose that Q is a family of seminorms on a locally convex space E which determines the topology of E will be denoted by τ_Q .

Let $D \subseteq B \subseteq E$ and P be a retraction of B onto D , that is, $Px = x$, for each $x \in D$. Then, P is said to be sunny if, for each $x \in B$ and $t \geq 0$ with $Px + t(x - Px) \in B$, we have

$$P(Px + t(x - Px)) = Px. \quad (3)$$

A subset D of B is said to be sunny Q -nonexpansive retract of B if there exists a sunny Q -nonexpansive retraction P of B onto D . We know that if E is a smooth Banach space and P a retraction of B onto D , then P is sunny nonexpansive if and only if

$$\langle x - Px, J(z - Px) \rangle \leq 0, \quad (4)$$

for each $x \in B$ and $z \in D$. For more details, see [18]. In the sequel, we will prove inequality (4) for a real locally convex space.

Now, we recall the following definitions:

- (1) The locally convex topology τ_Q is separated if and only if the family of seminorms Q possesses the following property: for each $x \in E \setminus \{0\}$, there exists $q \in Q$ such that $q(x) \neq 0$ (see [19]) or equivalently

$$\bigcap_{q \in Q} \{x \in E: q(x) = 0\} = \{0\}. \quad (5)$$

- (2) Let E be a locally convex topological vector space over $\mathbb{R}$ or $\mathbb{C}$. If $U \subset E$, the polar of U is denoted by

$$U^\circ = \{f \in E^*: |f(x)| \leq 1, \forall x \in U\}, \quad (6)$$

see [20].

- (3) Suppose G is a Hausdorff topological group, and (x_α) is a net in G defined on a directed set D . (x_α) is left Cauchy when the following happens: for every neighborhood B of the identity e of G , there exists an $\alpha_0 \in D$ such that

$$\begin{aligned} \forall \beta, \gamma \in D: \beta > \alpha_0, \\ \gamma > \alpha_0 \implies x_\beta^{-1} x_\gamma \in B, \end{aligned} \quad (7)$$

see [20].

Suppose that Q is a family of seminorms on a locally convex space X which determines the topology of X and $q \in Q$. Let Y be a subset of X and

$$q_Y^*(f) = \sup\{|f(y)|: y \in Y, q(y) \leq 1\}. \quad (8)$$

Also,

$$q^*(f) = \sup\{|f(x)|: x \in X, q(x) \leq 1\}, \quad (9)$$

for every linear functional f on X . Notice that if, for each $x \in X$, $q(x) \neq 0$, and $f \in X^*$, then

$$|\langle x, f \rangle| \leq q(x)q^*(f), \quad (10)$$

for more details, see [12].

We will make use of the following theorems.

Theorem 1 (see [12]). Suppose that Q is a family of seminorms on a real locally convex space X which determines the topology of X and $q \in Q$ is a continuous seminorm and Y is a vector subspace of X such that

$$Y \cap \{x \in X: q(x) = 0\} = \{0\}. \quad (11)$$

Let f be a real linear functional on Y such that $q_Y^*(f) < \infty$. Then, there exists a continuous linear functional h on X that extends f such that

$$q_Y^*(f) = q^*(h). \quad (12)$$

Theorem 2 (see [12]). Suppose that Q is a family of seminorms on a real locally convex space X which determines the topology of X and $q \in Q$ a nonzero continuous seminorm. Let x_0 be a point in X . Then, there exists a continuous linear functional f on X such that $q^*(f) = 1$ and $f(x_0) = q(x_0)$.

Next, we bring the following known results for easy reference.

Theorem 3 (see Proposition 2.5.2 in [1]). Let X be a topological space and $f: X \rightarrow (-\infty, +\infty]$ a function. Then, the following statements are equivalent: (a) f is lower semicontinuous; (b) for each $\alpha \in \mathbb{R}$, the level set $\{x \in X: f(x) \leq \alpha\}$ is closed.

Theorem 4 (see Theorem 3.26 (Banach-Alaoglu) in [20]). Suppose that X is a locally convex space and U is a neighborhood of zero in X . Then, U° , the polar set of U , is weak* compact.

For general topological spaces, we know the useful property as follows.

Theorem 5 (see Theorem 3.3.18 in [21]). For a topological space (X, τ) , the following are equivalent:

- (i) X is compact
- (ii) Each net in X has a convergent subnet

3. Main Result

First, we define our notation of the q -duality mapping as follows.

Suppose that Q is a family of seminorms on a real locally convex space X which determines the topology of X , $q \in Q$ is a continuous seminorm, and X^* is the dual space of X . A multivalued mapping $J_q: X \rightarrow 2^{X^*}$, defined by

$$J_q x = \{j \in X^*: \langle x, j \rangle = q(x)^2 = q^*(j)^2\}, \quad (13)$$

is called q -duality mapping. Obviously, $J_q(-x) = -J_q(x)$ and $J_q x \neq \emptyset$. Indeed, let $x \in X$; if $q(x) = 0$, then $j = 0 \in J_q x$, and if $q(x) \neq 0$, by applying Theorem 2, there exists a linear functional $f \in X^*$ such that $q^*(f) = 1$ and $\langle x, f \rangle = q(x)$. Putting $j = q(x)f$, then

$$\begin{aligned}
\langle x, j \rangle &= \langle x, q(x)f \rangle = q(x)\langle x, f \rangle = q(x)^2, \\
q^*(j) &= \sup\{|j(y)|: y \in X, q(y) \leq 1\} \\
&= \sup\{|q(x)f(y)|: y \in X, q(y) \leq 1\} \\
&= q(x)\sup\{|f(y)|: y \in X, q(y) \leq 1\} \\
&= q(x)q^*(f) = q(x),
\end{aligned} \tag{14}$$

and then, $j = q(x)f \in J_q x$.

Now, we present the following result that extends Proposition 4.1.1 in [1].

Lemma 1. *Let Q be a family of seminorms on a Hausdorff and complete locally convex space E which determines the topology of E , $\phi_q: E \rightarrow (-\infty, \infty]$ a bounded below and lower semicontinuous function, for each $q \in Q$, and $\{x_n\}$ a sequence in E such that $q(x_n - x_{n+1}) \leq \phi_q(x_n) - \phi_q(x_{n+1})$, for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $q \in Q$. Then, $\{x_n\}$ converges to a point $v \in E$, and for each $q \in Q$,*

$$q(x_n - v) \leq \phi_q(x_n) - \phi_q(v), \tag{15}$$

for all $n \in \mathbb{N}_0$.

Proof. Since $q(x_n - x_{n+1}) \leq \phi_q(x_n) - \phi_q(x_{n+1})$, for each $n \in \mathbb{N}_0$ and $q \in Q$, then $\{\phi_q(x_n)\}$ is a decreasing sequence for each $q \in Q$. Moreover,

$$\begin{aligned}
\sum_{n=0}^m q(x_n - x_{n+1}) &= q(x_0 - x_1) + q(x_1 - x_2) + \cdots + q(x_m - x_{m+1}) \\
&\leq \phi_q(x_0) - \phi_q(x_{m+1}) \leq \phi_q(x_0) - \inf_{n \in \mathbb{N}_0} \phi_q(x_n),
\end{aligned} \tag{16}$$

for $m \in \mathbb{N}_0$. Letting $m \rightarrow \infty$, we have

$$\sum_{n=0}^{\infty} q(x_n - x_{n+1}) < \infty, \tag{17}$$

and then, $\lim_n q(x_n - x_{n+1}) = 0$, for each $q \in Q$. This implies that $\{x_n\}$ is a left Cauchy sequence in E . Since E is Hausdorff and complete, there exists a unique $v \in E$ such that $\lim_{n \rightarrow \infty} x_n = v$. Let $m, n \in \mathbb{N}_0$ with $m > n$. Then,

$$\begin{aligned}
q(x_n - x_m) &\leq \sum_{i=n}^{m-1} q(x_i - x_{i+1}) \\
&\leq \phi_q(x_n) - \phi_q(x_m),
\end{aligned} \tag{18}$$

for each $q \in Q$. Since ϕ_q is lower semicontinuous and $q \in Q$ is continuous (Theorem 1.4 in [19]), letting $m \rightarrow \infty$, we conclude that

$$q(x_n - v) \leq \phi_q(x_n) - \lim_{m \rightarrow \infty} \phi_q(x_m) \leq \phi_q(x_n) - \phi_q(v), \tag{19}$$

for all $n \in \mathbb{N}_0$ and $q \in Q$. $\square$

Now, we state an extension of Banach contraction principle to locally convex spaces, and we call it Banach Q-contraction principle.

Theorem 6 (Banach Q-contraction principle). *Let Q be a family of seminorms on a separated and complete locally convex space E which determines the topology of E and $T: E \rightarrow E$ a Q-contraction mapping with Lipschitz constant $k \in (0, 1)$. Then,*

- (a) T has a unique fixed point $v \in E$.
- (b) For arbitrary $x_0 \in E$, the Picard iteration process defined by

$$x_{n+1} = T(x_n), \quad n \in \mathbb{N}_0, \tag{20}$$

converges to v .

- (c) $q(x_n - v) \leq (k^n / (1 - k))q(x_0 - x_1)$, for all $n \in \mathbb{N}_0$ and $q \in Q$.

Proof

- (a) For each $q \in Q$, let $\phi_q: E \rightarrow \mathbb{R}^+$ be a function defined by

$$\phi_q(x) = \frac{q(x - Tx)}{1 - k}, \tag{21}$$

for each $x \in E$. Since q is continuous, ϕ_q is also a continuous function. Also, T is a Q-contraction mapping; then,

$$q(Tx - T^2x) \leq kq(x - Tx), \tag{22}$$

for each $x \in E$ and $q \in Q$, which conclude that

$$q(x - Tx) - kq(x - Tx) \leq q(x - Tx) - q(Tx - T^2x). \tag{23}$$

Hence,

$$\begin{aligned}
q(x - Tx) &\leq \frac{1}{1 - k} [q(x - Tx) - q(Tx - T^2x)] \\
&\leq \phi_q(x) - \phi_q(Tx),
\end{aligned} \tag{24}$$

for each $q \in Q$. Let x be an arbitrary and fixed element in E . We define the sequence x_n in E by $x_n = T^n x$ ($n \in \mathbb{N}$). By using (24), we have

$$q(x_n - x_{n+1}) \leq \phi_q(x_n) - \phi_q(x_{n+1}), \tag{25}$$

for each $n \in \mathbb{N}$ and $q \in Q$. Since E is Hausdorff, by applying Lemma 1, there exists an element $v \in E$ such that

$$\begin{aligned}
\lim_{n \rightarrow \infty} x_n &= v, \\
q(x_n - v) &\leq \phi_q(x_n),
\end{aligned} \tag{26}$$

for each $n \in \mathbb{N}_0$ and $q \in Q$. Notice that T is continuous (page 3 in [19]) and $x_{n+1} = Tx_n$; therefore, $v = Tv$. Let $z \in E$ be another fixed point for T and $q(v - z) > 0$ for some $q \in Q$. Then,

$$0 < q(v - z) = q(Tv - Tz) \leq kq(v - z) < q(v - z), \tag{27}$$

which is a contradiction. Therefore, $q(v - z) = 0$, for each $q \in Q$. Since E is separated, we have $v = z$. Hence, T has unique fixed point $v \in E$.

(b) This assertion follows from part (a).

(c) By applying (22), we have $\phi_q(x_n) \leq k^n \phi_q(x_0)$, for each $q \in Q$. This implies from (26) that $q(x_n - v) \leq k^n \phi_q(x_0)$, for each $q \in Q$. $\square$

Next, we prove a generalization of Proposition 2.4.7 in [1] as follows.

Lemma 2. *Let E be a locally convex space. Then, for $x, y \in E$ and for each $q \in Q$ with $q(x) \neq 0$, the following statements are equivalent:*

- (a) $q(x) \leq q(x + ty)$, for all $t > 0$ with $q(x + ty) \neq 0$
- (b) There exists $j_q \in J_q x$ such that $\langle y, j_q \rangle \geq 0$

Proof

(a) $\implies$ (b). Let $t > 0$ and $f_t \in J_q(x + ty)$, and we define $g_t = f_t / q^*(f_t)$. It is clear that $q^*(g_t) = 1$ and $g_t \in (1/q^*(f_t))J_q(x + ty)$; then,

$$\begin{aligned} q(x) \leq q(x + ty) &= \frac{\langle x + ty, f_t \rangle}{q^*(f_t)} \\ &= \langle x + ty, g_t \rangle = \langle x, g_t \rangle + t \langle y, g_t \rangle \\ &\leq q(x) + t \langle y, g_t \rangle. \end{aligned} \quad (28)$$

Putting $U = \{x \in E : q(x) \leq 1\}$, it is clear that

$$\{g_t\}_t \subset U^\circ = \{f \in E^* : |f(x)| \leq 1, \forall x \in U\}, \quad (29)$$

from Theorem 4, U° is weak* compact; then, by applying Theorem 5, the net $\{g_t\}$ has a limit point $g \in E^*$ such that $q^*(g) \leq 1$. From the above equations $q(x) \leq \langle x + ty, g_t \rangle$, let $y = 0$; by limiting, we have $q(x) \leq \langle x, g \rangle$. Also, since $t > 0$, by using (28) and limiting, we have $\langle y, g \rangle \geq 0$. Therefore,

$$q(x) \leq \langle x, g \rangle \leq q(x)q^*(g) \leq q(x), \quad (30)$$

which conclude that $\langle x, g \rangle = q(x)$ and $q^*(g) = 1$. Set $j_q = gq(x)$; then, $j_q(x) = g(x)q(x) = q(x)^2$ and

$$\begin{aligned} q^*(j_q) &= \sup \left\{ |j_q(z)| : z \in X, q(z) \leq 1 \right\} \\ &= \sup \{ |q(x)g(z)| : z \in X, q(z) \leq 1 \} \\ &= q(x) \sup \{ |g(z)| : z \in X, q(z) \leq 1 \} \\ &= q(x)q^*(g) = q(x). \end{aligned} \quad (31)$$

Hence, $j_q \in J_q x$ and $\langle y, j_q \rangle = \langle y, gq(x) \rangle = q(x) \langle y, g \rangle \geq 0$.

(b) $\implies$ (a). Suppose, for $x, y \in X$ and $q \in Q$ with $q(x) \neq 0$, there exists $j_q \in J_q x$ such that $\langle y, j_q \rangle \geq 0$. We know that

$$|\langle x + ty, j_q \rangle| \leq q(x + ty)q^*(j_q) = q(x + ty)q(x). \quad (32)$$

Hence, for $t > 0$ with $q(x + ty) \neq 0$, we have

$$\begin{aligned} q(x)^2 &= \langle x, j_q \rangle \leq \langle x, j_q \rangle + \langle ty, j_q \rangle \\ &= \langle x + ty, j_q \rangle \leq q(x + ty)q(x), \end{aligned} \quad (33)$$

which implies that $q(x) \leq q(x + ty)$. This completes the proof. $\square$

The next theorem is significant in the sequel, and it extends Proposition 2.10.21 in [1].

Theorem 7. *Let E be a separated locally convex space, C a nonempty convex subset of E , and D a nonempty subset of C . Also, let $J_q : E \longrightarrow E^*$ be a single valued, for every $q \in Q$ and P a retraction of C onto D . Then, for each $q \in Q$, the following statements are equivalent:*

- (a) $\langle x - Px, J_q(y - Px) \rangle \leq 0$, ($x \in C, y \in D$)
- (b) P is sunny Q -nonexpansive

Proof

(a) $\implies$ (b). First, we show that P is sunny. For $x \in C$ and $t > 0$, set $x_t = Px + t(x - Px)$. Since C is convex, $x_t \in C$, for each $t \in (0, 1]$. Hence, from (a), we have

$$\begin{aligned} \langle x - Px, J_q(Px - Px_t) \rangle &\geq 0, \\ \langle x_t - Px_t, J_q(Px_t - Px) \rangle &\geq 0. \end{aligned} \quad (34)$$

Because $x_t - Px = t(x - Px)$ and $\langle t(x - Px), J_q(Px - Px_t) \rangle \geq 0$, we have

$$\langle x_t - Px, J_q(Px - Px_t) \rangle \geq 0. \quad (35)$$

Combining (34) and (35), we have

$$\begin{aligned} q(Px - Px_t)^2 &= \langle Px - x_t + x_t - Px_t, J_q(Px - Px_t) \rangle \\ &= -\langle x_t - Px, J_q(Px - Px_t) \rangle \\ &\quad + \langle x_t - Px_t, J_q(Px - Px_t) \rangle \leq 0. \end{aligned} \quad (36)$$

Then, $q(Px - Px_t) = 0$, for each $q \in Q$. Thus, from the fact that E is separated, $Px = Px_t$. This means that P is sunny. Now, we show that P is Q -nonexpansive. For $x, z \in C$ and $q \in Q$, we have from (a) that $\langle x - Px, J_q(Px - Px) \rangle \geq 0$ and $\langle z - Px, J_q(Pz - Px) \rangle \geq 0$. Hence,

$$\begin{aligned}
\langle x - z - (Px - Pz), J_q(Px - Pz) \rangle &\geq 0, \\
\langle x - z, J_q(Px - Pz) \rangle &= \langle x - Px + Px - Pz + Pz - z, J_q(Px - Pz) \rangle \\
&= \langle x - Px, J_q(Px - Pz) \rangle - \langle z - Pz, J_q(Px - Pz) \rangle \\
&\quad + \langle Px - Pz, J_q(Px - Pz) \rangle \geq \langle Px - Pz, J_q(Px - Pz) \rangle \\
&= q^2(Px - Pz).
\end{aligned} \tag{37}$$

(notice that $J_q(Px - Pz)$ is single valued). On the contrary,

$$\begin{aligned}
\langle x - z, J_q(Px - Pz) \rangle &\leq q(x - z)q^*(J_q(Px - Pz)) \\
&= q(x - z)q(Px - Pz).
\end{aligned} \tag{38}$$

Therefore, be applying (37) and (38), we have

$$q(Px - Pz) \leq q(x - z), \tag{39}$$

it follows that P is Q -nonexpansive.

(b) $\implies$ (a). Conversely, suppose that P is the sunny Q -nonexpansive retraction from C onto D , $x \in C$ and $y \in D$. Then, $Px \in D$ and $Py = y$. Since C is convex, $Px + t(x - Px) \in C$, for $0 < t \leq 1$. Also, P is sunny; then,

$$Px = P(Px + t(x - Px)), \tag{40}$$

and therefore,

$$\begin{aligned}
q(y - Px) &= q(Py - Px) = q(Py - P(Px + t(x - Px))) \\
&\leq q(y - (Px + t(x - Px))) \\
&= q(y - Px + t(Px - x)).
\end{aligned} \tag{41}$$

Hence, from Lemma 2, there exists $j_q \in J_q(y - Px)$ such that

$$\langle Px - x, j_q(y - Px) \rangle \geq 0, \quad x \in C, y \in D. \tag{42}$$

Since $J_q: E \longrightarrow E^*$ is single valued, we can say that

$$\langle x - Px, J_q(y - Px) \rangle \leq 0, \quad x \in C, y \in D. \tag{43}$$

□

Next, as a direct consequence of Theorem 6.5.3 in [18], we have the following result. To apply Theorem 6.5.3 in [18] in Theorems 8, 9, 11, and 12 and Corollary 1, it is considered that $\inf q(x) > 0$, for each $q \in Q$, when x is in K (28) and (34), C_0 , $\text{Fix}(T)$, and C , respectively.

Theorem 8. Let Q be a family of seminorms on a locally convex space E which determines the topology of E and K be a convex subset of E , $x \in E$ and $x_0 \in K$. Then, for each $q \in Q$, the following are equivalent:

$$(1) \quad q(x_0 - x) = \inf\{q(x - y): y \in K\}.$$

(2) There exists an $L_q \in E^*$ such that $L_q(y) \leq q(y)$, for each $y \in E$, $q^*(L_q) = 1$, and

$$\inf\{L_q(y - x): y \in K\} = q(x_0 - x). \tag{44}$$

(3) There exists an $L_q \in E^*$ such that $L_q(y) \leq q(y)$, for each $y \in E$, $q^*(L_q) = 1$, and

$$\begin{aligned}
\inf\{L_q(y): y \in K\} &= L_q(x_0), \\
L_q(x_0 - x) &= q(x_0 - x).
\end{aligned} \tag{45}$$

The next theorem is similar to Corollary 6.5.5 in [18].

Theorem 9. Let Q be a family of seminorms on a real locally convex space E which determines the topology of E and K be a convex subset of E , $x \in E$ and $x_0 \in K$. Then, for each $q \in Q$, the following statements are equivalent:

- (1) $q(x_0 - x) = \inf\{q(x - y): y \in K\}$
- (2) There exists an $f_q \in J_q(x - x_0)$ such that $f_q(x_0 - y) \geq 0$, for every $y \in K$

Proof

- (1) $(1) \implies (2)$. Let $q \in Q$ and $q(x_0 - x) = \inf\{q(x - y): y \in K\}$. Then, from Theorem 8, there exists an $L_q \in E^*$ such that $L_q(y) \leq q(y)$, for each $y \in E$, $q^*(L_q) = 1$, and

$$\inf\{L_q(y - x): y \in K\} = q(x_0 - x). \tag{46}$$

Set $f_q = -q(x_0 - x)L_q$. It is easy to see that $L_q(x_0 - x) = q(x_0 - x)$; therefore,

$$\begin{aligned}
f_q(x - x_0) &= -q(x_0 - x)L_q(x - x_0) \\
&= q^2(x_0 - x) = (q^*(f_q))^2.
\end{aligned} \tag{47}$$

Then, $f_q \in J_q(x - x_0)$. Hence, from (46) and (47), we have

$$\begin{aligned}
f_q(x_0 - y) &= f_q(x_0 - x + x - y) = -q^2(x_0 - x) \\
&\quad - q(x_0 - x)L_q(x - y) \\
&= -q^2(x_0 - x) + q(x_0 - x)L(y - x) \\
&\geq -q^2(x_0 - x) + q^2(x_0 - x) = 0.
\end{aligned} \tag{48}$$

(2) $\implies$ (1). Since $f_q \in J_q(x - x_0)$, then

$$f_q(x - x_0) = q^2(x_0 - x) = (q^*(f_q))^2. \quad (49)$$

Also, $f_q(x_0 - y) \geq 0$, for every $y \in K$; therefore,

$$\begin{aligned} q^2(x_0 - x) &= f_q(x - x_0) = f_q(x - y + y - x_0) \\ &\leq q(x - y)q^*(f_q) + f_q(y - x_0) \\ &\leq q(x - y)q^*(f_q) = q(x - y)q(x_0 - x), \end{aligned} \quad (50)$$

for every $y \in K$; hence, we have

$$q(x_0 - x) \leq q(x - y), \quad (51)$$

for all $y \in K$. This means that

$$q(x_0 - x) = \inf\{q(x - y) : y \in K\}. \quad (52)$$

Corollary 1. Let Q be a family of seminorms on a locally convex space E which determines the topology of E , C a nonempty closed convex subset of E , $C_0 \subseteq C$, and P be a sunny Q -nonexpansive retraction of C onto C_0 . Let $J_q : E \rightarrow E^*$ be single-valued duality mapping for each $q \in Q$; then,

$$\langle x - Px, J_q(y - Px) \rangle \leq 0, \quad (53)$$

for each $x \in C$ and $y \in C_0$.

Proof. Let $x \in C$ and $y \in C_0$. Set $x_t = Px + t(x - Px)$, for each $0 \leq t \leq 1$; then, $x_t \in C$ and

$$q(Px - y) = q(Px_t - Py) \leq q(x_t - y), \quad (54)$$

for each $q \in Q$. Now, from Theorem 9, there exists $f_q \in J_q(y - Px)$ such that $\langle Px - x_t, f_q \rangle = f_q(Px - x_t) \geq 0$. Since J_q is single valued, we can say that

$$\langle Px - x_t, J_q(y - Px) \rangle \geq 0, \quad (55)$$

and since $Px - x_t = t(Px - x)$, then

$$\langle x - Px, J_q(y - Px) \rangle \leq 0. \quad (56)$$

We now extend the result of Theorem 2.6.10 in [1] as follows.

Theorem 10. Let E be a locally convex space and $J_q : E \rightarrow E^*$ be a single-valued duality mapping. Then, J_q is continuous from τ_Q to weak* topology, for each $q \in Q$.

Proof. Let $q \in Q$ be fixed and arbitrary, $\{x_\alpha\}_{\alpha \in I} \subset E$, and $x \in E$ such that $x_\alpha \rightarrow x$ in τ_Q , and we prove that $J_q x_\alpha \rightarrow J_q x$ in the weak* topology. First, assume that $q(x) = 0$; then, $J_q x = 0$ (from (13)). It is clear that $q^*(J_q x_\alpha) = q(x_\alpha)$, also $q \in Q$ is continuous Theorem 1.4 in [19]; therefore,

$$\lim_\alpha q^*(J_q x_\alpha) = \lim_\alpha q(x_\alpha) = 0, \quad (57)$$

and hence, $\lim_\alpha \langle y, J_q x_\alpha \rangle = 0$, for every $y \in E$ that $q(y) \leq 1$ (from (9)). On the contrary, if $q(y) > 1$, then $q(y/q(y)) = 1$; therefore, $\lim_\alpha \langle y/q(y), J_q x_\alpha \rangle = 0$. So, we conclude that

$\lim_\alpha \langle y, J_q x_\alpha \rangle = 0$, for every $y \in E$. Hence, $J_q x_\alpha \rightarrow 0$ in the weak* topology. Second, assume that $q(x) \neq 0$. Since

$$J_q x_\alpha = \{j \in X^* : \langle x_\alpha, j \rangle = q(x_\alpha)^2 = q^*(j)^2\}, \quad (58)$$

and J_q is single-valued set $f_\alpha^q = J_q x_\alpha$. Then,

$$\langle x_\alpha, f_\alpha^q \rangle = q(x_\alpha)q^*(f_\alpha^q), \quad q(x_\alpha) = q^*(f_\alpha^q). \quad (59)$$

Since $\{x_\alpha\}_\alpha$ is bounded with respect to τ_Q , then there exists $N_q > 0$ such that $q(x_\alpha) \leq N_q$, for each $\alpha \in I$. Now, it is shown that $\{f_\alpha^q\}_{\alpha \in I}$ is bounded in E^* .

One can see that E^* is a locally convex space and the family of seminorms $\{p_y : y \in E\}$ defines the weak* topology on it, where $p_y(x^*) = |\langle y, x^* \rangle|$, for each $x^* \in E$ (Chapter 5, Definition 1.1 in [22]). Hence,

$$p_y(f_\alpha^q) = |\langle y, f_\alpha^q \rangle| \leq q(y)q^*(f_\alpha^q) = q(y)q(x_\alpha) \leq q(y)N_q, \quad (60)$$

when $q(y) \neq 0$. Also, if $q(y) = 0$,

$$\begin{aligned} p_y(f_\alpha^q) &= |\langle y, f_\alpha^q \rangle| \leq \sup\{|\langle y, f_\alpha^q \rangle| : y \in E, q(y) \leq 1\} \\ &= q^*(f_\alpha^q) = q(x_\alpha) \leq N_q, \end{aligned} \quad (61)$$

for each $\alpha \in I$. By applying (60) and (61), there exists $M_{q,y} > 0$ related to each $y \in E$ and $q \in Q$, such that $p_y(f_\alpha^q) \leq M_{q,y}$, for each $\alpha \in I$. This means that $\{f_\alpha^q\}_{\alpha \in I}$ is bounded in the weak* topology of E^* . Putting $U_q = \{z \in E : q(z) < 1\}$, we have $\{(1/M_{q,y})f_\alpha^q\}_{\alpha \in I} \subset U_q^\circ$, for each $y \in E$, where U_q° is the polar of U_q and it is weak* compact (Theorem 4). Then, from Theorem 5, there exists a subnet $\{(1/M_{q,y})f_{\alpha_\beta}^q\}_\beta$ of $\{(1/M_{q,y})f_\alpha^q\}_\alpha$ such that $(1/M_{q,y})f_{\alpha_\beta}^q \rightarrow f \in U_q^\circ$ in the weak* topology, equivalently

$$\langle x, M_{q,y}f - f_{\alpha_\beta}^q \rangle \rightarrow 0. \quad (62)$$

We know that the function q^* on E^* is lower semicontinuous in the weak* topology. Indeed, if $\{g_\beta\}_\beta \subset \{f \in E^* : q^*(f) \leq \alpha\}$ such that $g_\beta \rightarrow g$ in the weak* topology, then

$$|\langle y, g_\beta - g \rangle| = p_y(g_\beta - g) \rightarrow 0, \quad (63)$$

for each $y \in E$; therefore, $\langle y, g_\beta \rangle \rightarrow \langle y, g \rangle$ and $q^*(g) \leq \alpha$, i.e.,

$$g \in \{f \in E^* : q^*(f) \leq \alpha\}. \quad (64)$$

So, by applying Theorem 3, q^* is lower semicontinuous in the weak* topology on E^* . Hence, we have

$$q^*(f) \leq \liminf_\beta q^*\left(\frac{1}{M_{q,y}}f_{\alpha_\beta}^q\right) = \frac{1}{M_{q,y}} \liminf_\beta q(x_{\alpha_\beta}) = \frac{1}{M_{q,y}} q(x). \quad (65)$$

Since $x_{\alpha_\beta} \rightarrow x$ in τ_Q and $x_{\alpha_\beta} \rightarrow x$ in weak topology, then

$$\langle x - x_{\alpha_\beta}, f_{\alpha_\beta}^q \rangle \rightarrow 0. \quad (66)$$

Therefore, using (62) and (66) implies that

$$\begin{aligned} \left| \langle x, M_{q,y}f \rangle - q(x_{\alpha_\beta})^2 \right| &= \left| \langle x, M_{q,y}f \rangle - \langle x_{\alpha_\beta}, f_{\alpha_\beta}^q \rangle \right| \\ &\leq \left| \langle x, M_{q,y}f - f_{\alpha_\beta}^q \rangle \right| + \left| \langle x - x_{\alpha_\beta}, f_{\alpha_\beta}^q \rangle \right| \longrightarrow 0. \end{aligned} \quad (67)$$

On the contrary, $q(x)^2(x_{\alpha_\beta}) \longrightarrow q(x)^2(x)$. Thus,

$$\langle x, M_{q,y}f \rangle = q(x)^2. \quad (68)$$

Since $q(x) \neq 0$, using (65), we have

$$q(x)^2 = \langle x, M_{q,y}f \rangle \leq q^*(M_{q,y}f)q(x) \leq q(x)^2, \quad (69)$$

and therefore,

$$\begin{aligned} \langle x, M_{q,y}f \rangle &= q(x)^2, \\ q(x) &= q^*(M_{q,y}f). \end{aligned} \quad (70)$$

This means that $M_{q,y}f = J_q x$. Finally, since $f_{\alpha}^q = J_q x_{\alpha}$, from (62), we conclude that $J_q x_{\alpha} \longrightarrow J_q x$. This completes the proof. $\square$

In the next theorem, we prove an existence theorem of a sunny Q-nonexpansive retract.

Theorem 11. *Let Q be a family of seminorms on a real separated and complete locally convex space E which determines the topology of E , C a nonempty closed convex and bounded subset of E such that every sequence in C has a convergent subsequence, and $T: C \longrightarrow C$ be a Q -nonexpansive mapping such that $\text{Fix}(T) \neq \emptyset$, the fixed points set of T . Let $J_q: E \longrightarrow E^*$ be single valued for every $q \in Q$. Then, $\text{Fix}(T)$ is a sunny Q -nonexpansive retract of C and the sunny Q -nonexpansive retraction of C onto $\text{Fix}(T)$ is unique.*

Proof. Let $x \in C$ be fixed; then, there exists a sequence $\{z_n\}$ in C such that

$$z_n = \frac{1}{n}x + \left(1 - \frac{1}{n}\right)Tz_n, \quad n \in \mathbb{N}. \quad (71)$$

For this means, we define the function $K_n: C \longrightarrow C$, with

$$K_n(t) = \frac{1}{n}x + \left(1 - \frac{1}{n}\right)T(t), \quad (72)$$

for each $n \in \mathbb{N}$ and $t \in C$. It is clear that K_n is Q -nonexpansive because

$$q(K_n(t_1) - K_n(t_2)) \leq \left(1 - \frac{1}{n}\right)q(T(t_1) - T(t_2)) \leq q(t_1 - t_2), \quad (73)$$

for every $t_1, t_2 \in C$. Then, by applying Theorem 6, there exists a unique point z_n in C such that $K_n(z_n) = z_n$, for each $n \in \mathbb{N}$; therefore,

$$z_n = \frac{1}{n}x + \left(1 - \frac{1}{n}\right)Tz_n, \quad n \in \mathbb{N}. \quad (74)$$

Since C is bounded in τ_Q , there exists $M_q > 0$ such that $q(y) \leq M_q$, for each $y \in C$ and $q \in Q$. Also,

$$q(Tz_n - z_n) = \frac{1}{n}q(Tz_n - x) \leq \frac{2}{n}M_q, \quad (75)$$

and then,

$$\lim_n q(Tz_n - z_n) = 0, \quad (76)$$

for each $q \in Q$. Next, we show that the sequence $\{z_n\}$ converges to an element of $\text{Fix}(T)$. In the other words, we show that the limit set of $\{z_n\}$ (denoted by $\mathfrak{S}\{z_n\}$) is a subset of $\text{Fix}(T)$. First, for each $z \in \text{Fix}(T)$ and $n \in \mathbb{N}$, we have

$$\langle z_n - x, J_q(z_n - z) \rangle \leq 0 \quad (77)$$

because $Tz_n - x = n(Tz_n - z_n)$ and

$$\begin{aligned} \langle Tz_n - Tz, J_q(z_n - z) \rangle &\leq q(Tz_n - Tz)q^*(J_q(z_n - z)) \\ &\leq q(z_n - z)q(z_n - z) = q^2(z_n - z). \end{aligned} \quad (78)$$

Also, since J_q is single valued, from definition (13), we have

$$\langle z - z_n, J_q(z_n - z) \rangle = -\langle z_n - z, J_q(z_n - z) \rangle = -q^2(z_n - z). \quad (79)$$

Therefore,

$$\begin{aligned} \langle z_n - x, J_q(z_n - z) \rangle &= \langle z_n - Tz_n + Tz_n - x, J_q(z_n - z) \rangle \\ &= \langle z_n - Tz_n + n(Tz_n - z_n), J_q(z_n - z) \rangle \\ &= (n-1)\langle Tz_n - z_n, J_q(z_n - z) \rangle \\ &= (n-1)\langle Tz_n - Tz, J_q(z_n - z) \rangle \\ &\quad + (n-1)\langle z - z_n, J_q(z_n - z) \rangle \\ &\leq (n-1)(q(Tz_n - Tz)q(z_n - z) - q(z_n - z)^2) \\ &\leq (n-1)(q(z_n - z)^2 - q(z_n - z)^2) = 0. \end{aligned} \quad (80)$$

From our assumption, $\{z_n\}$ has a subsequence converges to a point in C . Let $\{z_{n_i}\}$ and $\{z_{n_j}\}$ be subsequences of $\{z_n\}$ such that $\{z_{n_i}\}$ and $\{z_{n_j}\}$ converge to y and z , respectively. Then, (76) implies that

$$\begin{aligned} q(y - Ty) &\leq q(y - z_{n_i}) + q(z_{n_i} - Tz_{n_i}) + q(Tz_{n_i} - Ty) \\ &\leq 2q(y - z_{n_i}) + q(z_{n_i} - Tz_{n_i}) \longrightarrow 0, \end{aligned} \quad (81)$$

for each $q \in Q$. Since E is separated, $Ty = y$; hence, $y \in \text{Fix}(T)$ and similarly $z \in \text{Fix}(T)$.

Now, $z_{n_i} \longrightarrow y$ and $z \in \text{Fix}(T)$, and we claim that

$$\langle y - x, J_q(y - z) \rangle = \lim_{i \rightarrow \infty} \langle z_{n_i} - x, J_q(z_{n_i} - z) \rangle. \quad (82)$$

$$\begin{aligned} & \left| \langle z_{n_i} - x, J_q(z_{n_i} - z) \rangle - \langle y - x, J_q(y - z) \rangle \right| \\ &= \left| \langle z_{n_i} - x, J_q(z_{n_i} - z) \rangle - \langle y - x, J_q(z_{n_i} - z) \rangle + \langle y - x, J_q(z_{n_i} - z) \rangle - \langle y - x, J_q(y - z) \rangle \right| \\ &\leq \left| \langle z_{n_i} - y, J_q(z_{n_i} - z) \rangle \right| + \left| \langle y - x, J_q(z_{n_i} - z) - J_q(y - z) \rangle \right| \\ &\leq q(z_{n_i} - y)q(z_{n_i} - z) + \left| \langle y - x, J_q(z_{n_i} - z) - J_q(y - z) \rangle \right| \\ &\leq 2q(z_{n_i} - y)M_q + \left| \langle y - x, J_q(z_{n_i} - z) - J_q(y - z) \rangle \right| \longrightarrow 0, \end{aligned} \quad (83)$$

and hence, by using (77),

$$\langle y - x, J_q(y - z) \rangle = \lim_{i \rightarrow \infty} \langle z_{n_i} - x, J_q(z_{n_i} - z) \rangle \leq 0. \quad (84)$$

Similarly, $\langle z - x, J_q(z - y) \rangle \leq 0$ since $J_q(y - z) = -J_q(z - y)$; then,

$$\begin{aligned} q(y - z)^2 &= \langle y - z, J_q(y - z) \rangle \\ &= \langle y - x, J_q(y - z) \rangle + \langle x - z, J_q(y - z) \rangle \\ &= \langle y - x, J_q(y - z) \rangle + \langle z - x, J_q(z - y) \rangle \leq 0, \end{aligned} \quad (85)$$

that is, $q(y - z) = 0$, for each $q \in Q$, and since E is separated, $y = z$. Thus, $\{z_n\}$ converges to an element of $\text{Fix}(T)$. Therefore, a mapping $P: C \longrightarrow C$ can be defined by $Px = \lim_n z_n$. Then, since $z_n \longrightarrow Px$, by using (84) and for each $z \in \text{Fix}(T)$,

$$\begin{aligned} \langle x - Px, J_q(z - Px) \rangle &= \langle Px - x, J_q(Px - z) \rangle, \\ &= \lim_n \langle z_n - x, J_q(z_n - z) \rangle \leq 0. \end{aligned} \quad (86) \quad (87)$$

It follows from Theorem 7 that P is a sunny Q -nonexpansive retraction of C onto $\text{Fix}(T)$. Let R be another sunny Q -nonexpansive retraction of C onto $\text{Fix}(T)$. Then, from Corollary 1,

$$\langle x - Rx, J_q(z - Rx) \rangle \leq 0, \quad (88)$$

for each $x \in C$ and $z \in \text{Fix}(T)$. Putting $z = Rx$ in (86) and $z = Px$ in (88), we have $\langle x - Px, J_q(Rx - Px) \rangle \leq 0$ and $\langle x - Rx, J_q(Px - Rx) \rangle \leq 0$; then,

$$q^2(Rx - Px) = \langle Rx - Px, J_q(Rx - Px) \rangle \leq 0, \quad (89)$$

for each $q \in Q$. Since E is separated, $Rx = Px$. This completes the proof. $\square$

Proposition 1. Let Q be a family of seminorms on a separated locally convex space E which determines the topology of E . Then,

$$q(x)^2 - q(y)^2 \geq 2\langle x - y, j \rangle, \quad (90)$$

Indeed, from the fact that J_q is single valued and continuous from τ_Q to weak* topology, we have

for all $x, y \in E$ and $j \in J_q y$ such that $q(y) \neq 0$.

Proof. Let $j \in J_q x$, $x \in E$. Then,

$$\begin{aligned} q(y)^2 - q(x)^2 - 2\langle y - x, j \rangle &= q(x)^2 + q(y)^2 - 2\langle y, j \rangle \\ &\geq q(x)^2 + q(y)^2 - 2q(x)q(y) \\ &\geq q(q(x) - q(y))^2 \geq 0. \end{aligned} \quad (91)$$

$\square$

Theorem 12. Let Q be a family of seminorms on a real separated and complete locally convex space E which determines the topology of E , C a nonempty closed convex and bounded subset of E such that every sequence in C has a convergent subsequence, and $T: C \longrightarrow C$ a Q -nonexpansive mapping such that $\text{Fix}(T) \neq \emptyset$. Let $J_q: E \longrightarrow E^*$ be single valued for each $q \in Q$ and f be a Q -contraction on C ; also, ε_n is a sequence in $(0, 1)$ such that $\lim_n \varepsilon_n = 0$. Then, there exists a unique $x \in C$ and sunny Q -nonexpansive retraction P of C onto $\text{Fix}(T)$ such that the following sequence $\{z_n\}$, generated by

$$z_n = \varepsilon_n f z_n + (1 - \varepsilon_n) T z_n, \quad n \in \mathbb{N}, \quad (92)$$

converges to Px .

Proof. Since f is a Q -contraction, there exists $0 \leq \beta < 1$ such that

$$q(f(x) - f(y)) \leq \beta q(x - y), \quad (93)$$

for each $x, y \in E$ and $q \in Q$. We divide the proof into five steps.

Step 1: the existence of z_n which satisfies (92).

This follows immediately from the fact that, for every $n \in \mathbb{N}$, the mapping $N_n: C \longrightarrow C$, given by

$$N_n x := \varepsilon_n f x + (1 - \varepsilon_n) T x, \quad x \in C, \quad (94)$$

is a Q -contraction. To see this, put $\beta_n = (1 + \varepsilon_n)(\beta - 1)$, $0 \leq \beta_n < 1$, $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
q(N_n x - N_n y) &\leq \varepsilon_n q(fx - fy) + (1 - \varepsilon_n)q(Tx - Ty) \\
&\leq \varepsilon_n \beta q(x - y) + (1 - \varepsilon_n)q(x - y) \\
&= (1 + \varepsilon_n(\beta - 1))q(x - y) \\
&= \beta_n q(x - y) < q(x - y).
\end{aligned} \tag{95}$$

Therefore, by Theorem 6, there exists a unique point $z_n \in C$ such that $N_n z_n = z_n$, that is,

$$z_n = \varepsilon_n f z_n + (1 - \varepsilon_n) T z_n. \tag{96}$$

Step 2: $\lim_n q(T z_n - z_n) = 0$, for each $q \in Q$.

Since C is bounded in τ_Q , there exists $M_q > 0$ such that $q(y) \leq M_q$, for each $y \in C$ and $q \in Q$.

Then,

$$\begin{aligned}
q(T z_n - z_n) &= \varepsilon_n q(T z_n - f z_n) \leq 2M_q \varepsilon_n, \\
\lim_n q(T z_n - z_n) &= 0,
\end{aligned} \tag{97}$$

for each $q \in Q$.

Step 3: $\mathfrak{S}(\{z_n\}) \subset \text{Fix}(T)$, where $\mathfrak{S}(\{z_n\})$ denotes the set of τ_Q -limit points of subsequences of $\{z_n\}$.

Let $z \in \mathfrak{S}(\{z_n\})$ and $\{z_{n_k}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_k} \rightarrow z$. By using (97), we have

$$\begin{aligned}
q(Tz - z) &\leq q(Tz - Tz_{n_k}) + q(Tz_{n_k} - z_{n_k}) + q(z_{n_k} - z) \\
&\leq 2q(z_{n_k} - z) + q(Tz_{n_k} - z_{n_k}) \rightarrow 0,
\end{aligned} \tag{98}$$

for each $q \in Q$. Hence, $q(Tz - z) = 0$, for each $q \in Q$, and since E is separated, we conclude that $z \in \text{Fix}(T)$.

Step 4: there exists a unique sunny Q -nonexpansive retraction P of C onto $\text{Fix}(T)$ and $x \in C$ such that

$$K := \limsup_n \langle x - Px, J_q(z_n - Px) \rangle \leq 0. \tag{99}$$

We know from Theorem 10 that J_q is continuous from τ_Q to weak* topology; then, by Theorem 11, there exists a unique sunny Q -nonexpansive retraction P of C onto $\text{Fix}(T)$. Theorem 6 guarantees that fP has a unique fixed point $x \in C$. Since

$$K = \limsup_n \langle x - Px, J_q(z_n - Px) \rangle, \tag{100}$$

by the definition of $\limsup$, there exists subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$K = \lim_i \langle x - Px, J_q(z_{n_i} - Px) \rangle. \tag{101}$$

On the contrary, every sequence in C has a convergent subsequence; then, there exists subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \rightarrow z$ and $z \in C$.

By Step 3, we have $z \in \text{Fix}(T)$. From Theorem 7, we have

$$\langle x - Px, J_q(z - Px) \rangle \leq 0, \tag{102}$$

and since J_q is continuous, then

$$\lim_k \langle x - Px, J_q(z_{n_k} - Px) \rangle = \langle x - Px, J_q(z - Px) \rangle. \tag{103}$$

Therefore, by applying (101)–(103), we have $K \leq 0$.

Step 5: $\{z_n\}$ converges to Px in τ_Q .

Since $fPx = x$, thus $(f - I)Px = x - Px$. Now, from Proposition 1 and our assumption, we have

$$\begin{aligned}
&\varepsilon_n(\beta - 1)q(z_n - Px)^2 \\
&\geq [\varepsilon_n \beta q(z_n - Px) + (1 - \varepsilon_n)q(z_n - Px)]^2 - q(z_n - Px)^2 \\
&\geq [\varepsilon_n q(fz_n - f(Px)) + (1 - \varepsilon_n)q(Tz_n - Px)]^2 - q(z_n - Px)^2 \\
&\geq 2\langle \varepsilon_n(fz_n - f(Px)) + (1 - \varepsilon_n)(Tz_n - Px) \\
&\quad - (z_n - Px), J_q(z_n - Px) \rangle \\
&= -2\varepsilon_n \langle (f - I)Px, J_q(z_n - Px) \rangle \\
&= -2\varepsilon_n \langle x - Px, J_q(z_n - Px) \rangle,
\end{aligned} \tag{104}$$

for each $n \in \mathbb{N}$. Hence,

$$q(z_n - Px)^2 \leq \frac{2}{1 - \beta} \langle x - Px, J_q(z_n - Px) \rangle, \tag{105}$$

for each $q \in Q$. Since $Px \in \text{Fix}(T)$, from (99) and (105), we conclude that

$$\limsup_n q(z_n - Px)^2 \leq \frac{2}{1 - \beta} \limsup_n \langle x - Px, J_q(z_n - Px) \rangle \leq 0, \tag{106}$$

for each $q \in Q$. That is, $z_n \rightarrow Px$ in τ_Q . $\square$

4. Numerical Example

The following examples illustrate Theorem 12. The first example is in the setting of a locally convex space that is not normable and the rest in finite dimensional spaces.

Example 1. Let ε be an arbitrary positive number, $E = C([0, 1])$, the set of all continuous complex valued functions, and $\varepsilon_n = (1/n)$. For each $x \in [0, 1]$, the seminorm q on E is defined by $q_x(f) = |f(x)|$, for each $f \in E$. Note that E is a locally convex space with the topology induced by Q . It can be proved by Urysohn's Lemma that it is not normable.

In Theorem 12, let $C = N(x_0, \varepsilon, 0) = \{f: q_{x_0}(f) = |f(x_0)| < \varepsilon\}$ be convex neighborhood of 0 with $x_0 \in [0, 1]$ and $|x_0| < (\varepsilon/2)$. Suppose that I is the identity mapping on the complex numbers, and let $T, F: E \rightarrow E$ be defined by $T(f)$. The following examples illustrate Theorem 12. The first example is in the setting of a locally convex space that is not normable and the rest in finite dimensional spaces.

Example 2. Let ε be an arbitrary positive number and $E = C([0, 1])$ be the set of all continuous complex valued functions and $\varepsilon_n = (1/n)$. For each $x \in [0, 1]$, we define the seminorm q_x on E by $q_x(f) = |f(x)|$, for each $f \in C_C([0, 1])$. Note that E is a locally convex space with the topology induced by Q . It can be proved by Urysohn's Lemma which is not normable.

In Theorem 12, let $C = N(x_0, \varepsilon, 0) = \{f: q_{x_0}(f) = |f(x_0)| < \varepsilon\}$ be convex neighborhood of 0 with $x_0 \in [0, 1]$ and $|x_0| < (\varepsilon/2)$. Suppose that I is the identity mapping on the complex numbers, and let $T, F: E \rightarrow E$ be defined by $T(f) = (1/2)(f + I)$ and $F(f) = (1/2)f$, respectively. It is obvious that T is Q -nonexpansive and F is Q -contraction mapping on C . It can be verified that $P(f) = I$ is a sunny Q -nonexpansive retraction from C onto $\text{Fix}(T) = \{I\}$; then, obviously, we have the sequence $\{f_n\}_{n \in \mathbb{N}} = \{(1 - (1/n))I\}_{n \in \mathbb{N}}$ generated by (92) which converges to $PI = I$ with respect to the topology induced by Q .

Example 3. In Theorem 12, let $E = \mathbb{R}$, $\varepsilon_n = (1/n)$, and $q(x) = |x|$ be the only seminorm on $\mathbb{R}$, i.e., $Q = \{|\cdot|\}$. Let $C = [0, 2]$ and $T(x) = (1/2)x + 1$ be a Q -nonexpansive mapping and $f(x) = (1/2)x$ be a Q -contraction mapping on $[0, 2]$. It is obvious $P(x) = 2$ is a sunny Q -nonexpansive retraction from $[0, 2]$ onto $\text{Fix}(T) = \{2\}$ and the sequence $\{z_n\}_{n \in \mathbb{N}} = \{2 - (2/n)\}_{n \in \mathbb{N}}$ generated by (92) converges to $P1 = 2$ and $F(f) = (1/2)f$, respectively. It is obvious that T is Q -nonexpansive and F is Q -contraction mapping on C . It can be verified that $P(f) = I$ is a sunny Q -nonexpansive retraction from C onto $\text{Fix}(T) = \{I\}$; then, obviously, we have the sequence $\{f_n\}_{n \in \mathbb{N}} = \{(1 - (1/n))I\}_{n \in \mathbb{N}}$ generated by (92) converges to $PI = I$ with respect to the topology induced by Q .

Example 4. In Theorem 12, let $E = \mathbb{R}$, $\varepsilon_n = (1/n)$, and $q(x) = |x|$ be the only seminorm on $\mathbb{R}$, i.e., $Q = \{|\cdot|\}$. Let $C = [0, 2]$ and $T(x) = (1/2)x + 1$ be a Q -nonexpansive mapping and $f(x) = (1/2)x$ be a Q -contraction mapping on $[0, 2]$. It is obvious $P(x) = 2$ is a sunny Q -nonexpansive retraction from $[0, 2]$ onto $\text{Fix}(T) = \{2\}$ and the sequence $\{z_n\}_{n \in \mathbb{N}} = \{2 - (2/n)\}_{n \in \mathbb{N}}$, generated by (92), converges to $P1 = 2$.

Example 5. In Theorem 12, assume $E = \mathbb{R}^2$ and $\varepsilon_n = (1/n)$, and $q_1(x, y) = |x|$ and $q_2(x, y) = |y|$ are two seminorms on $\mathbb{R}^2$, i.e., $Q = \{q_1, q_2\}$. Let $C = [0, 2] \times [0, 2]$ and $T(x, y) = ((1/2)x + 1, (1/3)y + 1)$ be a Q -nonexpansive mapping and $f(x, y) = ((1/2)x, (1/3)y)$ be a Q -contraction mapping on $[0, 2] \times [0, 2]$ in Theorem 11. It is easy to check that $P(x, y) = (2, (3/2))$ is a sunny Q -nonexpansive retraction from $[0, 2] \times [0, 2]$ onto $\text{Fix}(T) = \{(2, (3/2))\}$, and the sequence $\{z_n\}_{n \in \mathbb{N}} = \{(x_n, y_n)\}_{n \in \mathbb{N}} = \{(2 - (2/n), (3/2) - (3/2n))\}_{n \in \mathbb{N}}$ generated by (92) converges to $P(1, 1) = (2, (3/2))$.

Data Availability

The data used to support the findings of the study are available within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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